# Inverse Problem in a Weakly Horizontally Inhomogeneous Layered Medium

A.S. Blagovestchenskii [*]     Yaroslav Kurylev[†]

Victor Zalipaev[‡]

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## 1   Introduction

In this paper we consider an inverse boundary-value problem for an acoustic wave equation in the half-space $z > 0$,

$$Au_{tt} - \text{div}(B\nabla u) = 0, \quad u|_{t<0} = 0, \ u|_{z=0} = \delta(x)\delta(t), \tag{1}$$

with the inverse data of the form

$$u_z|_{z=0} = R(x,t). \tag{2}$$

Having in mind applications to geophysics and seismology, we consider the case when the dependence of $A$ and $B$ on horizontal coordinates, $x$, is much weaker than on the depth, $z$, i.e.

$$A = A(z, \varepsilon x) = A_0(z) + \varepsilon x A_1(z) + \ldots; \quad B = B(z, \varepsilon x) = B_0(z) + \varepsilon x B_1(z) + \ldots. \tag{3}$$

For conventional simplicity we assume that $x \in \mathbb{R}^1$, however, the method developed in the paper may be generalised for $x \in \mathbb{R}^n$, $n > 1$. We also restrict our considerations to the reconstruction of $A_0$, $B_0$ and $A_1$, $B_1$ only, although, in principle, the method may be extended to find $A_2$, $B_2$, etc. This is aimed not only at technical simplifications but reflects the reality of geophysical experiments. Indeed, decomposition (3) implies that the wave $u(z, x, t; \varepsilon)$ and, therefore, the measured data $R(x, t; \varepsilon)$ may be decomposed as

$$u = u_0(z, x, t) + \varepsilon u_1(z, x, t) + \ldots, \quad R = R_0(x, t) + \varepsilon R_1(x, t) + \ldots. \tag{4}$$

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[*]St-Petersburg University
[†]Loughborough University
[‡]Loughborough University

In practical applications, measured data are not given in form (4) so it is necessary to find, from $R(x,t)$, members $R_0$, $R_1$, etc. Since $R_k(-x,t) = (-1)^k R_k(x,t)$, we can find, up to an error of an order $O(\varepsilon^2)$, only the first two members, $R_0$ and $R_1$. Thus, in practical applications, the method could determine $A$ and $B$ only up to an error of an order $O(\varepsilon^2)$, for $x = O(1)$, cf. [8]. In this connection, we can restrict our considerations to the first two terms, with respect to $\varepsilon$, in $A$, $u$, $R$, etc. We use notations,

$$f(\cdot, \varepsilon) \sim g(\cdot, \varepsilon),$$

if $f - g = O(\varepsilon^2)$. Using the above, we rewrite (1) in the equivalent form

$$u_{tt} - \frac{1}{(1 + \varepsilon\alpha x)\sigma}\partial_y\left(\sigma(1 + \varepsilon\beta x)u_y\right) - \frac{C}{1 + \varepsilon\alpha x}\partial_x\left((1 + \varepsilon\beta x)u_x\right) \sim 0, \qquad (5)$$

while inverse data takes the form

$$R(x,t) \sim R_0(x,t) + \varepsilon R_1(x,t). \qquad (6)$$

Here

$$y = \int_0^z \frac{dz}{c(z)}, \quad c(z) = \sqrt{B_0(z)/A_0(z)}, \quad C(z) = c^2(z), \quad \sigma(z) = \sqrt{B_0(z)\,A_0(z)}, \quad (7)$$

with $c$ being the wave velocity and $\sigma$ – the acoustics stiffness of the layered medium corresponding to $\varepsilon = 0$. Also,

$$\alpha(z) = \frac{A_1(z)}{A_0(z)}, \quad \beta(z) = \frac{B_1(z)}{B_0(z)}. \qquad (8)$$

By rescaling $t$ we have $c(0) = 1$ and, without loss of generality, $\sigma(0) = 1$.

Summarising, the inverse problem under consideration is to determine $A_0$, $B_0$ and $A_1$, $B_1$ or, in other words, $c$ and $\sigma$ and $\alpha$, $\beta$ from $R_0(x,t)$, $R_1(x,t)$ given for $t \in (0, 2T)$. Causality arguments show that these data depend on the behaviour of the parameters of the medium up to the depth $z_T$,

$$z_T = \int_0^T c(y)\,dy \quad c(y) = c(z(y)), \qquad (9)$$

and, indeed, considerations in this paper provide algorithms to determine $c, \sigma, \alpha$ and $\beta$ for $z \leq z_T$.

The solvability of the dynamic inverse problem (1), (6) is well-known even for $A$, $B$ strongly depending on $(z,x)$. By means of the boundary control (BC) method developed in [1] it was shown that, when $B = 1$, $R(x,t)$, $t \in (0, 2T)$, determines $A(z,x)$. Later [2], [3] generalised the BC-method to general, including anisotropic, acoustic equations. In particular, it follows from [3], sections 4.2, 4.3 that $R(x,t)$, $t \in (0, 2T)$ determines $A(z,x)$, $B(z,x)$ for the points $(z,x)$ with the geodesic distance from the surface $z = 0$ less than $T$, where the geodesic

distance is measured in the metric $ds^2 = c^{-2}(z,x)\,(|dz|^2 + |dx|^2)$. An important feature of the BC-method is its algorithmic nature which makes it suitable, in principle to numerical implementation, e.g [4], [5]. Due to a very general nature of this method, the corresponding reconstruction algorithms turn out to be technically involved and prone to numerical instability. This instability is indeed a manifestation of the well-known ill-posedness of the multidimensional inverse problems. Even assuming a priori boundedness, in proper functional classes, of the unknown coefficients, i.e. stabilizing the inverse problem, the convergence rate may be very slow. For example, in the impedance tomography inverse problems the convergence rate is no better that of the logarithmic type [6]. For some dynamic inverse problems of the type (1), (6), one can achieve Hölder or Lipschitz convergence [7]. Examples considered in these papers have the corresponding fields of geodesics to be simple and we believe that, in the general case, in multidimensions, the rate of convergence could be only of a logarithmic type.

On the contrary, it is well-known, e.g. [8], [9] that one-dimensional inverse problems and inverse problems for a layered media often have a Lipschitz-type rate of convergence. As the inverse problem (5), (6) can be treated as a perturbation of the one for a layered medium, one can expect, at each step of a recurrent procedure to find $A_0$, $B_0$, $A_1$, $B_1$, etc better convergence properties than for a general multidimensional inverse problem.

In this paper we use the special structure (5) of the acoustic equation to derive a recurrent system of one-dimensional inverse problems for $A_0$, $B_0$ and $A_1$, $B_1$, which turns out to be Lipschitz stable. At the first stage, we come across a classical one-dimensional dynamic inverse problem to determine $A_0$, $B_0$ or equivalently $c, \sigma$ which goes back to the fundamental papers of Borg [10], Gelfand-Levitan [11], Marchenko [12] and Krein [13], [14] with further developments in e.g. [15], [16]. We use here the method of coupled integral equations [8], [9] intimately related, due to causality, to methods developed by Krein. It reduces the problem to a system of non-linear Volterra equations convenient for numerical implementation. Advancing the method, we derive a system of Volterra integral equations, this time linear but singular, for further approximations $A_1$, $B_1$. However, these singular Volterra equations can be transformed into regular ones. A word of warning: the linear system of Volterra equations for $A_1$, $B_1$ or, more precisely $\alpha, \beta$ is not a linearization of the non-linear system near $A_0$, $B_0$ due to the factor $\varepsilon x$ in front of $A_1$, $B_1$ on $x$.

As we obtain Volterra equations for the coefficients of the acoustic wave equation (5) it is easy to show that the convergence rate, for each step of the solution recurrent algorithm is Lipschitz. This seems to contradict the ill-posedness of the multidimensional inverse problem. However, if we look at the kernels of the integral equations, they depend on the lower-order members in the decomposition (3) of coefficients and their increasing number of derivatives. Moreover, trying to recover the full coefficients $A(z,\varepsilon x)$, $B(z,\varepsilon x)$ will end up with the classes of analytic, with respect to $x$, coefficients of the type studied in e.g. [17], [18].

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The principal goal of this paper is to develop analytically a reconstruction algorithm to find $A$, $B$ which may be used for numerical implementation. In this connection we provide some first results on the numerical solution of the inverse problem for a simpler model case $B = 1$.

As we have noted the model of a weakly horizontally inhomogeneous layered media is often used in applications to seismology and acoustics, e.g. [19], [20], [21], [22], [23], [24] where also some numerical results were obtained. This is due to the fact that, e.g. in seismology, large areas of the upper crust of the Earth is very well approximated as a stack of layers with parameters in each layer depending weakly on the horizontal coordinates and interfaces between layers being almost horizontal. However, this type of structures often describes areas important for the oil industry, e.g. the oil fields in in West Siberia (see the talk N17 in [25]). On the other side, the weakly lateral inhomogeneous medium is a standard model in oceanography well approximating the reality of the wave propagation in an open sea.

As for the mathematical analysis of the inverse problem for such model including rigorous derivation of a recurrent system of one-dimensional inverse problems and their solution, we refer to [8], [9] dealing with a plasma wave equation

$$u_{tt} - \Delta u + q(z, \varepsilon x)u = 0.$$

A similar model was used in [17], [18] dealing with a plasma wave equation and acoustic equation with $B = 1$, correspondingly. Papers [17], [18] are devoted to the existence and uniqueness of solution in the corresponding model rather than developing a reconstruction algorithm for $q$ or $A$ depending analytically on $x$. We note also that the case of two unknown coefficients, $A$ and $B$ is significantly more complicated. This is due to the fact that, in the case of one unknown coefficient, its first-order perturbation may be obtained from the first of the system of coupled integral equation necessary to find $A_1$ and $B_1$, namely equation which is easy to derive and study. For two coefficients we need the second equation of the pair, equation which is more technically involved. Moreover, in comparison with one equation, the nature of the coupled system provides additional difficulties in solving it.

## 2  Constant main coefficients

In this section we consider the model case of $c = 1$, $\sigma = 1$ when considerations leading to the integral equations for $\alpha, \beta$ are quite simple and provide an insight for the general case. Conditions $c = 1$, $\sigma = 1$ imply that equation (5) takes the form

$$u_{tt} - (1 + \varepsilon\alpha(z)x)^{-1} \partial_z \left[ (1 + \varepsilon\beta(z)x) \partial_z u \right] \tag{10}$$
$$- (1 + \varepsilon\alpha(z)x)^{-1} \partial_x \left[ (1 + \varepsilon\beta(z)x) \partial_x u \right] \sim 0.$$

**Theorem 2.1** *Let $R_0(x,t)$, $R_1(x,t)$ of form (6) be given for $t \in (0, 2T)$. Let, in addition, either $\partial_z \alpha(0)$ or $\partial_z \beta(0)$ is known. Then these data determine $\alpha(z)$, $\beta(z)$; $z \in (0,T)$. Namely, let*

$$\rho(t) = 2r_1^1(2t) - \frac{1}{3t}r_{1,z}^3(2t). \tag{11}$$

*Then*

$$\alpha(z) = \beta(z) + \rho(z), \tag{12}$$

$$\beta(t) = \beta(0) + Ct + t\int_0^t \frac{F(\eta)}{\eta^2}d\eta, \tag{13}$$

$$F(t) = \frac{1}{3}(r_1^3)''(2t) - 2(r_1^1)'(2t)t - 3r_1^1(2t)$$

*Here $r_1^j(t)$ is the $j$−th moment of $R_1(x,t)$,*

$$r_1^j(t) = \int_{-\infty}^{\infty} x^j R_1(x,t)\, dx, \tag{14}$$

*and the constant $C = \beta'(0)$ in (13) is arbitrary.*

**Proof** Introduce new variables

$$z \longrightarrow z + \varepsilon m(z)x, \quad x \longrightarrow x + \varepsilon n(z), \tag{15}$$

where

$$m(z) = -\frac{1}{2}\int_0^z \beta(\tilde{z})\, d\tilde{z}, \quad n(z) = -\int_0^z m(\tilde{z})d\tilde{z} = \frac{1}{2}\int_0^z \beta(\tilde{z})(z - \tilde{z})d\tilde{z}, \tag{16}$$

which makes (5) into

$$u_{tt} - \Delta u \sim \varepsilon \left( -\alpha x u_{tt} + \beta x u_{xx} + \frac{1}{2}\beta_z x u_z + \frac{3}{2}\beta u_x \right), \tag{17}$$

$$u|_{z=0} = \delta(x)\delta(t), \quad u|_{t<0} = 0.$$

Decomposition (4) yields that

$$u_{0,tt} - \Delta u_0 = 0, \quad u_0|_{z=0} = \delta(x)\delta(t), \tag{18}$$

$$u_{1,tt} - \Delta u_1 = -\alpha x u_{0,tt} + \beta x u_{0,xx} + \frac{1}{2}\beta_z x u_{0,z} + \frac{3}{2}\beta u_{0,x}, \quad u_1|_{z=0} = 0, \tag{19}$$

which imply, in particular, that

$$u_0(z, -x, t) = u_0(z, x, t), \quad u_1(z, -x, t) = -u_1(z, x, t). \tag{20}$$

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Note that transformation (15) makes the inverse data in (4) into

$$R_0(x,t) \longrightarrow R_0(x,t), \quad R_1(x,t) \longrightarrow R_1(x,t) + \frac{1}{2}\beta(0)xR_0(x,t). \tag{21}$$

Let $u_i^k(z,t)$ be the moments,

$$u_i^k(z,t) = \int_{-\infty}^{\infty} x^k u_i(z,x,t)\, dx, \quad i = 0, 1. \tag{22}$$

This makes equations (18), (19) together with inverse data (21) into the recurrent systems,

$$u_{0,tt}^k - u_{0,zz}^k - k(k-1)u_0^{k-2} = 0, \quad u_0^k|_{z=0} = \delta_0^k \delta(t), \quad u_{0,z}^k|_{z=0} = r_0^k(t); \tag{23}$$

$$u_{1,tt}^k - u_{1,zz}^k = \Omega^k(z,t) \tag{24}$$

$$= k(k-1)u_1^{k-2} - \alpha u_{0,tt}^{k+1} + k(k-1/2)\beta u_0^{k-1} + \frac{1}{2}\beta_z u_{0,z}^{k+1},$$

$$u_1^k|_{z=0} = 0, \quad u_{1,z}^k|_{z=0} = r_1^k(t).$$

It is clear from (23), (24) that $u_0^k = 0$ for odd $k$ and $u_1^k = 0$ for even $k$. As

$$u_0^0 = \delta(t-z), \quad u_0^2 = z\theta(t-z), \quad u_0^4 = \frac{3}{2}z(t+z)(t-z)_+, \tag{25}$$

equations (24) with $k = 1$ and $k = 3$ may be considered as inverse source problems to identify $\alpha$ and $\beta$. Indeed, by (24)

$$u_1^k(z,t) = \int\int G_0(z,\eta,t-\tau)\Omega^k(\eta,\tau)\, d\eta d\tau, \tag{26}$$

where Green's function, $G_0(z,\eta,t)$, is given by

$$G_0(z,\eta,t-\tau) = \frac{1}{2}\chi_{T(z,t)}(\eta,\tau) \tag{27}$$

$$= \frac{1}{2}\left(\theta(t-\tau-z+\eta)\theta(z-\eta) + \theta(t-\tau+z-\eta)\theta(\eta-z) - \theta(t-\tau-z-\eta)\right)$$

with $\chi_{T(z,t)}$ being the characteristic function of the domain, in the first quadrant, bounded by the characteristics $\tau - \eta = t - z$, $\tau + \eta = t - z$ and $\tau + \eta = t + z$ and the axis $\tau = 0$, and

$$G_{0,z}(z,\eta,t)|_{z=0} = \delta(t-\eta). \tag{28}$$

In particular,

$$u_1^1(z,t) = -\frac{1}{2}\int_0^z \alpha(\eta)\eta d\eta\, \delta(t-z) \tag{29}$$

$$+ \left\{ -\frac{1}{4}[(\alpha+\beta)(\eta)\eta]_{(t-z)/2}^{(t+z)/2} + \int_{(t-z)/2}^{(t+z)/2}\beta\, d\eta - \frac{1}{2}\int_0^z \beta\, d\eta \right\} \cdot \theta(t-z).$$

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Indeed, by (26)

$$u_1^1(z,t) = i_1(z,t) + i_2(z,t) + i_3(z,t).$$

Here

$$i_1(z,t) = -\frac{1}{2}\partial_t \int\int \chi_{T(z,t)}(\tau,\eta)\alpha(\eta)\eta\delta(\tau-\eta)\,d\eta d\tau \qquad (30)$$

$$= -\frac{1}{2}\partial_t \left(\int_{(t-z)/2}^{(t+z)/2} \alpha(\eta)\eta d\eta\, \theta(t-z)\right)$$

$$= -\frac{1}{2}\int_0^z \alpha(\eta)\eta d\eta\, \delta(t-z) - \frac{1}{4}[\alpha(\eta)\eta]_{(t-z)/2}^{(t+z)/2}\,\theta(t-z),$$

where, in the first equation we use (25) and the fact that $G_0(z,\eta,t,\tau) = G_0(z,\eta,t-\tau)$. Similarly,

$$i_2(z,t) = \frac{1}{4}\int\int \chi_{T(z,t)}(\tau,\eta)\beta(\eta)\delta(\tau-\eta)d\eta d\tau \qquad (31)$$

$$= \frac{1}{4}\int_{(t-z)/2}^{(t+z)/2} \beta(\eta)d\eta\, \theta(t-z).$$

At last,

$$i_3(z,t) = \qquad (32)$$

$$\frac{1}{4}\int\int \chi_{T(z,t)}(\tau,\eta)\beta_\eta(\eta)\theta(\tau-\eta)d\eta d\tau$$

$$-\frac{1}{4}\int\int \chi_{T(z,t)}(\tau,\eta)\beta_\eta(\eta)\eta\delta(\tau-\eta)d\eta d\tau$$

$$= \left(\frac{1}{2}\int_0^{(t-z)/2}\int_{(t-z)/2}^{(t+z)/2}\beta_s(s-r)ds - \frac{1}{4}\int_{(t-z)/2}^{(t+z)/2}\beta_\eta(\eta)\eta d\eta\right)\theta(t-z)$$

$$= \left(\frac{1}{2}\int_z^{(t+z)/2}\beta(\eta)d\eta - \frac{1}{2}\int_0^{(t-z)/2}\beta(\eta)d\eta\right)\theta(t-z)$$

$$\left(-\frac{1}{4}[\beta(\eta)\eta]_{(t-z)/2}^{(t+z)/2} + \frac{1}{4}\int_{(t-z)/2}^{(t+z)/2}\beta(\eta)d\eta\right)\theta(t-z)$$

Clearly, (30)-(32) provide (29).

As $r_1^1 = u_{1,z}^1|_{z=0}$, equation (29) immediately provides the first equation,

$$r_1^1(t) = u_{1,z}^1|_{z=0} = -\frac{t}{8}(\alpha_z + \beta_z)|_{z=t/2} - \frac{1}{4}\alpha(t/2) + \frac{3}{4}\beta(t/2) - \frac{1}{2}\beta(0). \qquad (33)$$

Note that the important term $[(\alpha+\beta)(\eta)\eta]_{(t-z)/2}^{(t+z)/2}$ in (29) which gives rize to the term $-\frac{t}{8}(\alpha_z + \beta_z)(t/2)$ in (33), comes from $i_1$ and $i_3$ or, more precisely, $\int\int \chi_{T(z,t)}(\tau,\eta)\beta_\eta(\eta)\eta\delta(\tau-\eta)d\eta d\tau$.

