Solitary gravity water waves with an arbitrary distribution of vorticity

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Abstract: This paper presents an existence theory for small-amplitude solitary-wave solutions to the classical water-wave problem in the absence of surface tension and with an arbitrary distribution of vorticity. The hydrodynamic problem is formulated as an infinite-dimensional Hamiltonian system in which the horizontal spatial direction is the time-like variable. A centre-manifold reduction technique is employed to reduce the system to a locally equivalent Hamiltonian system with one degree of freedom. The phase portrait of the reduced system contains a homoclinic orbit, and the corresponding solution of the water-wave problem is a solitary wave of elevation.

1 Introduction

The *water-wave problem* concerns the gravity-driven flow of a perfect fluid of unit density; the fluid is bounded below by a rigid horizontal bottom $\{y = 0\}$ and above by a free surface $\{y = \eta(x, t)\}$, where η depends upon the horizontal spatial coordinate x and time t. *Travelling waves* are waves which propagate from left to right with constant speed c and without change of shape, so that $\eta(x, t) = \eta(x - ct)$. The two principal classes of travelling waves are *Stokes waves*, which are periodic in a frame of reference moving with the wave, and *solitary waves*, which have the property that $\eta(x - ct) \to 0$ as $x - ct \to \pm\infty$. In this paper we construct a rigorous existence theory for solitary waves on flows with an arbitrary distribution of vorticity.

Working in a frame of reference moving with the wave, let us describe the velocity field $(u(x, y), v(x, y))$ within the fluid domain $D_\eta = \{(x, y) : x \in$

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$\mathbb{R}$, $0 < y < \eta(x)\}$ in terms of a stream function $\psi(x,y)$ which satisfies $\psi_x = -v$, $\psi_y = u - c$ and suppose that $u < c$, so that $\psi_y < 0$. The vorticity $\omega(x,y) = v_x(x,y) - u_y(x,y)$ is known under this condition to be a function of the stream function ψ, and we specify its distribution by prescribing a *vorticity function* γ such that $\omega = \gamma(\psi)$. The hydrodynamic problem is to solve the nonlinear elliptic equation

$$\Delta\psi = -\gamma(\psi), \qquad 0 < y < \eta(x) \tag{1}$$

subject to the boundary conditions

$$\psi(x,0) = 0, \tag{2}$$

$$\psi(x,\eta(x)) = m_0, \qquad \frac{1}{2}|\nabla\psi(x,\eta(x))|^2 + g(\eta(x) - d) = \frac{\lambda}{2} \tag{3}$$

and the asymptotic conditions

$$\eta(x) \to d \qquad \text{as } x \to \pm\infty. \tag{4}$$

Here g and d are respectively the acceleration due to gravity and the asymptotic depth of the water, and λ is a constant called the Bernoulli constant (e.g. see Constantin & Strauss [9]).

The mathematical study of irrotational solitary water waves began with small-amplitude bifurcation theories. Lavrentiev [22] constructed a solitary wave as the limit of a sequence of Stokes waves of increasing period, while Friedrichs & Hyers [14] gave an existence proof based upon a series expansion and Beale [3] used a Nash-Moser implicit-function theorem. A global branch of large-amplitude solutions was obtained by Amick & Toland [1,2] using a formulation of the problem as an integral equation, and Plotnikov [24] used a variational formulation of the problem to demonstrate the non-uniqueness of large-amplitude solitary waves.

Until recently the mathematical theory of solitary waves on flows with vorticity was restricted to approximate theories and numerical results. The Korteweg-de Vries approximation of solitary waves on shallow water with an arbitrary distribution of vorticity was studied by Benjamin [4] and Freeman & Johnson [13], while large-amplitude solitary waves with constant vorticity were computed numerically by Teles da Silva & Peregrine [10], Vanden-Broeck [26] and Sha & Vanden-Broeck [25].

