

An existence theory for three-dimensional periodic travelling gravity-capillary water waves with bounded transverse profiles

M. D. Groves

*Department of Mathematical Sciences, Loughborough University,
Loughborough, LE11 3TU, UK*

Abstract: This article presents a rigorous existence theory for three-dimensional gravity-capillary water waves which are uniformly translating and periodic in one horizontal spatial direction x and have a nontrivial transverse profile in the other z . The hydrodynamic equations are formulated as an infinite-dimensional Hamiltonian system in which z is the time-like variable, and a centre-manifold reduction technique is applied to demonstrate that the problem is locally equivalent to a finite-dimensional Hamiltonian system of ordinary differential equations.

A family of straight lines C_1, C_2, \dots in Bond number-Froude number parameter space is identified which are associated with codimension-one bifurcation phenomena: at each point on one of these lines two real eigenvalues of the linear problem become purely imaginary by passing through zero. There are also codimension-two points: the line C_k intersects each of the lines C_{k+1}, C_{k+2}, \dots in precisely one point.

General statements concerning the existence of waves which are periodic or quasiperiodic in z are made by applying standard tools in Hamiltonian-systems theory to the reduced equations. Moreover, a critical curve in parameter space is found at which a wave with a spatially localised and exponentially decaying transverse profile bifurcates from zero. This curve is piecewise linear: it contains one line segment from each of C_1, C_2, \dots .

1 Introduction

The method of ‘spatial dynamics’ for small-amplitude travelling capillary-gravity water waves was introduced by Kirchgässner [15], who suggested writing the governing equations as an infinite-dimensional, quasilinear dynamical system in which an unbounded horizontal spatial coordinate plays the role of the time-like variable. This formulation of the hydrodynamic problem can be studied using the centre-manifold reduction theorem of Mielke [17], which states that the infinite-dimensional dynamical system is locally equivalent to a system of ordinary differential equations whose solution set can, in theory, be analysed. Kirchgässner’s method is the basis for several existence theories for two-dimensional travelling gravity-capillary water waves. Particular attention has been devoted to the existence of *solitary waves* (waves which decay to the undisturbed state of the water far up- and downstream [2,11]) and *generalised solitary waves* (disturbances which decay to small-amplitude periodic waves far up- and downstream [12]).

The ‘spatial dynamics’ technique has been considerably enhanced by the use of methods from the theory of Hamiltonian systems. Several authors have noticed that the equations for two-dimensional travelling capillary-gravity water waves can be written as a Hamiltonian system, and Mielke [18] demonstrated that the reduced system of ordinary differential equations inherits the Hamiltonian structure. In contrast to the Hamiltonian formulation of the *time-dependent* water-wave problem discovered by Zakharov [21], the Hamiltonian formulation of the problem for travelling waves involves only *local* operators and is readily placed upon a secure functional-analytic foundation [8]. The Hamiltonian nature of the reduction process is a crucial ingredient in recent results which demonstrate the existence of infinite families of multi-crested solitary-wave solutions to the gravity-capillary water-wave problem [5,4].

Hamiltonian spatial dynamics methods have recently been extended to three-dimensional travelling gravity-capillary water waves by Groves and Mielke [7]. Introducing Cartesian coordinates (x, y, z) so that y points vertically upward and x points in the direction of the travelling wave, one finds that the three-dimensional hydrodynamic problem involves two independent unbounded horizontal spatial coordinates $x - ct$ and z , where c denotes the speed of the wave. Motivated by the arguments used for a model equation by Haragus-Courcelle and Illichev [9], Groves and Mielke formulated the hydrodynamic equations as a dynamical system in which $x - ct$ is the time-like variable and examined wave motions which are periodic in z . This approach represents a natural step from two to three dimensions since it includes all two-dimensional travelling waves as special cases. Groves and Mielke found several types of three-dimensional waves, in particular generalised solitary waves consisting of a three-dimensional disturbance which decays to a two-dimensional peri-

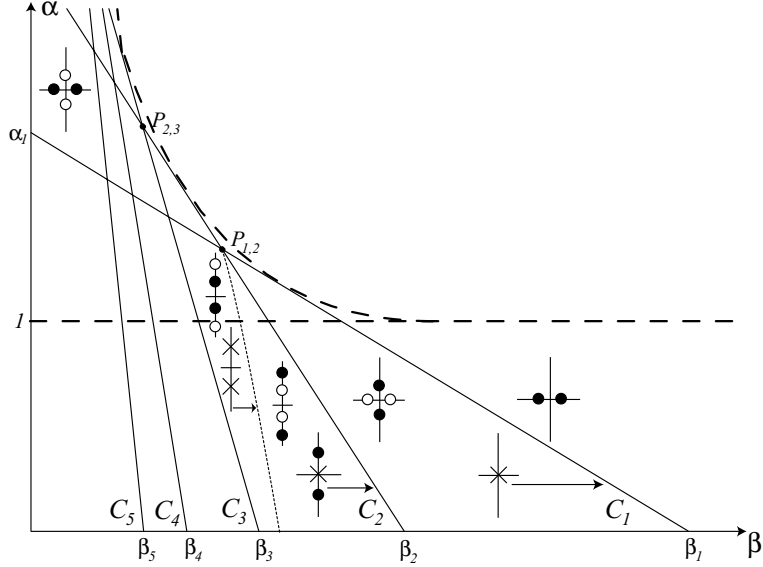


Fig. 1. The line C_k, \dots consists of points in (β, α) -parameter space at which two real eigenvalues in the k th Fourier mode become purely imaginary by passing through zero. It connects $(\beta_k, 0)$ with $(0, \alpha_k)$ and crosses C_{k+1}, C_{k+2}, \dots at the points $P_{k,k+1}, P_{k,k+2}, \dots$. The dashed curves delimit the regions of parameter space in which the corresponding two-dimensional hydrodynamic problem has zero, two or four purely imaginary eigenvalues. Eigenvalues in the first and second Fourier modes are denoted by respectively solid and hollow circles; they pass through each other on the imaginary axis at points on the dotted curve.

odic ripple far up- and downstream. In a study of another model equation for three-dimensional water waves (the KP-II equation) Haragus-Courcelle and Pego [10] recently took z as the time-like variable. This idea is applied to the hydrodynamic equations for three-dimensional travelling gravity-capillary water waves in the present article. In Section 2 below the problem is formulated as an infinite-dimensional Hamiltonian system in which z is the time-like variable; the wave motions are supposed to be periodic in $x - ct$. The centre-manifold reduction procedure is carried out in Section 3 and Section 4 presents some existence theories based upon a study of the reduced equations.

The hydrodynamic problem studied in this article depends upon two dimensionless parameters $\alpha = gh/c^2$ and $\beta = \sigma/hc^2$ where g, σ, h and c denote respectively the acceleration due to gravity, the coefficient of surface tension, the depth of the water in its undisturbed state and the velocity of the travelling wave; by assuming periodicity in the variable $x - ct$ with period hP , a third dimensionless parameter $P > 0$ is introduced. The eigenvalues λ of the linearised problem with corresponding eigenvectors in the k th Fourier mode satisfy the equation

$$(\alpha - \beta\sigma^2)\sigma \sin \sigma + \frac{4\pi^2 k^2}{P^2} \cos \sigma = 0, \quad \sigma^2 = \lambda^2 - 4\pi^2 k^2 / P^2,$$

which always has zero or two purely imaginary solutions. Figure 1 shows how the number of purely imaginary eigenvalues changes at each of a countably infinite number of straight lines C_1, C_2, \dots in (β, α) -parameter space; at each point of the line C_k two real eigenvalues in the k th Fourier mode become purely imaginary by passing through zero. The line C_k consists of those positive values of α and β which satisfy the equation

$$\left(\alpha + \frac{4\pi^2\beta k^2}{P^2}\right) \sinh\left(\frac{2\pi k}{P}\right) = \frac{2\pi k}{P} \cosh\left(\frac{2\pi k}{P}\right); \quad (1)$$

it connects a point $(\beta_k, 0)$ on the β -axis with a point $(0, \alpha_k)$ on the α -axis. Clearly

$$\beta_k = \frac{P}{2\pi k} \coth\left(\frac{2\pi k}{P}\right), \quad \alpha_k = \frac{2\pi k}{P} \coth\left(\frac{2\pi k}{P}\right),$$

so that $\beta_1 > \beta_2 > \beta_3 > \dots$ and $\alpha_1 < \alpha_2 < \alpha_3 < \dots$

Equation (1) arises in another context: it states that $\pm 2\pi ki/P$ are eigenvalues of the spatial dynamics formulation of the problem for two-dimensional travelling gravity-capillary water waves in which $x - ct$ is the time-like variable [15]. This two-dimensional hydrodynamic problem has two imaginary eigenvalues when (β, α) lies below the line $\{\alpha = 1\}$, four purely imaginary eigenvalues when (β, α) lies above the line $\{\alpha = 1\}$ and to the left of the curve

$$C = \left\{ (\beta, \alpha) = \left(\frac{-1}{2 \sinh^2(\nu)} + \frac{1}{2\nu \tanh(\nu)}, \frac{\nu^2}{2 \sinh^2(\nu)} + \frac{\nu}{2 \tanh(\nu)} \right) : \nu \geq 0 \right\}$$

and no purely imaginary eigenvalues when (β, α) lies above the line $\{\alpha = 1\}$ and to the right of the curve C . These three regions of (β, α) -parameter space are delimited by the dashed lines in Figure 1. Because C_k consists of those points (β, α) at which $\pm 2\pi ki/P$ are eigenvalues of the two-dimensional hydrodynamic problem it intersects the line segment $\{(\beta, \alpha) : 0 < \beta < 1/3, \alpha = 1\}$ in a point $(\beta_k^{(1)}, 1)$ and is tangent to the curve C at a point (β_k^C, α_k^C) . An examination of the behaviour of the eigenvalues of the two-dimensional problem on $\{\alpha = 1\}$ and C shows that $(\beta_{k+1}^{(1)}, 1)$ lies to the left of $(\beta_k^{(1)}, 1)$ and $(\beta_{k+1}^C, \alpha_{k+1}^C)$ lies further up the curve C than (β_k^C, α_k^C) . Starting from the β -axis and moving left, one concludes that the line C_k crosses $\{\alpha = 1\}$ to the left of $\beta = 1/3$, touches the curve C at a single point and then intersects each of the curves C_{k+1}, C_{k+2}, \dots in turn before arriving at the α -axis. This qualitative behaviour is shown in Figure 1. Notice that purely imaginary eigenvalues can cross on the imaginary axis; a straightforward calculation shows that the purely imaginary eigenvalues in the i th Fourier mode cross those in the j th Fourier mode at points on the curve $\{(\beta, \alpha) = (\beta_{ij}, \alpha_{ij}) : s \geq 0, \alpha > 0\}$, where

