BIFURCATIONS OF PERIODICS FROM HOMOCLINICS IN SINGULAR O.D.E.: APPLICATIONS TO DISCRETIZATIONS OF TRAVELLING WAVES OF P.D.E.

MICHAL FEČKAN
Department of Mathematical Analysis
Comenius University, Mlynská
dolina, 842 48 Bratislava, Slovakia
and
VASSILIS ROTHOS
Department of Mathematical Sciences
University of Loughborough
Loughborough LE11 3TU, UK

November 20, 2001

Abstract. Bifurcations of periodic solutions from homoclinic ones are investigated for certain singularly perturbed systems of autonomous ordinary differential equations in $\mathbb{R}^4$. Results are applied to discretization of travelling waves of certain p.d.e.

1. Introduction

F. Battelli [2], W. Eckhaus [3], J. M. Hammersley and G. Mazzarino [5], and C. Lazzari [8] examined the existence or nonexistence of homoclinic solutions of singular ordinary differential systems of the following type

$$\varepsilon^2 y^{(4)} + \ddot{y} - y + y^2 = 0 \,. \tag{1.1}$$

which arises in the theory of water-waves in the presence of surface tension [1].

Setting $v = y$, $u = \ddot{y} - y + y^2$, eq. (1.1) leads to [2]

$$\begin{aligned} \varepsilon^2 \ddot{u} + u &= \varepsilon^2 \left[2\dot{v}^2 - (1-2v)(u + v - v^2) \right] \,, \\ \ddot{v} &= u + v - v^2 \,. \end{aligned} \tag{1.2}$$

The present paper can be considered a direct continuation of the investigations [2, 8], on the systematic study of bifurcations of periodic solutions in more general systems of the form

$$\begin{aligned} \ddot{x} + h(x) &= f(x, \dot{x}, y, \varepsilon\dot{y}, \varepsilon) \,, \\ \varepsilon^2 \ddot{y} + y &= \varepsilon^2 g(x, \dot{x}, y, \varepsilon\dot{y}, \varepsilon) \,, \end{aligned} \tag{1.3}$$

AMS Subject Classification (2000) 34 C23, 34 C25, 34C37, 35 B99.
Key words and phrases. periodic solutions, singular perturbations, bifurcations, travelling waves.
M.F supported by Grant GA-MS 1/6179/01
V.R supported by EPSRC Grant No. GR/R02702/01 and LMS Grant Scheme 5.

where $\varepsilon > 0$ is a small parameter and we assume the following assumptions

(A1) $h, f, g \in C^1$, $f(x_1, x_2, 0, 0, 0) = 0$.

(A2) $f(x_1, x_2, y_1, y_2, \varepsilon)$, $g(x_1, x_2, y_1, y_2, \varepsilon)$ are even in the variables x_2 and y_2, i.e.

$$f(x_1, -x_2, y_1, -y_2, \varepsilon) = f(x_1, x_2, y_1, y_2, \varepsilon)$$
$$g(x_1, -x_2, y_1, -y_2, \varepsilon) = g(x_1, x_2, y_1, y_2, \varepsilon)\,.$$

(A3) $h(0) = 0$, $h'(0) = -a^2 < 0$ and there is a homoclinic solution ϕ of $\ddot{x} + h(x) = 0$ such that $\phi(t) = \phi(-t)$ and $\phi(t) \to 0$ as $t \to \pm\infty$.

F. Battelli has shown in [2] that bifurcation functions of homoclinic solutions of eq. (1.3) under the above assumptions are exponentially small in addition that h, f, g are analytical. W. Eckhaus [3], and J. M. Hammersley and G. Mazzarino [5] established the nonexistence of certain homoclinic solutions of (1.1).

In this paper, we study the existence of periodic solutions of (1.3) near $(\phi(t), 0)$. Substituting $y = 0, \varepsilon = 0$ into the equation (1.3), we get the equation

$$\ddot{x} + h(x) = 0\,. \tag{1.4}$$

Eq. (1.4) has a hyperbolic fixed point $(0, 0)$ with the homoclinic solution $(\phi, \dot{\phi})$ which is accumulated by periodic solutions with periods tending to infinity. We show that in spite of the fact that generally the homoclinic solution of (1.4) does not survive under the singular perturbation (1.3). The problem (1.3) has many layers of continuum periodic solutions near the solution $(\phi, 0)$: The smaller ε the more layers of continuum periodic solutions of (1.3) exist near $(\phi, 0)$ with very large periods.

In the last section, we study the (kink/antikink) travelling wave solutions in a chain of interacting particles. We prove the existence of discrete travelling waves near to the continuum limit for a large period, applying the centre manifold reduction and our Theorem.