Hur [20] has recently generalised Beale's application of Nash-Moser theory to obtain small-amplitude solitary waves on flows with an arbitrary distribution of vorticity; her solutions are solitary waves of elevation which decay exponentially far up- and downstream. In the present paper we present an alternative, more elementary construction of these waves using *spatial dynamics* methods. We formulate the physical problem as an ill-posed evolutionary equation

$$u_x = F(u), \tag{5}$$

in which the unbounded spatial coordinate x plays the role of 'time', and study it in an infinite-dimensional phase space consisting of functions of y. Notice that the hydrodynamic problem is conservative and isotropic in x, and these symmetries manifest themselves in the fact that its spatial dynamics formulation is *Hamiltonian* and *reversible*. In Section 2 we derive a formulation of the water-wave problem with an arbitrary choice of $\gamma \in L^2(m_0, 0)$ as a reversible Hamiltonian system and place it in a secure functional-analytic framework.

Under the hypothesis that

$$\int_0^1 \frac{1}{\sqrt{2\Gamma(s) - 2\Gamma_{\min}}} \, \mathrm{d}s > 1,$$

where

$$\Gamma(s) = \frac{d^2}{m_0} \int_s^1 \gamma(m_0 t) \, \mathrm{d}t, \qquad \Gamma_{\min} = \min_{s \in [0,1]} \Gamma(s),$$

we show that (5) admits an equilibrium (that is x independent) solution $u^\star$ corresponding to a horizontal laminar flow (which is in general not uniform), and we seek solutions of the form $u = u^\star + w$, so that our solitary waves 'ride' a horizontal laminar flow. Writing the equation for w as

$$w_x = Lw + N(w), \tag{6}$$

where L and N are the linear and nonlinear parts of the vector field for w, and assuming that $\gamma \in H^1(m_0, 0)$, we show that the spectrum of L consists of a countable number of simple, real eigenvalues which accumulate at $\pm\infty$, together with a pair of simple, purely imaginary eigenvalues whenever the physical parameter $\alpha = gd^3/m_0^2$ is greater than a critical value $\alpha_\star$. The purely imaginary eigenvalues are created in a *Hamiltonian* 0^2 *resonance* at $\alpha = \alpha_\star$: two real eigenvalues collide at the origin and become purely imaginary as α is varied through $\alpha_\star$ from below. In Section 3 we show that, for $\alpha \geq \alpha_\star$, equation (6) admits a two-dimensional invariant manifold called the *centre manifold* which contains all its small, bounded solutions. The flow on the centre manifold is controlled by a reduced system which inherits the Hamiltonian structure and reversibility of (6).

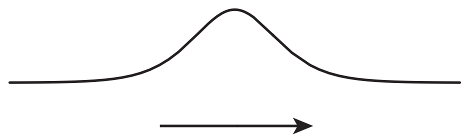

Fig. 1. A symmetric solitary wave of elevation is found for $\alpha = \alpha_\star - \delta$, $0 < \delta \ll 1$; it decays exponentially and monotonically to a horizontal laminar flow far up- and downstream.

In Section 4 we introduce a bifurcation parameter by writing $\alpha = \alpha_\star + \varepsilon$. The reduction procedure delivers an ε dependent two-dimensional centre manifold which captures the small-amplitude dynamics for small values of ε, and the flow on this manifold is controlled by the reversible Hamiltonian system

$$\begin{aligned} Q_X &= P + O(|\varepsilon|^{1/2}), \\ P_X &= -\operatorname{sgn}(\varepsilon)Q - \frac{3}{2}Q^2 + O(|\varepsilon|^{1/2}), \end{aligned}$$

where $X \sim |\varepsilon|^{1/2}x$. For small, negative values of ε, the phase portrait contains a reversible homoclinic orbit in the right half-plane; the corresponding hydrodynamic flow is a symmetric solitary wave of elevation which decays exponentially and monotonically to a horizontal laminar flow as $x \to \pm\infty$ and is sketched in Figure 1.

Of course one can also study the phase portrait of the above Hamiltonian system for $\varepsilon > 0$. In this case the zero equilibrium is a centre surrounded by periodic orbits; the corresponding hydrodynamic flows are small-amplitude symmetric Stokes waves. Rotational Stokes waves of this kind have recently been extensively studied by Constantin & Strauss [9], who in particular obtained a global branch of large-amplitude waves (see also Hur [21] for the corresponding theory for deep-water Stokes waves).