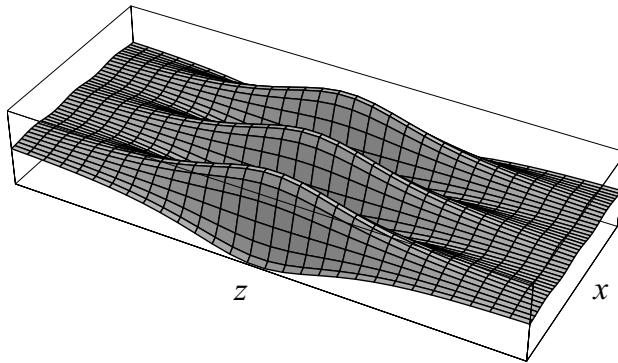


Fig. 2. A surface wave which is uniformly translating in the x -direction and spatially localised in the z -direction.

$$\beta_{ij} = \frac{j^2 \coth \gamma_j}{\gamma_j(j^2 - i^2)} - \frac{i^2 \coth \gamma_i}{\gamma_i(j^2 - i^2)},$$

$$\alpha_{ij} = -\frac{j^2 \gamma_i^2 \coth \gamma_j}{(j^2 - i^2)\gamma_j} + \frac{i^2 \gamma_i \coth \gamma_i}{j^2 - i^2} + \frac{4i^2 \pi^2 \coth \gamma_j}{P^2 \gamma_i}$$

and $\gamma_k^2 = s^2 + 4k^2\pi^2/P^2$.

The three-dimensional hydrodynamic problem is invariant under the transformation $(x - ct, \eta, \phi) \mapsto (-(x - ct), \eta, -\phi)$, and one may take advantage of this reflective symmetry to restrict to wave motions whose surface profiles are symmetric in the variable $x - ct$. Its eigenvalues become geometrically simple, and for values of (β, α) away from the lines C_1, C_2, \dots there is no zero eigenvalue. This fact, together with the Hamiltonian structure, may be used as the basis of an existence theory for waves which are periodic or quasiperiodic in z . The centre-manifold reduction procedure preserves both the Hamiltonian structure and the configuration of the purely imaginary eigenvalues. Introducing canonical coordinates for the reduced equations using Darboux's theorem, one can complete the existence theory by a direct application of the Lyapunov centre theorem or an appropriate result from the KAM theory.

The lines C_1, C_2, \dots are associated with bifurcation phenomena of codimension one except at the codimension-two points $P_{ij}, i < j$ where C_i crosses C_j . The simplest bifurcation occurs at those points where the number of purely imaginary eigenvalues increases from zero to two as two real eigenvalues become purely imaginary by passing through zero. Restricting to wave motions which are symmetric in the variable $x - ct$, one obtains a two-dimensional centre manifold which contains two homoclinic orbits corresponding to the same surface wave. This result may be stated more precisely as follows. Take $(\beta_0, \alpha_0) \in C_k$ to the right of $P_{1,2}$ if $k = 1$ or between $P_{k-1,k}$ and $P_{k,k+1}$ if $k > 1$ and let P be a positive number. There exists a positive real number μ_0 such that the hydrodynamic problem has a travelling wave of infinite spatial extent for each $(\beta, \alpha) = (\beta_0, \alpha_0 + \mu)$ with $\mu \in (0, \mu_0)$. This wave is periodic in $x - ct$

with period P and symmetric in $x - ct$ and z ; for each fixed value of $x - ct$ it decays exponentially and monotonically to zero as $z \rightarrow \pm\infty$; the surface profile of the water waves is given by

$$y = h + c_1^{1/2} \mu^{1/2} h \operatorname{sech} \left(\frac{c_2 \mu^{1/2} z}{h} \right) \cos \left(\frac{2\pi k(x - ct)}{P} \right) + O(\mu),$$

in which c_1, c_2 are positive dimensionless constants, and is sketched in Figure 2. Although this wave is spatially localised in the z -direction it seems inappropriate to refer to it as a ‘solitary wave’ since it does not decay to zero far up- and downstream *in the direction of propagation*.

2 The hydrodynamic problem as a Hamiltonian system

The physical problem in question concerns the three-dimensional flow of a perfect fluid of unit density subject to the forces of gravity and surface tension. Let (x, \tilde{y}, z) denote the usual Cartesian coordinates. The fluid occupies the domain $D_\rho = \{(x, \tilde{y}, z) : x, z \in \mathbb{R}, \tilde{y} \in (0, h + \rho(x, z, t))\}$, where $\rho > -h$ is a function of the spatial coordinates x, z and of time t and h represents the depth of the water in its undisturbed state. In terms of an Eulerian velocity potential ϕ , the mathematical problem is to solve Laplace’s equation

$$\phi_{xx} + \phi_{\tilde{y}\tilde{y}} + \phi_{zz} = 0 \quad \text{in } D_\rho, \quad (2)$$

with boundary conditions

$$\phi_{\tilde{y}} = 0 \quad \text{on } \tilde{y} = 0, \quad (3)$$

$$\rho_t = \phi_{\tilde{y}} - \rho_x \phi_x - \rho_z \phi_z \quad \text{on } \tilde{y} = h + \rho, \quad (4)$$

$$\begin{aligned} \phi_t = \frac{1}{2}(\phi_x^2 + \phi_{\tilde{y}}^2 + \phi_z^2) - g\rho \\ + \sigma \left[\frac{\rho_x}{\sqrt{1 + \rho_x^2 + \rho_z^2}} \right]_x + \sigma \left[\frac{\rho_z}{\sqrt{1 + \rho_x^2 + \rho_z^2}} \right]_z + B \quad \text{on } \tilde{y} = h + \rho, \end{aligned} \quad (5)$$

in which $\sigma > 0$ is the coefficient of surface tension and B is a constant called the Bernoulli constant (e.g. see Stoker [20, §§1, 2.1]). Equation (3) is the kinematic condition that the water cannot permeate the rigid horizontal boundary at $\tilde{y} = 0$, while (4), (5) are respectively the kinematic and dynamic conditions at the free surface.

This paper treats waves that are periodic and uniformly translating in the x -direction. These waves are described by solutions of (2)–(5) of the special

form $\rho(x, z, t) = \rho'(x - ct, z)$, $\phi(x, \tilde{y}, z, t) = \phi'(x - ct, \tilde{y}, z)$, in which ρ' and ϕ' are periodic in their first argument. Substituting this form of ρ , ϕ into (2)–(5) and introducing the dimensionless variables

$$(x', \tilde{y}', z') = \frac{1}{h}(x, \tilde{y}, z), \quad \rho'(x', z') = \frac{1}{h}\rho(x, z), \quad \phi'(x', \tilde{y}', z') = \frac{1}{ch}\phi(x, \tilde{y}, z),$$

one obtains the equations

$$\phi_{xx} + \phi_{\tilde{y}\tilde{y}} + \phi_{zz} = 0 \quad 0 < \tilde{y} < 1 + \rho, \quad (6)$$

$$\phi_{\tilde{y}} = 0 \quad \text{on } \tilde{y} = 0, \quad (7)$$

$$\phi_{\tilde{y}} = \rho_x \phi_x + \rho_z \phi_z - \rho_x \quad \text{on } \tilde{y} = 1 + \rho \quad (8)$$

and

$$-\phi_x + \frac{1}{2}(\phi_x^2 + \phi_{\tilde{y}}^2 + \phi_z^2) + \alpha\rho - \beta \left[\frac{\rho_x}{\sqrt{1 + \rho_x^2 + \rho_z^2}} \right]_x - \beta \left[\frac{\rho_z}{\sqrt{1 + \rho_x^2 + \rho_z^2}} \right]_z = 0 \quad \text{on } \tilde{y} = 1 + \rho, \quad (9)$$

in which the primes have been dropped and x is now a shorthand for the variable $x - ct$. Here $\alpha = gh/c^2$, $\beta = \sigma/hc$ and the Bernoulli constant has been set to zero to allow for waves which decay to zero as $z \rightarrow \infty$. The variables η , ϕ are assumed to be periodic in x ; their period will henceforth be denoted by P . Notice that the hydrodynamic problem (6)–(9) has certain symmetries. The equations are invariant under translations in the spatial coordinates x , z and in the velocity potential ϕ . There are also two discrete symmetries which play an important role in the following theory, namely $z \mapsto -z$ (*reflection in z*) and $(x, \eta, \phi) \mapsto (-x, \eta, -\phi)$ (*reflection in x*).