2. Preliminary Results

We take the linearization of the equation

$$\ddot{x} + h(x) = 0 \tag{2.1}$$

along $\phi(t)$ and consider the variational equation

$$\ddot{u} + h'(\phi(t))u = z(t), \quad 0 \le t \le T \tag{2.2}$$

with the boundary value conditions

$$\dot{u}(0) = 0, \quad \dot{u}(T) = b\,. \tag{2.3}$$

Since $h'(0) = -a^2 < 0$, $a > 0$, we have $\phi(t), \dot{\phi}(t) \sim \mathrm{e}^{-at}$ as $t \to +\infty$, i.e. it holds that

$$\phi(t)/\mathrm{e}^{-at} \to c_1 \neq 0 \quad \textit{and} \quad \dot{\phi}(t)/\mathrm{e}^{-at} \to c_2 \neq 0 \quad \textit{as} \quad t \to +\infty\,.$$

The homogeneous equation (2.3) with $z = 0$ has solutions $w_i(t)$, $i = 1, 2$ such that:

- w_1 is odd, $w_1(0) = 0$, $\dot{w}_1(0) = 1$, $w_1(t), \dot{w}_1(t) \sim \mathrm{e}^{-at}$ as $t \to +\infty$,
- w_2 is even, $w_2(0) = -1$, $\dot{w}_2(0) = 0$, $w_2(t), \dot{w}_2(t) \sim \mathrm{e}^{at}$ as $t \to +\infty$.

The general solution of (2.2) has the form

$$u(t) = L_T(z,b) \equiv c_1 w_1(t) + c_2 w_2(t) + z_1(t)\,,$$
$$z_1(t) = \int_0^t \big[w_2(t)w_1(s) - w_1(t)w_2(s)\big] z(s)\,ds\,.$$

The condition (2.3) gives $c_1 = 0$ and $c_2 = -\frac{\dot z_1(T)}{\dot w_2(T)} + \frac{b}{\dot w_2(T)}$. Hence, we get

$$u(t) = b\frac{w_2(t)}{\dot w_2(T)} - \int_t^T w_2(t)w_1(s)z(s)\,ds$$
$$+\frac{\dot w_1(T)}{\dot w_2(T)}\int_0^T w_2(t)w_2(s)z(s)\,ds - \int_0^t w_1(t)w_2(s)z(s)\,ds\,.$$

Then

$$\dot u(t) = b\frac{\dot w_2(t)}{\dot w_2(T)} - \int_t^T \dot w_2(t)w_1(s)z(s)\,ds$$
$$+\frac{\dot w_1(T)}{\dot w_2(T)}\int_0^T \dot w_2(t)w_2(s)z(s)\,ds - \int_0^t \dot w_1(t)w_2(s)z(s)\,ds\,.$$

By using the above asymptotic properties of w_1 and w_2 for t, s large, we get

$$w_2(t)/\dot w_2(T) \sim \mathrm{e}^{a(t-T)}, \quad w_2(t)w_1(s) \sim \mathrm{e}^{a(t-s)}\,,$$
$$\frac{\dot w_1(T)}{\dot w_2(T)} w_2(t)w_2(s) \sim \mathrm{e}^{a(-2T+t+s)}, \quad w_1(t)w_2(s) \sim \mathrm{e}^{a(s-t)}\,,$$
$$\dot w_2(t)w_1(s) \sim \mathrm{e}^{a(t-s)}, \quad \dot w_1(t)w_2(s) \sim \mathrm{e}^{a(s-t)}\,,$$
$$\dot w_2(t)/\dot w_2(T) \sim \mathrm{e}^{a(t-T)}, \quad \frac{\dot w_1(T)}{\dot w_2(T)}\dot w_2(t)w_2(s) \sim \mathrm{e}^{a(-2T+t+s)}.$$

These estimates imply the existence of a constant $c > 0$ such that

$$||u|| + ||\dot u|| \le c(|b| + ||z||)\,, \tag{2.4}$$

where $||x|| = \max_{[0,T]} |x(t)|$. Summarizing, we get the next result.

Lemma 2.1. *Problem* (2.2 − 3) *has a unique solution* $u = L_T(z,b)$ *satisfying* (2.4).

Now, we consider the problem

$$\varepsilon^2 \ddot v + v = \varepsilon z(t), \quad 0 \le t \le T\,, \qquad \dot v(0) = \dot v(T) = 0\,. \tag{2.5}$$

We can immediately see that the solution of eq. (2.5) is given by

$$v(t) = L_{\varepsilon,T}(z,b) \equiv \frac{1}{\sin(T/\varepsilon)} \int_0^T \cos\frac{T-s}{\varepsilon} z(s)\,ds \cos(t/\varepsilon) + \int_0^t \sin\frac{t-s}{\varepsilon} z(s)\,ds\,.$$

If T satisfies

$$\Big|\frac{T}{\varepsilon} - 2k\pi \pm \frac{\pi}{2}\Big| \le \pi/4, \quad k \in \mathbb{N} \tag{2.6}$$

then $1 \ge |\sin(T/\varepsilon)| \ge \sqrt{2}/2$, and we obtain the estimate

$$||v|| + ||\varepsilon\dot v|| \le 2T||z||(\sqrt{2}+1)\,. \tag{2.7}$$

Summarizing, we get the next result.