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Representation (29) together with (28) and (26), $k = 3$ provides the second equation for $\alpha, \beta$,

$$r_1^3(t) = u_{1,z}^3(t)|_{z=0} = \tag{34}$$

$$- \quad 3(\alpha + \beta)(t/2) \cdot \left(\frac{t}{2}\right)^2 - 3\beta(0)(t/2)^2 - 3\int_0^{t/2} \alpha\eta \, d\eta + 21\int_0^{t/2} \beta\eta \, d\eta.$$

Indeed, equations (24), $k = 3$, and (26), together with the condition (28) imply that

$$r_1^3(t) = u_{1,z}^3(0,t) = \sum_{j=1}^7 I_j(t).$$

The functions $I_j$ are due to different terms in the right-hand side of (24) with $I_1 - I_4$ generated by 4 terms in representation (29). Namely,

$$I_1(t) = -3\int\int \delta(t - \tau - \eta)\left(\int_0^\eta \alpha(\xi)\xi d\xi\right)\delta(\tau - \eta)d\eta d\tau = -\frac{3}{2}\int_0^{t/2} \alpha(\eta)\eta d\eta;$$

$$I_2(t) = -\frac{3}{2}\int\int \delta(t - \tau - \eta)\left[(\alpha(\xi) + \beta(\xi))\xi\right]_{(\tau-\eta)/2}^{(\tau+\eta)/2}\theta(\tau - \eta)d\eta d\tau \tag{35}$$

$$= -\frac{3}{2}\int_0^{t/2}\left[(\alpha(\xi) + \beta(\xi))\xi\right]_{(t/2-\eta)}^{t/2} d\eta$$

$$= -\frac{3}{2}[\alpha(t/2) + \beta(t/2)](t/2)^2 + \frac{3}{2}\int_0^{t/2}[\alpha(\eta) + \beta(\eta)]\eta d\eta;$$

$$I_3(t) = 6\int\int \delta(t - \tau - \eta)\left(\int_{(\tau-\eta)/2}^{(\tau+\eta)/2} \beta(\xi)d\xi\right) d\eta d\tau$$

$$= 6\int_0^{t/2}\left(\int_{t/2-\eta}^{t/2} \beta(\xi)d\xi\right) d\eta = 6\int_0^{t/2} \beta(\eta)\eta d\eta;$$

$$I_4(t) = -3\int\int \delta(t - \tau - \eta)\left(\int_0^\eta \beta(\xi)d\xi\right)\theta(\tau - \eta)d\eta d\tau$$

$$= -\frac{3t}{2}\int_0^{t/2} \beta(\eta)d\eta + 3\int_0^{t/2} \beta(\eta)\eta d\eta;$$

$$I_5(t) = -\frac{3}{2}\int\int \delta(t - \tau - \eta)\alpha(\eta)\eta\partial_\tau^2[(\tau - \eta)_+(\tau + \eta)]d\eta d\tau \tag{36}$$

$$= -3\int\int \delta(t - \tau - \eta)\alpha(\eta)\eta^2\delta(\tau - \eta)d\eta d\tau - 3\int\int \delta(t - \tau - \eta)\alpha(\eta)\eta\theta(\tau - \eta)d\eta d\tau$$

$$= -\frac{3}{2}\alpha(t/2)(t/2)^2 - 3\int_0^{t/2} \alpha(\eta)\eta d\eta;$$

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$$I_6(t) = \frac{15}{2} \int \int \delta(t - \tau - \eta)\beta(\eta)\eta\theta(\tau - \eta)d\eta d\tau = \frac{15}{2} \int_0^{t/2} \beta(\eta)\eta d\eta;$$

$$I_7(t) = \frac{3}{4} \int \int \delta(t - \tau - \eta)\beta_\eta(\eta)\partial_\eta[\tau^2\eta - \eta^3]\theta(\tau - \eta)\theta(\eta)d\eta d\tau \qquad (37)$$

$$= -\frac{3}{4} \int \int \delta(t - \tau - \eta)\beta(\eta)(\tau^2 - 3\eta^2)\delta(\eta)d\eta d\tau$$

$$+ \frac{3}{4}\partial_t \left( \int \int \delta(t - \tau - \eta)\beta(\eta)(\tau^2 - 3\eta^2)\theta(\eta)d\eta d\tau \right) - \frac{1}{2} \int \int \delta(t - \tau - \eta)\beta(\eta)u_{0,\eta\eta}^4 d\eta d\tau$$

$$= -\frac{3}{4}\beta(0)t^2 + \partial_t \left( \int_0^{t/2} \beta(\eta)(t^2 - 2t\eta - 2\eta^2)d\eta \right) - \frac{1}{2} \int \int \delta(t - \tau - \eta)\beta(\eta)u_{0,\tau\tau}^4 d\eta d\tau$$

$$+ 6 \int \int \delta(t - \tau - \eta)\beta(\eta)\eta\theta(\tau - \eta)d\eta d\tau$$

$$= -\frac{3}{4}\beta(0)t^2 - \frac{3}{4}\beta(t/2)(t/2)^2 + \frac{3t}{2} \int_0^{t/2} \beta(\eta)d\eta - \frac{3}{2} \int_0^{t/2} \beta(\eta)\eta d\eta$$

$$- \frac{3}{4}\beta(t/2)(t/2)^2 - \frac{3}{2} \int_0^{t/2} \beta(\eta)\eta d\eta + 6 \int_0^{t/2} \beta(\eta)\eta d\eta$$

$$= -3\beta(0)(t/2)^2 + \frac{3t}{2} \int_0^{t/2} \beta(\eta)d\eta - \frac{3}{2}\beta(t/2)(t/2)^2 + 3 \int_0^{t/2} \beta(\eta)\eta d\eta.$$

Using these equations we obtain (34). Observe for the future that the most relevant term, $-3/4t^2[\alpha(t/2) + \beta(t/2)]$, is obtained from 3 terms, namely, the term $-3/2 \int \int G_{0,z}|_{z=0}[(\alpha(\xi) + \beta(\xi)\xi]|_{(\tau-\eta)/2}^{(\tau+\eta)/2}$, i.e. $I_1$, the term $-\int \int G_{0,z}|_{z=0}\alpha u_{0,\tau\tau}^4$, i.e. $I_5$, and the term $1/2 \int \int G_{0,z}|_{z=0}\beta_\eta u_{0,\eta}^4$, i.e. $I_7$, which may be written as $-1/2 \int \int G_{0,z\eta}|_{z=0}\beta u_{0,\eta}^4 - 1/2 \int \int G_{0,z}|_{z=0}\beta u_{0,\eta\eta}^4$.

Simple calculations show that

$$r_1^1(0) = \frac{1}{4}[\beta(0) - \alpha(0)], \quad (r_1^3)''(0) = \frac{9}{4}[\beta(0) - \alpha(0)].$$

Thus it looks as if $r_1^1(t)$, $r_1^3(t)$ do not determine $\alpha(0)$, $\beta(0)$ uniquely. However, in view of (21), the actual measurements provide the functions $r_0^k(t)$, $r_1^k(t) - \frac{1}{2}\beta(0)r_0^{k+1}(t)$ rather than $r_0^k(t)$, $r_1^k(t)$. As $r_0^2(t) = 1$, $r_0^4(t) = \frac{3}{2}t^2$, the actual measurements provide $-\frac{1}{4}[\alpha(0) + \beta(0)]$ and $-\frac{3}{4}[3\alpha(0) + \beta(0)]$ making possible to determine $\alpha(0)$, $\beta(0)$.

Differentiating equation (34) and dividing the result by $t$, we see that $\alpha$ and $\beta$ satisfy relation (12).

This makes possible to solve (33) for $\alpha$ to obtain representation (13). Indeed, by (33)

$$\beta'(t)t - \beta(t) = -\frac{1}{2}\rho'(t)t - \frac{1}{2} - 2r_1^1(2t) - \beta(0) \qquad (38)$$

$$= F(t) - \beta(0),$$

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where we use definition (11). Let

$$\beta(t) = \beta(0) + \beta'(0)t + \widetilde{\beta}(t), \quad \widetilde{\beta}(t) = O(t^2).$$

This makes (38) an equation for $\widetilde{\beta}(t)$,

$$\left(\frac{\widetilde{\beta}(t)}{t}\right)' = \frac{F(t)}{t^2}.$$

Solving it with $\lim_{t\to 0}\frac{\widetilde{\beta}(t)}{t} = 0$ immediately implies (13). Observe that the constant $\beta'(0)$ remains undetermined.

**Remark 2.2** *Change of variables (15), (16) is not necessary but makes calculations simpler in the general case.*

**Remark 2.3** *For $\alpha(z), \beta(z) \in C^{1,1}$ the integrand in (13) is in $L^\infty_{loc}$ so that, in particular, the integral in the right-hand side of (13) is well-defined. In particular, it is easy to verify that $F(0) = F'(0) = 0$.*

**Remark 2.4** *It is clear from (13) that $R_1^1$, $R_1^3$ do not determine $\frac{d}{dz}\alpha$ at $z = 0$. To understand the nature of this non-uniqueness, let $u$, $\widetilde{u}$ be solutions of equations (10) with $\alpha$ and $\widetilde{\alpha} = \alpha + C\varepsilon zx$ and $\beta$ and $\widetilde{\beta} = \beta + C\varepsilon zx$. Let*

$$v = (1 + \frac{1}{2}C\varepsilon zx)^{-1}u.$$

*Then $v$ satisfies*

$$0 \sim v_{tt} - \frac{1}{(1 + \varepsilon\alpha x)(1 + \frac{1}{2}C\varepsilon zx)} div\left\{(1 + \varepsilon\beta x)\nabla[(1 + \frac{1}{2}C\varepsilon zx)v]\right\}$$
$$\sim v_{tt} - (1 + \varepsilon(\beta - \alpha)x)\Delta v - \varepsilon(\beta_z + C)xv_z - \varepsilon(\beta + Cz)v_x,$$

*so that*

$$v \sim \widetilde{u}$$

*Also*

$$\widetilde{u}_z|_{z=0} \sim v_z|_{z=0} \sim u_z|_{z=0}$$

*and $R_1^1$ and $R_1^3$ do not distinguish between $\alpha$, $\beta$ and $\widetilde{\alpha}$, $\widetilde{\beta}$. This is a special case of the non-uniqueness in inverse problems due to gauge transformations, e.g. [3]. It is likely that $\alpha'(0)$, $\beta'(0)$ can be found from the next term, $R_1^5(t)$, however, this lies beyond the scope of this work.*

*Clearly, similar phenomenon would take place in the general case but we will not dwell on this issue anymore.*

# 3    General case: Preliminary constructions

We return to the general case of equations (1), (3) where $A_0$, $B_0$ are not necessary constants or, equivalently of equations (5), (7), (8) where $c(y)$, $\sigma(y)$ are not necessary equal to 1 with, however, $c(0) = \sigma(0) = 1$. As in the case of the constant main coefficients we introduce new coordinates (15). Keeping the terms up to order $O(\varepsilon)$ we obtain the equation

$$u_{tt} \quad - \quad \frac{1}{\sigma}\partial_y(\sigma u_y) - C u_{xx} \sim \tag{39}$$
$$\varepsilon\left\{-\alpha x u_{tt} + x C_{xx} u_{xx} + x C_y u_y + C_x u_x\right\},$$

where

$$C_{(xx)} = (C\beta - m\partial_y C), \qquad C_{(y)} = \left(\frac{\partial_y(\beta\sigma)}{2\sigma} - m\partial_y\left(\frac{\partial_y\sigma}{\sigma}\right)\right), \tag{40}$$

$$C_{(x)} \quad = \quad \left(\frac{3}{2}C\beta - \frac{\partial_y(C\sigma)m}{\sigma}\right),$$

cf. (16). Introduce the Liouville transformation,

$$u \longrightarrow v = \sqrt{\sigma}u, \quad q = \frac{\partial_y^2(\sqrt{\sigma})}{\sqrt{\sigma}} \tag{41}$$

This makes (39) into

$$v_{tt} - v_{yy} \quad + \quad qv - C v_{xx} \sim \tag{42}$$
$$\varepsilon\left\{-\alpha x v_{tt} + x C_{xx} v_{xx} + x C_y v_y + C_x v_x - x\frac{\partial_y\sigma}{2\sigma}C_y v\right\}.$$

Using decomposition (4) for $v$, we see that

$$v_{0,tt} - v_{0,yy} + qv_0 = C v_{0,xx} \quad v_0|_{y=0} = \delta(x)\delta(t), \quad v_{0,y}|_{y=0} = R_0(x,t); \tag{43}$$

$$v_{1,tt} - v_{1,yy} - qv_1$$
$$= \quad C v_{1,xx} - \alpha x v_{0,tt} + x C_{xx} v_{0,xx} + x C_y v_{0,y} + C_x v_{0,x} - x\frac{\partial_y\sigma}{2\sigma}C_y v_0,$$
$$v_1|_{y=0} = 0, \quad v_{1,y}|_{y=0} = R_1(x,t) + \frac{1}{2}\beta(0)x R_0(x,t). \tag{44}$$

Similar to the case of constant main coefficients we introduce the moments,

$$v_i^k(y,t) = \int_{-\infty}^{\infty} x^k v_i(y,x,t)\,dx, \quad r_0^k(t) = \int_{-\infty}^{\infty} x^k R_0(x,t)\,dx, \tag{45}$$

$$r_1^k(t) = \int_{-\infty}^{\infty} x^k\left(R_1(x,t) + \frac{1}{2}\beta(0)x R_0(x,t)\right)dx,$$

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cf. (22). Making use of (43), (44) we obtain a recurrent system for $v_0^k$, $v_1^k$, correspondingly,

$$v_{0,tt}^k - v_{0,yy}^k + qv_0^k = k(k-1)Cv_0^{k-2}, \quad v_0^k|_{y=0} = \delta_0^k \delta(t), \quad v_{0,y}|_{y=0} = r_0^k(t), \ (46)$$

cf. (23), and

$$v_{1,tt}^k - v_{1,yy}^k + qv_1^k = \Omega^k(y,t) = k(k-1)Cv_1^{k-2} \tag{47}$$

$$-\alpha v_{0,tt}^{k+1} + k(k+1)C_{xx}v_0^{k-1} + C_y v_{0,y}^{k+1} - kC_x v_0^{k-1} - \frac{\partial_y \sigma}{2\sigma}C_y v_0^{k+1},$$

$$v_1^k|_{y=0} = 0, \quad v_{1,y}^k|_{y=0} = r_1^k(t),$$

cf. (24). Note that (46), (47) imply that $u_0^k = 0$ for odd $k$ while $u_1^k = 0$ for even $k$.

Following the scheme of section 2 we represent

$$v_0^0(y,t) = \delta(t-y) - \int\int_{\eta>0} G(y,\eta,t-\tau)q(\eta)\delta(\tau-\eta)\,d\eta d\tau \cdot \theta(t-y), \tag{48}$$

$$v_0^k(y,t) = k(k-1)\int\int_{\eta>0} G(y,\eta,t-\tau)C(\eta)v_0^{k-2}(\eta,\tau)\,d\eta d\tau \cdot \theta(t-y), \quad k>0,$$

and

$$v_1^k(y,t) = \int\int_{\eta>0} G(y,\eta,t-\tau)\Omega^k(\eta,\tau)\,d\eta d\tau \cdot \theta(t-y), \ k>0, \tag{49}$$

where $G(y,\eta,t)$ is fundamental solution for the string equation in the first quadrant, $\mathbb{R}_+^2 = \{x, t > 0\}$,

$$G_{tt} - G_{yy} + qG = \delta(y-\eta)\delta(t), \quad G|_{y=0} = 0, \quad G|_{t<0} = 0, \tag{50}$$

cf. (25)–(3.2).

We finish this section by deriving some representations and estimates of Green's function $G(y,\eta,t)$ and its derivatives. In the future we assume that

$$|q| \le M, \tag{51}$$

and use notation $\mathcal{A}^k(y,\eta,t)$, $\mathcal{A}^k(y,t)$ for various functions of $(y,\eta,t)$, $(y,t)$ such that

$$|\mathcal{A}^k(y,\eta,t)| \le Mc(T,M)\left(y^k + \eta^k + t^k\right) \quad \text{for } y+t+\eta \le T. \tag{52}$$

**Proposition 3.1** *Green's function $G(y,\eta,t)$ of problem (50) may be represented as*

$$G(y,\eta,t) = G_0(y,\eta,t) + g(y,\eta,t). \tag{53}$$

Here $G_0(y, \eta, t)$ is Green's function corresponding to the ?free? string equation, $q = 0$,

$$G_{0,tt} - G_{0,yy} = \delta(y - \eta)\delta(t), \quad G_0|_{y=0} = 0, \quad G_0|_{t<0} = 0.$$

It is given by the characteristic function of the semi-infinite layer $L(\eta, 0) \subset \mathbb{R}_+^2$ bounded by the characteristics $t - y + \eta = 0$, $t + y - \eta = 0$ and $t - y - \eta = 0$,

$$G_0(y, \eta, t) = \chi_{L(\eta,0)}(y, t) = \tag{54}$$

$$\frac{1}{2} \left[ \theta(t - y + \eta)\theta(y - \eta) + \theta(t + y - \eta)\theta(\eta - y) - \theta(t - y - \eta) \right]$$

The remainder $g(y, \eta, t)$ has the form,

$$g(y, \eta, t) = -\frac{1}{4} \left[ Q(y) - Q(\eta) \right] (t - y + \eta)_+ \theta(y - \eta) \tag{55}$$

$$+\frac{1}{4} \left[ Q(y) - Q(\eta) \right] (t + y - \eta)_+ \theta(\eta - y)$$

$$+\frac{1}{4} \left[ Q(y) + Q(\eta) \right] (t - y - \eta)_+ + \widetilde{g}(y, \eta, t),$$

where $\widetilde{g}(y, \eta, t) \in C_{loc}^{1,1}(\mathbb{R}_+^2)$ and

$$Q(y) = \int_0^y q(\xi) \, d\xi, \tag{56}$$

Moreover, representation (53) may be differentiated with respect to $y$ and $t$ with

$$\partial_y g(y, \eta, t) = \frac{1}{4} [Q(y) - Q(\eta)]\theta(t - y + \eta)\theta(y - \eta) \tag{57}$$

$$+\frac{1}{4} [Q(y) - Q(\eta)]\theta(t + y - \eta)\theta(\eta - y) - \frac{1}{4} [Q(y) + Q(\eta)]\theta(t - y - \eta) + \widetilde{g}_y(y, \eta, t).$$

$$\partial_t g(y, \eta, t) = -\frac{1}{4} [Q(y) - Q(\eta)]\theta(t - y + \eta)\theta(y - \eta) \tag{58}$$

$$+ \quad \frac{1}{4} [Q(y) - Q(\eta)]\theta(t + y - \eta)\theta(\eta - y) + \frac{1}{4} [Q(y) + Q(\eta)]\theta(t - y - \eta) + \widetilde{g}_t(y, \eta, t).$$

The functions $\widetilde{g}_y, \widetilde{g}_t \in C^{0,1}(\mathbb{R}_+^2)$ and

$$\widetilde{g}_t(y, \eta, t), \ \widetilde{g}_y(y, \eta, t) = \mathcal{A}^0(y, \eta, t)(t - |y - \eta|)_+. \tag{59}$$

At last,

$$\partial_{tt} g(y, \eta, t) = \frac{1}{4} [Q(\eta) - Q(y)]\delta(t - y + \eta)\theta(y - \eta) \tag{60}$$

$$-\frac{1}{4} [Q(\eta) - Q(y)]\delta(t + y - \eta)\theta(\eta - y) + \frac{1}{4} [Q(\eta) + Q(y)]\delta(t - y - \eta) \quad + \quad \widetilde{g}_{tt}(y, \eta, t),$$

with $\widetilde{g}_{tt}(y, \eta, t) = \mathcal{A}^0(y, \eta, t)\theta(t - |y - \eta|)$ being piecewise continuous with only possible jump discontinuities across the above characteristics.