Let us now consider a formal, but physically motivated and direct argument which suggests the correct choice of Hamiltonian function for the above three-dimensional problem. The starting point is the observation that equations (6)–(9) follow from the formal variational principle

$$\delta \left\{ \int \int_0^P \left(\int_0^{1+\rho} [-\phi_x + \frac{1}{2}(\phi_x^2 + \phi_{\tilde{y}}^2 + \phi_z^2)] d\tilde{y} + \frac{1}{2}\alpha\rho^2 + \beta(\sqrt{1 + \rho_x^2 + \rho_z^2} - 1) \right) dz dx \right\} = 0, \quad (10)$$

where the variation is taken in (η, ϕ) . A more satisfactory version of this variational principle is obtained using the change of variable

$$\phi(x, \tilde{y}, z) = \Phi(x, y, z), \quad (11)$$

where $\tilde{y} = y(1 + \rho(x, z))$. One finds that

$$\delta\mathcal{L} = 0, \quad \mathcal{L} = \int L(\rho, \Phi, \rho_z, \Phi_z) dz,$$

in which

$$\begin{aligned} L(\rho, \Phi, \rho_x, \Phi_x) = & \int_S \left\{ \frac{1}{2} \alpha \rho^2 + \beta [\sqrt{1 + \rho_x^2 + \rho_z^2} - 1] \right\} dx \\ & + \int_{\Sigma} \left(- \left[\Phi_x - \frac{\Phi_y y \rho_x}{1 + \rho} \right] + \frac{1}{2} \left[\Phi_x - \frac{\Phi_y y \rho_x}{1 + \rho} \right]^2 + \frac{\Phi_y^2}{2(1 + \rho)^2} \right. \\ & \left. + \frac{1}{2} \left[\Phi_z - \frac{\Phi_y y \rho_z}{1 + \rho} \right]^2 \right) (1 + \rho) dy dx \end{aligned}$$

$S = (0, P)$ and $\Sigma = (0, 1) \times (0, P)$. The next step is to carry out a Legendre transform. Define new variables ω and Φ by the formulae

$$\begin{aligned} \omega = \frac{\delta\mathcal{L}}{\delta\rho_z} &= - \int_0^1 \left(\Phi_z - \frac{\Phi_y y \rho_z}{1 + \rho} \right) \frac{y \Phi_y}{1 + \rho} dy + \frac{\beta \rho_z}{\sqrt{1 + \rho_x^2 + \rho_z^2}}, \\ \Psi = \frac{\delta\mathcal{L}}{\delta\Phi_z} &= \left(\Phi_z - \frac{\Phi_y y \rho_z}{1 + \rho} \right) (1 + \rho), \end{aligned}$$

in which the variational derivatives are taken formally in respectively $L^2(S)$ and $L^2(\Sigma)$, and set

$$\begin{aligned} H(\rho, \omega, \Phi, \Psi) &= \int_{\Sigma} \Psi \Phi_z dy dx + \int_S \omega \rho_z dx - L(\rho, \Phi, \rho_z, \Phi_z) \\ &= \int_{\Sigma} \left\{ (1 + \rho) \Phi_x - \Phi_y y \rho_x + \frac{\Psi^2 - \Phi_y^2}{2(1 + \rho)} - \frac{(1 + \rho)}{2} \left(\Phi_x - \frac{\Phi_y y \rho_x}{1 + \rho} \right)^2 \right\} dy dx \\ &+ \int_S \left\{ -\frac{1}{2} \alpha \rho^2 + \beta - (\beta^2 - W^2)^{\frac{1}{2}} (1 + \rho_x^2)^{\frac{1}{2}} \right\} dx, \quad (12) \end{aligned}$$

where

$$W = \omega + \frac{1}{1 + \rho} \int_0^1 \Psi y \Phi_y \, dy. \quad (13)$$

The above procedure suggests that the equations

$$\rho_z = \frac{\delta H}{\delta \omega}, \quad \omega_z = -\frac{\delta H}{\delta \rho}, \quad \Phi_z = \frac{\delta H}{\delta \Psi}, \quad \Psi_z = -\frac{\delta H}{\delta \Phi}$$

formally represent Hamilton's equations for a formulation of the above steady water-wave problem as a Hamiltonian system. To confirm this suggestion rigorously let us introduce the Hilbert spaces

$$\begin{aligned} H_{\text{per}}^t(S) &= \{u \in H_{\text{loc}}^t(\mathbb{R}) : u(x + P) = u(x), x \in \mathbb{R}\}, \\ H_{\text{per}}^t(\Sigma) &= \{u \in H_{\text{loc}}^t((0, 1) \times \mathbb{R}) : u(x + P, y) = u(x, y), x \in \mathbb{R}, y \in (0, 1)\}, \end{aligned}$$

in which $S = (0, P)$, $\Sigma = (0, 1) \times (0, P)$ and t is a nonnegative real number. The following results concerning these spaces were proved by Groves and Mielke [7, Appendix A].

Lemma 1 *Let t be a non-negative real number.*

(i) *The mapping*

$$u \mapsto \int_0^1 u(\cdot, y) \, dy$$

defines a bounded linear operator $H_{\text{per}}^t(\Sigma) \rightarrow H_{\text{per}}^t(S)$.

(ii) *The natural extension of $u \in H_{\text{per}}^t(S)$ to Σ defines a bounded linear operator $H_{\text{per}}^t(S) \rightarrow H_{\text{per}}^t(\Sigma)$.*

(iii) *The trace mapping $u \mapsto u|_S$ is a bounded operator $H_{\text{per}}^{t+1/2}(\Sigma) \rightarrow H_{\text{per}}^t(S)$.*

(iv) *The spaces $H_{\text{per}}^{s_1}(S)$ and $H_{\text{per}}^{s_2}(\Sigma)$ are Banach algebras for $s_1 > 1/2$, $s_2 > 1$.*

(v) *Take $u_1 \in H_{\text{per}}^{s+1}(\Sigma)$, $u_2 \in H_{\text{per}}^s(\Sigma)$ for some $s \in (0, 1/2)$. The function $u_1 u_2$ belongs to $H_{\text{per}}^s(\Sigma)$ and satisfies the estimate*

$$\|u_1 u_2\|_s \leq c \|u_1\|_{1+s} \|u_2\|_s,$$

in which c is a positive constant. The same result holds when Σ is replaced by S .

Take $s \in (0, 1/2)$, define $X_s = H_{\text{per}}^{s+1}(S) \times H_{\text{per}}^s(S) \times H_{\text{per}}^{s+1}(\Sigma) \times H_{\text{per}}^s(\Sigma)$ and consider the symplectic manifold (M, Ω) , where $M = X_s$ and Ω is the position-independent 2-form on M given by

$$\begin{aligned} \Omega|_m((\rho_1, \omega_1, \Phi_1, \Psi_1), (\rho_2, \omega_2, \Phi_2, \Psi_2)) = \\ \int_S (\omega_2 \rho_1 - \rho_2 \omega_1) dz + \int_\Sigma (\Psi_2 \Phi_1 - \Phi_2 \Psi_1) dy dx \end{aligned} \quad (14)$$

(the canonical 2-form with respect to the $L^2(S) \times L^2(S) \times L^2(\Sigma) \times L^2(\Sigma)$ -inner product). Observe that the set

$$N = \{(\rho, \omega, \Phi, \Psi) \in X_{s+1} : |W(x)| < \beta, \rho(x) > -1 \text{ for each } x \in [0, P]\},$$

in which W is defined by (13), is a manifold domain of M : the pointwise constraints are valid since $\rho \in H_{\text{per}}^{s+1}(S) \subset C(\bar{S})$ and it follows from the results in Lemma 1 that the same is true of W . These results also show that the function H given by (12) belongs to $C^\infty(N, \mathbb{R})$; a direct calculation shows that

$$\begin{aligned} \mathbf{d}H|_n(v_1|_n) = \\ \int_S \omega_1 W \left(\frac{1 + \rho_x^2}{\beta^2 - W^2} \right)^{\frac{1}{2}} dz + \int_\Sigma \rho_{1x} \left(\Phi_x - \frac{y \Phi_y \rho_x}{\rho + 1} \right) \Phi_y y dy dx \\ - \int_\Sigma \rho_1 \left\{ \frac{\Psi^2 - \Phi_y^2}{2(\rho + 1)^2} + \left(\Phi_x - \frac{y \Phi_y \rho_x}{\rho + 1} \right)^2 + \left(\Phi_x - \frac{y \Phi_y \rho_x}{\rho + 1} \right) \frac{\Phi_y y \rho_x}{\rho + 1} \right\} dy dx \\ - \int_S \alpha \rho \rho_1 dx - \int_S \rho_{1x} \rho_x \left(\frac{\beta^2 - W^2}{1 + \rho_x^2} \right)^{\frac{1}{2}} dx \\ - \int_\Sigma \frac{\rho_1 W \Psi y \Phi_y}{(\rho + 1)^2} \left(\frac{1 + \rho_x^2}{\beta^2 - W^2} \right)^{\frac{1}{2}} dy dx + \int_\Sigma (\rho_1 \Phi_x + \Phi_y y \rho_{1x}) dy dx \\ + \int_\Sigma \Phi_{1y} \left\{ \left(\Phi_x - \frac{y \Phi_y \rho_x}{\rho + 1} \right) y \rho_x + \frac{W y \Psi}{\rho + 1} \left(\frac{1 + \rho_x^2}{\beta^2 - W^2} \right)^{\frac{1}{2}} - \frac{\Phi_y}{\rho + 1} \right\} dy dx \\ - \int_\Sigma \Phi_{1x} \left(\Phi_x - \frac{y \Phi_y \rho_x}{\rho + 1} \right) (\rho + 1) dy dx + \int_\Sigma ((1 + \rho) \Phi_{1x} - \Phi_{1y} y \rho_x) dy dx \\ + \int_\Sigma \Psi_1 \left\{ \frac{\Psi}{\rho + 1} + \frac{W y \Phi_y}{\rho + 1} \left(\frac{1 + \rho_x^2}{\beta^2 - W^2} \right)^{\frac{1}{2}} \right\} dy dx \end{aligned} \quad (15)$$

for $n = (\rho, \omega, \Phi, \Psi) \in N$ and $v_1|_n = (\rho_1, \omega_1, \Phi_1, \Psi_1) \in TN|_n \cong X_{s+1}$. The triple (M, Ω, H) is therefore a Hamiltonian system.