The above results represent a complete analysis of the local bifurcation picture for α near $\alpha_\star$ (solitary waves for $\alpha < \alpha_\star$ and Stokes waves for $\alpha > \alpha_\star$). This ‘unfolding’ has previously been recorded for irrotational waves by Dias & Iooss [11, §4.1] in a review of spatial dynamics and centre-manifold methods for water waves. Spatial dynamics and centre-manifold reduction techniques have indeed been used in a variety of existence proofs for other types of water waves, in particular for three-dimensional irrotational gravity-capillary water waves (Groves & Haragus [16]) and two-dimensional gravity-capillary water waves with arbitrary distributions of vorticity (Groves & Wahlén [19]). Some of the technical results in the present paper are similar to those in the latter reference, to which we defer for their proofs.

2 Formulation as a Hamiltonian system

We begin by writing the hydrodynamic problem (1)–(4) in terms of the dimensionless variables

$$(x', y') = \frac{1}{d}(x, y), \quad \eta'(x') = \frac{1}{d}\eta(x), \quad \psi'(x', y') = -\frac{1}{m_0}\psi(x, y)$$

and dimensionless vorticity function

$$\gamma'(\psi') = -\frac{d^2}{m_0}\gamma(\psi).$$

One finds that

$$\Delta\psi = -\gamma(\psi), \qquad 0 < y < \eta(x), \tag{7}$$

with boundary conditions

$$\psi(x, 0) = 0, \tag{8}$$

$$\psi(x, \eta(x)) = -1, \qquad \frac{1}{2}|\nabla\psi(x, \eta(x)|^2 + \alpha(\eta(x) - 1) = \frac{\mu}{2} \tag{9}$$

and asymptotic conditions

$$\eta(x) \to 1 \qquad \text{as } x \to \pm\infty, \tag{10}$$

in which

$$\alpha = \frac{gd^3}{m_0^2}, \qquad \mu = \frac{\lambda d^2}{m_0^2}$$

are dimensionless parameters and the primes have been dropped for notational simplicity.

The next step is to map the unknown fluid domain D_η into a fixed strip $\mathbb{R} \times (0, 1)$ using a transformation devised by Dubreil-Jacotin [12]. We define $s = -\psi(x, y)$, $h = y$ and treat $(x, s) \in \mathbb{R} \times (0, 1)$ as independent variables and $h(x, s)$ as the dependent variable. A straightforward calculation shows that equations (7)–(9) are transformed into

$$\left[\frac{h_x}{h_s}\right]_x - \left[\frac{1 + h_x^2}{2h_s^2}\right]_s + \gamma(-s) = 0, \qquad 0 < s < 1, \tag{11}$$

$$h(x, 0) = 0, \tag{12}$$

$$\frac{1 + h_x^2(x, 1)}{2h_s^2(x, 1)} + \alpha(h(x, 1) - 1) = \frac{\mu}{2}, \tag{13}$$

and we seek solutions with $h_s > 0$, a condition which is implied by the assumption $\psi_y < 0$ (Constantin & Strauss [9]), and $h(x,1) \to 1$ as $x \to \pm\infty$. The variable η is recovered from the formula $\eta(x) = h(x,1)$; note that $\eta(x) > 0$ because $h(x,0) = 0$ and $h_s(x,s) > 0$ for $s \in [0,1]$. The following proposition, which is proved by straightforward arguments from the theory of elliptic boundary-value problems, relates solutions of the transformed equations to those of (7)–(9).

Proposition 2.1 *Define $I = (x_1, x_2)$, $I' = (x_1', x_2')$ with $x_1 < x_1' < x_2' < x_2$ and let $D_{\eta,I} = \{(x,y) : x \in I,\, 0 < y < \eta(x)\}$.*

(i) Suppose that $\gamma \in L^2(-1,0)$. Any solution $h \in H^2(I \times (0,1)) \cap C^1(I \times [0,1])$ and $\eta = h|_{\{s=1\}} \in C^1(I)$ of (11)–(13) defines a solution $\psi \in H^2(D_{\eta,I'}) \cap C^1(\overline{D_{\eta,I'}})$, $\eta \in C^1(I)$ of (7)–(9).

(ii) The additional regularity $\gamma \in C^{k,\alpha}[-1,0]$ and $\eta \in C^{k+2,\alpha}(I)$ for some $\alpha \in (0,1)$ and some nonnegative integer k implies that $\psi \in C^{k+2,\alpha}(\overline{D_{\eta,I'}})$.

