Realizations of behavior for generalized chain-scattering representations

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Abstract

This paper introduces a notion of *realization of behavior* which is shown to be a generalization of the classical concept of a realization of transfer function. By using this approach, the input-output structures of the generalized chain-scattering representations (**GCSR**s) and the dual generalized chain-scattering representations (**DGCSR**s) are investigated in a behavioral theory context. Subsequently the corresponding autoregressive-moving-average (**ARMA**) representations are proposed and are proved to be realizations of behavior for any GCSR. Realization of behavior is particularly suitable for situations in which the coefficients are symbolic rather than numerical due to the fact that no numerical computation is involved in this approach.

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1 Introduction

In classical network theory, a circuit representation called the *chain matrix* [1] was widely used to deal with the cascade connection of circuits arising in analysis and synthesis problems. Based on this, Kimura [2] developed the *chain-scattering representation* which was subsequently used to provide a unified framework for H^∞ control theory. Kimura's approach is however only available to the special cases where the matrices P_{21} and P_{12} (refer to (1)) satisfy the assumption of full rank. Recently, in [3] this approach has been extended to the general case in which such conditions are essentially relaxed. From an input-output consistency point of view, the *generalised chain-scattering representation* (**GCSR**) and the *dual generalised chain-scattering representation* (**DGCSR**) emerge and are there successfully used to characterize the cascade structure property and the symmetry of general plants in a general setting.

Latterly *behavioral approach* (see e.g. [4]) has received broad acceptance as an approach for modelling dynamical systems. This approach has been shown [5] to be powerful in system modelling and analysis. However in many control contexts it is often found that the system models can easily be formulated into the input/state/output models such as Kalman state space descriptions and Rosenbrock polynomial matrix descriptions (**PMD**s). Based on such classical input/state/output representations, the action of the controller can usually be explained in a natural manner and the control aims can usually be attained effectively.

If the *physical description* of a system is known, the recently developed automated modelling approaches can be applied to find a set of equations to describe the *dynamical behavior* of the given system. It is seen however that in many cases (eg. in an electrical circuit or, more generally, in an interconnection of blocks) such a physical description is more conveniently specified through the *frequency behavior* of the system. It turns out that the mechanism of this system modelling approach can be interpreted through the notion of *realization of behavior* which we shall introduce in this paper. In fact, as we shall see later, realization of behavior, in many cases, amounts to the introduction of latent variables in the time domain. From this point of view, realization of behavior can be understood to be a converse procedure of the latent variable elimination theorem [4] in a particular sense. It should also be noted that realization of behavior also generalizes the notion of transfer function matrix realization in the classical control theory framework.

Recently in [9], a realization approach was suggested that reduces high-order linear differential equations to the first-order system representations by using the method of "linearization". From the point of view of *realization* in a physical sense one is, however, forced to start from the system frequency behavior description into which system behavior is generally described rather than from the high-order linear differential equations in the time domain. One of the main aims of this paper is to present a new notion of *realization of behavior*. Further to the results of [3], the input-output structures of the GCSRs are thus clarified by using this approach. These results are interesting in that they provide a good insight into the natural relationship between the (frequency) behavior of any GCSR and the (dynamical) behavior of the corresponding ARMA representations.

Consider a plant P (Fig 1) with two kinds of inputs $(w(s), u(s))$ and two kinds of outputs $(z(s), y(s))$ represented by

$$\begin{bmatrix} z(s) \\ y(s) \end{bmatrix} = P(s) \begin{bmatrix} w(s) \\ u(s) \end{bmatrix} = \begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix} \begin{bmatrix} w(s) \\ u(s) \end{bmatrix}, \tag{1}$$

where $P_{ij}(s)$ $(i = 1, 2;\ j = 1, 2)$ are all rational matrices with dimensions $m_i \times k_j$ $(i = 1, 2;\ j = 1, 2)$.

Definition 1 [3] : An input-output pair $(u(s), y(s))$ is said to be *consistent* about w for the plant P if there exists at least one input $w(s)$ satisfying $P_{21}(s)w(s) = y(s) - P_{22}(s)u(s)$. In this case, P is said to be *input-output consistent* about w with respect to the input-output pair $(u(s), y(s))$.

Theorem 1 [3]: The plant P is input-output consistent about w with respect to the input-output pair $(u(s), y(s))$ if and only if

$$\left(I - P_{21}(s)P_{21}^{-}(s)\right)\left[-P_{22}(s), I\right]\begin{bmatrix} u(s) \\ y(s) \end{bmatrix} = 0,$$

where $P_{21}^{-}(s)$ is any $\{1\}$-inverse of the rational matrix $P_{21}(s)$.