Lemma 2.2. *If condition* (2.6) *holds then problem* (2.5) *has a unique solution* $v = L_{\varepsilon,T}(z)$ *satisfying* (2.7).

3. Periodic Solutions

We are looking for periodic solutions of (1.3) near $(\phi, 0)$. For this reason, we make the change of variables

$$x(t) = \phi(t) + \varepsilon^{1/4}u(t), \quad y(t) = \sqrt{\varepsilon}v(t)\,,$$

and we get

$$\begin{aligned} \varepsilon^2\ddot v + v &= \varepsilon^{3/2} g(\phi + \varepsilon^{1/4}u, \dot\phi + \varepsilon^{1/4}\dot u, \sqrt{\varepsilon}v, \varepsilon^{3/2}\dot v, \varepsilon) \\ \ddot u + h'(\phi)u &= -\frac{1}{\varepsilon^{1/4}}\Big\{h(\phi+\varepsilon^{1/4}u) - h(\phi) - h'(\phi)\varepsilon^{1/4}u\Big\} \\ &\quad + \frac{1}{\varepsilon^{1/4}} f(\phi + \varepsilon^{1/4}u, \dot\phi + \varepsilon^{1/4}\dot u, \sqrt{\varepsilon}v, \varepsilon^{3/2}\dot v, \varepsilon)\,. \end{aligned} \tag{3.1}$$

We are looking for solutions of (1.3) satisfying $\dot x(0) = \dot x(T) = 0$, $\dot y(0) = \dot y(T) = 0$. This gives

$$\begin{aligned} \dot u(0) = 0, &\quad \dot u(T) = -\dot\phi(T)/\varepsilon^{1/4} \\ \dot v(0) = 0, &\quad \dot v(T) = 0\,. \end{aligned} \tag{3.2}$$

The next results deals with this problem.

Theorem 3.1. *For any* $k_0 \in \mathbb{N}$ *there is an* $\varepsilon_0 > 0$ *such that for any* $0 < \varepsilon < \varepsilon_0$ *and* $T = \varepsilon\big(2k[1/\varepsilon^{3/2}]\pi + \tau\big)$ *with* $k \in \mathbb{N}$, $k \le k_0$, $\tau \in [\pi/4, 3\pi/4] \cup [5\pi/4, 7\pi/4]$, *system* (1.3) *has a* $2T$*-periodic solution near* $(\phi(t), 0)$, $-T \le t \le T$. *Here* $[1/\varepsilon^{3/2}]$ *is the integer part of* $1/\varepsilon^{3/2}$.

Proof. First of all, we show the existence of a solution of (3.1-2). We take the Banach space $X_\varepsilon = C^1([0,T],\mathbb{R})^2$ with the norm $|||(v,u)||| = ||u|| + ||\dot u|| + ||v|| + ||\varepsilon\dot v||$. By using Lemmata 2.1 and 2.2, we rewrite (3.1-2) in the form

$$\begin{aligned} v &= L_{\varepsilon,T}\sqrt{\varepsilon} g(\phi + \varepsilon^{1/4}u, \dot\phi + \varepsilon^{1/4}\dot u, \sqrt{\varepsilon}v, \varepsilon^{3/2}\dot v, \varepsilon) \\ u &= L_T\Big(-\frac{1}{\varepsilon^{1/4}}\Big\{h(\phi+\varepsilon^{1/4}u) - h(\phi) - h'(\phi)\varepsilon^{1/4}u\Big\} \\ &\quad + \frac{1}{\varepsilon^{1/4}} f(\phi + \varepsilon^{1/4}u, \dot\phi + \varepsilon^{1/4}\dot u, \sqrt{\varepsilon}v, \varepsilon^{3/2}\dot v, \varepsilon), -\dot\phi(T)/\varepsilon^{1/4}\Big)\,. \end{aligned} \tag{3.3}$$