Note that estimates of $\widetilde{g}$, $\widetilde{g}_y$, etc. remains valid when $q \in L^\infty(\mathbb{R}_+)$. When $|q(y)| \le M$ for $y \le 2T$ these estimates are valid for $y + \eta + t \le T$.

**Proof** $G(y, \eta, t)$ is the solution of the integral equation

$$G(y, \eta, t) \;=\; G_0(y, \eta, t) - \int\int_{\xi > 0} G_0(y, \xi, t - \tau)q(\xi)G(\xi, \eta, \tau)\, d\xi d\tau$$

$$= G_0(y, \eta, t) \;-\; G_0 \circ qG(y, \eta, t) = G_0 - G_0 \circ qG_0 - G_0 \circ qg. \tag{61}$$

Representation (61) and the fact that $G_0$ has at most jump discontinuities (54) show that $g(y, \eta, t)$ is Lipschitz continuous and satisfies (55). Equations (57)–(60) follow by differentiating (61) and using representation (54). Indeed, consider e.g. $\partial_t g$,

$$\partial_t g(y, \eta, t) = -G_{0,t} \circ qG_0 - G_{0,t} \circ qg. \tag{62}$$

Then, due to (55), the second term in the rhs satisfies (59). To analyse the first term in the rhs, i.e.

$$-\int\int_{\xi>0} G_{0,t}(y, \xi, t - \tau)q(\xi)G_0(\xi, \eta, \tau), \tag{63}$$

observe that $G_{0,t}(y, \xi, t - \tau)$ is a sum of 3 $\delta-$ functions on characteristics, namely $\frac{1}{2}\delta(t - \tau - y + \xi)$ for $\xi \in (0, y)$; $\quad \frac{1}{2}\delta(t - \tau + y - \xi)$ for $\xi > y$ and $-\frac{1}{2}\delta(t - \tau - y - \xi)$ for $\xi > 0$. As $G_0(\xi, \eta, \tau)$ is the characteristic function of $L(\eta, 0)$, the integral in (63) consists of 3 one-dimensional integrals, some of them probably empty, along the parts of the above characteristics inside $L(\eta, 0)$. Depending on whether $y + \eta > t$ or $y + \eta < t$,

$$- \quad G_{0,t} \circ qG_0(y, \eta, t) = \tag{64}$$

$$- \quad \frac{1}{4}\left[Q((t + y + \eta)/2) - Q((y + \eta - t)/2)\right] \cdot \theta(y + \eta - t)\theta(t - |y - \eta|)$$

$$- \quad \frac{1}{4}\left[Q((t + y + \eta)/2) + Q((t - y - \eta)/2)\right] \cdot \theta(t - y - \eta)\theta(t - |y - \eta|)$$

$$+ \quad \frac{1}{4}\left[Q((t + y - \eta)/2) + Q((t - y + \eta)/2)\right] \cdot \theta(t - y - \eta)\theta(t - |y - \eta|).$$

Thus, the jumps of $-G_{0,t} \circ qG_0$ across the bicharacteristic $t + y - \eta = 0$ are equal to those in the rhs of (58) proving (59). Similar arguments prove (57).

At last, as representation (61) is a convolution with respect to $\tau$,

$$G_{tt}(y, \eta, t) \;=\; G_{0,tt}(y, \eta, t) - \int\int G_{0,t}(y, \xi, t - \tau)q(\xi)G_\tau(\xi, \eta, \tau)\, d\xi d\tau$$

$$= G_{0,tt}(y, \eta, t) \;-\; \int\int G_{0,t}(y, \xi, t - \tau)q(\xi)G_{0,\tau}(\xi, \eta, \tau)\, d\xi d\tau$$

$$- \quad \int\int G_{0,t}(y, \xi, t - \tau)q(\xi)g_\tau(\xi, \eta, \tau)\, d\xi d\tau. \tag{65}$$

Taking into account (54), (58) provide equations (60).

**Remark 3.2** *Green's function $G(y, \eta, t)$ satisfies*

$$|G(y, \eta, t)| \leq K(\sqrt{M(t^2 - (y-\eta)^2)}) \, \theta(t - |y - \eta|), \tag{66}$$

*where $K(z) = J_0(iz)$ where $J_0$ is the $0-$th Bessel function and $M$ is given by (51). Indeed, as $G_0 \geq 0$ it follows from (61) that*

$$|G| \leq G_0 + MG_0 \circ G_0 + M^2 G_0 \circ G_0 \circ G_0 + \dots$$
$$\leq K(\sqrt{M(t^2 - (y-\eta)^2)}))\theta(t - |y - \eta|)$$

*with $K(\sqrt{M(t^2 - (y-\eta)^2)}))\theta(t - |y - \eta|)$ being the Green's function for the equation*

$$K_{tt} - K_{yy} + MK = \delta(y - \eta)\delta(t), \quad K|_{t<0} = 0,$$

*on the whole real axis $y \in \mathbb{R}$.*

# 4   General case: terms $v_0^k$

As we see later, equations for $\alpha$, $\beta$ can be obtained from the analysis of $v_{1,y}^1$, $v_{1,y}^3$ at $y = 0$, i.e. from $r_1^1$, $r_1^3$, cf. (25), (26). As follows from (47), expressions for these moments involve $v_0^k$, $k = 0, 2, 4$. Following [8], we also obtain integral equations for $q(y)$ and $C(y)$ from the singularity analysis of $v_{0,y}^0$, $v_{0,y}^2$ at $y = 0$, .

**Lemma 4.1** *The function $v_0^0(y, t)$ is of the form*

$$v_0^0(y, t) = \delta(t - y) - \frac{1}{2}\left[Q(\frac{t+y}{2}) - Q(\frac{t-y}{2})\right]\theta(t - y) + \widetilde{v}_0^0(y, t), \tag{67}$$

*with*

$$\widetilde{v}_0^0(y, t) = \mathcal{A}^1(y, t)(t - y)_+. \tag{68}$$

**Proof.** By (46) with $k = 0$,

$$v_0^0(y, t) = \delta(t - y) - \int \int G(y, \eta, t - \tau)q(\eta)\delta(\tau - \eta) \, d\eta d\tau \tag{69}$$

$$= \delta(t - y) - \frac{1}{2}\left[Q(\frac{t+y}{2}) - Q(\frac{t-y}{2})\right]\theta(t - y) + \int g(y, \eta, t - \eta)q(\eta) \, d\eta,$$

where we use representation (53), (54). This proves (67) with $\widetilde{v}_0^0$ given by the integral in rhs of (69). Properties of $\widetilde{v}_0^0$ follows from the properties of $g(y, \eta, t)$ described in Proposition 3.1.

**Remark 4.2** *By employing the progressive wave expansion method to problem (46) with $k = 0$, it is easy to see that*

$$v_0^0(y,t) = \delta(t-y) + \nu_0(y)\theta(t-y) + \mathcal{B}^1(y,t)(t-y)_+, \quad \nu_0(y) = -\frac{1}{2}Q(y), \quad (70)$$

**Here and in the sequel we denote by $\mathcal{B}^k$ the class of $C^\infty-$functions which satisfy the estimate**

$$|\partial^\alpha \mathcal{B}^k| \leq C_\alpha (y+t)^{k-|\alpha|}, \quad |\alpha| \leq k. \quad (71)$$

**Proposition 4.3** *Denote by*

$$w_0^0(y,t) = v_{0,t}^0(y,t) + v_{0,y}^0(y,t). \quad (72)$$

*Then, for $0 < y < t$, the vector function $(v_0^0(y,t), w_0^0(y,t), q(y))$ satisfies the non-linear Volterra equations*

$$v_0^0(y,t) = \int_0^y w_0^0(\eta, \eta + t - y)\, d\eta,$$

$$w_0^0(y,t) = \int_0^y q(\eta)v_0^0(\eta, t + y - \eta)\, d\eta + r_0^0(t+y),$$

$$q(y) = -2\int_0^y q(\eta)v_0^0(\eta, 2y - \eta)\, d\eta - 2r_0^0(2y). \quad (73)$$

Note that though $r_0^0(t)$ has a singularity at $t = 0$, for $t > 0$ this is a continuous function making sense in the rhs of (73).

**Proof.** It follows from (46) with $k = 0$ that $(v_0^0, w_0^0)$ satisfies a system of the first-oder equations,

$$v_{0,t}^0 + v_{0,y}^0 = w_0^0; \quad w_{0,t}^0 - w_{0,y}^0 = -qv_0^0,$$

with

$$v_0^0|_{y=0} = \delta(t); \quad w_0^0|_{y=0} = \delta'(t) + r_0^0(t) = -\frac{1}{2}q(y)\theta(t-y) + \mathcal{A}^0(y,t)(t-y)_+,$$

where the last equation follows from (67), (68). Integrating along bicharacteristics $\tau + \eta = t + y$, $\tau - \eta = t - y$, we obtain the first two equations in (73). As by (67), (68) $w_0^0(y, y+0) = -\frac{1}{2}q(y)$ we obtain from the second equation in (73) the third one.

It is easy to show, for details see section later that the integral operator in the rhs of (73) is a contraction for $t < t_0$, where $t_0$ depends only on $M$ in (51), and may be solved by iterations with Lipschitz dependence of the solution on the variation of $r_0^0$ in $L^\infty-$norm.

We turn now to $v_0^2(y,t)$ which, together with $v_{0,y}^2(y,t)$, $v_{0,t}^2(y,t)$ will be used later to find $v_1^1$, $v_1^3$. Also, an expression for $v_{0,y}^2|_{y=0}$ will be used to derive an integral equation for $C(y)$.

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**Lemma 4.4** *The function $v_0^2(y,t)$ is of the form*

$$v_0^2(y,t) = \left[ C_1(\frac{t+y}{2}) - C_1(\frac{t-y}{2}) \right] \theta(t-y) + \int_0^{(t+y)/2} \mathcal{A}^1(y,\eta,t)C(\eta)\,d\eta \cdot (t-y)_+$$

$$= C_1(y)\,\theta(t-y) + \mathcal{B}^1(y,t)\,(t-y)_+, \tag{74}$$

*where*

$$C_1(y) = \int_o^y C(\eta)\,d\eta. \tag{75}$$

*Also,*

$$v_{0,y}^2(y,t) = -\int_0^y C(\eta)\,d\eta \cdot \delta(t-y) + \frac{1}{2}\left[ C\left(\frac{t+y}{2}\right) + C\left(\frac{t-y}{2}\right) \right] \cdot \theta(t-y)$$

$$+ \int_0^{(t+y)/2} \mathcal{A}^1(y,\eta,t)C(\eta)\,d\eta \cdot \theta(t-y); \tag{76}$$

$$v_{0,t}^2(y,t) = \int_0^y C(\eta)\,d\eta\,\delta(t-y) + \frac{1}{2}\left[ C\left(\frac{t+y}{2}\right) - C\left(\frac{t-y}{2}\right) \right] \theta(t-y)$$

$$+ \int_0^{(t+y)/2} \mathcal{A}^1(y,\eta,t)C(\eta)\,d\eta\,\theta(t-y); \tag{77}$$

*Here and later, we denote by $\mathcal{B}^k(y,t)$ (or $(y,\eta,t)$) the functions which satisfy the same estimates (52) as functions $\mathcal{A}^k$ but with the constant $M$ satisfying, instead of (51),*

$$|q|,\ |\frac{dC}{dy}| \le M. \tag{78}$$

*Note that if $C = 1$, $q = 0$, so that $\mathcal{A}^1 = 0$ in (74), then*

$$v_0^2(y,t) = y\theta(t-y),$$

cf. (25). Also the main two terms in rhs of (76)–(77) are obtained by differentiating, with respect $y$ and $t$ correspondingly, of the main term in rhs of (74).

**Proof.** By (46) with $k = 2$, together with representations (53), (54) and (67).

$$v_0^2(y,t) = 2\int\int G(y,\eta,t-\tau)C(\eta)v_0^0(\eta,\tau)\,d\eta d\tau = \int_{(t-y)/2}^{(t+y)/2} C(\eta)\,d\eta\,\theta(t-y) + \widetilde{v}_0^2(y,t),$$

where

$$\widetilde{v}_0^2(y,t) = 2\int C(\eta)\left\{ g(y,\eta,t-\eta) + \int G(y,\eta,t-\tau)\left(v_0^0(\eta,\tau) - \delta(\tau-\eta)\right)d\tau \right\}d\eta.$$

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It then follows from Proposition 3.1 and Lemma 4.1 that

$$g(y, \eta, t - \eta) + \int G(y, \eta, t - \tau) \left( v_0^0(\eta, \tau) - \delta(\tau - \eta) \right) d\tau = \mathcal{A}^1(y, \eta, t) (t - y)_+, \quad (79)$$

since the domain of integration with respect to $\tau$ in the rhs of (81), i.e. $\eta < \tau$, $|y - \eta| < t - \tau$ shrinks to 0 when $t \to y + 0$. Note also that the function (81) equals to 0 for $\eta > \frac{t+y}{2}$ so that the integral defining $\widetilde{v}_0^2$ is to $\eta = \frac{t+y}{2}$.

Turning to $v_{0,y}^2$, we have

$$v_{0,y}^2(y, t) = 2 \int \int G_y(y, \eta, t - \tau) C(\eta) v_0^0(\eta, \tau) \, d\eta d\tau =$$

$$- \int_0^y C(\eta) \, d\eta \, \delta(t - y) + \frac{1}{2} \left[ C \left( \frac{t+y}{2} \right) + C \left( \frac{t-y}{2} \right) \right] \theta(t - y) + \widetilde{v}_{0,}^2(y, t),$$

where

$$\widetilde{v}_{0,y}^2(y, t) = 2 \int C(\eta) \left\{ g_y(y, \eta, t - \eta) + \int G_y(y, \eta, t - \tau) \left( v_0^0(\eta, \tau) - \delta(\tau - \eta) \right) d\tau \right\} d\eta.$$

It follows from Proposition 3.1 and Lemma 4.1 that

$$g_y(y, \eta, t - \eta) + \int G_y(y, \eta, t - \tau) \left( v_0^0(\eta, \tau) - \delta(\tau - \eta) \right) d\tau = \mathcal{A}^1(y, \eta, t) \theta(t - y).$$

Similar considerations are valid for $v_{0,t}^2$.

**Remark 4.5** *Similar to $v_0^0$ we can show that (46) with $k = 2$ implies*

$$v_0^2(y, t) = \kappa_0(y)\theta(t - y) + \kappa_1(y)(t - y)_+ + \mathcal{B}^1(y, t)(t - y)_+^2; \quad (80)$$

$$\kappa_0(y) = C_1(y), \quad \kappa_1(y) = -\frac{1}{2}Q(y)C_1(y) + \frac{1}{2}\left[ C(y) - C(0) \right].$$

**Corollary 4.6** *The function $C(t)$ satisfies a linear Volterra equation*

$$C(t) = r_0^2(2t) + \int_0^t \mathcal{K}(t, \eta)C(\eta) \, d\eta, \quad (81)$$

*where the kernel $\mathcal{K}$ has the form*

$$\mathcal{K}(t, \eta) = -g_y(0, \eta, 2t - \eta) - \int G_y(0, \eta, 2t - \tau) \left( v_0^0(\eta, \tau) - \delta(\tau - \eta) \right) d\tau.$$

Observe that $G(y, \eta, t)$ and $v_0^0(y, t)$ depend only on $q$ and thus can be constructed having recovered $q$. Summarizing, Proposition 4.3 and Corollary 4.6 provide a method to recover unknown $q(t)$, $C(t)$, for $t \in (0, t_0)$ with sufficiently small $t_0$. To this end, we should know $r_0^0(t)$, $r_0^2(t)$ for $t \in (0, 2t_0)$. Later we will show that iterating this procedure it is possible to find $q(t)$, $C(t)$, for $t \in (0, T)$ from $r_0^0(t)$, $r_0^2(t)$ known for $t \in (0, 2T)$ with an arbitrary $T > 0$. Clearly, having $q$ and $C$ we can find $A_0$, $B_0$.

We end this section with representations of $v_0^4$ and its derivatives.

**Lemma 4.7** *Let $v_0^4(y,t)$ be the solution to initial-value problem (46) with $k = 4$. Then*

$$v_0^4(y,t) = \quad 6\left[\int_0^{t+y/2} C_1^2(\eta)\,d\eta - \int_0^{t-y/2} C_1^2(\eta)\,d\eta - \int_0^y C_1^2(\eta)\,d\eta\right]\theta(t-y)$$
$$+\mathcal{B}^2(y,t)(t-y)_+^2; \tag{82}$$

$$v_{0,y}^4(y,t) = 3\left[C_1^2(\frac{t+y}{2}) + C_1^2(\frac{t-y}{2}) - 2C_1^2(y)\right]\theta(t-y) + \mathcal{B}^2(y,t)(t-y)_+; \tag{83}$$

$$v_{0,tt}^4(y,t) = \quad 3C_1^2(y)\delta(t-y) + 3\left[CC_1(\frac{t+y}{2}) - CC_1^2(\frac{t-y}{2})\right]\theta(t-y)$$
$$+\mathcal{B}^2(y,t)\theta(t-y) \tag{84}$$

Note that, for $C = 1$, $q = 0$ equation (82) takes the form (25) for $u_0^k$ while (83), (84) become the corresponding derivatives of $u_0^k$. Also as

$$\int_0^{t+y/2} C_1^2(\eta)\,d\eta - \int_0^{t-y/2} C_1^2(\eta)\,d\eta - \int_0^y C_1^2(\eta)\,d\eta \longrightarrow 0, \quad y \to t+0,$$

$$v_0^4(y,t) = \mathcal{B}^3(y,t)(t-y)_+. \tag{85}$$

**Proof.** We obtain the main term in (82) from the integral representation,

$$v_0^4(y,t) = 12\int\int G(y,\eta,t-\tau)\,C(\eta)v_0^2(\eta,\tau)\,d\eta d\tau, \tag{86}$$

taking into account Proposition 3.1 and Lemma 4.4. They imply that the main term in (82) is given by

$$\int\int_{\Pi(y,t)} C(\eta)C_1(\eta)\,d\eta d\tau,$$

where $\Pi(y,t)$ is the rectangular bounded by the characteristics $\tau - \eta = 0$, $\tau - \eta = t - y$, $\tau + \eta = t - y$ and $\tau + \eta = t + y$ providing the desired main term. An estimates for the remainder in (82) follows from the estimates for remainder in equations in Proposition 3.1 and Lemma 4.4.