Theorem 2 [3]: If the plant P is consistent about w with respect to the input-output pair $(u(s), y(s))$, then (1) can be written as

$$\begin{bmatrix} z(s) \\ w(s) \end{bmatrix} = GCHAIN(P; P_{21}^{-})\begin{bmatrix} u(s) \\ y(s) \\ h(s) \end{bmatrix}, \tag{2}$$

where we denote the matrix

$$\begin{aligned} GCHAIN(P; P_{21}^{-}) &:= \left[\begin{array}{c|c} GCHAIN^{*}(P; P_{21}^{-}) & \Delta GCHAIN(P; P_{21}^{-}) \end{array}\right] \\ &= \left[\begin{array}{cc|c} P_{12}(s) - P_{11}(s)P_{21}^{-}(s)P_{22}(s) & P_{11}(s)P_{21}^{-}(s) & P_{11}(s)(I - P_{21}^{-}(s)P_{21}(s)) \\ -P_{21}^{-}(s)P_{22}(s) & P_{21}^{-}(s) & I - P_{21}^{-}(s)P_{21}(s) \end{array}\right], \end{aligned} \tag{3}$$

where $h(s)$ is arbitrary rational vector, and $P_{21}^{-}(s)$ is any $\{1\}$-inverse of $P_{21}(s)$, i.e. any matrix satisfying $(\cdot)(\cdot)^{-}(\cdot) = (\cdot)$.

Definition 2 [3]: The relation (2) is called a *generalised chain-scattering representation* (**GCSR**) of the plant P and any matrix $GCHAIN(P; P_{21}^{-})$ is termed a *GCSR matrix*.

The GCSR (2) is schematically shown in Fig. 2.

It should be noted that, under the condition that P is consistent about u with respect to $(z(s), w(s))$, the *dual generalised chain-scattering representation* (DGCSR) exists and can be formulated [3] in a similar manner. It should also be noted that unlike the Kimura approach [2], the formulations of the GCSRs and the DGCSRs are not unique due to the fact that the $\{1\}$-inverses of a matrix are not unique.

2 Behavior realization

This section introduces the concept of *realization of behavior*. Recall that in the behavioral framework the (dynamical) behavioural equations of an autoregressive-moving-average (**ARMA**) representation [4] are

$$R_1(\rho)u_1(t) + R_2(\rho)y_1(t) = S(\rho)\xi(t), \tag{4}$$

where $w' := [(y_1(t))^T, (u_1(t))^T]^T$ stands for the external variables representing the dynamical behavior of the underlying dynamical system, $\xi(t)$ which are called latent variables corresponding to auxiliary variables resulting from the modelling procedure. $R_1(\rho)$, $R_2(\rho)$ and $S(\rho)$ are polynomial matrices containing the differential operator $\rho = d/dt$. In order to distinguish them from the existing notation y, u of (1), the external variables are denoted by y_1 and u_1. When $S(\rho) = 0$, (4) is termed [4] an autoregressive (**AR**) representation.

In the following approach we are interested in the external behavior of the system (4), where we choose the underlying function space to be $\mathcal{C}^\infty := \mathcal{C}^\infty(\mathcal{R}, \Re)$, this function space consists of all the infinitely differentiable functions which are defined for all time $\mathcal{R} := [0, +\infty)$ and take values in the real number field $\Re$. For brevity, we write $\mathcal{C}_k^\infty := \mathcal{C}^\infty(\mathcal{R}, \Re^k)$. Then the dynamical external behavior of (4) is given by

$$\begin{aligned}
\mathcal{B}_d(R_1, R_2; S) : &= \left\{ \begin{bmatrix} y_1(t) \\ u_1(t) \end{bmatrix} \in \mathcal{C}_{m+p}^\infty \;\middle|\; \exists \xi(t) \in \mathcal{C}_n^\infty \text{ so that (4) is valid} \right\} \\
&= \left\{ \begin{bmatrix} y_1(t) \\ u_1(t) \end{bmatrix} \in \mathcal{C}_{m+p}^\infty \;\middle|\; [R_2(\rho), R_1(\rho)] \begin{bmatrix} y_1(t) \\ u_1(t) \end{bmatrix} \in ImS(\rho) \right\}.
\end{aligned} \tag{5}$$

From the above, to avoid the trivial case that the external behavior is empty, i.e., to ensure that there exist latent variables, every pair $(y_1(t), u_1(t))$ in the external behavior must be consistent, that is

$$(I - S(\rho)S^-(\rho))(R_1(\rho)u_1(t) + R_2(\rho)y_1(t)) = 0, \; \forall \begin{bmatrix} y_1(t) \\ u_1(t) \end{bmatrix} \in \mathcal{B}_d(R_1, R_2; S),$$

where $\{1\}$-inverse is arbitrary. It is immediately noted that, when $S(\rho)$ is invertible or more specially when $S(\rho) = I$, the above condition is automatically satisfied.

In many real cases (for example, electrical circuits), however, system behavior is usually described in the frequency domain as

$$A(s)u^*(s) + B(s)y^*(s) = C(s)\eta(s), \tag{6}$$

where $A(s) \in \Re(s)^{q\times p}$, $B(s) \in \Re(s)^{q\times m}$ and $C(s) \in \Re(s)^{q\times n}$, as the following example suggests, are not polynomial but rational matrices. The vector-valued signals $u^*(s)$,

$y^*(s)$ and $\eta(s)$ live in the square (Lebesgue) integrable functional spaces $\mathbf{L}_2^p$, $\mathbf{L}_2^m$ and $\mathbf{L}_2^n$ respectively.