It remains to compute Hamilton's equations for (M, Ω, H) and to confirm that a solution to them defines a solution of the hydrodynamic problem. Lemma 1(iv) shows that $\mathbf{d}H|_n$ extends to an element of $T^*M|_n \cong X_s^*$ for each $n \in N$.

Recall that the point $n \in N$ belongs to $\mathcal{D}(v_H)$ with $v_H|_n = \bar{v}|_n$ if and only if

$$\Omega|_n(\bar{v}|_n, v_1|_n) = \mathbf{d}H|_n(v_1|_n)$$

for all tangent vectors $v_1|_n \in TM|_n$. Using this criterion, equations (14), (15) and integrating by parts, one finds that $\mathcal{D}(v_H)$ is the set of functions $(\eta, \omega, \Phi, \Psi) \in N$ that satisfy the natural boundary conditions

$$\Phi_y = 0 \quad \text{on } y = 0, \quad (16)$$

$$\rho_x + \frac{\Phi_y}{\rho + 1} = \rho_x \left(\Phi_x - \frac{\Phi_y \rho_x}{\rho + 1} \right) - \frac{W\Psi}{\rho + 1} \left(\frac{1 + \rho_x^2}{\beta^2 - W^2} \right)^{\frac{1}{2}} \quad \text{on } y = 1 \quad (17)$$

and that Hamilton's equations

$$u_z = v_H(u)$$

are given explicitly by

$$\rho_z = W \left(\frac{1 + \rho_x^2}{\beta^2 - W^2} \right)^{\frac{1}{2}}, \quad (18)$$

$$\begin{aligned} \omega_z = & \frac{W}{(\rho + 1)^2} \left(\frac{1 + \rho_z^2}{\beta^2 - W^2} \right)^{\frac{1}{2}} \int_0^1 \Psi y \Phi_y \, dy - \left[\rho_x \left(\frac{\beta^2 - W^2}{1 + \rho_x^2} \right)^{\frac{1}{2}} \right]_x \\ & + \int_0^1 \left\{ \frac{\Psi^2 - \Phi_y^2}{2(\rho + 1)^2} + \frac{1}{2} \left(\Phi_x - \frac{y\Phi_y \rho_x}{\rho + 1} \right)^2 + \left[\left(\Phi_x - \frac{y\Phi_y \rho_x}{\rho + 1} \right) y \Phi_y \right]_x \right. \\ & \left. + \left(\Phi_x - \frac{y\Phi_y \rho_x}{\rho + 1} \right) \frac{\Phi_y y \rho_x}{\rho + 1} \right\} \, dy + \alpha \rho - \Phi_x|_{y=1}, \quad (19) \end{aligned}$$

$$\Phi_z = \frac{\Psi}{(\rho + 1)} + \frac{W y \Phi_y}{\rho + 1} \left(\frac{1 + \rho_x^2}{\beta^2 - W^2} \right)^{\frac{1}{2}}, \quad (20)$$

$$\begin{aligned} \Psi_z = & -\frac{\Phi_{yy}}{\rho + 1} - \left[(\rho + 1) \left(\Phi_x - \frac{y\Phi_y \rho_x}{\rho + 1} \right) \right]_x + \left[\left(\Phi_x - \frac{y\Phi_y \rho_x}{\rho + 1} \right) y \rho_x \right]_y \\ & + \frac{W(y\Psi)_y}{\rho + 1} \left(\frac{1 + \rho_x^2}{\beta^2 - W^2} \right)^{\frac{1}{2}}. \quad (21) \end{aligned}$$

The results in Lemma 1 show that the right-hand sides of equations (18)–(21) define a smooth function $N \rightarrow X_s$ (the terms in square brackets may be expanded using the product rule in the usual fashion) and that the boundary conditions (16), (17) are also well defined. Moreover, an explicit calculation shows that a solution to (16)–(21) defines a solution of the hydrodynamic problem (6)–(9) (cf. Lemma 1 in ref. [7]).

The Hamiltonian system (M, Ω, H) has the conserved quantities

$$H(\rho, \omega, \Phi, \Psi), \quad \int_S \omega \rho_x dx + \int_\Sigma \Psi \Phi_x dy dx, \quad \int_\Sigma \Psi dy dx,$$

which are associated with continuous symmetries, namely the invariance of Hamilton's equations under translations in z , x and Φ . Hamilton's equations also inherit the discrete symmetries of the hydrodynamic problem (6)–(9). They are invariant under the reflection $R : X_s \rightarrow X_s$ given by

$$R(\rho(x), \omega(x), \Phi(x), \Psi(x)) = (\rho(-x), \omega(-x), -\Phi(-x), -\Psi(-x)) \quad (22)$$

and are reversible; the reverser $S : X_s \rightarrow X_s$ is defined by $S(\rho, \omega, \Phi, \Psi) = (\rho, -\omega, \Phi, -\Psi)$.

3 Centre-manifold reduction

Let us begin by stating the centre-manifold reduction theorem for Hamiltonian systems. This result is a Hamiltonian version of a reduction theorem for quasilinear systems obtained by Mielke [17]; its extension to Hamiltonian systems was discussed by Mielke [18, Theorem 4.1] and Buffoni, Groves and Toland [5, Theorem 4.1].

Theorem 2 *Consider the differential equation*

$$\dot{u} = Ku + n(u, \lambda), \quad (23)$$

which represents Hamilton's equations for the Hamiltonian system (M, Ω, H) . Here u belongs to a Hilbert space X , $\lambda \in \mathbb{R}^n$ is a parameter and $K : \mathcal{D}(K) \subset X \rightarrow X$ is a densely defined, closed linear operator. Regarding $\mathcal{D}(K)$ as a Hilbert space equipped with the graph norm, suppose that 0 is an equilibrium point of (23) when $\lambda = 0$ and that

(H1) X admits a direct-sum decomposition $X = X_1 \oplus X_2$, where X_1, X_2 are closed, K -invariant subspaces, so that (23) can be written as the linearly-decoupled system

$$\dot{u}_1 = K_1 u_1 + n_1(u_1 + u_2, \lambda), \quad (24)$$

$$\dot{u}_2 = K_2 u_2 + n_2(u_1 + u_2, \lambda), \quad (25)$$

in which $K_i = K|_{\mathcal{D}(K) \cap X_i} : \mathcal{D}(K) \cap X_i \subset X_i \rightarrow X_i$, $i = 1, 2$;

- (H2) X_1 is finite dimensional and the spectrum of K_1 lies on the imaginary axis;
(H3) The imaginary axis lies in the resolvent set of K_2 and

$$\|(K_2 - iaI)^{-1}\| \leq \frac{C}{1 + |a|}, \quad a \in \mathbb{R},$$

for some constant C that is independent of a ;

- (H4) There exists a natural number k and neighbourhoods $\Lambda \subset \mathbb{R}^n$ of 0 and $U \subset \mathcal{D}(K)$ of 0 such that n is $(k + 1)$ times continuously differentiable on $U \times \Lambda$, its derivatives are bounded and uniformly continuous on $U \times \Lambda$ and

$$n(0, 0) = 0, \quad d_1 n[0, 0] = 0.$$

Under these hypotheses there exist neighbourhoods $\tilde{\Lambda} \subset \Lambda$ of 0 and $\tilde{U}_1 \subset U \cap X_1$, $\tilde{U}_2 \subset U \cap X_2$ of 0 and a reduction function $h : \tilde{U}_1 \times \tilde{\Lambda} \rightarrow \tilde{U}_2 \subset X_2$ with the following properties. The reduction function h is k times continuously differentiable on $\tilde{U}_1 \times \tilde{\Lambda}$, its derivatives are bounded and uniformly continuous on $\tilde{U}_1 \times \tilde{\Lambda}$ and

$$h(0, 0) = 0, \quad d_1 h[0, 0] = 0. \quad (26)$$

The graph

$$M_C^\lambda = \{u_1 + h(u_1, \lambda) \in \tilde{U}_1 \times \tilde{U}_2 : u_1 \in \tilde{U}_1\}$$

is a Hamiltonian centre manifold for (23), so that

- (1) M_C^λ is a locally invariant manifold of (23): through every point in M_C^λ there passes a unique solution of (23) that remains on M_C^λ as long as it remains in $\tilde{U}_1 \times \tilde{U}_2$;
- (2) Every small bounded solution $u(t)$, $t \in \mathbb{R}$ of equation (23) that satisfies $(u_1(t), u_2(t)) \in \tilde{U}_1 \times \tilde{U}_2$ lies completely in M_C^λ ;
- (3) Every solution $\tilde{u} : (a, b) \rightarrow \tilde{U}_1$ of the reduced equation

$$\dot{u}_1 = K_1 u_1 + n_1(u_1 + h(u_1, \lambda), \lambda) \quad (27)$$

generates a solution

$$u(t) = u_1(t) + h(u_1(t), \lambda)$$

of the full equation (23).

- (4) M_C^λ is a symplectic submanifold of M and the flow determined by the Hamiltonian system $(M_C^\lambda, \tilde{\Omega}^\lambda, \tilde{H}^\lambda)$, where the tilde denotes restriction to

M_C^λ , coincides with the flow on M_C^λ determined by $(M, \Omega^\lambda, H^\lambda)$. The reduced equation (27) represents Hamilton's equations for $(M_C^\lambda, \tilde{\Omega}, \tilde{H})$.