Similar arguments using representations and estimates for $G_y$, $G_t$ in Proposition 3.1 and for $v_0^2$, $v_{0,t}^2$ in Lemma 4.4 yield equations (83), (82). We need also to take into account that the integral (86) is a convolution with respect to $t$, so that

$$v_{0,tt}^4(y,t) = 12\int\int G_t(y,\eta,t-\tau)\,C(\eta)v_{0,\tau}^2(\eta,\tau)\,d\eta d\tau.$$

**Remark 4.8** *Similar to $v_0^0$, $v_0^2$ we can show that (46) with $k = 4$ implies*

$$v_0^4(y,t) = \mu_1(y)(t-y) + \frac{1}{2}\mu_2(y)(t-y)_+^2 + \mathcal{B}^1(y,t)(t-y)_+^3, \quad \mu_1(y) = 3C_1^2(y), \text{ (87)}$$

$$\mu_2(y) = 3C(y)C_1(y) + 3\int_0^y C(\eta)\left[C(\eta) - C(0)\right]d\eta - \frac{3}{2}C_1^2(y)Q(y).$$

## 5   First integral equation

Recall that, by (47),

$$r_1^1(t) = u_{1,y}^1|_{y=0} = \int\int G_y(0,\eta,t-\tau)\Omega^1(\tau,\eta)d\tau d\eta, \tag{88}$$

with $G_y(0,\eta,t) = v_0^0(\eta,t)$. Using representation (47) for $\Omega_1^1$, we see that

$$r_1^1(t) = \mathcal{U}_1 + \mathcal{U}_2 + \mathcal{U}_3, \tag{89}$$

where

$$\mathcal{U}_1(t) = -\int\int G_y(0,\eta,t-\tau)\alpha(\eta)v_{0,\tau\tau}^2(\tau,\eta)d\tau d\eta \tag{90}$$

$$\mathcal{U}_2(t) = \int\int G_y(0,\eta,t-\tau)\left(2C_{(xx)}(\eta) - C_{(x)}(\eta)\right)v_0^0(\tau,\eta)d\tau d\eta; \tag{91}$$

$$\mathcal{U}_3(t) = \int\int G_y(0,\eta,t-\tau)C_{(y)}(\eta)\sqrt{\sigma}\partial_\eta\left(\frac{v_0^0}{\sqrt{\sigma}}\right)(\tau,\eta)\,d\tau d\eta, \tag{92}$$

where the coefficients $C_{(x)}$, $C_{(xx)}$, $C_{(y)}$ are given by (40).

We start with $\mathcal{U}_1$. Employing equations (70) for $v_0^0$ and (80) for $v_0^2$, we obtain

$$\mathcal{U}_1 = \mathcal{U}_{11} + \mathcal{U}_{12} + \mathcal{U}_{13} + \mathcal{U}_{14}.$$

Here

$$\mathcal{U}_{11}(t) = -\int\int \delta(t-\tau-\eta)a(\eta)\kappa_0(\eta)\delta'(\tau-\eta)d\tau d\eta =$$

$$-\partial_t\int\int \delta(t-\tau-\eta)a(\eta)\kappa_0(\eta)\delta(\tau-\eta)d\tau d\eta$$

$$= -\frac{1}{4}\alpha'(t/2)C_1(t/2) - \frac{1}{4}\alpha(t/2)C(t/2);$$

$$\mathcal{U}_{12}(t) = -\int\int g_y(t-\tau,0,\eta)\alpha(\eta)\kappa_0(\eta)\delta'(\tau-\eta)d\tau d\eta =$$

$$-\partial_t \int_0^{t/2} g_y(t-\eta,0,\eta)\alpha(\eta)\kappa_0(\eta)d\eta$$

$$-\frac{1}{2}\alpha(t/2)\nu_0(t/2)\kappa_0(t/2) - \int_0^{t/2} g_{yt}(t-\eta,0,\eta)\alpha(\eta)\kappa_0(\eta)d\eta$$

$$\frac{1}{4}\alpha(t/2)Q(t/2)C_1(t/2) + \int_0^{t/2} \mathcal{B}^2(t,\eta)\alpha(\eta)d\eta;$$

$$\mathcal{U}_{13}(t) = -\int\int \delta(t-\tau-\eta)a(\eta)\left(\kappa_1(\eta)\delta(\tau-\eta) + \mathcal{B}^1(\tau,\eta)\theta(\tau-\eta)\right)d\tau d\eta$$

$$= -\frac{1}{2}\alpha(t/2)\kappa_1(t/2) + \int_0^{t/2} \mathcal{B}^1(t,\eta)\alpha(\eta)d\eta$$

$$= \frac{1}{4}\alpha(t/2)Q(t/2)C_1(t/2) - \frac{1}{4}\alpha(t/2)\left(C(t/2) - C(0)\right) + \int_0^{t/2} \mathcal{B}^1(t,\eta)\alpha(\eta)d\eta.$$

At last

$$\mathcal{U}_{14}(t) = -\int\int g_y(t-\tau,0,\eta)\alpha(\eta)\left(\kappa_1(\eta)\delta(\tau-\eta) + \mathcal{B}^1(\tau,\eta)\theta(\tau-\eta)\right)d\tau d\eta$$

$$= \int_0^{t/2} \mathcal{B}^2(t,\eta)\alpha(\eta)d\eta.$$

Summarising, we see that

$$\mathcal{U}_1(t) = -\frac{1}{4}\alpha'(t/2)C_1(t/2) \tag{93}$$

$$-\frac{1}{4}\alpha(t/2)\left\{C(t/2) + 2\nu_0(t/2)\kappa_0(t/2) + 2\kappa_1(t/2)\right\} + \int_0^{t/2} \mathcal{B}^1(t,\eta)\alpha(\eta)d\eta$$

$$= -\frac{1}{4}\alpha'(t/2)C_1(t/2) - \frac{1}{4}\alpha(t/2)\left\{2C(t/2) - Q(t/2)C_1(t/2) - C(0)\right\}$$

$$+ \int_0^{t/2} \mathcal{B}^1(t,\eta)\alpha(\eta)d\eta.$$

We now proceed to $\mathcal{U}_2$. Let

$$B_\beta(y) = \int_0^y \beta(\eta)d\eta, \tag{94}$$

so that

$$2C_{(xx)}(y) - C_{(x)}(y) = \frac{1}{2}\left[C(y)\beta(y) + \sigma(y)\partial_y\left(C/\sigma\right)(y)B_\beta(y)\right].$$

Then

$$\mathcal{U}_2 = \mathcal{U}_{21} + \mathcal{U}_{22} + \mathcal{U}_{23},$$

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where

$$\mathcal{U}_{21}(t) = \frac{1}{2}\int\int\int \delta(t-\tau-\eta)[C\beta + \sigma\partial_\eta\left(C/\sigma\right)B_\beta](\eta)\theta(\tau-\eta)d\tau d\eta$$

$$= \frac{1}{4}\beta(t/2)C(t/2) + \frac{1}{4}\sigma(t/2)\partial_\eta\left(C\sigma\right)|_{\eta=t/2}\int_0^{t/2}\beta(\eta)d\eta;$$

$$\mathcal{U}_{22}(t) = \frac{1}{2}\int\int\int g_y(t-\tau,0,\eta)[C\beta + \sigma\partial_\eta\left(C\sigma\right)B_\beta](\eta)v_0^0(\tau,\eta)d\tau d\eta$$

$$= \frac{1}{2}\int_0^{t/2} g_y(t-\eta,0,\eta)[C\beta + \sigma\partial_\eta\left(C\sigma\right)B_\beta](\eta)d\eta$$

$$+ \frac{1}{2}\int\int g_y(t-\tau,0,\eta)[C\beta + \sigma\partial_\eta\left(C\sigma\right)B_\beta](\eta)[v_0^0(\tau,\eta) - \delta(\tau-\eta)]d\tau d\eta$$

$$= \int_0^{t/2} \mathcal{B}^1(t,\eta)\beta(\eta)d\eta;$$

$$\mathcal{U}_{23}(t) = \frac{1}{2}\int\int \delta(t-\tau-\eta)[C\beta + \sigma\partial_\eta\left(C\sigma\right)B_\beta](\eta)[v_0^0(\tau,\eta) - \delta(\tau-\eta)]d\tau d\eta$$

$$= \frac{1}{2}\int_0^{t/2} [C\beta + \sigma\partial_\eta\left(C\sigma\right)B_\beta](\eta)[\nu_0(\eta) + \mathcal{B}^1(t-\eta,\eta)(t-2\eta)]d\eta$$

$$= \int_0^{t/2} \mathcal{B}^1(t,\eta)\beta(\eta)d\eta.$$

Summarising, we see that

$$\mathcal{U}_2(t) = \frac{1}{4}\beta(t/2)C(t/2) + b(t/2)\int_0^{t/2}\beta(\eta)d\eta + \int_0^{t/2}\mathcal{B}^1(t,\eta)\beta(\eta)d\eta, \qquad (95)$$

where

$$b(\eta) = \frac{1}{4}\sigma(\eta)\partial_\eta\left(C\sigma\right)(\eta). \qquad (96)$$

We term now to the term $\mathcal{U}_3$,

$$\mathcal{U}_3(t) = \int\int G_y(t-\tau,0,\eta)C_{(y)}(\eta)\sqrt{\sigma}\partial_\eta\left(\frac{v_0^2}{\sqrt{\sigma}}\right)(\tau,\eta)\,d\tau d\eta \qquad (97)$$
$$= \mathcal{U}_{31}(t) + \mathcal{U}_{32}(t) + \mathcal{U}_{33}(t) + \mathcal{U}_{34}(t) + \mathcal{U}_{35}(t),$$

with

$$C_{(y)}(\eta) = \frac{1}{2}\left[\partial_\eta\beta(\eta) + \partial_\eta\left(B_\beta(\eta)\partial_\eta\log(\sigma(\eta))\right)\right]. \qquad (98)$$

Taking into the account that $G_y(t, 0, \eta) = v_0^0(t, \eta)$ and equations (70), (80), we obtain for $\mathcal{U}_{3k}$:

$$\mathcal{U}_{31}(t) = -\frac{1}{2} \int \int \delta(t - \tau - \eta) \left[ \partial_\eta \beta(\eta) + \partial_\eta \left( B_\beta(\eta) \partial_\eta \log(\sigma(\eta)) \right) \right] \kappa_0(\eta) \delta(\tau - \eta)$$

$$= -\frac{1}{4} \left[ \beta'(t/2) + \left( B_\beta \frac{\sigma'}{\sigma} \right)' (t/2) \right] \kappa_0(t/2)$$

$$= -\frac{1}{4} \beta'(t/2) C_1(t/2) - \frac{1}{4} \beta(t/2) \frac{\sigma'}{\sigma}(t/2) C_1(t/2) + \int_0^{t/2} \mathcal{B}^0(t, \eta) \beta(\eta) d\eta;$$

$\mathcal{U}_{32}(t)$

$$= \frac{1}{2} \int \int \delta(t - \tau - \eta) \left[ \beta + B_\beta \left( \frac{\sigma'}{\sigma} \right) \right]' (\eta) \sqrt{\sigma(\eta)} \left[ \left( \frac{\kappa_0}{\sqrt{\sigma}} \right)' - \frac{\kappa_1}{\sqrt{\sigma}} \right] (\eta) \theta(\tau - \eta)$$

$$= \frac{1}{2} \int_0^{t/2} \left[ \beta + B_\beta \left( \frac{\sigma'}{\sigma} \right) \right]' (\eta) \sqrt{\sigma(\eta)} \left[ \left( \frac{\kappa_0}{\sqrt{\sigma}} \right)' - \frac{\kappa_1}{\sqrt{\sigma}} \right] (\eta) d\eta$$

$$= \frac{1}{2} \beta(t/2) \sqrt{\sigma(t/2)} \left[ \left( \frac{\kappa_0}{\sqrt{\sigma}} \right)' - \frac{\kappa_1}{\sqrt{\sigma}} \right] (t/2) - \frac{1}{2} \beta(0) \kappa_0'(0) + \int_0^{t/2} \mathcal{B}^0(t, \eta) \beta(\eta) d\eta$$

$$= \frac{1}{4} \beta(t/2) \left[ C(t/2) + C(0) - \left( \frac{\sigma'}{\sigma} \right) (t/2) C_1(t/2) + Q(t/2) C_1(t/2) \right]$$

$$- \frac{1}{2} \beta(0) C(0) + \int_0^{t/2} \mathcal{B}^0(t, \eta) \beta(\eta) d\eta;$$

$\mathcal{U}_{33}(t)$

$$= -\frac{1}{2} \int \int \nu_0(\eta) \theta(t - \tau - \eta) \left[ \beta + B_\beta \left( \frac{\sigma'}{\sigma} \right) \right]' (\eta) \kappa_0(\eta) \delta(\tau - \eta)$$

$$= -\frac{1}{2} \beta(t/2) \nu_0(t/2) \kappa_0(t/2) + \int_0^{t/2} \mathcal{B}^1(t, \eta) \beta(\eta) d\eta$$

$$= \frac{1}{4} \beta(t/2) Q(t/2) C_1(t/2) + \int_0^{t/2} \mathcal{B}^1(t, \eta) \beta(\eta) d\eta;$$

$\mathcal{U}_{34}(t)$

$$= \frac{1}{2} \int \int \theta(t - \tau - \eta) \nu_0 \left[ \beta + B_\beta \left( \frac{\sigma'}{\sigma} \right) \right]' \left[ \sqrt{\sigma} \left( \frac{\kappa_0}{\sqrt{\sigma}} \right)' - \frac{\kappa_1}{\sqrt{\sigma}} \right] (\eta) \theta(\tau - \eta)$$

$$= \frac{1}{2} \int_0^{t/2} \left[ \beta + B_\beta \left( \frac{\sigma'}{\sigma} \right) \right]' \nu_0 \left[ \sqrt{\sigma} \left( \frac{\kappa_0}{\sqrt{\sigma}} \right)' - \frac{\kappa_1}{\sqrt{\sigma}} \right] (\eta) \left( \int_\eta^{t-\eta} d\tau \right) d\eta$$

$$= \int_0^{t/2} \mathcal{B}^1(t, \eta) \beta(\eta) d\eta.$$

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At last, all the remaining terms are summed up to

$$\mathcal{U}_{35}(t)$$

$$= \frac{1}{2}\int \left[\beta + B_\beta\left(\frac{\sigma'}{\sigma}\right)\right]'(\eta)\mathcal{B}^1(t - \tau - \eta)_+ \sqrt{\sigma}\partial_\eta\left(\frac{v_0^2}{\sqrt{\sigma}}\right)$$

$$+ \frac{1}{2}\int \left[\beta + B_\beta\left(\frac{\sigma'}{\sigma}\right)\right]'(\eta)G_y|_{y=0}\mathcal{A}^0(\tau - \eta)_+$$

$$= \beta(0)\mathcal{A}^1(t) + \int_0^{t/2}\mathcal{A}^0(t,\eta)\beta(\eta)d\eta).$$

A closer look at the coefficient in front of $\beta(0)$ shows that this terms appears from $\frac{1}{2}\int\int \delta(t - \tau - \eta)\beta'(\eta)v_{0,\eta}^2(\tau,\eta)$ providing the term

$$-\frac{1}{2}\beta(0)v_{0,y}^2(t,0) = -\frac{1}{2}\beta(0)r_0^2(t).$$

Summarising,

$$\mathcal{U}_3(t) = -\frac{1}{2}\beta(0)r_0^2(t) - \frac{1}{4}\beta'(t/2)C_1(t/2) \tag{99}$$

$$+ \frac{1}{2}\beta(t/2)\left[\frac{C(t/2) + C(0)}{2} - \left(\frac{\sigma'}{\sigma}\right)(t/2)C_1(t/2) + Q(t/2)C_1(t/2)\right]$$

$$+ \int_0^{t/2}\mathcal{A}^0(t,\eta)\beta(\eta)d\eta$$

Thus it follows from equations (89), (93), (95) and (99) that

$$r_1^1(t) = \tag{100}$$

$$= -\frac{1}{4}\left[\alpha'(t/2) + \beta'(t/2)\right]C_1(t/2) - \frac{1}{4}\alpha(t/2)\left[2C(t/2) - C(0) - Q(t/2)C_1(t/2)\right]$$

$$+ \frac{1}{4}\beta(t/2)\left[2C(t/2) + C(0) + Q(t/2)C_1(t/2) - 2\left(\frac{\sigma'}{\sigma}\right)(t/2)\right]$$

$$- \frac{1}{2}\beta(0)r_0^2(t) + \int_0^{t/2}\left[\mathcal{A}_\alpha^0(t,\eta)\alpha(\eta) + [\mathcal{A}_\beta^0(t,\eta)\beta(\eta)]\right]d\eta$$

dealing with constant coefficients, $C(y) = 1$, $\sigma(y) = 1, r_0^2(t) = \theta(t)$, equation (100) takes the form (33).

# 6 Second integral equation : terms due to $v_1^1$

We turn now to the equation for $r_1^3(t)$. Recall that due to (47), (49) with $k = 3$

$$r_1^3(t) = \int\int G_y|_{y=0} \circ \Omega^3 = \sum_{i=1}^4 \mathcal{V}_i(t), \tag{101}$$

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where

$$\mathcal{V}_1(t) = \int \int G_y(t - \tau, 0, \eta) C(\eta) v_1^1(\tau, \eta) \, d\tau d\eta, \tag{102}$$

$$\mathcal{V}_2(t) = -\int \int G_y(t - \tau, 0, \eta) \alpha(\eta) v_{0,\tau\tau}^4(\tau, \eta) \, d\tau d\eta, \tag{103}$$

$$\mathcal{V}_3(t) = 6 \int \int G_y(t - \tau, 0, \eta) \left[ 12 C_{(xx)}(\eta) - 3 C_{(x)}(\eta) \right] v_0^2(\tau, \eta) \, d\tau d\eta, \tag{104}$$

$$\mathcal{V}_4(t) = \int \int G_y(t - \tau, 0, \eta) C_{(y)}(\eta) \sqrt{\sigma}(\eta) \partial_\eta \left( \frac{v_0^4}{\sqrt{\sigma}} \right) (\tau, \eta) \, d\tau d\eta. \tag{105}$$

Integrals (103)-(105) may be studied, similar to the terms $\mathcal{U}_i$ in (89), using the propagative wave expansions (80), (87). However, term $\mathcal{V}_1$ which can also be written in the form of a propagative wave expansion, requires more attention. This is due to the fact that the coefficients in this propagative wave expansion, starting from the third one, will contain derivatives of coefficients $\alpha$, $\beta$ so that extra care should be taken to guarantee that the resulting formulae do not contain these derivatives. Instead, we directly employ representation (49) for $v_1^1$ with $\Omega^1$ of form (47). Thus,

$$\mathcal{V}_1(t) = \mathcal{V}_{11}(t) + \mathcal{V}_{12}(t) + \mathcal{V}_{13}(t), \tag{106}$$

where

$$\mathcal{V}_{11}(t) = \tag{107}$$
$$-6 \int G_y(t - \tau, 0, \eta) C(\eta) G(\tau - s, \eta, \xi) \alpha(\xi) v_{0,ss}^2(s, \xi),$$

$$\mathcal{V}_{12}(t) = \tag{108}$$
$$6 \int G_y(t - \tau, 0, \eta) C(\eta) G(\tau - s, \eta, \xi) \left[ 2 C_{(xx)}(\xi) - C_{(x)}(\xi) \right] v_0^0(s, \xi),$$

$$\mathcal{V}_{13}(t) = \tag{109}$$
$$6 \int G_y(t - \tau, 0, \eta) C(\eta) G(\tau - s, \eta, \xi) C_{(y)}(\xi) \sqrt{\sigma}(\xi) \partial_\eta \left( \frac{v_0^2}{\sqrt{\sigma}} \right) (s, \xi).$$