As a special case of (6) when $C(s) = 0$ and $B(s)$ is invertible, (6) determines a transfer function $G(s) = -B^{-1}(s)A(s)$. The frequency behavior of (6) is given by

$$\begin{aligned}\mathcal{B}_f(A, B; C) :=: \; &= \left\{ \begin{bmatrix} y^*(s) \\ u^*(s) \end{bmatrix} \in \mathbf{L}_2^{m+p} \mid \exists \eta(s) \in \mathbf{L}_2^n \text{ so that (6) is valid} \right\} \\ &= \left\{ \begin{bmatrix} y^*(s) \\ u^*(s) \end{bmatrix} \in \mathbf{L}_2^{m+p} \mid [B(s), A(s)] \begin{bmatrix} y^*(s) \\ u^*(s) \end{bmatrix} \in Im\ C(s) \right\}. \end{aligned} \quad (7)$$

In relation to (4) denoting

$$\mathcal{L}(\mathcal{B}_d)(R_1, R_2; S) = \left\{ \begin{bmatrix} \hat{y}_1(s) \\ \hat{u}_1(s) \end{bmatrix} \Bigg| \begin{bmatrix} \hat{y}_1(s) \\ \hat{u}_1(s) \end{bmatrix} = \mathcal{L}(\begin{bmatrix} y_1(t) \\ u_1(t) \end{bmatrix}), \right.$$
$$\left. \begin{bmatrix} y_1(t) \\ u_1(t) \end{bmatrix} \in \mathcal{B}_d(R_1, R_2; S), \rho^i y_1(0) = 0, \rho^i u_1(0) = 0, \rho^i \xi(0) = 0, i = 0, 1, \cdots \right\},$$

where $\mathcal{L}(f(t))$ denotes the Laplace transformation of $f(t)$, the definition of realization of behavior follows.

Definition 3 : Given a frequency behavior description (6), if there exists an ARMA representation (4), i.e., there exist polynomial matrices $R_1(\rho)$, $R_2(\rho)$ and $S(\rho)$ such that

$$\mathcal{L}(\mathcal{B}_d)(R_1, R_2; S) = \mathcal{B}_f(A, B; C),$$

then the ARMA representation (4) is said to be a *realization of behavior* for (6).

Remark 1 : The above concept is a generalization to the classical notion of realization of transfer function matrix.

To see this, let us consider the special case, when $B(s)$ is invertible and $C(s) = 0$. Then (6) determines a transfer matrix and can be written as

$$y^*(s) = -B^{-1}(s)A(s)u^*(s). \quad (8)$$

If there exist polynomial matrices $T(\rho)$, $U(\rho)$, $V(\rho)$ and $W(\rho)$ of appropriate dimensions such that $T(\rho)$ is invertible and

$$-B^{-1}(s)A(s) = V(s)T^{-1}(s)U(s) + W(s), \quad (9)$$

then by definition 3, it is easy to verify that the following ARMA representation

$$\begin{bmatrix} I \\ 0 \end{bmatrix} y_1(t) + \begin{bmatrix} -W(\rho) \\ U(\rho) \end{bmatrix} u_1(t) = \begin{bmatrix} V(\rho) \\ T(\rho) \end{bmatrix} x(t) \quad (10)$$

is a realization of behavior for the frequency behavior description (6). It is noted that (10) is nothing but the Rosenbrock PMD

$$\left\{ \begin{aligned} T(\rho)x(t) &= U(\rho)u_1(t) \\ y_1(t) &= V(\rho)x(t) + W(\rho)u_1(t). \end{aligned} \right.$$

The condition of consistency is seen to be satisfied because of the invertibility of $T(\rho)$.

When $T(\rho) = \rho E - A$ with E singular, the above PMD is termed a singular system, while when $T(\rho) = \rho I - A$, the above description is known as the conventional state space system. It is clearly seen that in the above special cases, realization of behavior is equivalent to realization of transfer function in the classical sense.

3 Realizations of behavior for GCSRs

Before developing the realization of behavior for GCSRs, we will establish a realization of behavior for the general plant. Given the general plant described by

$$\left[\begin{array}{c} z(s) \\ y(s) \end{array}\right] = P(s)\left[\begin{array}{c} w(s) \\ u(s) \end{array}\right] = \left[\begin{array}{cc} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{array}\right]\left[\begin{array}{c} w(s) \\ u(s) \end{array}\right], \tag{11}$$

where $P_{ij}(s)$ $(i = 1, 2;\ j = 1, 2)$ are all rational matrices with dimensions $m_i \times k_j$ $(i = 1, 2;$ $j = 1, 2)$, consider the rational matrix $P(s) \in \Re(s)^{(m_1+m_2)\times(k_1+k_2)}$. It is well-known [7] that there always exists a non-unique polynomial matrix pair $P_1(s) \in \Re[s]^{(m_1+m_2)\times(m_1+m_2)}$ and $P_2(s) \in \Re[s]^{(m_1+m_2)\times(k_1+k_2)}$ such that

$$P(s) = P_1^{-1}(s)P_2(s). \tag{12}$$

It should be noted in here $P_1(s)$ and $P_2(s)$ do not need to be coprime. In this way, the following result is obtained.