The task in hand is to write the Hamiltonian formulation of the hydrodynamic problem introduced in Section 2 in a form to which Theorem 2 is applicable. Write

$$\alpha = \alpha_0 + \lambda_1, \quad \beta = \beta_0 + \lambda_2,$$

where α_0, β_0 are fixed and λ_1, λ_2 lie in neighbourhoods Λ_1, Λ_2 of the origin in \mathbb{R} , and consider trajectories of Hamilton's equations for (M, Ω, H) which lie in a neighbourhood V of the origin in X_{s+1} . Choosing V and Λ_2 small enough so that

$$\rho(x) > -\frac{1}{2} > -1, \quad |\lambda_2| < \frac{1}{4}\beta_0, \quad |W(x)| < \frac{\beta_0}{2} < \beta_0 + \lambda_2$$

for all $x \in [0, P]$, one arrives at the equation

$$u_z = f^\lambda(u), \tag{28}$$

where $f^\lambda : V \rightarrow X_s$ is the smooth function defined by the right-hand sides of (18)–(21) with (α, β) replaced by $(\alpha_0 + \lambda_1, \beta_0 + \lambda_2)$, together with the boundary conditions

$$\Phi_y = 0, \tag{29} \quad y = 0,$$

$$\rho_x + \frac{\Phi_y}{\rho + 1} = \rho_x \left(\Phi_x - \frac{\Phi_y \rho_x}{\rho + 1} \right) - \frac{W\Psi}{\rho + 1} \left(\frac{1 + \rho_x^2}{(\beta^2 + \lambda_2)^2 - W^2} \right)^{\frac{1}{2}}, \quad y = 1. \tag{30}$$

Formula (28) clearly represents Hamilton's equations for the Hamiltonian system (M, Ω, H^λ) , where H^λ is the smooth functional defined upon the manifold domain V of M by

$$\begin{aligned} H^\lambda(\rho, \omega, \Phi, \Psi) &= \\ &= \int_{\Sigma} \left\{ (1 + \rho)\Phi_x - \Phi_y y \rho_x + \frac{\Psi^2 - \Phi_y^2}{2(1 + \rho)} - \frac{1 + \rho}{2} \left(\Phi_x - \frac{\Phi_y y \rho_x}{1 + \rho} \right)^2 \right\} dy dx \\ &\quad + \int_S \left(-\frac{1}{2}(\alpha_0 + \lambda_1)\rho^2 + \beta_0 + \lambda_2 - ((\beta_0 + \lambda_2)^2 - W^2)^{\frac{1}{2}}(1 + \rho_x^2)^{\frac{1}{2}} \right) dx; \end{aligned}$$

the domain $\mathcal{D}(v_H^\lambda)$ of the Hamiltonian vector field v_H^λ is the subset of V specified by (29), (30) and $v_H^\lambda(u) = f^\lambda(u)$ for each $u \in \mathcal{D}(v_H^\lambda)$.

Theorem 2 cannot be applied directly to (28) because of the nonlinear boundary condition (30). It is therefore necessary to introduce a change of variable which leads to an equivalent problem in a linear space. To this end, define $F : V \times \Lambda_2 \subset X_{s+1} \times \mathbb{R} \rightarrow H_{\text{per}}^{s+1}(\Sigma)$ by the formula

$$F(\rho, \omega, \Phi, \Psi, \lambda_2)(y) = [(\Phi_x - 1)(\rho + 1) - y\Phi_y\rho_x]y\rho_x + \left(\frac{1 + \rho_x^2}{(\beta_0 + \lambda_2)^2 - W^2} \right)^{\frac{1}{2}} W y \Psi,$$

so that the boundary conditions (29), (30) are equivalent to

$$\Phi_y = F(\rho, \omega, \Phi, \Psi, \lambda_2) \quad \text{on } y = 0, 1, \quad (31)$$

and consider the function $G^{\lambda_2} : V \rightarrow X_{s+1}$ given by $G^{\lambda_2}(\rho, \omega, \Phi, \Psi) = (\rho, \nu, \Gamma, \Psi)$, where

$$\nu = \omega - \int_0^1 y\Phi_y dy + (\beta_0 + \lambda_2)\rho_x,$$

$$\Gamma = \Phi + \chi_y$$

and $\chi \in H_{\text{per}}^{s+3}(\Sigma)$ is the unique solution of the elliptic boundary-value problem

$$\begin{aligned} -\chi_{xx} - \chi_{yy} &= F(\rho, \omega, \Phi, \Psi, \lambda_2) && \text{in } \Sigma, \\ \chi &= 0 && \text{on } y = 0, 1. \end{aligned}$$

The next lemma, whose proof is analogous to that of Lemma 4 in ref. [7], shows that G_{λ_2} defines a valid change of variable.

Lemma 3

- (i) For each $\lambda_2 \in \Lambda_2$ the mapping G^{λ_2} is a smooth diffeomorphism from the neighbourhood V of 0 in X_{s+1} onto a neighbourhood \tilde{V} of 0 in X_{s+1} . The mappings G^{λ_2} , $(G^{\lambda_2})^{-1}$ and their derivatives depend smoothly upon $\lambda_2 \in \Lambda_2$.
- (ii) For each $(v, \lambda_2) \in V \times \Lambda_2$ the operator $dG^{\lambda_2}[v] : X_{s+1} \rightarrow X_{s+1}$ extends to an isomorphism $\widetilde{dG}^{\lambda_2}[v] : X_s \rightarrow X_s$. The operators $\widetilde{dG}^{\lambda_2}[v]$, $\widetilde{dG}^{\lambda_2}[v]^{-1} \in \mathcal{L}(X_s, X_s)$ depend smoothly upon $(v, \lambda_2) \in V \times \Lambda_2$.

The diffeomorphism G^{λ_2} transforms (28) into

$$u_z = g^\lambda(u), \quad (32)$$

where $g^\lambda : \tilde{V} \rightarrow X_s$ is the smooth vector field defined by

$$g^\lambda(u) = \widetilde{dG}^{\lambda_2}[(G^{\lambda_2})^{-1}(u)](f^\lambda((G^{\lambda_2})^{-1}(u))).$$

Notice that

$$\begin{aligned} \Gamma_y &= \Phi_y + \chi_{yy} \\ &= \Phi_y - \chi_{xx} - F(\rho, \omega, \Phi, \Psi, \lambda_2), \end{aligned}$$

and since $\chi_{xx} = 0$ on $y = 0, 1$ it follows from (31) that the boundary conditions (29), (30) are transformed into

$$\Gamma_y = 0 \quad \text{on } y = 0, 1. \quad (33)$$

Equation (32) can be written as

$$u_{1z} = K_s u + n(u, \lambda), \quad (34)$$

where $K_s = dg^0[0]$, $\mathcal{D}(K_s) = \{(\rho, \omega, \Gamma, \Psi) \in X_{s+1} : \Gamma_y|_{y=0} = \Gamma_y|_{y=1} = 0\}$ and $n(u, \lambda) = g^\lambda(u) - K_s u$. This equation is in a form suitable for the application of Theorem 2. Writing $X = X_s$, $K = K_s$, $\Lambda = \Lambda_1 \times \Lambda_2$ and $U = \tilde{V}$, observe that K_s is densely defined and that (H4) is satisfied for any natural number k . Moreover, formula (32) represents Hamilton's equations for the Hamiltonian system $(M, \Omega_2^{\lambda_2}, H_2^\lambda)$, where $\Omega_2^{\lambda_2}$ and H_2^λ are defined on the manifold domain \tilde{V} of M by

$$\Omega_2^{\lambda_2}|_m(v_1, v_2) = \Omega(\widetilde{dG}^{\lambda_2}[(G^{\lambda_2})^{-1}(m)]^{-1}(v_1), \widetilde{dG}^{\lambda_2}[(G^{\lambda_2})^{-1}(m)]^{-1}(v_2))$$

for $\lambda_2 \in \Lambda_2$, $v_1, v_2 \in TM|_m$ and

$$H_2^\lambda(m) = H^\lambda((G^{\lambda_2})^{-1}(m)).$$

The domain $\mathcal{D}(v_{H_2}^\lambda)$ of the Hamiltonian vector field is the subset of \tilde{V} given by the linear boundary conditions (33) and $v_{H_2}^\lambda(u) = g^\lambda(u)$ for each $u \in \mathcal{D}(v_{H_2}^\lambda)$.

To discuss the spectrum of K_s it is helpful to introduce the operator $L_s : \mathcal{D}(L_s) \subset X_s \rightarrow X_s$ given by

$$L_s \begin{pmatrix} \rho \\ \omega \\ \Phi \\ \Psi \end{pmatrix} = \begin{pmatrix} \frac{1}{\beta_0} \omega \\ -\Phi_x|_{y=1} + \alpha_0 \rho - \beta_0 \rho_{xx} \\ \Psi \\ -\Phi_{xx} - \Phi_{yy} \end{pmatrix}; \quad (35)$$

its domain $\mathcal{D}(L_s)$ is the subspace of elements of X_{s+1} that satisfy the boundary conditions

$$\Phi_y = 0 \quad \text{on } y = 0, \quad (36)$$

$$\Phi_y = -\rho_x \quad \text{on } y = 1. \quad (37)$$

(The operator L_s is the formal linearisation of $f^0(\rho, \omega, \Phi, \Psi)$ at 0.) Observe that $dG^0[0]$ is a homeomorphism of $\mathcal{D}(L_s)$ onto $\mathcal{D}(K_s)$ and that

$$K_s = \widetilde{dG^0[0]} L_s (dG^0[0])^{-1} \quad (38)$$

for $s \in (0, 1/2)$, where $K_s : \mathcal{D}(K_s) \subset X_s \rightarrow X_s$ is the linear operator in equation (34). Equation (38) shows that the topological properties of L_s and K_s are identical; in particular their spectra coincide and L_s is densely defined. The following result concerning the spectral properties of K_s is obtained by performing explicit calculations with the operator L_s (cf. [7, §3.3]).