We start with $\mathcal{V}_{11}(t)$. Using decompositions (70) for $G_y|_{y=0} = v_0^0$, (54), (55) for $G$, and (80) for $v_0^2$, we see that

$$\mathcal{V}_{11}(t) = J_1(t) + J_2(t) + J_3(t) + J_4(t) + J_5(t). \tag{110}$$

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Here

$$J_1(t) = \tag{111}$$

$$-6 \int G_{0,y}(t-\tau,0,\eta)C(\eta)G_0(\tau-s,\eta,\xi)\alpha(\xi)\kappa_0(\xi)\delta'(s-\xi)$$

$$= -3 \int C(\eta) \left[\delta(t-2\eta)\theta(\eta-\xi) + \delta(t-2\xi)\theta(\xi-\eta) - \delta(t-2\xi-2\eta)\right]\alpha(\xi)\kappa_0(\xi)$$

$$= -\frac{3}{2}\alpha(t/2)C_1^2(t/2) - \frac{3}{2}C(t/2)\int \alpha(\eta)C_1(\eta)d\eta + \frac{3}{2}\int \alpha(\eta)C_1(\eta)C(t/2-\eta)d\eta,$$

where we have used that the integral (112) is a convolution with respect to $s$ so that we can trasfer the $s-$differentiation from $\delta(s-\xi)$ to $G_0(\tau-s,\eta,\xi)$. Next terms, $J_2 - -J_4$ are obtained when we keep two most singular terms in the integrand in (107) but use the next order term in the third one, namely,

$$J_2(t)$$

$$= -6 \int \delta(t-\tau-\eta)C(\eta)g_\tau(\tau-s,\eta,\xi)\alpha(\xi)\kappa_0(\xi)\delta(s-\xi)$$

$$= -6 \int C(\eta)g_\tau(\tau-\eta-\xi,\eta,\xi)\alpha(\xi)\kappa_0(\xi)$$

$$= -6 \int_0^{t/2} \alpha(\xi)\kappa_0(\xi)\left(\int_0^{t/2} C(\eta)g_\tau(\tau-\eta-\xi,\eta,\xi)d\eta\right)d\xi$$

where we use that $g_t(\tau,\eta,\xi) = 0$ for $\tau < |\eta-\xi|$. We note that $g_\tau(\tau-\eta-\xi,\eta,\xi)$ has a jump discontinuity at $\eta = t/2-\xi$. We rewrite $J_2(t)$ in the following form convenient for the future,

$$J_2(t) = \int_0^{t/2} \mathcal{J}_2(t,\xi)\alpha(\xi)\,d\xi. \tag{112}$$

where $\mathcal{J}_2(t,\xi) = \mathcal{A}^3(t,\xi)$ and, due to (58),

$$\mathcal{J}_2(t,\xi)|_{\xi=t/2} = \frac{3}{2}C_1(t/2)\int_0^{t/2} [Q(t/2)-Q(\eta)]\,C(\eta)d\eta. \tag{113}$$

Next,

$$J_3(t) \tag{114}$$

$$= -3 \int G_{0,y}(t-\tau,0,\eta)G_0(\tau-s,\eta,\xi)C(\eta)\alpha(\xi)\kappa_1(\xi)\delta(s-\xi)$$

$$= -3 \int [\theta(t-2\eta)\theta(\eta-\xi) - \theta(t-2\xi)\theta(\xi-\eta)]\,C(\eta)\alpha(\xi)\kappa_1(\xi)$$

$$+3 \int \theta(t-2\xi-2\eta)C(\eta)\alpha(\xi)\kappa_1(\xi)$$

$$= -3 \int_0^{t/2} \alpha(\xi)\kappa_1(\xi)\left[\int_\xi^{t/2} C(\eta)d\eta + \int_0^\xi C(\eta)d\eta - \int_0^{t/2-\xi} C(\eta)d\eta\right]d\xi$$

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$$= \int_0^{t/2} \mathcal{J}_3(t,\xi)\alpha(\xi)d\xi,$$

with

$$\mathcal{J}_3(t,\xi) = \mathcal{A}^2(t,\xi), \tag{115}$$

$$\mathcal{J}_3(t,\xi)|_{\xi=t/2} = \frac{3}{2}C_1(t/2)\left[Q(t/2)C_1(t/2) - C(t/2) + C(0)\right],$$

and we use equation (80) for $\kappa_1$. Eventually,

$$J_4(t) = -6 \int \nu_0(\eta)\theta(t - \tau - \eta)C(\eta)G_{0,\tau}(\tau - s, \eta, \xi)\alpha(\xi)\kappa_0(\xi)\delta(s - \xi) \tag{116}$$

$$= -3 \int C(\eta)\left[\theta(t - 2\eta)\theta(\eta - \xi) - \theta(t - 2\xi)\theta(\xi - \eta) - \theta(t - 2\xi - 2\eta)\right]\alpha(\xi)\kappa_0(\xi)$$

$$= -3 \int_0^{t/2} \alpha(\xi)\kappa_0(\xi)\left[\int_\xi^{t/2} \nu_0(\eta)C(\eta)d\eta + \int_0^\xi \nu_0(\eta)C(\eta)d\eta\right]d\xi$$

$$+3 \int_0^{t/2} \alpha(\xi)\kappa_0(\xi)\left[\int_0^{t/2-\xi} \nu_0(\eta)C(\eta)d\eta\right]d\xi$$

$$= -3 \int_0^{t/2} \alpha(\xi)\kappa_0(\xi)\left(\int_{t/2-\xi}^{t/2} \nu_0(\eta)C(\eta)d\eta\right)d\xi = \int_0^{t/2} \mathcal{J}_4(t,\xi)\alpha(\xi)d\xi,$$

with

$$\mathcal{J}_4(t,\xi) = \mathcal{A}^3(t,\xi), \quad \mathcal{J}_4(t,\xi)|_{\xi=t/2} = \frac{3}{2}C_1(t/2)\int_0^{t/2} Q(\eta)C(\eta)d\eta. \tag{117}$$

We combine all the remaining terms in $\mathcal{V}_{11}$ into $J_5(t)$. Our aim is to show that

$$J_5(t) = \int_0^{t/2} \mathcal{J}_5(t,\xi)\alpha(\xi)d\xi, \tag{118}$$

$$\mathcal{J}_5(t,\xi) = \mathcal{A}^2(t,\xi), \quad \mathcal{J}_5(t,\xi)|_{\xi=t/2} = 0.$$

To this end observe that the integrals in $J_5$ are of two kinds, the first kind consists of terms with only one of three factors containing the main $\delta-$type singularity when the second kind may contain two $\delta-$type singularities with the third factor being continuous across the corresponding charateristic. A typical term of the first kind is

$$-6 \int G_{0,y}(t - \tau, 0, \eta)C(\eta)[G_\tau - G_{0,\tau}](\tau - s, \eta, \xi)\alpha(\xi)[v_{0,s}^2(s,\xi) - \delta(s - \xi)]$$

$$= -6 \int C(\eta)[G_\tau - G_{0,\tau}](t - \eta - s, \eta, \xi)\alpha(\xi)[v_{0,s}^2(s,\xi) - \delta(s - \xi)]\, d\xi d\eta ds.$$

Now all terms in the integrand are bounded functions. Moreover, their support is the intersection of the domains $s \geq 0$, $\xi \geq 0$, $\eta \geq 0$, $s \geq \xi$ and $t - \eta - s \geq |\eta - \xi|$.

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Clearly, this implies that, making the external integration with respect to $\xi$, its domain is $0 \leq \xi \leq t/2$, so that the corresponding terms are of the form $\int_0^{t/2} \mathcal{A}^2(t,\xi)\alpha(\xi)d\xi$. What is more, the kernel $\mathcal{A}^2(t,\xi)$ is defined as a double integral, with respect to $\eta$, $s$ over the region $s \geq 0$, $\eta \geq 0$, $s \geq \xi$, $t-\eta-s \geq |\eta-\xi|$. In particular, when $\xi = t/2$ this region shrinks to the interval $s = t/2$, $0 \leq \eta \leq t/2$, so that $\mathcal{A}^2(t,t/2) = 0$.

Similarly, a typical term of the second kind is

$$-6 \int G_{0,y}(t - \tau, 0, \eta)C(\eta)G_{0,\tau}(\tau - s, \eta, \xi)\alpha(\xi)\widetilde{v}(s,\xi) \tag{119}$$

$$= -3 \int_0^{t/2} \alpha(\xi)\left[\int_\xi^{t/2} C(\eta)\widetilde{v}(t - 2\eta + \xi, \xi)d\eta\right]d\xi$$

$$-3\int_0^{t/2}\alpha(\xi)\left[\widetilde{v}(t - \xi, \xi)\int_0^\xi C(\eta)d\eta\right]d\xi$$

$$+3\int_0^{t/2}\alpha(\xi)\left[\int_0^{t/2-\xi} C(\eta)\widetilde{v}(t - 2\eta - \xi, \xi)d\eta\right]d\xi$$

where

$$\widetilde{v}(s,\xi) = v_{0,s}^2(s,\xi) - \delta(s - \xi) - \kappa_0(\xi)\theta(s - \xi)$$

When $\xi = t/2$, the first and the third integrals with respect to $\eta$ in the right-hand side of (119) become 0. As $\widetilde{v}(s,\xi) = 0$ for $s = \xi$, the second integral in the right-hand side of (119) is multiplied by 0. Thus it has the form $\int_0^{t/2} \mathcal{A}^2(t,\xi)\alpha(\xi)d\xi$ with $\mathcal{A}^2(t,t/2) = 0$.

Summarizing equations (112)–(118), we obtain with $\mathcal{J}_{11}(t,\xi) = \mathcal{A}^2(t,\xi)$,

$$\mathcal{V}_{11}(t) = -\frac{3}{2}\alpha(t/2)C_1^2(t/2) + \int_0^{t/2} \mathcal{J}_{11}(t,\xi)\alpha(\xi)d\xi, \tag{120}$$

$$\mathcal{J}_{11}(t,\xi)|_{\xi=t/2} = 3C_1(t/2)\left[C(0) - C(t/2) + \int_0^{t/2} Q(\eta(C(\eta)d\eta\right].$$

Next we turn to $\mathcal{V}_{12}$, see (108). As, by (40),

$$2C_{(xx)}(\xi) - C_{(x)}(\xi) = \frac{1}{2}C(\xi)[\beta + D_\beta](\xi),$$

$$D_\beta(\xi) = \partial_\xi\left[\log\left(\frac{C}{\sigma}\right)(\xi)\right]\int_0^\xi \beta(\eta)d\eta,$$

$$\mathcal{V}_{12}(t) = J_1(t) + J_2(t). \tag{121}$$

Here $J_1$ contains the leading singularities of the integrand in (103),

$$J_1(t) = \tag{122}$$

$$= 3 \int \delta(t - \tau - \eta) C(\eta) C(\xi) [\beta + D_\beta](\xi) G_0(\tau - s, \eta, \xi) \delta(s - \xi)$$

$$= \frac{3}{2} \int \delta(t - \tau - \eta) C(\eta) C(\xi) [\beta + D_\beta](\xi) \theta(\tau - s - \eta + \xi) \theta(\xi - \eta) \delta(s - \xi)$$

$$+ \frac{3}{2} \int \delta(t - \tau - \eta) C(\eta) C(\xi) [\beta + D_\beta](\xi) \theta(\tau - s + \eta - \xi) \theta(\eta - \xi) \delta(s - \xi)$$

$$- \frac{3}{2} \int \delta(t - \tau - \eta) C(\eta) C(\xi) [\beta + D_\beta](\xi) \theta(\tau - s - \eta - \xi) \delta(s - \xi),$$

with $J_2$ built of less singular terms. thus, we see that

$$J_1(t) = -\frac{3}{2} \int C(\eta) C(\xi) [\beta + D_\beta](\xi) \theta(t - 2\eta - 2\xi) \tag{123}$$

$$+ \frac{3}{2} \int C(\eta) C(\xi) [\beta + D_\beta](\xi) \left[ \theta(t - 2\xi) \theta(\xi - \eta) + \theta(t - 2\eta) \theta(\eta - \xi) \right]$$

$$= -\frac{3}{2} \int_0^{t/2} C(\xi) [\beta + D_\beta](\xi) \left( \int_0^{t/2 - \xi} C(\eta) d\eta \right) d\xi$$

$$+ \frac{3}{2} \int_0^{t/2} C(\xi) [\beta + D_\beta](\xi) \left( \int_0^{\xi} C(\eta) d\eta \right) d\xi$$

$$+ \frac{3}{2} \int_0^{t/2} C(\eta) \left( \int_0^{\eta} C(\xi) [\beta + D_\beta](\xi) d\xi \right) d\eta$$

$$= \frac{3}{2} \int_0^{t/2} C(\xi) [\beta + D_\beta](\xi) \left( C_1(t/2) - C_1(t/2 - \xi) \right) d\xi$$

$$= \int_0^{t/2} \mathcal{J}_1(t, \xi) \beta(\xi) d\xi.$$

Here

$$\mathcal{J}_1(t, \xi) = \mathcal{A}^2(t, \xi), \quad \mathcal{J}_1(t, \xi)|_{\xi = t/2} = \frac{3}{2} C_1(t/2) C(t/2) \tag{124}$$

and, in deriving (124), we use that

$$\frac{3}{2} \int_0^{t/2} C(\xi) D_\beta(\xi) \left( C_1(t/2) - C_1(t/2 - \xi) \right) d\xi = \int_0^{t/2} \mathcal{A}^3(t, \xi) \beta(\xi) d\xi,$$

where

$$\mathcal{A}^3(t, \xi)|_{\xi = t/2} = 0.$$

Let us show that the terms constituting $J_2$ are of the form

$$J_2(t) = \int_0^{t/2} \mathcal{A}^2(t, \xi) \beta(\xi) d\xi, \quad \mathcal{A}^2(t, \xi)|_{\xi = t/2} = 0. \tag{125}$$

To this end we observe that the terms in $J_2$ are of three different types, one containing $\delta(t - \tau - \eta)$, the second -$\delta(s - \xi)$ and the third one having bounded

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integrand. Consider e.g the term with $\delta(t - \tau - \eta)$ reducing the quadruple integral to a triple one with respect to $(s, \eta, \xi)$. This triple integral extends over three possible regions,

$$\Omega_1 = \{(s, \eta, \xi) : s + \xi \le t, \; \xi \le s, \; \eta \le \xi\};$$
$$\Omega_2 = \{(s, \eta, \xi) : s + 2\eta \le t + \xi, \; \xi \le s, \; \xi \le \eta\};$$
$$\Omega_3 = \{(s, \eta, \xi) : s + \xi + 2\eta \le t, \; \xi \le s\}.$$

These conditions always imply that $0 \le \xi \le t/2$, so that the corresponding terms in $J_2(t)$ is of the form $\int_0^{t/2} \mathcal{A}^2(t, \xi)[\beta + D_\beta](\xi)d\xi$ and, as we know

$$\int_0^{t/2} \mathcal{A}^2(t, \xi)D_\beta(\xi)d\xi = \int_0^{t/2} \mathcal{A}^3(t, \xi)\beta(\xi)d\xi, \quad \mathcal{A}^3(t, \xi)|_{\xi=t/2} = 0.$$

To analyse $\mathcal{A}^2(t, \xi)|_{\xi=t/2}$, consider the double integral with respect to $(s, \eta)$ over $\Omega_i \cap \{\xi = t/2\}$, $i = 1, 2, 3$. Clearly, $\Omega_i \cap \{\xi = t/2\} = (t/2.t/2)$, $i = 2, 3$, while $\Omega_1 \cap \{\xi = t/2\} = \{s = t/2, \; 0 \le \eta \le t/2\}$. This implies that these double integrals are taken over a region of $0-$measure.

Similar considerations are valid having the factor $\delta(s - \xi)$. At last, dealing with the genuinely quadruple integral, observe that the domain of integration consists of three regions

$$\widetilde{\Omega}_1 = \{(\tau, s, \eta, \xi) : \tau + \eta \le t, \; \xi \le s, \; \eta \le \xi, \; s + \xi \le \tau + \eta\};$$
$$\widetilde{\Omega}_2 = \{(\tau, s, \eta, \xi) : \tau + \eta \le t, \; \xi \le s, \; \xi \le \eta, \; s + \eta \le \tau + \xi\};$$
$$\widetilde{\Omega}_3 = \{(\tau, s, \eta, \xi) : \tau + \eta \le t, \; s + \xi + \eta \le \tau, \; \xi \le s\},$$

implying that $\xi \le t/2$. Then $\widetilde{\Omega}_1 \cap \{\xi = t/2\} = \{s = t/2, \; \tau + \eta = t$, i.e. an interval, while $\widetilde{\Omega}_2 \cap \{\xi = t/2\} = \{s = t/2, \eta = t/2, \tau = t/2\}$, i.e. a point, and $\widetilde{\Omega}_2 \cap \{\xi = t/2\} = \{s = t/2, \eta = 0, \tau = t\}$, i.e. another point.

The above considerations prove (125) so that equations (121) together with (123)–(125) imply that

$$\mathcal{V}_{12}(t) = \int_0^{t/2} \mathcal{J}_{12}(t, \xi)\beta(\xi)d\xi, \tag{126}$$

$$\mathcal{J}_{12}(t, \xi) = \mathcal{A}^2(t, \xi), \quad \mathcal{J}_{12}(t, \xi)|_{\xi=t/2} = \frac{3}{2}C_1(t/2)C(t/2)$$

We note that the impact of the terms with $D_\beta$ into integral (103) is of the form

$$\int_0^{t/2} \mathcal{A}^2(t, \xi)\beta(\xi)d\xi, \quad \mathcal{A}^2(t, \xi)|_{\xi=t/2} = 0.$$

Our next goal is to evaluate the term $\mathcal{V}_{13}(t)$ given by (109). As, due to (40),

$$C_{(y)}(\xi) = \frac{1}{2}\partial_\xi[\beta + B_\beta](\xi), \quad B_\beta(\xi) = \left(\frac{\sigma'}{\sigma}\right)(\xi)\int_0^\xi \beta(\eta)d\eta,$$

integrating by parts, with respect to $\xi$, in (109) and taking into the account that there is actually a factor $\theta(\xi)$ in the integrand of (109), we find that

$$
\mathcal{V}_{13}(t)
$$
$$
= -3 \int G_y(t-\tau,0,\eta)C(\eta)[\beta + B_\beta](\xi)\partial_\xi \left[ G(\tau - s,\eta,\xi)\sqrt{\sigma}(\xi)\partial_\xi \left( \frac{v_0^2}{\sqrt{\sigma}} \right)(s,\xi) \right]
$$
$$
-3 \int G_y(t-\tau,0,\eta)C(\eta)[\beta + B_\beta](\xi)G(\tau - s,\eta,\xi)\sqrt{\sigma}(\xi)\partial_\xi \left( \frac{v_0^2}{\sqrt{\sigma}} \right)(s,\xi)\delta(\xi).
$$

First we show that the integral with $\delta(\xi)$ is equal to 0. To this end observe that the strongest singularity in this integral is a $\delta$−type one and $G(\tau - s,\eta,0) = 0$ due to the Dirichlet boundary condition. thus,

$$
\mathcal{V}_{13}(t) = \tag{127}
$$
$$
-3 \int G_y|_{y=0} C[\beta + B_\beta] \left[ G_\xi \sqrt{\sigma}\partial_\xi \left( \frac{v_0^2}{\sqrt{\sigma}} \right) + \frac{1}{2}G\frac{\sigma'}{\sigma}\left( \frac{v_0^2}{\sqrt{\sigma}} \right) + G\sqrt{\sigma}\partial_\xi^2 \left( \frac{v_0^2}{\sqrt{\sigma}} \right) \right]
$$

To further analyze equation (127), observe that

$$
\partial_\xi^2 v_0^2(s,\xi) = \partial_s^2 v_0^2(s,\xi) + q(\xi)v_0^2(s,\xi) - 2C(\xi)v_0^0(s,\xi).
$$

This yields that the expression in $[\ldots]$ in (127) has the form

$$
G_\xi \sqrt{\sigma}\partial_\xi \left( \frac{v_0^2}{\sqrt{\sigma}} \right) + \frac{1}{2}G\frac{\sigma'}{\sigma}\left( \frac{v_0^2}{\sqrt{\sigma}} \right) + G\frac{\sigma'}{\sigma}\partial_\xi^2 \left( \frac{v_0^2}{\sqrt{\sigma}} \right) = \tag{128}
$$
$$
G_\xi v_{0,\xi}^2 + G_\tau v_{0,s}^2 - \frac{\sigma'}{2\sigma}\left( G_\xi v_0^2 + Gv_{0,\xi}^2 \right) + Gv_0^2 \left[ q - \frac{1}{2}\left( \frac{\sigma'}{\sigma} \right)' \right] - 2CGv_0^0.
$$