Theorem 3 : The following AR representation

$$-P_2(\rho)u_1(t) + P_1(\rho)y_1(t) = 0 \tag{13}$$

is a realization of behavior for the general plant (11), where the external variables are denoted by

$$\left[\begin{array}{c} y_1(t) \\ u_1(t) \end{array}\right] := \left[\begin{array}{c} z(t) \\ y(t) \\ \hline w(t) \\ u(t) \end{array}\right],$$

and the polynomial matrices $P_1(\rho)$, $P_2(\rho)$ satisfy (12).

Proof : Under the decomposition (12), (11) can be written into

$$P_1(s)\left[\begin{array}{c} z(s) \\ y(s) \end{array}\right] = P_2(s)\left[\begin{array}{c} w(s) \\ u(s) \end{array}\right],$$

the above frequency behavior is seen to be $\mathcal{B}_f = Ker([P_1(s), -P_2(s)])$, while the dynamical external behavior of the AR representation (13) is

$$\mathcal{B}_d = \left\{\left[\begin{array}{c} y_1(t) \\ u_1(t) \end{array}\right] \in \mathcal{C}^{\infty}_{m_1+m_2+k_1+k_2} \ \mid\ [P_1(\rho), -P_2(\rho)]\left[\begin{array}{c} y_1(t) \\ u_1(t) \end{array}\right] = 0\right\}.$$

The Laplace transformation of (13) with zero initial conditions yields

$$-P_2(s)\hat{u}_1(s) + P_1(s)\hat{y}_1(s) = 0, \tag{14}$$

where $\hat{u}_1(s) := \int_0^{+\infty} u_1(t)e^{-st}dt$, $\hat{y}_1(s) := \int_0^{+\infty} y_1(t)e^{-st}dt$. Thus (14) gives

$$\mathcal{L}(\mathcal{B}_d) = \mathcal{B}_f.$$

Hence the theorem follows from Definition 3. $\square$

Remark 2 The above realization is not unique due to the fact that the decomposition (12) is not unique.

Recall now Theorem 2. If the input-output pair $(u(s), y(s))$ is consistent about w to the plant P, then the GCSR is represented by

$$\begin{aligned}\begin{bmatrix} z(s) \\ w(s) \end{bmatrix} &= GCHAIN(P; P_{21}^{-}) \begin{bmatrix} u(s) \\ y(s) \\ h(s) \end{bmatrix} \\ &= GCHAIN^{*}(P; P_{21}^{-}) \begin{bmatrix} u(s) \\ y(s) \end{bmatrix} + \Delta GCHAIN(P; P_{21}^{-}) h(s). \qquad (15)\end{aligned}$$

The above GCSR gives rise to the frequency behavior

$$\mathcal{B}_f(I, -GCHAIN(P; P_{21}^{-})) := \left\{ \begin{bmatrix} y^*(s) \\ u^*(s) \end{bmatrix} := \begin{bmatrix} z(s) \\ w(s) \\ \hline u(s) \\ y(s) \end{bmatrix} \middle| \begin{array}{c} y^*(s) = GCHAIN^*(P; P_{21}^{-}) u^*(s) + \Delta GCHAIN(P; P_{21}^{-}) h(s), \\ \\ h(s) \text{ is arbitrary}, (u(s), y(s)) \text{ is consistent to } P \end{array} \right\}.$$

It can be proved that every GCSR gives rise to the same frequency behavior, in other words, the frequency behavior of GCSRs is independent of the particular $\{1\}$-inverse. The following theorem establishes this observation.

Theorem 4 : Given any two GCSRs $GCHAIN(P; P_{21}^{-})$, $GCHAIN(P; P_{21}^{g})$ which are formulated in terms of two $\{1\}$-inverses of P_{21} respectively, one has

$$\mathcal{B}_f(I, -GCHAIN(P; P_{21}^{-})) = \mathcal{B}_f(I, -GCHAIN(P; P_{21}^{g})).$$

Proof: One only needs to prove that $\mathcal{B}_f(I, -GCHAIN(P; P_{21}^{-})) \subseteq \mathcal{B}_f(I, -GCHAIN(P; P_{21}^{g}))$. The converse statement $\mathcal{B}_f(I, -GCHAIN(P; P_{21}^{g})) \subseteq \mathcal{B}_f(I, -GCHAIN(P; P_{21}^{-}))$ can be proved similarly.

From [6], there exists a matrix $K(s)$ such that

$$P_{21}^{-}(s) = P_{21}^{g}(s) + K(s) - P_{21}^{g}(s) P_{21}(s) K(s) P_{21}(s) P_{21}^{g}(s). \qquad (16)$$

Also the input-output pair $(u(s), y(s))$, if being consistent to the plant P, must satisfy (Theorem 1)

$$(I - P_{21}(s) P_{21}^{g}(s)) \left[-P_{22}(s), I\right] \begin{bmatrix} u(s) \\ y(s) \end{bmatrix} = 0.$$

This follows that

$$y(s) - P_{22}(s) u(s) = P_{21}(s) P_{21}^{g}(s) (y(s) - P_{22}(s) u(s)). \qquad (17)$$