Theorem 4 *The spectrum $\sigma(K_s)$ of K_s consists of entirely of isolated eigenvalues of finite algebraic multiplicity and $\sigma(K_s) \cap i\mathbb{R}$ is a finite set. A complex number λ is an eigenvalue of K_s with corresponding eigenvectors in the k th Fourier mode if and only if*

$$(\alpha - \beta\sigma^2)\sigma \sin \sigma + \frac{4\pi^2 k^2}{P^2} \cos \sigma = 0 \quad \text{with } \sigma^2 = \lambda^2 - 4\pi^2 k^2 / P^2. \quad (39)$$

There exist real constants $C, \xi_0 > 0$ such that

$$\|(K_s - i\xi I)^{-1}\|_{X_s \rightarrow X_s} \leq \frac{C_2}{|\xi|}$$

for each real number ξ with $|\xi| > \xi_0$.

Let us now verify the remaining hypotheses in Theorem 2 which ensure that it is applicable to equation (34). First note that K_s is closed because it has a non-empty resolvent set. Define the spectral projection P corresponding to the subset $\sigma(K_s) \cap i\mathbb{R}$ of $\sigma(K_s)$ by

$$Px = \frac{1}{2\pi i} \int_C (K_s - \lambda I)^{-1} x \, d\lambda,$$

where C is a closed curve in the resolvent set of K_s which contains $\sigma(K_s) \cap i\mathbb{R}$ and no other point of $\sigma(K_s)$. Writing $X_1 = P(X_s)$, $X_2 = (I - P)(X_s)$, one

finds that hypotheses (H1) and (H2) of Theorem 2 are satisfied (see Kato [14, Theorem III.6.17]), and Theorem 4 shows that (H3) is also satisfied.

The equation

$$u_{1z} = PKu_1 + Pn(u_1 + h(u_1, \lambda))$$

represents Hamilton's equations for the reduced system $(M_C^\lambda, \tilde{\Omega}^\lambda, \tilde{H}^\lambda)$, where M_C^λ is equipped with the single coordinate chart $\tilde{U}_1 \subset X_1$ and coordinate map $\chi : M_C^\lambda \rightarrow \tilde{U}_1$ defined by $\chi^{-1}(u_1) = u_1 + h(u_1, \lambda)$. In the following calculations it is more convenient to change to the coordinate chart $\tilde{W}_1 = (dG^0[0])^{-1}(\tilde{U}_1)$, so that \tilde{W}_1 is a neighbourhood of the origin in the centre subspace of the linear operator L_s . The coordinate map $\zeta : M_C^\lambda \rightarrow \tilde{W}_1$ can be written as $\zeta^{-1}(w_1) = w_1 + k(w_1, \lambda)$, where $k : \tilde{W}_1 \times \tilde{\Lambda} \rightarrow V$ is given by

$$k(w_1, \lambda) = (G^{\lambda_2})^{-1}(dG^0[0](w_1) + h(dG^0[0](w_1, \lambda)) - w_1).$$

In this coordinate system M_C^λ is $\{w_1 + k(w_1, \mu) : w_1 \in \tilde{W}_1\}$; it lies in $\mathcal{D}(v_H^\lambda)$ and defines a centre manifold for the Hamiltonian system (M, Ω, H^λ) . The reduced Hamiltonian function \tilde{H}^λ and reduced 2-form $\tilde{\Omega}^\lambda$ are given by the formulae

$$\begin{aligned} \tilde{H}^\lambda(w_1) &= H_2^\lambda(w_1 + k(w_1, \lambda)), \\ \tilde{\Omega}^\lambda|_{w_1}(v^1, v^2) &= \Omega|_{w_1+k(w_1, \lambda)}(v^1 + d_1k[w_1, \lambda](v^1), v^2 + d_1k[w_1, \lambda](v^2)) \\ &= \Omega(v^1, v^2) + O(|(\lambda, w_1)|) \end{aligned}$$

as $(w_1, \lambda) \rightarrow 0$.

The next step is to choose a symplectic basis for X_1 consisting of generalised eigenvectors of L_s . Observe that $\lambda = 0$, $k = 0$ is always a solution of (39), so that there is always a zero eigenvector in the 0th Fourier mode; this eigenvector has a Jordan chain of length 2. Choosing

$$e = \frac{1}{\sqrt{P}} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad f = \frac{1}{\sqrt{P}} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

one finds that $L_s e = 0$, $L_s f = e$ and $\Omega(e, f) = 1$. According to equation (39), additional zero eigenvectors in the k th Fourier mode ($k \neq 0$) appear if

$$\left(\alpha + \frac{4\pi^2 \beta k^2}{P^2} \right) \sinh \left(\frac{2\pi k}{P} \right) = \frac{2\pi k}{P} \cosh \left(\frac{2\pi k}{P} \right), \quad (40)$$

and this equation has zero, one or two solutions $k \in \mathbb{N}$. (The significance of equation (40) is discussed in Section 1.) Zero eigenvectors corresponding to such solutions are obviously geometrically double, and each eigenvector has a Jordan chain of length 2. The vectors

$$e_+^{0,k} = \frac{1}{\sqrt{\gamma}} \begin{pmatrix} 2 \sinh\left(\frac{2k\pi}{P}\right) \cos\left(\frac{2k\pi x}{P}\right) \\ 0 \\ 2 \cosh\left(\frac{2k\pi y}{P}\right) \sin\left(\frac{2k\pi x}{P}\right) \\ 0 \end{pmatrix}, \quad e_-^{0,k} = \frac{1}{\sqrt{\gamma}} \begin{pmatrix} -2 \sinh\left(\frac{2k\pi}{P}\right) \sin\left(\frac{2k\pi x}{P}\right) \\ 0 \\ 2 \cosh\left(\frac{2k\pi y}{P}\right) \cos\left(\frac{2k\pi x}{P}\right) \\ 0 \end{pmatrix},$$

$$f_+^{0,k} = \frac{1}{\sqrt{\gamma}} \begin{pmatrix} 0 \\ 2\beta \sinh\left(\frac{2k\pi}{P}\right) \cos\left(\frac{2k\pi x}{P}\right) \\ 0 \\ 2 \cosh\left(\frac{2k\pi y}{P}\right) \sin\left(\frac{2k\pi x}{P}\right) \end{pmatrix}, \quad f_-^{0,k} = \frac{1}{\sqrt{\gamma}} \begin{pmatrix} 0 \\ -2\beta \sinh\left(\frac{2k\pi}{P}\right) \sin\left(\frac{2k\pi x}{P}\right) \\ 0 \\ 2 \cosh\left(\frac{2k\pi y}{P}\right) \cos\left(\frac{2k\pi x}{P}\right) \end{pmatrix},$$

where

$$\gamma = P + 2P\beta \sinh^2\left(\frac{2k\pi}{P}\right) + \frac{P^2}{4k\pi} \sinh\left(\frac{4k\pi}{P}\right),$$

satisfy $Le_{\pm}^{0,k} = 0$, $Lf_{\pm}^{0,k} = e_{\pm}^{0,k}$; moreover $e_+^{0,k}$, $f_+^{0,k}$ are symmetric with respect to the reflection R (see equation (22)), $e_-^{0,k}$, $f_-^{0,k}$ are antisymmetric with respect to this reflection, $\Omega(e_+^{0,k}, f_+^{0,k}) = 1$, $\Omega(e_-^{0,k}, f_-^{0,k}) = 1$ and the symplectic product of any other combination is zero. The generalised eigenspace E_0 is therefore either two-, six- or ten-dimensional and has the symplectic basis $\{e, e_+^{0,k}, e_-^{0,k}, f, e_+^{0,k}, e_-^{0,k}\}$. The coordinates $q, q_+^{0,k}, q_-^{0,k}, p, p_+^{0,k}, p_-^{0,k}$ in the $e, e_+^{0,k}, e_-^{0,k}, f, f_+^{0,k}, f_-^{0,k}$ directions are canonical coordinates for E_0 and the actions of the reflector R , the reverser S and the translator $T_a : x \mapsto x + a$, $a \in \mathbb{R}$ on this generalised eigenspace are given by

$$R(q, q_+^{0,k}, q_-^{0,k}, p, p_+^{0,k}, p_-^{0,k}) = (q, q_+^{0,k}, -q_-^{0,k}, p, p_+^{0,k}, -p_-^{0,k}),$$

$$S(q, q_+^{0,k}, q_-^{0,k}, p, p_+^{0,k}, p_-^{0,k}) = (q, q_+^{0,k}, q_-^{0,k}, -p, -p_+^{0,k}, -p_-^{0,k}),$$

$$T_a(q, q_+^{0,k}, q_-^{0,k}, p, p_+^{0,k}, p_-^{0,k}) = (q, \mathcal{R}_{2\pi a/k}(q_+^{0,k}, q_-^{0,k}), p, \mathcal{R}_{2\pi a/k}(p_+^{0,k}, p_-^{0,k})),$$

where \mathcal{R}_θ is the 2×2 matrix representing a rotation through the angle θ .

All other purely imaginary eigenvalues are also geometrically double but have no Jordan chains. They occur in pairs $\pm is_k$ with $L_s e_{\pm}^k = is_k e_{\pm}^k$, $L_s \bar{e}_{\pm}^k = -is_k \bar{e}_{\pm}^k$, where the eigenvectors can be chosen so that $R(e_+^k) = e_+^k$, $R(e_-^k) = -e_-^k$, $\Omega(e_+^k, \bar{e}_+^k) = \pm i$, $\Omega(e_-^k, \bar{e}_-^k) = \pm i$ and the symplectic product of any other

combination is zero. It follows that $\{e_+^k, e_-^k, \bar{e}_+^k, \bar{e}_-^k\}$ is a symplectic basis for $E_{is_k} \oplus E_{-is_k}$; canonical coordinates for $E_{is_k} \oplus E_{-is_k}$ are given by the complex coordinates C_+^k, C_-^k in the e_+^k, e_-^k directions, and the actions of R, S and T_a on $E_{is_k} \oplus E_{-is_k}$ are respectively $(C_+^k, C_-^k) \mapsto (C_+^k, -C_-^k)$, $(C_+^k, C_-^k) \mapsto (\bar{C}_+^k, \bar{C}_-^k)$ and $(C_+^k, C_-^k) \mapsto \mathcal{R}_{2\pi k/a}(C_+^k, C_-^k)$.