Using this equation, we obtain

$$
\mathcal{V}_{13}(t) = J_1(t) + J_2(t) + J_3(t) + J_4(t) + J_5(t) + J_6(t) + J_7(t), \tag{129}
$$

The term $J_1$ is the main term obtained by substitution $G_{0,\xi}$, $G_{0,\tau}$ and $-\kappa_0\delta(s - \xi)$, $\kappa_0\delta(s - \xi)$ instead of $G_\xi$, $G_\tau$ and $v_{0,\xi}^2$, $v_{0,s}^2$, correspondingly,

$$
J_1(t) = \tag{130}
$$
$$
-3 \int \delta(t-\tau-\eta)C(\eta)[\beta + B_\beta](\xi)\kappa_0(\xi)\delta(\tau - s + \eta - \xi)\theta(\xi - \eta)\delta(s - \xi)
$$
$$
+3 \int \delta(t-\tau-\eta)C(\eta)[\beta + B_\beta](\xi)\kappa_0(\xi)\delta(\tau - s - \eta - \xi)\delta(s - \xi)
$$
$$
= -3 \int C(\eta)[\beta + B_\beta](\xi)\kappa_0(\xi) \left[ \delta(t - 2\xi)\theta(\xi - \eta) - \delta(t - 2\eta - 2\xi) \right]
$$
$$
= -\frac{3}{2}[\beta + B_\beta](t/2)\kappa_0(t/2)\int_0^{t/2} C(\eta)d\eta + \frac{3}{2}\int_0^{t/2} [\beta + B_\beta](\xi)\kappa_0(\xi)C(t/2 - \xi)d\xi
$$
$$
= -\frac{3}{2}\beta(t/2)C_1^2(t/2) + \frac{3}{2}\int_0^{t/2} \beta(\xi)\left[ C_1(\xi)C(t/2 - \xi) - \left( \frac{\sigma'}{\sigma} \right)(t/2)C_1^2(t/2) \right]
$$

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$$+\frac{3}{2}\int_0^{t/2}\beta(\eta)\left[\int_\eta^{t/2}\left(\frac{\sigma'}{\sigma}\right)(\xi)C_1(\xi)C(t/2-\xi)d\xi\right]d\eta$$

$$=-\frac{3}{2}\beta(t/2)C_1^2(t/2)+\int_0^{t/2}\mathcal{J}_1(t,\xi)\beta(\xi)d\xi,$$

where

$$\mathcal{J}_1(t,\xi)=\mathcal{A}^1(t,\xi),\quad \mathcal{J}_1(t,\xi)|_{\xi=t/2}=\frac{3}{2}C_1(t/2)\left[C(0)-\left(\frac{\sigma'}{\sigma}\right)(t/2)C_1(t/2)\right]\ (131)$$

As in the case of $\mathcal{V}_{11}$, the next three terms correspond to taking exactly one non-leading term in the integrand in (127). Namely,

$$J_2(t)=\tag{132}$$

$$-3\int g_y(t-\tau,0,\eta)C(\eta)[\beta+B_\beta](\xi)\kappa_0(\xi)\delta(\tau-s+\eta-\xi)\theta(\xi-\eta)\delta(s-\xi)$$

$$+3\int g_y(t-\tau,0,\eta)C(\eta)[\beta+B_\beta](\xi)\kappa_0(\xi)\delta(\tau-s-\eta-\xi)\delta(s-\xi)$$

$$=-3\int_0^{t/2}[\beta+B_\beta](\xi)\kappa_0(\xi)\left(\int_\xi^{2\xi}C(2\xi-\tau)g_y(t-\tau,0,2\xi-\tau)d\tau\right)d\xi$$

$$+3\int_0^{t/2}[\beta+B_\beta](\xi)\kappa_0(\xi)\left(\int_{2\xi}^{t/2+\xi}C(\tau-2\xi)g_y(t-\tau,0,\tau-2\xi)d\tau\right)d\xi$$

$$=\int_0^{t/2}\mathcal{J}_2(t,\xi)\beta(\xi)d\xi,$$

where

$$\mathcal{J}_2(t,\xi)=\mathcal{A}^2(t,\xi),\tag{133}$$

$$\mathcal{J}_2(t,\xi)|_{\xi=t/2}=-3\kappa_0(t/2)\int_{t/2}^t C(t-\tau)g_y(t-\tau+0,0,t-\tau)d\tau$$

$$=\frac{3}{2}C_1(t/2)\int_0^{t/2}Q(\eta)C(\eta)d\eta$$

and we use that

$$g_y(\eta+0,0,\eta)=v_0^0(\eta+0,\eta)=\nu_0(\eta).$$

For the term $J_3$ we take $[-g_\xi+g_\tau](\tau-s,\eta,\xi)\kappa_0(\xi)\delta(s-\xi)$. Then

$$J_3(t)=$$

$$-3\int C(\eta)[\beta+B_\beta](\xi)\kappa_0(\xi)[-g_\xi+g_\tau](\tau-s,\eta,\xi)d\eta d\xi.$$

The support of the integrand is $t-\eta-\xi\geq|\eta-\xi|$, so that

$$J_3(t)=$$

$$-3\int_0^{t/2} C(\eta)\left(\int_0^{\eta}[\beta+B_\beta](\xi)\kappa_0(\xi)[-g_\xi+g_\tau](t-\eta-\xi,\eta,\xi)d\xi\right)d\eta$$

$$-3\int_0^{t/2}[\beta+B_\beta](\xi)\kappa_0(\xi)\left(\int_0^{\xi} C(\eta)[-g_\xi+g_\tau](t-\eta-\xi,\eta,\xi)d\eta\right)d\xi$$

$$=-3\int_0^{t/2}[\beta+B_\beta](\xi)\kappa_0(\xi)\left(\int_0^{t/2} C(\eta)[-g_\xi+g_\tau](t-\eta-\xi,\eta,\xi)d\eta\right)d\xi$$

Observe that the function $[-g_\xi+g_\tau](t-\eta-\xi,\eta,\xi)$ has a jump across $\eta=t/2-\xi$. Summarizing,

$$J_3(t)=\int_0^{t/2}\mathcal{J}_3(t,\xi)\beta(\xi)d\xi,\quad \mathcal{J}_3(t,\xi)=\mathcal{A}^2(t,\xi),\qquad(134)$$

$$\mathcal{J}_3(t,\xi)|_{\xi=t/2}=-3\kappa_0(t/2)\int_0^{t/2}C(\eta)[-g_\xi+g_\tau](t/2-\eta+0,\eta,t/2)d\eta$$

$$=\frac{3}{2}Q(t/2)C_1^2(t/2)-\frac{3}{2}C_1(t/2)\int_0^{t/2}Q(\eta)C(\eta)d\eta,$$

where we use that, due to (55),

$$[-g_\xi+g_\tau](t/2-\eta+0,\eta,t/2)=\frac{1}{2}[Q(\eta)-Q(t/2)]$$

The term $J_4$ is obtained taking main terms for $G_y$ and $G_\xi$, $G_\tau$ but $v_{0,\xi}+\kappa_0\delta(s-\xi)$ and $v_{0,s}-\kappa_0\delta(s-\xi)$ instead of $v_{0,\xi}$, $v_{0,s}$,

$$J_4(t)=-\frac{3}{2}\int\delta(t-\tau-\eta)C(\eta)[\beta+B_\beta](\xi)\mathcal{F}_4(\tau,s,\eta,\xi),$$

where

$$\mathcal{F}_4(\tau,s,\eta,\xi)$$
$$=[\delta(\tau-s+\xi-\eta)\theta(\eta-\xi)-\delta(\tau-s-\xi+\eta)\theta(\xi-\eta)]\left(v_{0,\xi}^2(s,\xi)+\kappa_0(\xi)\delta(s-\xi)\right)$$
$$+\delta(\tau-s-\xi-\eta)\left(v_{0,\xi}^2(s,\xi)+\kappa_0(\xi)\delta(s-\xi)\right)$$
$$+[\delta(\tau-s+\xi-\eta)\theta(\eta-\xi)+\delta(\tau-s-\xi+\eta)\theta(\xi-\eta)]\left(v_{0,s}^2(s,\xi)-\kappa_0(\xi)\delta(s-\xi)\right)$$
$$-\delta(\tau-s-\xi-\eta)\left(v_{0,s}^2(s,\xi)-\kappa_0(\xi)\delta(s-\xi)\right).$$

Therefore,

$$J_4(t,\xi)=$$
$$-\frac{3}{2}\int C(\eta)[\beta+B_\beta](\xi)[v_{0,\xi}^2+v_{0,s}^2](s,\xi)](t-2\eta+\xi,\xi)\theta(\eta-\xi)$$
$$+\frac{3}{2}\int C(\eta)[\beta+B_\beta](\xi)[v_{0,\xi}^2(t-\xi,\xi)+2\kappa_0(\xi)\delta(t-2\xi)-v_{0,s}^2(t-\xi,\xi)]\theta(\xi-\eta)$$
$$-\frac{3}{2}\int C(\eta)[\beta+B_\beta](\xi)[v_{0,\xi}^2(t-2\eta-\xi,\xi)+2\kappa_0(\xi)\delta(t--2\eta2\xi)-v_{0,s}^2(t--2\eta\xi,\xi)]$$

Note that $v_{0,\xi} + \kappa_0\delta(s-\xi)$ and $v_{0,s} - \kappa_0\delta(s-\xi)$ are smooth except for a jump discontinuity across $s = \xi$ and have support at $s \geq \xi$. Together with considerations above, this implies that terms containing $B_\beta$ have the form

$$\int_0^{t/2} \mathcal{A}^2(t,\xi)\beta(\xi)d\xi, \quad \mathcal{A}^2(t,\xi)|_{\xi=t/2} = 0.$$

Thus,

$$J_4(t) = \tag{135}$$

$$-\frac{3}{2}\int_0^{t/2} \beta(\xi)\left(\int_\xi^{t/2} C(\eta)[v_{0,\xi}^2 + v_{0,s}^2](t - 2\eta + \xi, \xi)d\eta\right)d\xi$$

$$+\frac{3}{2}\int_0^{t/2} \beta(\xi)\left(\int_0^\xi C(\eta)[v_{0,\xi}^2(t - \xi, \xi) - v_{0,s}^2(t - \xi, \xi) + 2\kappa_0(\xi)\delta(t - 2\xi)]d\eta\right)d\xi$$

$$-\frac{3}{2}\int_0^{t/2} \beta(\xi)\left(\int_0^{t/2-\xi} C(\eta)[v_{0,\xi}^2(t - 2\eta - \xi, \xi) - v_{0,s}^2(t - 2\eta - \xi, \xi)]d\eta\right)d\xi$$

$$-3\int_0^{t/2} \beta(\xi)\left(\int_0^{t/2-\xi} C(\eta)\kappa_0(\xi)\delta(t - 2\eta - 2\xi)d\eta\right)d\xi$$

$$+\int_0^{t/2} \mathcal{A}^2(t,\xi)\beta(\xi)d\xi = \int_0^{t/2} \mathcal{J}_4(t,\xi)\beta(\xi)d\xi,$$

with

$$\mathcal{J}_4(t,\xi) = \mathcal{A}^1(t,\xi), \tag{136}$$

$$\mathcal{J}_4(t,\xi)|_{\xi=t/2} = \frac{3}{2}\int_0^{t/2} C(\eta)[v_{0,\xi}^2(t/2 + 0, t/2) - v_{0,s}^2(t/2 + 0, t/2)]d\eta$$

$$= \frac{3}{2}[\kappa_0'(t/2) - 2\kappa_1(t/2)]\int_0^{t/2} C(\eta)d\eta = \frac{3}{2}C_1(t/2)[C(0) + Q(t/2)C_1(t/2)].$$

Two more terms which may have the form $\int_0^{t/2} \mathcal{A}^2(t,\xi)\beta(\xi)d\xi$ with $\mathcal{A}^2(t,\xi)|_{\xi=t/2} \neq 0$ are

$$J_5(t) = \frac{3}{4}\int \delta(t - \tau - \eta)C(\eta)\frac{\sigma'}{\sigma}(\xi)[\beta + B_\beta](\xi)\delta(\tau - s - \eta + \xi)\theta(\eta - \xi)v_0^2(s,\xi)$$

$$+\frac{3}{4}\int \delta(t - \tau - \eta)\left[-\delta(\tau - s + \eta - \xi)\theta(\xi - \eta) + \delta(\tau - s - \eta - \xi)\right]v_0^2(s,\xi)$$

$$= \frac{3}{4}\int C(\eta)\frac{\sigma'}{\sigma}(\xi)[\beta + B_\beta](\xi)\left[v_0^2(t - 2\eta + \xi, \xi)\theta(\eta - \xi) - v_0^2(t - \xi, \xi)\theta(\xi - \eta)\right]$$

$$+\frac{3}{4}\int C(\eta)\frac{\sigma'}{\sigma}(\xi)[\beta + B_\beta](\xi)v_0^2(t - 2\eta - \xi, \xi).$$

Taking into the account that $v_0^2(s,\xi) = 0$ for $s < \xi$, we see that

$$J_5(t) = \frac{3}{4}\int_0^{t/2} \frac{\sigma'}{\sigma}(\xi)\beta(\xi)\left[\int_\xi^{t/2} C(\eta)v_0^2(t - 2\eta + \xi, \xi)d\eta\right]d\xi \tag{137}$$

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$$-\frac{3}{4}\int_0^{t/2}\frac{\sigma'}{\sigma}(\xi)v_0^2(t-\xi,\xi)\beta(\xi)\left[\int_0^\xi C(\eta)d\eta\right]d\xi$$

$$+\frac{3}{4}\int_0^{t/2}\frac{\sigma'}{\sigma}(\xi)\left[\int_0^{t/2-\xi}C(\eta)v_0^2(t-2\eta-\xi,\xi)d\eta\right]d\xi+\int_0^{t/2}\mathcal{A}^3(t,\xi)\beta(\xi)d\xi,$$

where the last integral, with $\mathcal{A}^3(t,\xi)|_{\xi=t/2}=0$, is due to $B_\beta$. Equation (137) immediately imply that

$$J_5(t)=\int_0^{t/2}\mathcal{J}_5(t,\xi)\beta(\xi)d\xi,\quad \mathcal{J}_5(t,\xi)=\mathcal{A}^2(t,\xi),\tag{138}$$

$$\mathcal{J}_5(t,\xi)|_{\xi=t/2}=-\frac{3}{4}C_1(t/2)\frac{\sigma'}{\sigma}(t/2)v_0^2(t/2+0,t/2)=-\frac{3}{4}C_1^2(t/2)\frac{\sigma'}{\sigma}(t/2).$$

The last important term is $J_6$,

$$J_6(t)=-\frac{3}{2}\int\delta(t-\tau-\eta)C(\eta)\frac{\sigma'}{\sigma}(\xi)[\beta+B_\beta](\xi)G(\tau-s,\eta,\xi)\kappa_0(\xi)\delta(s-\xi)$$

$$=-\frac{3}{2}\int C(\eta)\frac{\sigma'}{\sigma}(\xi)\kappa_0(\xi)[\beta+B_\beta](\xi)G(t-\tau-\xi,\eta,\xi)$$

The support of the integrand, due to $G(t-\tau-\xi,\eta,\xi)$ is given by

$$\{0\le\eta\le\xi,\,\xi-\eta\le t-\eta-\xi\}\cup\{0\le\xi\le\eta,\,\eta-\xi\le t-\eta-\xi\}$$
$$=\{0\le\xi\le t/2,\,0\le\eta\le\xi\}\cup\{0\le\eta\le t/2,\,0\le\xi\le\eta\}$$

Therefore,

$$J_6(t)=-\frac{3}{2}\int_0^{t/2}\frac{\sigma'}{\sigma}(\xi)\kappa_0(\xi)\beta(\xi)\left[\int_0^\xi C(\eta)G(t-\tau-\xi,\eta,\xi)d\eta\right]d\xi$$

$$=-\frac{3}{2}\int_0^{t/2}\frac{\sigma'}{\sigma}(\xi)\kappa_0(\xi)\beta(\xi)\left[\int_\xi^{t/2}C(\eta)G(t-\tau-\xi,\eta,\xi)d\eta\right]d\xi\tag{139}$$

$$=\int_0^{t/2}\mathcal{J}_6(t,\xi)\beta(\xi)d\xi,$$

where

$$\mathcal{J}_6(t,\xi)|_{\xi=t/2}=-\frac{3}{2}\kappa_0(t/2)\frac{\sigma'}{\sigma}(t/2)\int_0^{t/2}C(\eta)G(t/2-\tau+0,\eta,t/2)d\eta$$

$$=-\frac{3}{4}C_1^2(t/2)\frac{\sigma'}{\sigma}(t/2),\quad \mathcal{J}_6(t,\xi)=\mathcal{A}^2(t,\xi).\tag{140}$$

The last term important for the future is

$$J_7(t)=6\int\delta(t-\tau-\eta)C(\eta)C(\xi)[\beta+B_\beta](\xi)G_0(\tau-s,\eta,\xi)\delta(s-\xi),$$

which is the leading term due to $-2Cv_0^0$ in (128). Clearly, this term is analogous to the term (122) in $\mathcal{V}_{12}$ with $B_\beta$ instead of $D_\beta$. Using the remark after (126) and taking into the account that similar behaviour is valid for the $B_\beta$−term, we obtain that

$$J_7(t) = \int_0^{t/2} \mathcal{J}_7(t,\xi)\beta(\xi)d\xi. \tag{141}$$

Here

$$\mathcal{J}_7(t,\xi) = \mathcal{A}^2(t,\xi), \quad \mathcal{J}_7(t,\xi)|_{\xi=t/2} = 3C_1(t/2)C(t/2). \tag{142}$$

We combine all remaining terms into $J_8$. Similar to $\mathcal{V}_{11}(t)$, $\mathcal{V}_{12}(t)$, we can show that

$$J_8(t) = \int_0^{t/2} \mathcal{J}_7(t,\xi)\beta(\xi)d\xi, \quad \mathcal{J}_8(t,\xi) = \mathcal{A}^2(t,\xi), \quad \mathcal{J}_8(t,\xi)|_{\xi=t/2} = 0. \tag{143}$$

Summarizing equations (130)− (142),

$$\mathcal{V}_{13}(t) = -\frac{3}{2}C_1^2(t/2)\beta(t/2) + \int_0^{t/2} \mathcal{J}_{13}(t,\xi)\beta(\xi)d\xi, \tag{144}$$

$$\mathcal{J}_{13}(t,\xi) = \mathcal{A}^1(t,\xi),$$

$$\mathcal{J}_{13}(t,\xi)|_{\xi=t/2} =$$

$$3C_1(t/2)\left[C(0) + C(t/2) + C_1(t/2)Q(t/2) - C_1(t/2)\left(\frac{\sigma'}{\sigma}\right)(t/2)\right]$$

We finish this section with the equation for $\mathcal{V}_1$,

$$\mathcal{V}_1(t) = \tag{145}$$

$$-\frac{3}{2}C_1^2(t/2)[\alpha(t/2) + \beta(t/2)] + \int_0^{t/2} [\mathcal{J}_{1\alpha}(t,\xi)\alpha(\xi) + \mathcal{J}_{1\beta}(t,\xi)\beta(\xi)] \, d\xi,$$

where

$$\mathcal{J}_{1,\alpha}(t,\xi)|_{\xi=t/2} = 3C_1(t/2)\left[C(0) - C(t/2) + \int_0^{t/2} Q(\eta)C(\eta)d\eta\right]. \tag{146}$$

$$\mathcal{J}_{1,\beta}(t,\xi)|_{\xi=t/2} =$$

$$\frac{3}{2}C_1(t/2)\left[2C(0) + 3C(t/2) + 2C_1(t/2)Q(t/2) - 2C_1(t/2)\left(\frac{\sigma'}{\sigma}\right)(t/2)\right].$$

Comparing (145)-(146) with the first 4 terms $I_1(t) - I_4(t)$ in (37) which are responsible for the impact of $u_1^1$ on $u_{1,y}^3|_{y=0}$ and can be written in form (145), we see that they coincide.