Given any $\begin{bmatrix} y^*(s) \\ u^*(s) \end{bmatrix} \in \mathcal{B}_f(I, -GCHAIN(P; P_{21}^{-}))$, there should be a rational vector $h_1(s)$ such that

$$\begin{bmatrix} z(s) \\ w(s) \end{bmatrix} = y^*(s) = GCHAIN^*(P; P_{21}^{-}) u^*(s) + \Delta GCHAIN(P; P_{21}^{-}) h_1(s)$$

$$
\begin{aligned}
&= \begin{bmatrix} P_{12}(s) - P_{11}(s)P_{21}^{-}(s)P_{22}(s) & P_{11}(s)P_{21}^{-}(s) \\ -P_{21}^{-}(s)P_{22}(s) & P_{21}^{-}(s) \end{bmatrix} \begin{bmatrix} u(s) \\ y(s) \end{bmatrix} \\
&\quad + \begin{bmatrix} P_{11}(s)(I - P_{21}^{-}(s)P_{21}(s)) \\ I - P_{21}^{-}(s)P_{21}(s) \end{bmatrix} h_1(s). \qquad (18)
\end{aligned}
$$

By substituting (16) and (17) into (18), one yields

$$
\begin{aligned}
z(s) &= (P_{12}(s) - P_{11}(s)P_{21}^{-}(s)P_{22}(s))u(s) + P_{11}(s)P_{21}^{-}(s)y(s) + P_{11}(s)(I - P_{21}^{-}(s)P_{21}(s))h_1(s) \\
&= [P_{12}(s) - P_{11}(s)(P_{21}^{g}(s) + K(s) - P_{21}^{g}(s)P_{21}(s)K(s)P_{21}(s)P_{21}^{g}(s))P_{22}(s)]u(s) + \\
&\quad +P_{11}(s)(P_{21}^{g}(s) + K(s) - P_{21}^{g}(s)P_{21}(s)K(s)P_{21}(s)P_{21}^{g}(s))y(s) + \\
&\quad +P_{11}(s)[I - (P_{21}^{g}(s) + K(s) - P_{21}^{g}(s)P_{21}(s)K(s)P_{21}(s)P_{21}^{g}(s))P_{21}(s)]h_1(s) \\
&= (P_{12}(s) - P_{11}(s)P_{21}^{g}(s)P_{22}(s))u(s) + P_{11}(s)P_{21}^{g}(s)y(s) + \\
&\quad +P_{11}(s)(I - P_{21}^{g}(s)P_{21}(s))[K(s)(y(s) - P_{22}(s)u(s)) + (I - K(s)P_{21}(s))h_1(s)]; \\
w(s) &= -P_{21}^{-}(s)P_{22}(s)u(s) + P_{21}^{-}(s)y(s) + (I - P_{21}^{-}(s)P_{21}(s))h_1(s) \\
&= -(P_{21}^{g}(s) + K(s) - P_{21}^{g}(s)P_{21}(s)K(s)P_{21}(s)P_{21}^{g}(s))P_{22}(s)u(s) + \\
&\quad +(P_{21}^{g}(s) + K(s) - P_{21}^{g}(s)P_{21}(s)K(s)P_{21}(s)P_{21}^{g}(s))y(s) + \\
&\quad +[I - (P_{21}^{g}(s) + K(s) - P_{21}^{g}(s)P_{21}(s)K(s)P_{21}(s)P_{21}^{g}(s))P_{21}(s)]h_1(s) \\
&= -P_{21}^{g}(s)P_{22}(s)u(s) + P_{21}^{g}(s)y(s) + \\
&\quad +(I - P_{21}^{g}(s)P_{21}(s))[K(s)(y(s) - P_{22}(s)u(s)) + (I - K(s)P_{21}(s))h_1(s)].
\end{aligned}
$$

By letting $h_2(s) := K(s)(y(s)-P_{22}(s)u(s))+(I-K(s)P_{21}(s))h_1(s)$, the above formulations about $z(s)$ and $w(s)$ can be written into the following matrix form

$$
\begin{bmatrix} z(s) \\ w(s) \end{bmatrix} = GCHAIN^{*}(P; P_{21}^{g})u^{*}(s) + \Delta GCHAIN(P; P_{21}^{g})h_2(s),
$$

which displays the fact that $\begin{bmatrix} y^{*}(s) \\ u^{*}(s) \end{bmatrix} \in \mathcal{B}_f(I, -GCHAIN(P; P_{21}^{g}))$, subsequently $\mathcal{B}_f(I, -GCHAIN(P; P_{21}^{-})) \subseteq \mathcal{B}_f(I, -GCHAIN(P; P_{21}^{g}))$. This finishes the proof. □

By virtue of the above theorem, the frequency behavior of any GCSR thus can be simply denoted by $\mathcal{B}_f(I, -GCHAIN(P))$.