We consider (3.3) as a fixed point problem in X_ε. Now we fix $k_0 \in \mathbb{N}$ and take $T = \varepsilon\big(2k[1/\varepsilon^{3/2}]\pi + \tau\big)$ with $k \in \mathbb{N}$, $k \le k_0$, $\tau \in [\pi/4, 3\pi/4] \cup [5\pi/4, 7\pi/4]$ and $[1/\varepsilon^{3/2}]$ is the integer part of $1/\varepsilon^{3/2}$. We take a sufficiently large ball $B_K = \big\{(v,u) \in X_\varepsilon \mid |||(v,u)||| \le K\big\}$ in X_ε. Since $T \sim 1/\sqrt{\varepsilon}$ and $\dot\phi(T) \sim \mathrm{e}^{-aT}$, we get $\dot\phi(T)/\varepsilon^{1/4} \sim \mathrm{e}^{-a/\sqrt{\varepsilon}}/\varepsilon^{1/4} = O(\varepsilon)$. From the C^1-smoothness of f, g, h, it follows the existence of a constant $M > 0$ such that for any $K > 0$ there is an $\varepsilon_0 > 0$ such that for any $0 < \varepsilon \le \varepsilon_0$, $(v,u) \in B_K$, it holds that

$$\begin{aligned} &\big|g(\phi + \varepsilon^{1/4}u, \dot\phi + \varepsilon^{1/4}\dot u, \sqrt{\varepsilon}v, \varepsilon^{3/2}\dot v, \varepsilon)\big| \le M\,, \\ &\Big|\frac{1}{\varepsilon^{1/4}}\Big\{h(\phi + \varepsilon^{1/4}u) - h(\phi) - h'(\phi)\varepsilon^{1/4}u\Big\}\Big| \le 1\,, \\ &\Big|\frac{1}{\varepsilon^{1/4}}f(\phi + \varepsilon^{1/4}u, \dot\phi + \varepsilon^{1/4}\dot u, \sqrt{\varepsilon}v, \varepsilon^{3/2}\dot v, \varepsilon)\Big| \le 1\,. \end{aligned}$$

For any $(u,v) \in B_K$, $0 < \varepsilon \le \varepsilon_0$, we put

$$\begin{aligned} u_1 &= L_T\Big(-\frac{1}{\varepsilon^{1/4}}\Big\{h(\phi + \varepsilon^{1/4}u) - h(\phi) - h'(\phi)\varepsilon^{1/4}u\Big\} \\ &\quad + \frac{1}{\varepsilon^{1/4}}f(\phi + \varepsilon^{1/4}u, \dot\phi + \varepsilon^{1/4}\dot u, \sqrt{\varepsilon}v, \varepsilon^{3/2}\dot v, \varepsilon), -\dot\phi(T)/\varepsilon^{1/4}\Big)\,, \\ v_1 &= L_{\varepsilon,T}\sqrt{\varepsilon}g(\phi + \varepsilon^{1/4}u, \dot\phi + \varepsilon^{1/4}\dot u, \sqrt{\varepsilon}v, \varepsilon^{3/2}\dot v, \varepsilon)\,. \end{aligned}$$

Then estimate (2.4) implies

$$||u_1|| + ||\dot u_1|| \le c\big(2 + O(\varepsilon)\big)\,,$$

and estimates (2.6-7) imply

$$||v_1|| + ||\varepsilon\dot v_1|| \le 2\sqrt{\varepsilon}TM(1+\sqrt{2}) \le 2M(\sqrt{2}+1)\big(2\pi k_0 + \frac{7\pi}{4}\varepsilon^{3/2}\big)\,.$$

By choosing K such that $2c + 4M(\sqrt{2}+1)\pi k_0 < K$ and $\varepsilon_0 > 0$ sufficiently small, we see that B_K is mapped to itself by the compact operator generated by the right-hand side of (3.3). We apply the Schauder fixed point theorem to get a solution of (3.2-3) in X_ε, i.e. there is a solution of (1.3) satisfying $\dot x(0) = \dot x(T) = 0$, $\dot y(0) = \dot y(T) = 0$. Since h, f, g are C^1, we get the uniqueness of the Cauchy problem for (1.3). Then the evenness of f, g in x_2, y_2 and the conditions $\dot x(0) = 0$, $\dot y(0) = 0$ imply that x, y are even functions. This implies $x(-T) = x(T)$, $\dot x(-T) = -\dot x(T) = 0$, $y(-T) = x(T)$, $\dot y(-T) = -\dot y(T) = 0$. Consequently, the uniqueness of the Cauchy problem for (1.3) implies that x and y are $2T$-periodic. This completes the proof of Theorem. □

Remark 3.2. We note that the derived T-periodic solutions $x_{T,\varepsilon}$ and $y_{T,\varepsilon}$ in Theorem 3.1 of equation (1.3) are near to $(\phi(t), 0)$ in the sense that $x_{T,\varepsilon}(t) - \phi(t) = O(\varepsilon^{1/4})$, $\dot x_{T,\varepsilon}(t) - \dot\phi(t) = O(\varepsilon^{1/4})$, $y_{T,\varepsilon}(t) = O(\sqrt{\varepsilon})$, $\varepsilon\dot y_{T,\varepsilon}(t) = O(\sqrt{\varepsilon})$ uniformly for $-T \le t \le T$ and T satisfying the assumption of Theorem 3.1 for a fixed k_0. This estimates are consistent with the form of (1.3).