# 7 Second integral equation

In this section we analyze terms $\mathcal{V}_2(t) - \mathcal{V}_4(t)$ in (101) and derive the second integral equation for $\alpha(y)$, $\beta(y)$.

We start with $\mathcal{V}_2(t)$ given by (103),

$$\mathcal{V}_2(t) = -\int G_y(t-\tau, 0, \eta)\alpha(\eta)v_{0,\tau\tau}^4(\tau, \eta)$$
$$= J_1(t) + J_2(t) + J_3(t) + J_4(t).$$

Here the term $J_1$ contains the main singularities of $G_y$ and $)v_{0,\tau\tau}^4$,

$$J_1(t) = -\int \delta(t-\tau-\eta)\alpha(\eta)\mu_1(\eta)\delta(\tau-\eta) \tag{147}$$
$$= -\frac{1}{2}\alpha(t/2)\mu_1(t/2) = -\frac{3}{2}C_1^2(t/2)\alpha(t/2).$$

The terms $J_2(t)$, $J_3(t)$ contain the next order singularity,

$$J_2(t) = -\int \nu_0(\eta)\theta(t-\tau-\eta)\alpha(\eta)\mu_1(\eta)\delta(\tau-\eta) \tag{148}$$
$$= -\int_0^{t/2} \nu_0(\eta)\alpha(\eta)\mu_1(\eta)d\eta = \frac{3}{2}\int_0^{t/2} C_1^2(\eta)Q(\eta)\alpha(\eta)d\eta$$
$$= \int_0^{t/2} \mathcal{J}_2(t,\eta)\alpha(\eta)d\eta,$$

with

$$\mathcal{J}_2(t,\eta)|_{\eta=t/2} = \frac{3}{2}C_1^2(t/2)Q(t/2), \tag{149}$$

while

$$J_3(t) = -\int \delta(t-\tau-\eta)\alpha(\eta)\mu_2(\eta)\theta(\tau-\eta) \tag{150}$$
$$= -\int_0^{t/2} \alpha(\eta)\mu_2(\eta)d\eta = \int_0^{t/2} \mathcal{J}_3(t,\eta)\alpha(\eta)d\eta,$$

with

$$\mathcal{J}_3(t,\eta)|_{\eta=t/2} = -\mu_2(t/2) \tag{151}$$
$$= -3C_1(t/2)C(t/2) - 3\int_0^{t/2} C(\eta)[C(\eta) - C(0)]d\eta + \frac{3}{2}C_1^2(t/2)Q(t/2).$$

We combine all remaining terms into $J_4$. It contains terms of three different types. The first consists of integrals of the form

$$\int \delta(t-\tau-\eta)\mathcal{A}^1(\tau,\eta)(\tau-\eta)_+\alpha(\eta) = \int_0^{t/2} \mathcal{A}^1(t,\eta)(t-2\eta)_+\alpha(\eta)d\eta,$$

which is clearly of the form

$$\int_0^{t/2} \mathcal{A}^2(t,\eta)\alpha(\eta)d\eta, \quad \mathcal{A}^2(t,\eta)|_{\eta=t/2} = 0.$$

The second has integrals

$$\int \mathcal{A}^1(t,\tau,\eta)\theta(t-\tau-\eta)\theta(\tau-\eta)\alpha(\eta) = \int_0^{t/2} \alpha(\eta)\left(\int_\eta^{t-\eta} \mathcal{A}^1(t,\tau,\eta)d\tau\right)d\eta \quad (152)$$

which is also of the form

$$\int_0^{t/2} \mathcal{A}^2(t,\eta)\alpha(\eta)d\eta, \quad \mathcal{A}^2(t,\eta)|_{\eta=t/2} = 0.$$

The last is of the form

$$\int \mathcal{A}^1(t,\tau,\eta)(t-\tau-\eta)_+\alpha(\eta)\delta(\tau-\eta) = \int_0^{t/2} \mathcal{A}^1(t,\eta)(t-2\eta)_+\alpha(\eta)d\eta,$$

i.e. of the same form as the first one. Thus

$$J_4(t) = \int_0^{t/2} \mathcal{J}_4(t,\eta)\alpha(\eta)d\eta, \quad \mathcal{J}_4(t,\eta)|_{\eta=t/2} = 0, \tag{153}$$

and $\mathcal{J}_4(t,\eta) = \mathcal{A}^2(t,\eta)$.

Summarizing (147)- (153),

$$\mathcal{V}_2(t) = -\frac{3}{2}C_1^2(t/2)\alpha(t/2) + \int_0^{t/2} \mathcal{J}_{2,\alpha}(t,\eta)\alpha(\eta)d\eta, \tag{154}$$

$$\mathcal{J}_{2,\alpha}(t,\eta)|_{\eta=t/2}$$

$$= -3C_1(t/2)C(t/2) - 3\int_0^{t/2} C(\eta)[C(\eta)-C(0)]d\eta + 3C_1^2(t/2)Q(t/2).$$

We turn now to $\mathcal{V}_3$ given by (104). Observe that by (40),

$$12C_{(xx)}(\eta) - 3C_{x)}(\eta) = \frac{15}{2}C(\eta)\beta(\eta) + E_\beta(\eta), \tag{155}$$

$$E_\beta(\eta) = \left[6C' - \frac{3}{2}\frac{(C\sigma)'}{\sigma}\right](\eta)\int_0^\eta \beta)\xi)d\xi.$$

Thus

$$\mathcal{V}_3(t) = J_1(t) + J_2(t),$$

and we put the most singular term into $J_1$,

$$J_1(t) = \int \delta(t-\tau-\eta)\left[\frac{15}{2}C\beta + E_\beta\right](\eta)v_0^2(\tau,\eta)$$

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$$= \frac{15}{2} \int_0^{t/2} C(\eta) v_0^2(t - \eta, \eta) \beta(\eta) d\eta$$

$$+ \int_0^{t/2} \beta(\xi) \left( \int_\eta^{t-\eta} \left[ 6C' - \frac{3}{2} \frac{(C\sigma)'}{\sigma} \right] (\eta) C(\eta) d\eta \right) d\xi$$

$$= \int_0^{t/2} \mathcal{J}_1(t, \eta) \beta(\eta) d\eta,$$

with

$$\mathcal{J}_1(t, \eta) = \mathcal{A}^1(t, \eta), \quad \mathcal{J}_1(t, \eta)|_{\eta=t/2} = \frac{15}{2} C_1(t/2) C(t/2),$$

where we take into the account that, by (80), $v_0^2(t/2 + 0, t/2) = C_1(t/2)$. This makes $J_2$ into

$$J_2(t) = \int (G_y(t - \tau, 0, \eta) - \delta(t - \tau - \eta)) \left[ \frac{15}{2} C\beta + E_\beta \right] (\eta) v_0^2(\tau, \eta)$$

By (70) and (80) this integral has the same form as the second term, (152) in the study of $\mathcal{J}_4$ of $\mathcal{V}_2$. Thus

$$J_2(t) = \int_0^{t/2} \mathcal{A}^2(t, \eta) \beta(\eta) d\eta, \quad \mathcal{A}^2(t, \eta)|_{\eta=t/2} = 0.$$

These give

$$\mathcal{V}_3(t) = \int_0^{t/2} \mathcal{J}_{3,\beta}(t, \eta) \beta(\eta) d\eta, \quad \mathcal{J}_{3,\beta}(t, \eta) = \mathcal{A}^1(t, \eta), \quad (156)$$

$$\mathcal{J}_{3,\beta}(t, \eta)|_{\eta=t/2} = \frac{15}{2} C_1(t/2) C(t/2).$$

The last remaining term in representation (101) is $\mathcal{V}_4$ of form (105). Using (40),

$$C_{(y)}(\eta) = \frac{1}{2} [\beta + B_\beta]' (\eta), \quad B_\beta(\eta) = \left( \frac{\sigma'}{\sigma} \right) (\eta) \int_0^\eta \beta(\xi) d\xi.$$

Therefore,

$$\mathcal{V}_4(t) = \frac{1}{2} \int G_y(t - \tau, 0, \eta) [\beta + B_\beta]' (\eta) \sqrt{\sigma}(\eta) \partial_\eta \left( \frac{v_0^4}{\sqrt{\sigma}} \right) (\tau, \eta) \quad (157)$$

$$= -\frac{1}{2} \int [\beta + B_\beta] (\eta) \partial_\eta \left[ G_y(t - \tau, 0, \eta) \sqrt{\sigma}(\eta) \partial_\eta \left( \frac{v_0^4}{\sqrt{\sigma}} \right) (\tau, \eta) \right]$$

$$- \frac{1}{2} \int [\beta + B_\beta] (\eta) G_y(t - \tau, 0, \eta) \sqrt{\sigma}(\eta) \partial_\eta \left( \frac{v_0^4}{\sqrt{\sigma}} \right) (\tau, \eta) \delta(\eta)$$

$$= J_0(t) + J_1(t) + J_2(t) + J_3(t) + J_4(t) + J_5(t) + J_6(t) + J_7(t),$$

where we take into the account that the integral defining $\mathcal{V}_4$ is taken over the region $\eta \geq 0$, i.e. contains the factor $\theta(\eta)$, with the term $J_0$ in the right-hand side of (157) corresponding to this "non-integral" term. Due to $\delta(\eta)$,

$$J_0(t) = -\frac{1}{2} \int [\beta + B_\beta](0)G_y(t - \tau, 0, 0)\sqrt{\sigma}(0)\partial_\eta\left(\frac{v_0^4}{\sqrt{\sigma}}\right)(\tau, 0).$$

As $G_y(t, 0, \eta) = v_0^0(t, \eta)$ and $v_0^0(t, 0) = \delta(t)$ and also $v_0^4(t, 0) = 0$ and $B_\beta(0) = 0$, it follows that

$$J_0(t) = -\frac{1}{2}\beta(0)\int \delta(t - \tau)v_0^4(\tau, 0)$$

$$= -\frac{1}{2}\beta(0)v_0^4(t, 0).$$

Observe that in the case of the constant main coefficients $v_0^4(t, \eta) = \frac{3}{2}\eta(t^2 - \eta^2)_+$ so that $v_{0,\eta}^4(t, 0) = \frac{3}{2}t^2$, $J_0(t)$ coincide with the term with $\beta(0)$ in (37).

Returning to the integral involving $\eta-$integration, we first note that

$$\partial_\eta\left[G_y\sqrt{\sigma}\partial_\eta\left(\frac{v_0^4}{\sqrt{\sigma}}\right)\right] = \partial_\eta\left[G_y v_{0,\eta}^4 - \frac{1}{2}\left(\frac{\sigma'}{\sigma}\right)G_y v_0^4\right]$$

$$= G_{y\eta}v_{0,\eta}^4 + G_y v_{0,\eta\eta}^4 - \frac{1}{2}\left(\frac{\sigma'}{\sigma}\right)G_{y\eta}v_0^4 - \frac{1}{2}\left(\frac{\sigma'}{\sigma}\right)G_y v_{0,\eta}^4 - \frac{1}{2}\left(\frac{\sigma'}{\sigma}\right)'G - yv_0^4.$$

We recall that, as $G_y|_{y=0} = v_0^0$,

$$G_{y\eta}(t - \tau, 0, \eta) = -\delta'(t - \tau - \eta) - \nu_0(\eta)\delta(t - \tau - \eta) + \mathcal{A}^0(t - \tau, \eta)\theta(t - \tau - \eta).$$

Using (87),

$$v_{0,\eta}^4(\tau, \eta) = -\mu_1(\eta)\theta(\tau - \eta) - (\mu_2(\eta) - \mu_1'(\eta))(\tau - \eta)_+ + \mathcal{A}^0(\tau, \eta)(\tau - \eta)_+^2,$$

$$v_{0,\eta\eta}^4(\tau, \eta) = \mu_1(\eta)\delta(\tau - \eta) + (\mu_2(\eta) - 2\mu_1'(\eta))\theta(\tau - \eta) + \mathcal{A}^0(\tau, \eta)(\tau - \eta)_+.$$

This makes possible to carry out the singularity analysis of the integrand in the first integral in the right-hand side of (157) with the term due to the most singular part being given by

$$J_1(t) = -\frac{1}{2}\int [\beta + B_\beta](\eta)\mu_1(\eta)\left(\delta'(t - \tau - \eta)\theta(\tau - \eta) + \delta(t - \tau - \eta)\delta(\tau - \eta)\right)$$

$$= -\frac{1}{2}\partial_t\int_0^{t/2} [\beta + B_\beta](\eta)\mu_1(\eta)d\eta$$

$$-\frac{1}{2}\int [\beta + B_\beta](\eta)\mu_1(\eta)\delta(t - 2\eta)d\eta$$

$$= -\frac{1}{2}[\beta + B_\beta](t/2)\mu_1(t/2)$$

$$= -\frac{3}{2}C_1^2(t/2)\beta(t/2) - \frac{3}{2}C_1^2(t/2)\left(\frac{\sigma'}{\sigma}\right)(t/2)\int_0^{t/2}\beta(\eta)d\eta.$$

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As usual, we obtain the next order terms when taking the most singular part in one of the factors in the integral in the right-hand side of (157) and the next order part in the second factor. Namely,

$$J_2(t) = -\frac{1}{2} \int \left[\beta + B_\beta\right](\eta) \left[\mu_2 - \mu_1'\right](\eta)\delta'(t - \tau - \eta)(\tau - \eta)_+$$

$$-\frac{1}{2} \int \left[\beta + B_\beta\right](\eta)\mu_1(\eta)\nu_0(\eta)\delta(t - \tau - \eta)\theta(\tau - \eta)$$

$$= -\frac{1}{2} \int \left[\beta + B_\beta\right](\eta) \left[\mu_2 - \mu_1' + \mu_1\nu_0\right](\eta)\delta(t - \tau - \eta)\theta(\tau - \eta)$$

$$= \int_0^{t/2} \mathcal{J}_2(t, \eta)\beta(\eta)d\eta,$$

where

$$\mathcal{J}_2(t, \eta)|_{\eta=t/2} = -\frac{1}{2} \left[\mu_2 - \mu_1' + \mu_1\nu_0\right](t/2)\delta(t - \tau - \eta)\theta(\tau - \eta)$$

$$= \frac{3}{2} \left[C_1(t/2)C(t/2) - \int_0^{t/2} C(\eta)[C(\eta) - C(0)]d\eta + C_1^2(t/2)Q(t/2)\right]$$

Similar, the next order term originated from $G_y v_{0,\eta\eta}^4$ is of the form,

$$J_3(t) = -\frac{1}{2} \int \left[\beta + B_\beta\right](\eta) \left[\mu_2 - 2\mu_1'\right](\eta)\delta(t - \tau - \eta)\theta(\tau - \eta)$$

$$-\frac{1}{2} \int \left[\beta + B_\beta\right](\eta)\mu_1(\eta)\nu_0(\eta)\theta(t - \tau - \eta)\delta(\tau - \eta)$$

$$= -\frac{1}{2} \int_0^{t/2} \left[\beta + B_\beta\right](\eta) \left[\mu_2 - 2\mu_1'\right](\eta)d\eta$$

$$-\frac{1}{2} \int_0^{t/2} \left[\beta + B_\beta\right](\eta)\mu_1(\eta)\nu_0(\eta)d\eta$$

$$= \int_0^{t/2} \mathcal{J}_3(t, \eta)\beta(\eta)d\eta,$$

with

$$\mathcal{J}_3(t, \eta)|_{\eta=t/2} = -\frac{1}{2} \left[\mu_2 - 2\mu_1' + \mu_1\nu_0\right](t/2)$$

$$= \frac{3}{2} \left[3C_1(t/2)C(t/2) - \int_0^{t/2} C(\eta)[C(\eta) - C(0)]d\eta + C_1^2(t/2)Q(t/2)\right]$$

We combine the remaining terms in $-\frac{1}{2}\int[\beta+B_\beta]\left[G_{y\eta}v_{0,\eta}^4 + G_y v_{0,\eta\eta}^4\right]$ into $J_4(t)$. Our immediate goal is to show that

$$J_4(t) = \int_0^{t/2} \mathcal{A}^2(t, \eta)\beta(\eta)d\eta, \quad \mathcal{A}^2(t, \eta)|_{\eta=t/2} = 0. \tag{158}$$

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To this end we observe that the integrands in the integrals for $J_4$ are of four types. Firstly, they contain terms of the form $\mathcal{A}^1(t,\tau,\eta)\delta?(t-\tau-\eta)(\tau-\eta)^2_+$, secondly ?terms $\mathcal{A}^1(t,\tau,\eta)\delta(t-\tau-\eta)(\tau-\eta)_+$, then terms with $\mathcal{A}^1(t,\tau,\eta)\theta(t-\tau-\eta)\theta(\tau-\eta)$, and, at last, terms with $\mathcal{A}^1(t,\tau,\eta)(t-\tau-\eta)\delta(\tau-\eta)$. All these terms are of the same type as those in $\mathcal{J}_4$ for $\mathcal{V}2$, so that the Same considerations prove (158). Further integrals to be taken into the account are those occurred due to $\left(\frac{\sigma'}{\sigma}\right)G_{y\eta}v_0^4$ and $\left(\frac{\sigma'}{\sigma}\right)G_yv_{0,\eta}^4$, i.e. $\mathcal{J}_5$ and $\mathcal{J}_6$ with

$$\mathcal{J}_5(t) = -\frac{1}{4}\int \delta?(t-\tau-\eta)[\beta+B_\beta](\eta)\left(\frac{\sigma'}{\sigma}\right)(\eta)\mu_1(\eta)(\tau-\eta)_+$$

$$= -\frac{1}{4}\partial_t\int[\beta+B_\beta](\eta)\left(\frac{\sigma'}{\sigma}\right)(\eta)\mu_1(\eta)(t-2\eta)$$

$$= -\frac{1}{4}\int_0^{t/2}[\beta+B_\beta](\eta)\left(\frac{\sigma'}{\sigma}\right)(\eta)\mu_1(\eta)d\eta$$

$$= \int_0^{t/2}\mathcal{J}_5(t,\eta)\beta(\eta)d\eta, \quad \mathcal{J}_5(t,\eta) = \mathcal{A}^2(t,\eta),$$

with

$$\mathcal{J}_5(t,\eta)|_{\eta=t/2} = -\frac{3}{4}C_1^2(t/2)\left(\frac{\sigma'}{\sigma}\right)(t/2).$$

Similarly, $J_4$

$$J_6(t) = -\frac{1}{4}\int \delta(t-\tau-\eta)[\beta+B_\beta](\eta)\left(\frac{\sigma'}{\sigma}\right)(\eta)\mu_1(\eta)\theta(\tau-\eta)$$

$$= -\frac{1}{4}\int_0^{t/2}[\beta+B_\beta](\eta)\left(\frac{\sigma'}{\sigma}\right)(\eta)\mu_1(\eta)d\eta$$

$$= \int_0^{t/2}\mathcal{J}_6(t,\eta)\beta(\eta)d\eta, \quad \mathcal{J}_6(t,\eta) = \mathcal{A}^2(t,\eta),$$

with

$$\mathcal{J}_6(t,\eta)|_{\eta=t/2} = -\frac{3}{4}C_1^2(t/2)\left(\frac{\sigma'}{\sigma}\right)(t/2).$$