One of the remaining aims of this section is to show how the frequency behavior of GCSRs can be realised as the dynamical behavior of an ARMA representation through the approach of realization of behavior. To this end, the general plant (11) is rewritten into

$$
\begin{bmatrix} z(s) \\ y(s) \end{bmatrix} = \begin{bmatrix} \dfrac{P_{11}^{*}(s)}{g(s)} & \dfrac{P_{12}^{*}(s)}{g(s)} \\ \dfrac{P_{21}^{*}(s)}{g(s)} & \dfrac{P_{22}^{*}(s)}{g(s)} \end{bmatrix} \begin{bmatrix} w(s) \\ u(s) \end{bmatrix}, \qquad (19)
$$

where $g(s)$ is the least common (monic) multiple of the denominator polynomials of all the entries in $P(s)$, and $P_{ij}^{*}(s)/g(s) = P_{ij}(s)$, $i = 1, 2; j = 1, 2$. It is immediately noted that the above decomposition of P is a special case of (12). By letting

$$
\begin{bmatrix} z_c(s) \\ y_c(s) \end{bmatrix} = g(s)I_{m_1+m_2} \begin{bmatrix} z(s) \\ y(s) \end{bmatrix}, \qquad (20)
$$

$$P^*(s) = \begin{bmatrix} P_{11}^*(s) & P_{12}^*(s) \\ P_{21}^*(s) & P_{22}^*(s) \end{bmatrix}, \tag{21}$$

where $I_{m_1+m_2}$ is the identity matrix with dimension $m_1 + m_2$. (19) thus takes the form

$$\begin{bmatrix} z_c(s) \\ y_c(s) \end{bmatrix} = P^*(s) \begin{bmatrix} w(s) \\ u(s) \end{bmatrix}. \tag{22}$$

It should be noted that in here $P^*(s)$ is a polynomial matrix and $g(s)$ is a polynomial.

As it is seen before that the realization of behavior for a general plant is rather straightforward, while the realization of behavior for GCSR is much more obscure, as in this case the introduction of latent variables is necessary. To propose a realization of behavior for GCSRs, consider the following ARMA representation

$$\left[\begin{array}{cc} P_{22}^*(\rho) & -g(\rho)I \\ \hline P_{12}^*(\rho) & 0 \\ 0 & 0 \end{array}\right] \begin{bmatrix} u(t) \\ y(t) \end{bmatrix} + \left[\begin{array}{cc} 0 & 0 \\ \hline -g(\rho)I & 0 \\ 0 & -I \end{array}\right] \begin{bmatrix} z(t) \\ w(t) \end{bmatrix} = \left[\begin{array}{c} P_{21}^*(\rho) \\ \hline P_{11}^*(\rho) \\ I \end{array}\right] x(t),\ t \geq 0. \tag{23}$$

The above ARMA representation is in fact

$$\left\{ \begin{array}{rcl} P_{21}^*(\rho)x(t) & = & [P_{22}^*(\rho), -g(\rho)I] \begin{bmatrix} u(t) \\ y(t) \end{bmatrix} \\ \begin{bmatrix} g(\rho)I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} z(t) \\ w(t) \end{bmatrix} & = & \begin{bmatrix} -P_{11}^*(\rho) \\ -I \end{bmatrix} x(t) + \begin{bmatrix} P_{12}^*(\rho) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u(t) \\ y(t) \end{bmatrix}, \end{array} \right. \tag{24}$$

where all the identity matrices and all the zero block matrices are of appropriate dimensions. $x(t) \in \mathcal{C}_{k_1}^\infty$ are the latent variables. It is noted that the condition every pair

$$\begin{bmatrix} y_1(t) \\ u_1(s) \end{bmatrix} := \left[\begin{array}{c} z(t) \\ w(t) \\ \hline u(t) \\ y(t) \end{array}\right] \in \mathcal{B}_d$$

is consistent about the latent variables $x(t)$ is equivalent to that every input-output pair $(u(s), y(s))$ is consistent about w.

Now we are ready to state and prove the following important result.

Theorem 5 : The ARMA representation (23) is a realization of behavior for any GCSR $GCHAIN(P; P_{21}^-)$.

Proof: Given any GCSR, its frequency behavior is $\mathcal{B}_f(I, -GCHAIN(P))$. The dynamical external behavior of the ARMA representation (23) is

$$\mathcal{B}_d = \left\{ \begin{bmatrix} y_1(t) \\ u_1(t) \end{bmatrix} = \left[\begin{array}{c} z(t) \\ w(t) \\ \hline u(t) \\ y(t) \end{array}\right] \in \mathcal{C}_{m_1+m_2+k_1+k_2}^\infty \;\middle|\; \exists x(t) \in \mathcal{C}_{k_1}^\infty \text{ so that (23) is valid} \right\}.$$

$\forall \begin{bmatrix} y_1(t) \\ u_1(t) \end{bmatrix} \in \mathcal{B}_d$, to ensure that there exists $x(t)$ such that (23) i.e., (24) is valid, $u_1(t)$ must be consistent to (24). This suggests that

$$[I - P_{21}^*(\rho)(P_{21}^*(\rho))^-]\,[P_{22}^*(\rho), -g(\rho)I] \begin{bmatrix} u(t) \\ y(t) \end{bmatrix} = 0.$$

it is easily seen to be equivalent to

$$\begin{bmatrix} \hat{u}(s) \\ \hat{y}(s) \end{bmatrix} \in Ker\{(I - P_{21}(s)P_{21}^{-}(s))\,[P_{22}(s), -I]\}$$