Remark 3.3. If $h \in C^2$ then, we can apply the uniform contraction mapping principle to (3.3) for getting a unique $2T$-periodic and even solution of (1.3) near $(\phi(t), 0)$ for $-T \le t \le T$.

4. Travelling Waves in Spatially Discretized P.D.E.

In order to demonstarte how our general theory can be applied to a particular physical problem we consider a chain of coupled particles subjected to an external on-site potential with two or more degenerate minima. It is known that in some limiting cases, this chain supports moving topological solitons as monotonically increasing (kink) or decreasing (antikink) travelling wave solutions. By travelling waves, we mean waves of stationary profile moving with constant velocity. In general, while propagating along the chain, the kink radiates small–amplitude waves and finally stops because of the existence of the so–called Peierls-Nabarro potential barrier. The topological soliton solutions appear to be well–defined travelling waves of stationary profile while they are moving on an appropriate oscillating background.

The dimensionless Hamiltonian H of such system can be written as:

$$H=\sum_{n\in\mathbb{Z}}\Big(\frac{1}{2}p_n^2+\frac{1}{2\epsilon^2}(u_{n+1}-u_n)^2-\mathcal{F}(u_n)\Big)\,, \tag{4.1}$$

where $p_n=\dot u_n$ is the conjugate momentum of the n-th particle in the chain, u_n is the displacement of the n–th particle from its equilibrium position. The Hamiltonian H gives the discrete nonlinear Klein-Gordon eqn:

$$\ddot u_n-\frac{1}{\varepsilon^2}(u_{n+1}-2u_n+u_{n-1})-h(u_n)=0\,, \tag{4.2}$$

where $h(u_n)=\mathcal{F}'(u_n), n\in\mathbb{Z}$.

Equation (4.2) can be considered as a spatial discretization of the p.d.e.

$$u_{tt}-u_{xx}-h(u)=0\,, \tag{4.3}$$

where $h\in C^1$ has the property (A3) (cf. section 1) and admits travelling waves solutions

$$u(x,t)=\phi\Big(\frac{x-\nu t}{\sqrt{1-\nu^2}}\Big),\quad 0<\nu<1$$

We consider for equation (4.2) travelling wave solutions of stationary profile moving with constant velocity ν/ε. For this type of solutions, one can write

$$u_n(t)=V\Big(n-\frac{\nu}{\varepsilon}t\Big)\equiv V(z),\quad z=n-\frac{\nu}{\varepsilon}t,\quad 0<\nu<1$$

The equation (4.2) is reduced to the following functional differential equation:

$$\nu^2V''(z)-V(z+1)+2V(z)-V(z-1)-\varepsilon^2h(V(z))=0\,. \tag{4.4}$$

By using the method of center manifolds like in [7] and Theorem 3.1, we study the existence of solutions of (4.4) near ϕ and the relationship between travelling wave solutions of (4.2) and (4.3) for $\varepsilon>0$ small.

We introduce a new variable $v\in[-1,1]$ and functions $X(t,v)=x(t+v)$. The notation $U(t)(v)=\big(x(t),\xi(t),X(t,v)\big)$ indicates our intention to construct V as a map from $\mathbb{R}$ into some function space living on the v-interval $[-1,1]$. Equation (4.4) can be written as follows

$$U_t = LU + \frac{\varepsilon^2}{\nu^2} M(U)\,, \qquad (4.5)$$
$$U(t,v) = \big(x(t), \xi(t), X(t,v)\big)\,, \quad v \in [-1,1]\,,$$

where

$$L = \begin{pmatrix} 0 & 1 & 0 \\ -\frac{2}{\nu^2} & 0 & \frac{1}{\nu^2}\delta^1 + \frac{1}{\nu^2}\delta^{-1} \\ 0 & 0 & \partial_v \end{pmatrix}$$
$$M(u) = \big(0, h(x), 0\big)\,, \quad \delta^{\pm} X(v) = X(\pm 1)\,.$$

We introduce the Banach spaces $\mathbb{H}$ and $\mathbb{D}$ for $U(v) = \big(x, \xi, X(v)\big)$

$$\mathbb{H} = \mathbb{R}^2 \times C[-1,1]\,,$$
$$\mathbb{D} = \big\{ U \in \mathbb{R}^2 \times C^1[-1,1] \mid X(0) = x \big\}$$

with the usual maximum norms. Then $L \in \mathcal{L}(\mathbb{D}, \mathbb{H})$ and $M \in C^\infty(\mathbb{D}, \mathbb{D})$. We consider (4.5) on $\mathbb{D}$. The spectrum $\sigma(L)$ is given by the resolvent equation

$$(\lambda I - L)U = F\,, \quad F \in \mathbb{H},\ \lambda \in \mathbb{C},\ U \in \mathbb{D}\,.$$