The fact that E_r and E_s are orthogonal with respect to Ω whenever $r + s \neq 0$ implies that Ω is the canonical symplectic 2-form Υ on X_1 in the coordinate system introduced above. The reduced 2-form $\tilde{\Omega}^\lambda$ for the symplectic submanifold $(M_C^\lambda, \tilde{\Omega}^\lambda)$ is position- and parameter-dependent, but can be transformed into Υ by a near-identity Darboux change of coordinates (cf. [4, Theorem 4]). The centre-manifold reduction procedure preserves reversibility and symmetries, and it is also possible to select the Darboux transformation to preserve such characteristics of the original equations [18]. These observations identify a convenient coordinate system in which to study Hamilton's equations for $(M_C^\lambda, \tilde{\Omega}, \tilde{H}^\lambda)$. Note in particular that Hamilton's equations are reversible and invariant under the continuous symmetry T_a and the discrete symmetry R . They therefore have $O(2)$ symmetry and Fix R is an invariant subspace, solutions in which correspond to water waves which are symmetric in x . The variable q is cyclic, so that its conjugate p is a conserved quantity (which will be set to zero in the following analysis), and the dimension of the system of equations can always be reduced by two.

4 The reduced equations

One advantage of the Hamiltonian structure of the reduced equations is that certain existence results for small-amplitude solutions of the hydrodynamic problem under consideration can be obtained by direct applications of standard results in the theory of Hamiltonian systems. Let us now give two examples of such results which concern fixed values (β_0, α_0) of (β, α) which do not lie on any of the lines C_1, C_2, \dots shown in Figure 1, so that the bifurcation parameters λ_1 and λ_2 are both set to zero. Introduce coordinates for the centre manifold using the procedure explained in Section 3 and restrict attention to the invariant subspace Fix R which corresponds to water waves that are symmetric in x . The zero eigenvalue, which in the present setting is geometrically simple and has a Jordan chain of length 2, can be removed using the procedure explained at the end of Section 3, and all other eigenvalues $\pm i\omega_1, \dots, \pm i\omega_n$ are simple.

The following result is obtained from the Lyapunov centre theorem (e.g. see Ambrosetti and Prodi [1, ch. 7, Theorem 3.4]).

Theorem 5 Assume that $\omega_i/\omega_j \notin \mathbb{Z}$ for all $i \neq j$.

- (i) Suppose that (β_0, α_0) lies below the line C_1 , to the left of C_j and to the right of C_{j+1} for some $j \in \mathbb{N}$. The reduced equations on the centre manifold have j geometrically distinct periodic orbits on the energy surface $\{\tilde{H}^0 = \epsilon\}$ for each sufficiently small value of $\epsilon > 0$.
- (ii) Suppose that (β_0, α_0) lies above the line C_i and below the line C_{i+1} for some $i \in \mathbb{N}$, to the left of C_j and to the right of C_{j+1} for some $j \geq i$. The reduced equations on the centre manifold have j geometrically distinct periodic orbits on the energy surface $\{\tilde{H}^0 = \epsilon\}$ for each sufficiently small value of $\epsilon > 0$.

Now suppose that $\omega_1, \dots, \omega_n$ satisfy no resonance relation of order three or four, that is any triple or quartet of frequencies is linearly independent over the field \mathbb{Q} . According to classical dynamical-systems theory the reduced Hamiltonian system given by the centre-manifold reduction procedure can be put into *fourth-order Birkhoff normal form* by a near identity symplectic transformation (e.g. see Arnold [3, Appendix 7A]). The transformed Hamiltonian function takes the form

$$\tilde{H}^0(w_1) = \frac{1}{2}\omega_1 I_1 + \dots + \frac{1}{2}\omega_n I_n + \frac{1}{8}(I_1, \dots, I_n)Q(I_1, \dots, I_n)^T + O(\|w_1\|^5)$$

as $w_1 \rightarrow 0$, in which I_1, \dots, I_n are action variables associated with the eigenvalues $\pm i\omega_1, \dots, \pm i\omega_n$ and Q is an $n \times n$ matrix. Theorem 6 below follows from a result in KAM theory given by Pöschel [19, Theorem 2.1].

Theorem 6 Suppose that $\omega_1, \dots, \omega_n$ satisfy no resonance relation of order three or four and that $\det Q \neq 0$.

- (i) The reduced equations on the centre manifold have j -dimensional invariant Kronecker system whenever (β_0, α_0) lies below the line C_1 , to the left of C_j and to the right of C_{j+1} for some $j \in \mathbb{N}$.
- (ii) The reduced equations on the centre manifold have a j -dimensional invariant Kronecker system in each neighbourhood of the origin whenever (β_0, α_0) lies above the line C_i and below the line C_{i+1} for some $i \in \mathbb{N}$, to the left of C_j and to the right of C_{j+1} for some $j \geq i$.

(An n -dimensional Kronecker system (T^n, ω) consists of an n -torus T^n and a frequency vector ω which describes linear quasiperiodic motion upon it. A $2n$ -dimensional system of ordinary differential equations is said to have an invariant Kronecker system (T^n, ω) if there exists an embedding $\phi \in C^1(T^n, \mathbb{R}^{2n})$ such that $\phi(\omega t + x_0)$ solves the system for each $x_0 \in T^n$.)

The lines C_1, C_2, C_3, \dots are associated with bifurcation phenomena of codimension one except at the codimension-two points $P_{ij} = (P_{ij}^\beta, P_{ij}^\alpha)$, $i < j$ where

C_i crosses C_j . The simplest bifurcation occurs at those points where the number of purely imaginary eigenvalues increases from zero to two as two real eigenvalues pass through zero. Such points are located on C_k between $P_{k-1,k}$ and $P_{k,k+1}$ if $k > 1$ and to the right of $P_{1,2}$ if $k = 1$. Introducing a bifurcation parameter $\mu \geq 0$ by writing $(\beta, \alpha) = (\beta_0, \alpha_0 + \mu)$ (so that $\lambda_1 = \mu$, $\lambda_2 = 0$ in the notation of Section 3), one obtains a six-dimensional centre manifold M_C^μ for $\mu \in \tilde{\Lambda}$. The manifold M_C^μ is modelled upon the single coordinate chart \tilde{W}_1 and one writes

$$w_1 = qe + pf + q_+^{0,k} e_+^{0,k} + p_+^{0,k} f_+^{0,k} + q_-^{0,k} e_-^{0,k} + p_-^{0,k} f_-^{0,k};$$

a near-identity change of variable transforms $(q, q_+^{0,k}, q_-^{0,k}, p, p_+^{0,k}, p_-^{0,k})$ into canonical coordinates for the reduced Hamiltonian system. The dimension of the centre manifold can be reduced by eliminating the cyclic variable q and restricting to the invariant subspace $\text{Fix } R = \{q_-^{0,k} = p_-^{0,k} = 0\}$ corresponding to water waves which are symmetric in x . This procedure breaks the $O(2)$ symmetry of the reduced equations, for which $T_a(q, q_+^{0,k}, q_-^{0,k}, p, p_+^{0,k}, p_-^{0,k}) \mapsto (q, \mathcal{R}_{2\pi a/k}(q_+^{0,k}, q_-^{0,k}), p, \mathcal{R}_{2\pi a/k}(p_+^{0,k}, p_-^{0,k}))$ is now only a symmetry when $a = nP/2$, $n \in \mathbb{Z}$. One obtains a two-dimensional Hamiltonian system $(\mathcal{M}, \Upsilon, \tilde{H}^\mu)$, where \mathcal{M} is a neighbourhood of the origin in \mathbb{R}^2 and

$$\Upsilon((q^1, p^2), (q^2, p^2)) = q^1 p^2 - p^1 q^2;$$

here the sub- and superscripts on the symbols $q_+^{0,k+}$, $p_+^{0,k}$, $e_+^{0,k+}$, $f_+^{0,k}$ have been dropped for notational simplicity. Recall that $(\mathcal{M}, \Upsilon, \tilde{H}^\mu)$ is a reversible Hamiltonian system whose reverser S maps (q, p) to $(q, -p)$; it is also invariant under the transformation $T(q, p) \mapsto (-q, -p)$ (the remnant of the $O(2)$ -symmetry).

The solution set of the one-degree-of-freedom Hamiltonian system $(\mathcal{M}, \Upsilon, \tilde{H}^\mu)$ is readily analysed. The following lemma is a straightforward application of the normal-form theory developed by Elphick *et al.* [6] and extended to Hamiltonian systems by Meyer and Hall [16, ch. VII]).

Lemma 7 *There is a near-identity, analytic, symplectic change of coordinates with the property that*

$$\tilde{H}^\mu = \frac{1}{2}p^2 + H_{\text{NF}}(q^2, \mu) + O(|(q, p)|^{N+2})$$

as $(q, p) \rightarrow 0$ in the new coordinates. The term $H_{\text{NF}}(q^2, \mu)$ is a polynomial of order $N + 1$ in q with coefficients which depend analytically upon μ ; it satisfies

$$H_{\text{NF}}(q, \mu) = O(q^2 |(\mu, q^2)|).$$

The change of coordinates preserves the reversibility and the invariance under the transformation $(q, p) \mapsto (-q, -p)$.