The Laplace transformation of (24) (with zero initial conditions) yields

$$P_{21}^{*}(s)\hat{x}(s) \;=\; [P_{22}^{*}(s), -g(s)I] \begin{bmatrix} \hat{u}(s) \\ \hat{y}(s) \end{bmatrix}, \tag{25}$$

and

$$\begin{bmatrix} g(s)I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \hat{z}(s) \\ \hat{w}(s) \end{bmatrix} \;=\; \begin{bmatrix} -P_{11}^{*}(s) \\ -I \end{bmatrix} \hat{x}(s) + \begin{bmatrix} P_{12}^{*}(s) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{u}(s) \\ \hat{y}(s) \end{bmatrix}, \tag{26}$$

where $\hat{f}(s) := \int_0^{+\infty} f(t)e^{-st}dt$.

Due to the consistency of $\begin{bmatrix} \hat{u}(s) \\ \hat{y}(s) \end{bmatrix}$, the equation (25) determines the latent variables $\hat{x}(s)$. By solving the latent variables $\hat{x}$ in (25) and then substituting into (26), one obtains

$$\begin{aligned} \begin{bmatrix} g(s)I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \hat{z}(s) \\ \hat{w}(s) \end{bmatrix} \;&=\; \begin{bmatrix} P_{12}^{*}(s) - P_{11}^{*}(s)(P_{21}^{*})^{-}(s)P_{22}^{*}(s) & g(s)P_{11}^{*}(s)(P_{21}^{*})^{-}(s) \\ -(P_{21}^{*})^{-}(s)P_{22}^{*}(s) & g(s)(P_{21}^{*})^{-}(s) \end{bmatrix} \begin{bmatrix} \hat{u}(s) \\ \hat{y}(s) \end{bmatrix} \\ &\quad + \begin{bmatrix} P_{11}^{*}(s)[I - (P_{21}^{*})^{-}(s)P_{21}^{*}(s)] \\ I - (P_{21}^{*})^{-}(s)P_{21}^{*}(s) \end{bmatrix} h(s), \end{aligned} \tag{27}$$

where $h(s)$ is any rational vector. By noticing that $P_{ij}^{*}(s) = g(s)P_{ij}(s)$, $i = 1, 2, j = 1, 2$, and that $P_{21}^{-}(s) = g(s)(P_{21}^{*})^{-}(s)$, (27) can also be written into

$$\begin{aligned} \begin{bmatrix} \hat{z}(s) \\ \hat{w}(s) \end{bmatrix} \;&=\; \begin{bmatrix} P_{12}(s) - P_{11}(s)P_{21}^{-}(s)P_{22}(s) & P_{11}(s)P_{21}^{-}(s) \\ -P_{21}^{-}(s)P_{22}(s) & P_{21}^{-}(s) \end{bmatrix} \begin{bmatrix} \hat{u}(s) \\ \hat{y}(s) \end{bmatrix} \\ &\quad + \begin{bmatrix} P_{11}(s)(I - P_{21}^{-}(s)P_{21}(s)) \\ I - P_{21}^{-}(s)P_{21}(s) \end{bmatrix} h(s). \end{aligned} \tag{28}$$

It is thus seen that

$$\begin{bmatrix} y^{*}(s) \\ u^{*}(s) \end{bmatrix} = \begin{bmatrix} \hat{z}(s) \\ \hat{w}(s) \\ \hline \hat{u}(s) \\ \hat{y}(s) \end{bmatrix} \in \mathcal{B}_f(I, -GCHAIN(P)).$$

So far it has been proved that $\mathcal{L}(\mathcal{B}_d) \subseteq \mathcal{B}_f(I, -GCHAIN(P))$. To prove the statement $\mathcal{B}_f(I, -GCHAIN(P)) \subseteq \mathcal{L}(\mathcal{B}_d)$, let

$$\begin{bmatrix} y^{*}(s) \\ u^{*}(s) \end{bmatrix} \in \mathcal{B}_f(I, -GCHAIN(P)),$$

there should be a rational vector $h_1(s)$ such that

$$y^{*}(s) = GCHAIN^{*}(P; P_{21}^{-})u^{*}(s) + \Delta GCHAIN(P; P_{21}^{-})h_1(s), \tag{29}$$

for any $\{1\}$-inverse of P_{21}. Furthermore the input-output pair $(u(s), y(s))$ must be consistent to the plant P. Now let

$$x(s) = (P_{21}^*)^-(s)[P_{22}^*(s), -g(s)I]u^*(s) + [I - (P_{21}^*)^-(s)P_{21}^*(s)]h_1(s),$$

using the consistency of $(u(s), y(s))$, it is easy to verify that the variables y^*, u^* and x satisfy (25) and (26), this is to say that

$$\begin{bmatrix} y^*(s) \\ u^*(s) \end{bmatrix} \in \mathcal{L}(\mathcal{B}_d).$$