The resolvent equation is solvable if and only if $N(\lambda) = 0$ for

$$N(\lambda) = \lambda^2 + \frac{2}{\nu^2}(1 - \cosh\lambda)\,.$$

Clearly $\sigma(L)$ is invariant under $\lambda \to \bar{\lambda}$ and $\lambda \to -\lambda$. The central part $\sigma_0(L) = \sigma(L) \cap \imath\mathbb{R}$ is determined by the equation

$$q^2 + \frac{2}{\nu^2}(\cos q - 1) = 0\,, \quad q \in \mathbb{R},. \qquad (4.6)$$

The basic properties of $\sigma(L)$ are given in Lemma 1 of [6] and we refer the reader to that paper for more details. In this paper, we assume that $\nu_1 < \nu < 1$ where $\nu = \nu_1$ is the first value from the left of 1 for which the equations

$$\lambda^2 + \frac{2}{\nu^2}(\cos\lambda - 1) = 0, \quad \lambda - \frac{1}{\nu^2}\sin\lambda = 0$$

have a common nonzero solution $\lambda \neq 0$. Then equation $N(\imath q) = 0$ has the double root 0 and simple roots $\pm q$. Hence we have $\sigma_0(L) = \{0, \pm\imath q\}$.

The linear operator on the 4th-dimensional central subspace $\mathbb{H}_c$ has the form

$$L_c = L/\mathbb{H}_c = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q \\ 0 & 0 & -q & 0 \end{pmatrix}\,.$$

in the basis $(\xi_1, \xi_2, \xi_3, \xi_4)$ defined by

$$\xi_1 = (1,0,1)\,, \quad \xi_2 = (0,1,v)$$
$$\xi_3 = (1,0,\cos qv)\,, \quad \xi_4 = (0,q,\sin qv)\,.$$

and which satisfies $L\xi_1 = 0$, $L\xi_2 = \xi_1$, $L\xi_3 = -q\xi_4$, $L\xi_4 = q\xi_3$.

The projection $P_c\colon \mathbb{H} \to \mathbb{H}_c$ is given by

$$P_c(U) = P_1(U)\xi_1 + P_2(U)\xi_2 + P_3(U)\xi_3 + P_4(U)\xi_4\,,$$

where

$$P_1(U) = \frac{\nu^2}{\nu^2-1}x - \frac{1}{\nu^2-1}\int_0^1 (1-s)\big[X(s)+X(-s)\big]\,ds\,,$$

$$P_2(U) = \frac{\nu^2}{\nu^2-1}\xi + \frac{1}{\nu^2-1}\int_0^1 \big[X(-s)-X(s)\big]\,ds\,,$$

$$P_3(U) = \Big(\nu^2 qx - \int_0^1 \sin q(1-s)\big[X(s)+X(-s)\big]\,ds\Big)/\big(q\nu^2-\sin q\big)\,,$$

$$P_4(U) = \Big(\nu^2 \xi + \int_0^1 \cos q(1-s)\big[X(-s)-X(s)\big]\,ds\Big)/\big(q\nu^2-\sin q\big)\,.$$

These projections are derived as the residues of the inverse $(\lambda I - L)^{-1}$ at $\lambda = 0, \pm\imath q$, respectively, of the resolvent operator [7].

For any bounded ball Ω of $\mathbb{H}_c$ centered at 0, we can apply the procedure of a center manifold method [7] to get for ε small the reduced equation of (4.5) over Ω given by

$$\begin{aligned}\dot u_c &= L_c u_c + \frac{\varepsilon^2}{\nu^2}P_c M\big(u_c + \varepsilon^2\Phi_\varepsilon(u_c)\big)\\ &= L_c u_c + \frac{\varepsilon^2}{\nu^2}P_c(M(u_c)) + O(\varepsilon^4)\,,\end{aligned} \tag{4.7}$$

where $u_c = u_1\xi_1 + u_2\xi_2 + u_3\xi_3 + u_4\xi_4$ and Φ_ε is the graph map of the center manifold. Then (4.7) has the form

$$\dot u_1 = u_2, \quad \dot u_2 = \frac{\varepsilon^2}{\nu^2-1}\tilde h(u_1,u_2,u_3,u_4)$$
$$\dot u_3 = qu_4, \quad \dot u_4 = -qu_3 + \frac{\varepsilon^2}{q\nu^2-\sin q}\tilde h(u_1,u_2,u_3,u_4)\,,$$

for a C^1-function $\tilde h$. Let us consider

$$x(t) = x_1(t) = u_1(t/\varepsilon)\,, \quad x_2(t) = u_2(t/\varepsilon)/\varepsilon\,,$$
$$y(t) = y_1(t) = u_3(t/\varepsilon)\,, \quad y_2(t) = u_4(t/\varepsilon)\,.$$