At last we show that all the remaining terms in the integrals with $\left(\frac{\sigma'}{\sigma}\right)G_{y\eta}v_0^4$, $\left(\frac{\sigma'}{\sigma}\right)G_yv_{0,\eta}^4$, as well as the integral with $\left(\frac{\sigma'}{\sigma}\right)G_yv_0^4$ give rize to the integral

$$J_7(t) = \int_0^{t/2}\mathcal{A}^2(t,\eta)\beta(\eta)d\eta, \quad \mathcal{A}^2(t,\eta)|_{\eta=t/2} = 0.$$

The proof follows the same lines as that for the term $J_4$ in $\mathcal{V}_2$. Summarizing $J_0, \ldots, J_7$, we get

$$\mathcal{V}_4(t) = -\frac{1}{2}v_{0,\eta}^4(t,0)\beta(0) - \frac{3}{2}C_1^2(t/2)\beta(t/2) \tag{159}$$

$$+ \int_0^{t/2} \mathcal{J}_{4,\beta}(t,\eta)\beta(\eta)d\eta, \quad \mathcal{J}_{4,\beta}(t,\eta) = \mathcal{A}^1(t,\eta),$$

and

$$\mathcal{J}_{4,\beta}(t,\eta)|_{\eta=t/2} = -3 \int_0^{t/2} C(\eta)[C(\eta - C(0)]d\eta \qquad (160)$$

$$+3 \left[ 2C_1(t/2)C(t/2) - C_1^2(t/2)\left(\frac{\sigma'}{\sigma}\right)(t/2) + C_1^2(t/2)Q(t/2)\right].$$

For constant main coefficients, equations (159) for $\mathcal{V}_4(t)$ coincide, if we take into the account formula (25) for $v_0^4$ in this case, with integral (37) with $\mathcal{J}_{4,\beta}(t,\eta)|_{\eta=t/2}$ given by (160) coinciding with the kernel in (37) evaluated at $\eta = t/2$. We are ready now to write $r_1^3 = u_{1,y}^3|_{y=0}$. Using formula (101) together with equations (145), (146) for $\mathcal{V}_1(t)$, (154), (155) $\mathcal{V}_2(t)$, (156) for $\mathcal{V}_3(t)$, and (159), (160) for $\mathcal{V}_4(t)$, we see that

$$r_1^3(t) = -\frac{1}{2}v_{0,\eta}^4(t,0)\beta(0) - 3\left[\alpha(t/2) + \beta(t/2)\right]C_1^2(t/2) \qquad (161)$$

$$+ \int_0^{t/2} \left(\mathcal{J}_\alpha(t,\eta)\alpha(\eta) + \mathcal{J}_\beta(t,\eta)\beta(\eta)\right) d\eta,$$

where $\mathcal{J}_\alpha(t,\eta)$, $\mathcal{J}_\beta(t,\eta) = \mathcal{A}^1(t,\eta)$ and

$$\mathcal{J}_\alpha(t,\eta)|_{\eta=t/2} = -3 \int_0^{t/2} C(\eta) \left[C(\eta) - C(0)\right] d\eta \qquad (162)$$

$$+3C_1(t/2) \left[C(0) - 2C(t/2) + \int_0^{t/2} C(\eta)Q(\eta)d\eta + C_1(t/2)Q(t/2)\right];$$

$$\mathcal{J}_\beta(t,\eta)|_{\eta=t/2} = -3 \int_0^{t/2} C(\eta) \left[C(\eta) - C(0)\right] d\eta \qquad (163)$$

$$+3C_1(t/2) \left[C(0) + 6C(t/2) + 2C_1(t/2)Q(t/2) - 2C_1(t/2)\left(\frac{\sigma'}{\sigma}\right)(t/2)\right].$$

For constant main coefficients, representations (161)-(163) become the corresponding terms in equation (34) if we compare the values of the integral kernels at $\eta = t/2$.

## 8 Reconstruction of $\alpha$ and $\beta$

In this section we will provide an algorithm to evaluate unknown $\alpha$ and $\beta$ from $r_1^1$ and $r_1^3$. To this end, observe that equations (100) and (161) for $r_1^1$ and $r_1^3$, respectively, may be treated as integro-differentil equations for unknown $\alpha$ and

$\beta$. Let us show how to reduce these equations to a recurrent system of regular Volterra integral equations of the second kind for $\alpha$ and $\beta$.

Observe that, taking into account formulae (56) and (75) of $C_1$ and $Q$ and definition (52) of the classes $\mathcal{A}^k$ and changing the variable into $y = t/2$, we obtain from (100) and (161) that

$$\frac{dr_1^3}{dy}(y) - yr_1^1(y) = -\frac{1}{2}\beta(0)\left[\frac{dv_{0,\eta}^4(y,0)}{dy} - yr_0^2(y)\right] \tag{164}$$

$$+2\alpha(y)[y + \mathcal{A}_\alpha^2(y)] - 2\beta(y)[y + \mathcal{A}_\beta^2(y)] + \int_0^y \left(\mathcal{A}_\alpha^1(y,\eta)\alpha(\eta) + \mathcal{A}_\beta^1(y,\eta)\beta(\eta)\right) d\eta.$$

These equation may be considered as a regular Volterra integral equation of the second kind for, say, $\alpha$ in terms of $\beta$,

$$\alpha(y) = \int_0^y \mathcal{B}_\alpha^0(y,\eta)\alpha(\eta) \, d\eta$$

$$+\widehat{\alpha}^0(y) + \beta(y)[1 + \mathcal{B}^1(y)] + \int_0^y \mathcal{B}_\beta^0(y,\eta)\beta(\eta) \, d\eta,$$

with known functions $\widehat{\alpha}^0, \mathcal{B}_\alpha^0, \mathcal{B}_\beta^0$ from $\mathcal{A}^0$ and $\mathcal{B}^1$ from $\mathcal{A}^0$. Solving this equation we obtain a representation of $\alpha$ in terms of $\beta$,

$$\alpha(y) = a(y) + \beta(y)[1 + \mathcal{C}^1(y)] + \int_0^y \mathcal{C}^0(y,\eta)\beta(\eta) \, d\eta, \tag{165}$$

where $a$, $\mathcal{C}^0$ and $\mathcal{C}^1$ are known.

Substitution of this representation for $\alpha$ into (161) provides a Volterra integral equation for $\beta$. To analyse the resulting equation, we use representations (162) and (163) for $\mathcal{J}_\alpha$ and $\mathcal{J}_\beta$. As $c(0) = 1$, we see that

$$\beta(y)y^2 = b^2(y) + \int_0^y \left(\eta + \mathcal{C}^2(y,\eta)\right)\beta(\eta) \, d\eta, \tag{166}$$

where $b^2$ and $\mathcal{C}^2$ are known functions from $\mathcal{A}^0$ and $\mathcal{A}^2$, correspondingly. Observe that this equation uniquely determines $\beta(0)$,

$$\beta(0) = -\lim_{y \to 0} y^{-2}b^2(y).$$

However, it does not determine $\beta'(0)$, cf. Remark 2.4, providing a necessary condition for $b^2$ and $\mathcal{C}^2$ for equation (166) to be valid in the third order for $y \to 0$.

Assuming that $\beta'(0)$ is known, introduce

$$\widetilde{\beta}(y) = \beta(y) - \beta(0) - \beta'(0)y.$$

Then equation (166) takes the form

$$\widetilde{\beta}(y)y^2 = b^{(4)}(y) + 3\int_0^y \left(\eta + \mathcal{D}^2(y,\eta)\right)\widetilde{\beta}(\eta) \, d\eta, \tag{167}$$

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with known $b^{(4)}$ and $\mathcal{D}^{(2)}$ from $\mathcal{A}^4$ and $\mathcal{A}^2$, correspondingly. As the integral kernel in (167) is of the order $O(y)$ near $y = 0$, this equation is a singular one. To regularize it, let

$$W(y) = y^{-4} \int_0^y \widetilde{\beta}(\eta)\eta \, d\eta, \quad \text{i.e. } \widetilde{\beta}(y) = 4y^2 W(y) + y^3 W'(y).$$

This makes equation (167) into the integro-differential equation for $W$

$$\frac{b^{(4)}(y)}{y^4} = W(y) + yW'(y) + \mathcal{F}^{(1)}(y)W(y) + \int_0^y \frac{\mathcal{F}^{(2)}(y,\eta)\eta^2}{y^4} W(\eta) \, d\eta.$$

At last, let

$$Z(y) = yW(y),$$

so that the above equation turns into

$$\frac{b^{(4)}(y)}{y^4} = Z'(y) + \mathcal{G}^{(0)}(y)Z(y) + \int_0^y \frac{\mathcal{G}^{(2)}(y,\eta)\eta}{y^4} Z(\eta) \, d\eta,$$

with known coefficients $\mathcal{G}^{(0)}$ and $\mathcal{G}^{(2)}$ from $\mathcal{A}^{(0)}$ and $\mathcal{A}^{(2)}$, correspondingly. Integrating this equation, we see that

$$Z(y) = \int_0^y \frac{b^{(4)}(\eta)}{\eta^4} \, d\eta + \int_0^y \mathcal{H}^{(0)}(y,\eta)Z(\eta) \, d\eta, \tag{168}$$

where

$$\mathcal{H}^{(0)}(y,\eta) = \mathcal{G}^{(0)}(\eta) + \eta \int_\eta^y \frac{\mathcal{G}^{(2)}(\eta,\xi)}{\xi^4} \, d\xi.$$

Clearly, $\mathcal{H}^{(0)}(y,\eta)$ is bounded for bounded $(t,\eta)$, so that equation (168) is a regular Volterra integral equation of the second kind for $Z(y)$.

# 9   Numerical Results

Let us consider an example of the wave propagation into the inhomogeneous half-plane described by the wave equation

$$n^2(z,\epsilon x)u_{tt} - \rho(z)\text{div}(\frac{1}{\rho(z)}\nabla u) = 0,$$

$$n^2(z,\epsilon x) = n_0^2(z) + \epsilon x n_1(z) + O(\varepsilon^2),$$

where $n(z,\epsilon x)$ is the refracting index and $\rho(z)$ is the density of the medium. In this case

$$\rho(z) = n_0(z)/\sigma(z)$$

In this section we apply not exactly the above method to solving numerically the inverse problem of reconstruction of $n_{0,1}(z)$ and $\rho(z)$, but its slight modification based on usage of the Fourier transform

$$u(z, x, t) = \frac{1}{4\pi^2} \int_{R^2} e^{-i\xi x} U(z, \xi, t) d\xi.$$

To generate the inverse data, $r_0(\xi, t)$ and $r_1(\xi, t)$, where

$$r_0(t, \xi) = \int_{\mathbb{R}^2} \cos(<\xi, x>) R_0(x, t) dx = \int_{\mathbb{R}^2} \cos(<\xi, x>) R(x, t) dx + O(\varepsilon^2),$$

$$r_1(t, \xi) = \int_{\mathbb{R}^2} \sin(<\xi, x>) R_1(x, t) dx = \varepsilon^{-1} \int_{\mathbb{R}^2} \sin(<\xi, x>) R(x, t) dx + O(\varepsilon^2),$$

we use a finite-difference method to solve the initial boundary value corresponding problems. in Fig. 1 and 2 the reconstructed profiles of the refracting index (zero-order approximation) and density are presented to compare with the corresponding original profiles given by $n_0 = 1 - 0.02z + 0.2\sin z$, $\sigma = 1 + 0.05z + 0.12\sin 0.6z$, and $n_0 = 1 + 0.02z - 0.2\sin z$ and $\sigma = 1 - 0.05z - 0.12\sin 0.6z$ with discritization step $\delta y = 0.04$. It may be seen that we have a good agreement between both types of data.

The method has shown to be quite stable, fast and accurate. When solving Volterra-type integral equations, both non-linear and linear, the iteration processes need just a few iterations (for all graphs the number of iterations was chosen 10). Numerous computer experiments have shown that for a better accuracy and fast convergence of the iteration process it is reasonable to use for the chosen profiles the segment $|\xi| < 0.5$. It is worth noting that the parameter $|\xi|$, the maximum depth $T$ and the maximum of $n_0''(z)$ are interconnected. For example, for the larger values of $T$ and the maximum of $n_0''(z)$, while computing the profiles we were forced to take smaller values of $|\xi|$. Moreover, due to the non-linearity of the Volterra system of the zero-order problem, when we increase $T$ and/or $n_0''(z)$ and $|\xi|$, a blow up effect can occur, i.e. the iterations stop to converge. This may be remedied by a variant of the layer-stripping. For a particular case, when $\rho = const$, these questions are analyzed in details in ([9]).

In applications to geophysics, the unity of the refraction index corresponds to the average speed of the wave propagation $c =$2-2.5km/sec. Thus, the dimensionless depth coordinate $z$ must be multiplied by $2 - 2.5km$.

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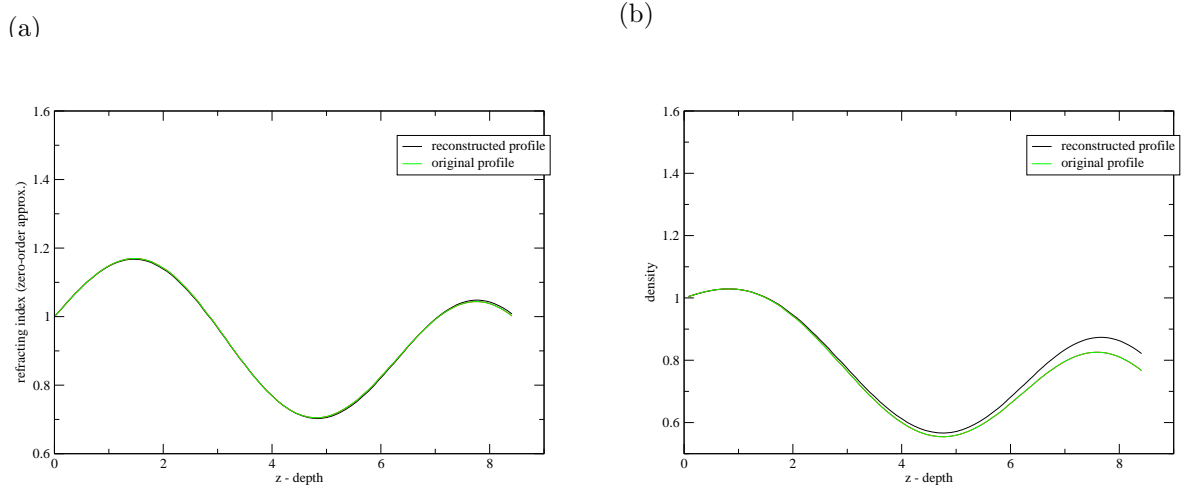
(a)  (b)



Figure 1: Numerical values of the refracting index (zero-order approximation) $n_0$ - (a) and density $\rho = n_0(z)/\sigma(z)$ - (b) against original profiles given by $n_0 = 1 - 0.02z + 0.2\sin z$ and $\sigma = 1 + 0.05z + 0.12\sin 0.6z$ with $\delta y = 0.04$.

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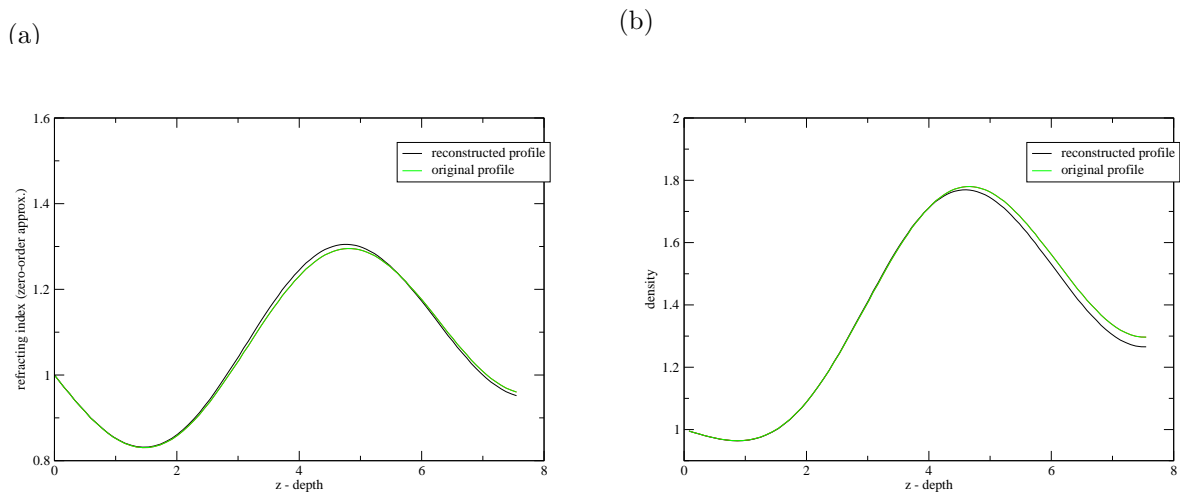
(a)

(b)



Figure 2: Numerical values of the refracting index (zero-order approximation) $n_0$ - (a) and density $\rho = n_0(z)/\sigma(z)$ - (b) against original profiles given by $n_0 = 1 + 0.02z - 0.2\sin z$ and $\sigma = 1 - 0.05z - 0.12\sin 0.6z$ with $\delta y = 0.04$.

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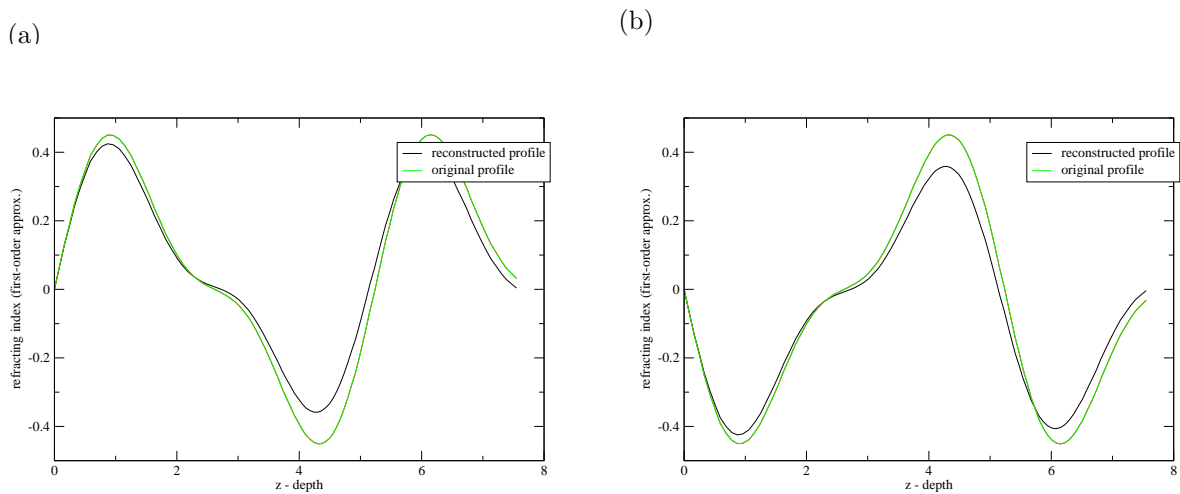(a)                                                    (b)

Figure 3: Numerical values of the refracting index (first-order approximation) $n_1$ against original profiles given by $n_1 = 0.37 \sin 1.2 + 0.15 \sin 2.4z$ - (a) and $n_1 = -0.37 \sin 1.2z - 0.15 \sin 2.4z$ - (b) with $\delta y = 0.04$, where $n_0$ and $\sigma$ were taken such as in Fig. 1 and Fig. 2, correspondingly.

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