□

On noticing (20), one can write the ARMA representation (23) into the following Rosenbrock PMD

$$\begin{cases} P_{21}^*(\rho)x(t) = [P_{22}^*(\rho), -I]\begin{bmatrix} u(t) \\ y_c(t) \end{bmatrix} \\ \begin{bmatrix} z_c(t) \\ w(t) \end{bmatrix} = \begin{bmatrix} -P_{11}^*(\rho) \\ -I \end{bmatrix} x(t) + \begin{bmatrix} P_{12}^*(\rho) & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} u(t) \\ y_c(t) \end{bmatrix}. \end{cases} \tag{30}$$

It is easily seen that $P_{21}^-(s) = g(s)(P_{21}^*)^-(s)$. By substituting the above and $P_{ij}(s) = P_{ij}^*(s)/g(s)$, $i = 1, 2$; $j = 1, 2$, into the GCSR

$$\begin{bmatrix} z(s) \\ w(s) \end{bmatrix} =$$

$$= \left[\begin{array}{cc|c} P_{12}(s) - P_{11}(s)P_{21}^-(s)P_{22}(s) & P_{11}(s)P_{21}^-(s) & P_{11}(s)(I - P_{21}^-(s)P_{21}(s)) \\ -P_{21}^-(s)P_{22}(s) & P_{21}^-(s) & I - P_{21}^-(s)P_{21}(s) \end{array}\right]\begin{bmatrix} u(s) \\ y(s) \\ h(s) \end{bmatrix},$$

using the notations of $z_c(s) = g(s)z(s)$ and $y_c(s) = g(s)y(s)$, one can find that

$$\begin{bmatrix} z_c(s) \\ w(s) \end{bmatrix} = GCHAIN(P^*; (P_{21}^*)^-)\begin{bmatrix} u(s) \\ y_c(s) \\ h(s) \end{bmatrix},$$

where we denote the matrix

$$GCHAIN(P^*; (P_{21}^*)^-) :=$$
$$\left[\begin{array}{cc|c} P_{12}^*(s) - P_{11}^*(s)(P_{21}^*)^-(s)P_{22}^*(s) & P_{11}^*(s)(P_{21}^*)^-(s) & P_{11}^*(s)(I - (P_{21}^*)^-(s)P_{21}^*(s)) \\ -(P_{21}^*)^-(s)P_{22}^*(s) & (P_{21}^*)^-(s) & I - (P_{21}^*)^-(s)P_{21}^*(s) \end{array}\right].$$

A further result concerning the realization of behavior for any GCSR $GCHAIN(P^*; (P_{21}^*)^-)$ is the following theorem.

Theorem 6 : The Rosenbrock PMD (30) is a realization of behavior for any GCSR $GCHAIN(P^*, (P_{21}^*)^-)$

Proof: This follows readily from Theorem 5 on noting that

$$\begin{bmatrix} z_c(s) \\ w(s) \end{bmatrix} = GCHAIN(P^*; (P_{21}^*)^-)\begin{bmatrix} u(s) \\ y_c(s) \\ h(s) \end{bmatrix},$$

where $h(s)$ is arbitrary rational vector, and that in the matrix P^* the least common (monic) multiple of the denominator polynomials of all the entries is 1. □

Remark 3 The realization of behavior for the dual generalised chain-scattering representations of the plant P can be proposed in a completely analogous manner.

Remark 4 The above theorems are interesting, not least for the way in which they clarify the input-output structure of GCSRs and that of DGCSRs. More importantly than this, however, is the observation that the frequency behavior of any GCSR or any DGCSR can be completely recovered, in a precise way by introducing latent variables, to the dynamical behavior of the ARMA representations via the approach of realization of behavior.

4 Conclusions

In this paper, the input-output structures of the chain-scattering representation approach [2], [3] has been investigated in a behavioral framework. A new notion of realization of behavior has been presented. It has been shown that realization of behavior generalizes the classical concept of realization of transfer function matrix by virtue of that the input consistency essentially relaxed the condition of full rank which is put on the relevant matrix to ensure the existency of transfer function. The basic idea in this approach is to find an ARMA representation for a given frequency behavior description such that the known frequency behavior is completely recovered to the corresponding dynamical behavior. From this point of view, realization of behavior is seen to be a converse procedure to the latent variable elimination process [4]. Such a realization approach is believed to be highly significant in modelling dynamical system in certain real cases where the system behavior is conveniently described in the frequency domain. Since no numerical computation is needed, the realization of behavior procedure is believed to be particularlly suitable for situations in which the coefficients are symbolic rather than numerical.

Based on this approach, the behavior structures of GCSRs have been clarified. It has been shown that the behavior is independant of the GCSR parametrisation. Subsequently corresponding autoregressive-moving-average (**ARMA**) representations are proposed and are proved to be realizations of behavior for any GCSR. Specifically, certain Rosenbrock PMDs are found to be the realizations of behavior for any GCSR $GCHAIN(P^*; (P^*_{21})^-)$. Once these **ARMA** representations are proposed, one can further find the corresponding first-order system representations by using the method of [9] or other well-developed realization approaches such as [10]. These results are therefore interesting in that they provide a natural linkage between the relatively new chain-scattering approach [2], the well-developed Rosenbrock PMD theory [8], and the developing theory of behaviors [4].

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