Then (4.7) has the form

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = \frac{1}{\nu^2 - 1}\tilde{h}(x_1, \varepsilon x_2, y_1, y_2)$$
$$\dot{y}_1 = \frac{q}{\varepsilon} y_2, \quad \dot{y}_2 = -\frac{q}{\varepsilon} y_1 + \frac{\varepsilon}{q\nu^2 - \sin q}\tilde{h}(x_1, \varepsilon x_2, y_1, y_2)\,,$$

which gives

$$\ddot{x} = \frac{1}{1-\nu^2} f(x, \varepsilon\dot{x}, y, \varepsilon\dot{y}/q, \varepsilon)\,, \tag{4.8}$$
$$\varepsilon^2\ddot{y} + q^2 y = \frac{\varepsilon^2 q}{\sin q - \nu^2 q} f(x, \varepsilon\dot{x}, y, \varepsilon\dot{y}/q, \varepsilon)\,,$$

where $f(x_1, x_2, y_1, y_2, \varepsilon) = -h(x_1 + y_1) + O(\varepsilon^2)$. For $\varepsilon = 0$ and $y = 0$, the limit equation of (4.8) has the form $(1-\nu^2)\ddot{x} + h(x) = 0$ which is precisely the travelling wave equation of (4.3). Equation $(1-\nu^2)\ddot{x} + h(x) = 0$ has a homoclinic solution $x(t) = \phi(t/\sqrt{1-\nu^2})$.

We consider the symmetry $S(U) = \big(x, -\xi, X(-v)\big)$ on $\mathbb{H}$. Then (4.5) is reversible with respect to S, i.e. $S \circ L = -L \circ S$, $M \circ S = -S \circ M$. Moreover, we have $P_c \circ S = S \circ P_c$ and $S\xi_1 = \xi_1$, $S\xi_2 = -\xi_2$, $S\xi_3 = \xi_3$, $S\xi_4 = -\xi_4$. Hence

$$S_c = S/\mathbb{H}_c = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Since S_c is unitary, the map Φ_ε can be chosen [6] in such a way that $S \circ \Phi_\varepsilon = \Phi_\varepsilon \circ S_c$. This implies

$$L_c S_c u_c + \frac{\varepsilon^2}{\nu^2} P_c M\big(S_c u_c + \varepsilon^2 \Phi_\varepsilon(S_c u_c)\big) = -S_c\Big(L_c u_c + \frac{\varepsilon^2}{\nu^2} P_c M\big(u_c + \varepsilon^2\Phi_\varepsilon(u_c)\big)\Big)\,.$$

Hence (4.7) is reversible with respect to S_c. Moreover, S_c has in the coordinates $(x_1, , x_2, y_1, y_2)$ on $\mathbb{H}_c$ the form $S_c(x_1, x_2, y_1, y_2) = (x_1, -x_2, y_1, -y_2)$. Consequently we get that assumptions (1)-(3) are satisfied for (4.8). The results of the papers [2, 3, 5, 8] can not be applied since (4.8) is not analytical even if h is analytical. But we can apply our result Theorem 3.1.

Hence (4.8) has T-periodic solutions $x_{T,\varepsilon}(t)$ and $y_{T,\varepsilon}(t)$ near $(\phi(t/\sqrt{1-\nu^2}), 0)$, $-T \le t \le T$ for any T satisfying the assumption of Theorem 3.1. They have the form

$$u_c^{T,\varepsilon}(t) = x_{T,\varepsilon}(\varepsilon t)\xi_1 + \varepsilon\dot{x}_{T,\varepsilon}(\varepsilon t)\xi_2 + y_{T,\varepsilon}(\varepsilon t)\xi_3 + \varepsilon(\dot{y}_{T,\varepsilon}(\varepsilon t)/q)\xi_4$$

in (4.7). Remark 3.2 gives that $u_c^{T,\varepsilon}(t)$ lies in a large ball Ω. Furthermore, we have $U(t,\cdot) = u_c(t) + \varepsilon^2\Phi_\varepsilon(u_c(t)) = u_c(t) + O(\varepsilon^2)$ for (4.5) on the center manifold considered in (4.7). We also note that the $x(t)$-coordinate of $U(t,v)$ in (4.5) satisfies (4.4). Consequently, if $x^{T,\varepsilon}(\varepsilon t)$ is the x-coordinate of $u_c^{T,\varepsilon}(t) + \varepsilon^2\Phi_\varepsilon(u_c^{T,\varepsilon}(t))$, then the travelling wave solution of (4.2) corresponding to $x_{T,\varepsilon}(t)$, $y_{T,\varepsilon}(t)$ has the form

$$u_n^{T,\varepsilon}(t) = x^{T,\varepsilon}\Big(\varepsilon\Big(n - \frac{\nu}{\varepsilon}t\Big)\Big) = x^{T,\varepsilon}(\varepsilon n - \nu t) =$$
$$x_{T,\varepsilon}(\varepsilon n - \nu t) + y_{T,\varepsilon}(\varepsilon n - \nu t) + O(\varepsilon^2)\,.$$

$u_n^{T,\varepsilon}(t)$ is T/ν-periodic in t with the velocity ν. Then Remark 3.2 gives

$$u_n^{T,\varepsilon}(t) = \phi\Big(\frac{\varepsilon n - \nu t}{\sqrt{1-\nu^2}}\Big) + O(\varepsilon^{1/4})$$

uniformly for $-T \le t \le T$ and T satisfying the assumption of Theorem 3.1 for a fixed k_0.