Writing

$$H_{\text{NF}} = c_1 \mu q^2 + c_2 q^4 + \dots,$$

one finds that Hamilton's equations for $(\mathcal{M}, \Upsilon, \tilde{H}^\mu)$ are

$$\dot{q} = p + \mathcal{R}_1(q, p, \mu), \tag{41}$$

$$\dot{p} = -2c_1 \mu q - 4c_2 q^3 + \mathcal{R}_2(q, p, \mu), \tag{42}$$

where \mathcal{R}_1 is odd in its first two arguments, \mathcal{R}_2 is even in its first two arguments and $\mathcal{R}_1 = O(|(q, p)|^{N+1})$, $\mathcal{R}_2 = O(|(q, p)|^{N+1}) + O(|q^3(\mu, q^2)|)$ as $(q, p) \rightarrow 0$. Introducing the scaled variables

$$Z = \mu^{1/2} z, \quad q(z) = \mu^{1/2} Q(Z), \quad p(z) = \mu P(Z),$$

one finds from (41), (42) that

$$Q_Z = P + \tilde{\mathcal{R}}_1(q, p, \mu), \tag{43}$$

$$P_Z = C_1 Q - C_2 Q^3 + \tilde{\mathcal{R}}_2(q, p, \mu), \tag{44}$$

where $C_1 = -2c_1$, $C_2 = 4c_2$ are positive constants (see below), the remainder terms $\tilde{\mathcal{R}}_1$ and $\tilde{\mathcal{R}}_2$ are respectively odd and even in their first two arguments and $\tilde{\mathcal{R}}_1, \tilde{\mathcal{R}}_2 = O(\mu^{1/2})$. In the limit $\mu \rightarrow 0$ equations (43), (44) are equivalent to

$$\begin{aligned} Q_Z &= P, \\ P_Z &= C_1 Q - C_2 Q^3, \end{aligned}$$

whose phase portrait is easily calculated by elementary methods and is depicted in Figure 3. Notice in particular that it has two homoclinic orbits (Q_+, P_+) , (Q_-, P_-) given by the explicit formulae

$$Q_\pm(Z) = \pm \left(\frac{2C_1}{C_2} \right)^{1/2} \operatorname{sech}(C_1^{1/2} Z), \quad P_\pm(Z) = \frac{d}{dZ}(Q_\pm(Z)).$$

One can exploit the reversibility of (43), (44) to deduce that its phase portrait for small positive values of μ is qualitatively the same as that for $\mu = 0$. For $\mu = 0$ the stable manifold W_s^0 of the zero equilibrium is known explicitly (it consists of the points on the homoclinic orbits), and an explicit calculation

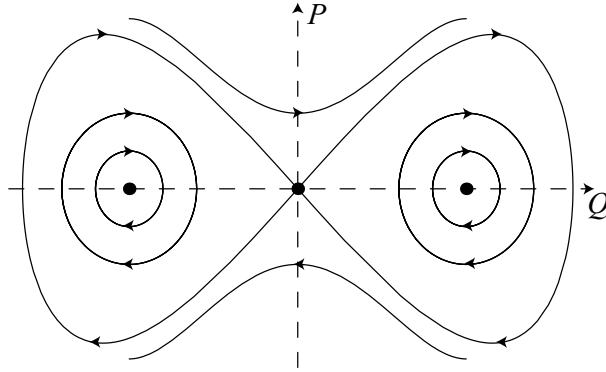


Fig. 3. Phase portrait of the scaled reduced system of equations.

shows that it intersects the symmetric section $\text{Fix } S = \{P = 0\}$ transversally in the two points $(Q_+(0), 0)$, $(Q_-(0), 0)$. The stable manifold theorem states that W_s^μ depends uniformly smoothly upon μ , and since the symmetric section is independent of μ it follows that W_s^μ and $\text{Fix } S$ intersect transversally in two points, one near $(Q_+(0), 0)$, one near $(Q_-(0), 0)$, for sufficiently small positive values of μ . One concludes that that equations (43), (44) have two homoclinic orbits, one in the left half-plane, one in the right half-plane, for sufficiently small positive values of μ . A similar ‘persistence’ argument is readily constructed for the periodic orbits in the phase diagram for $\mu = 0$, each of which intersects the symmetric section transversally in two distinct points. Standard results from the theory of ordinary differential equations assert that the trajectory \mathcal{O}_0 of one of these periodic solutions is smoothly deformed into another trajectory \mathcal{O}_μ as μ is increased; both trajectories are parameterised by the time-like variable Z and the deformation is uniform over compact subsets of Z . Regarding \mathcal{O}_0 being parameterised over one period, one finds that it is smoothly deformed into a compact subset $\mathcal{O}_\mu^{\text{per}}$ of \mathcal{O}_μ as μ is increased. The transversality argument used above for the homoclinic solutions may now be applied and shows that $\mathcal{O}_\mu^{\text{per}}$, and hence \mathcal{O}_μ , intersect $\text{Fix } S$ in two distinct points for sufficiently small positive values of μ . The fact that \mathcal{O}_μ is periodic follows from the observation that the reversible periodic orbits of a reversible dynamical system are precisely those orbits which intersect the symmetric section in two distinct points [13, p. 76].

The ‘persistence’ arguments exploit the fact that the phase portrait is symmetric with respect to the Q -axis (because of the reversibility); it is also symmetric with respect to the P -axis (because of the symmetry $(Q, P) \mapsto (-Q, -P)$). Notice that two solutions (Q_1, P_1) and (Q_2, P_2) related by $(Q_2, P_2) = (-Q_1, -P_1)$ correspond to the same surface wave (up to a translation of half a period in the direction of propagation) since the invariance of the reduced equations (41), (42) corresponds to the invariance of the original equations under the transformation $(x, \rho, \omega, \Phi, \Psi) \mapsto (x + P/2, \rho, \omega, -\Phi, -\Psi)$.

Tracing back the various changes of variable, one finds that the surface profile

of the water corresponding to one of the homoclinic orbits in Figure 3 is given by

$$\rho = \pm 2 \left(\frac{2C_1}{C_2} \right)^{1/2} \sinh \left(\frac{2k\pi}{P} \right) \mu^{1/2} \operatorname{sech}(C_1^{1/2} \mu^{1/2} z) \cos \left(\frac{2k\pi x}{P} \right) + O(\mu)$$

and is sketched in Figure 2. In keeping with the above remarks, this formula shows that the change of sign is equivalent to a translation in x through $P/2$.

It remains to compute the coefficients c_1 and c_2 . One determines c_1 using the fact that the eigenvalues of the linearised version of Hamilton's equations for $(\mathcal{M}, \Upsilon, \tilde{H}^\mu)$, which are given explicitly by $\pm(-\partial_1 H_{\text{NF}}(0, \mu))^{1/2}$, are solutions of equation (39). It follows that

$$c_1 = \frac{-2s \sinh(s) (2s \cosh(s) + \sinh(s))}{-2s\beta + 2s(1 + \beta) \cosh(2s) + \sinh(2s)},$$

where $s = 2k\pi/P$, and clearly $c_1 < 0$. The coefficient c_2 may be computed using the procedure explained by Groves and Mielke [7, Appendix B]. Recall that $u_1 + k(u_1, \mu)$ is a solution of Hamilton's equations for (M, Ω, H^λ) whenever u_1 is a solution of the reduced system $(\mathcal{M}, \Upsilon, \tilde{H}^\mu)$ and that the form taken by Hamilton's equations for the reduced system in the final coordinates is known. One can therefore substitute $u = u_1 + k(u_1, \mu)$ into equations (28) and (31) and equate powers of (q, p) to determine the coefficients in H_{NF} and the Taylor polynomials of k in a systematic fashion; the Hamiltonian structure is used to simplify the process. One finds that

$$c_2 = H_4^0(e_1, e_1, e_1, e_1) + \frac{3}{2} H_3^0(e_1, e_1, k_{0,20}),$$

where $k_{0,20}$ is the coefficient of q^2 in the power-series expansion of k ; it is found by solving the inhomogeneous linear equation

$$f_1^0 k_{0,20} = -f_2^0(e, e)$$

with inhomogeneous boundary conditions

$$\begin{aligned} \Phi_y^{k_{0,20}} &= 0 && \text{on } y = 0, \\ \Phi_y^{k_{0,20}} + \rho_x^{k_{0,20}} &= F_2(e, e) && \text{on } y = 1. \end{aligned}$$

(The notation $f_n^0 = \frac{1}{n!} d^n f^0[0]$, $F_n = \frac{1}{n!} d^n F[0]$ and $H_n = \frac{1}{n!} d^n H[0]$ has been used in the above equations.) The result is

$$\begin{aligned}
c_2 = & \frac{1}{P} \left(2s + 4s\beta \sinh(s)^2 + \sinh(2s) \right)^{-2} \\
& \times \left(-2s\beta(1 + 2 \cosh(2s)) + (1 + 3s^2\beta^2) \sinh(2s) \right)^{-1} \\
& \times \left(-2s^5(9 + \beta + 3s^2\beta^2) - 3s^5(6 + \beta) \cosh(2s) \right. \\
& \quad + 2s^5(-5 + 3\beta(1 + s^2\beta)) \cosh(4s) - s^5(2 + \beta) \cosh(6s) \\
& \quad + s^4(5 + 6s^2\beta(-3 + 2\beta)) \sinh(2s) + 2s^4(1 - 3s^2\beta^2) \sinh(4s) \\
& \quad \left. + s^4(1 + 2s^2\beta) \sinh(6s) \right),
\end{aligned}$$

and it can be verified that $c_2 > 0$ whenever $\beta > P_{1,2}^\beta$ if $k = 1$ or $P_{k,k+1}^\beta < \beta < P_{k-1,k}^\beta$ if $k > 1$.

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