Finally, we note that we get (4.2) from (4.3) by putting

$$u_n(t) = u(\varepsilon n, t)\,,$$
$$u_{xx}(\varepsilon n, t) \sim \frac{u(\varepsilon(n+1),t) - 2u(\varepsilon n, t) + u(\varepsilon(n-1),t)}{\varepsilon^2}\,.$$

Summarizing we get the following result.

Theorem 4.1. *If $h \in C^1$ satisfies the assumption $(A3)$ then travelling wave solution $u(x,t) = \phi\Big(\frac{x-\nu t}{\sqrt{1-\nu^2}}\Big)$ for $0 < \nu < 1$ of (4.3) can be approximated by periodic travelling wave solutions of (4.2) with very large periods and with the velocity ν.*

We also note that for a C^∞-smooth h, the center manifold graph Φ_ε is C^k-smooth for any fixed $k \in \mathbb{N}$, and then (4.8) is also C^k-smooth. Hence the bifurcation function of homoclinics for (4.8) is of order $O(\varepsilon^k)$. So it is flat at $\varepsilon = 0$. Since (4.8) is not analytical, we do not get further information of this flatness. Hence it seems that the center manifold method is not fruitful for detecting bounded solutions of (4.8) near $(\phi, 0)$ on $\mathbb{R}$.

Finally, the discrete sine-Gordon equation for $h(u) = -\sin u$ in (4.2) of the form

$$\ddot{u}_n = u_{n+1} - 2u_n + u_{n-1} - \Gamma^2 \sin u_n \tag{4.9}$$

has been numerically investigated by J.C. Eilbeck [4, 9]: As $\Gamma \to 0$, we get the continuum sine-Gordon equation with the supporting moving kinks of the form

$$4 \arctan\Big[\exp\Big(\Gamma \frac{x - \nu t}{\sqrt{1-\nu^2}}\Big)\Big]\,. \tag{4.10}$$

Thus it was natural for J.C. Eilbeck to seek numerically solutions of

$$\nu^2 U''(z) = U(z+1) - 2U(z) + U(z-1) - \Gamma^2 \sin U(z)\,, \tag{4.11}$$

where $U(z) = U(n - \nu t) = u_n(t)$, with the boundary conditions $U(z) \to 0 \bmod 2\pi$ as $z \to \pm\infty$. He did not find such solutions. His closest result is that the numerical solution of (4.11) near (4.10) has tails of periodic waves of small amplitude. But according to the form of (4.8), that result is consistent with our result, since the y-part of (4.8) is oscillatory with small amplitude. We note that Theorem 3.1 can not be applied to (4.9) since now the limit reduced equation is a pendulum-like equation with a heteroclinic connection, while we consider in (1.4) a homoclinic solution. We intend to study (4.9) in future paper.

References

1. C.J. AMICK & K. KIRCHGÄSSNER, *A theory of solitary water-waves in the presence of surface tension*, Arch. Ration. Mech. Anal. (1989), 1-49.
2. F. BATTELLI, *Exponentially small bifurcation functions in singular systems of O.D.E.*, Diff. Int. Equations **9** (1996), 1165-1181.
3. W. ECKHAUS, *Singular perturbations of homoclinic orbits in* $\mathbb{R}^4$, SIAM J. Math. Anal. **23** (1992), 1269-1290.
4. J. C. EILBECK & R.FLESCH, *Calculation of families of solitary waves on discrete lattices*, Physics Letters A. **149** (1990), 200-202.
5. J.M. HAMMERSLEY & G. MAZZARINO, *Computational aspects of some autonomous differential equations*, Proc. Royal Soc. London Sect. A **424** (1989), 19-37.
6. G. IOOSS, *Trawelling waves in the Fermi-Pasta Ulam lattice*, Nonlinearity **13 (3)**, 849–866.
7. G. IOOSS & K. KIRCHGÄSSNER, *Travelling waves in a chain of coupled nonlinear oscillators*, Comm. Math. Phys. **211(2)** (2000), 439-464.
8. C. LAZZARI, *Symmetries and exponentional smallness of bifurcation functions of a class of singular reversible systems*, Nonlin. Anal. Th. Meth. Appl. **33** (1998), 759-772.
9. A. SCOTT, *"Nonlinear Sciences: Emergence and Dynamics of Coherent Structures"*, Oxford Appl. Eng. Math. Nonl. Sciences, 1999.