Equations (11)–(13) follow from the formal variational principle

$$\delta\mathcal{J} = 0, \qquad \mathcal{J} = \int \left\{ \int_0^1 \left(\frac{1 + h_x^2}{2h_s^2} - \alpha(h-1) + \frac{\mu}{2} + \Gamma(s) \right) h_s \, \mathrm{d}s \right\} \mathrm{d}x,$$

where

$$\Gamma(s) = -\int_s^1 \gamma(-u)\, du, \qquad s \in [0,1]$$

and the variations are taken with respect to $h(x,s)$ such that $h(x,0) = 0$ (see Constantin, Sattinger & Strauss [8]). We exploit this variational principle by regarding $\mathcal{J}$ as an action functional of the form

$$\mathcal{J} = \int J(h, h_x)\, \mathrm{d}x$$

and deriving a Hamiltonian formulation of (11)–(13) by means of the Legendre transform. To this end, let us introduce a new variable

$$w = \frac{\delta\mathcal{J}}{\delta h_x} = \frac{h_x}{h_s}, \tag{14}$$

in which the variational derivative is taken in $L^2(\mathbb{R} \times (0,1))$, and define the Hamiltonian function by

$$
\begin{aligned}
&H(h,w)\\
&= \int_0^1 w h_x \,\mathrm{d}s - J(h,h_x)\\
&= \int_0^1 \left\{ \frac{1}{2}\left(h_s w^2 - \frac{1}{h_s}\right) - \Gamma(s) h_s \right\} \mathrm{d}s + \frac{1}{2}\alpha(h(1)-1)^2 - \frac{1}{2}\alpha - \frac{\mu}{2}h(1). \quad (15)
\end{aligned}
$$

This procedure suggests that the equations

$$
h_x = \frac{\delta H}{\delta w}, \quad w_x = -\frac{\delta H}{\delta h}
$$

formally represent Hamilton's equations for a formulation of the hydrodynamic problem (11)–(13) as a Hamiltonian system, an irrotational version of which has previously been recorded by Groves [15] and Benjamin [5, Appendix B].

In order to make the above suggestion rigorous, we define the Hilbert spaces

$$
\begin{aligned}
X &= \{(h,w) \in H^1(0,1) \times L^2(0,1) : h(0) = 0\},\\
Y &= \{(h,w) \in H^2(0,1) \times H^1(0,1) : h(0) = 0\}
\end{aligned}
$$

and consider the symplectic manifold (X,Ω), where Ω is the position-independent 2-form on X given by

$$
\Omega|_{(h,w)}((h_1,w_1),(h_2,w_2)) = \int_0^1 (w_2 h_1 - w_1 h_2)\,\mathrm{d}s
$$

(the canonical 2-form with respect to the $L^2(0,1) \times L^2(0,1)$-inner product). Choose $\gamma \in L^2(-1,0)$, so that $\Gamma \in H^1(-1,0)$, and observe that the set

$$
M = \{(h,w) \in Y : h_s(s) > 0 \text{ for each } s \in [0,1]\}
$$

is a manifold domain of X and that the function H given by (15) belongs to $C^\infty(M,\mathbb{R})$.The triple (X,Ω,H) is therefore a Hamiltonian system.

Recall that a point $m \in M$ belongs to the domain $\mathcal{D}(v_H)$ of the Hamiltonian vector field v_H corresponding to (M,Ω,H) with $v_H|_m = \bar{v}|_m$ if and only if

$$
\Omega_m(\bar{v}|_m, v_1|_m) = \mathrm{d}H|_m(v_1|_m)
$$

for all tangent vectors $v_1|_m \in TM|_m \subset TX|_m$. Using this fact and the calculation

$$\mathrm{d}H|_m(v_1|_m) = \int_0^1 \left\{ -\frac{1}{2}\left(w^2 + \frac{1}{h_s^2}\right)_s + \gamma(-s) \right\} h_1 \,\mathrm{d}s + \int_0^1 h_s w w_1 \,\mathrm{d}s$$
$$+ \frac{1}{2}\left(w^2(1) + \frac{1}{h_s^2(1)} + \alpha(h(1)-1) - \frac{\mu}{2}\right) h_1(1)$$

for $m = (h, w) \in M$ and $v_1|_m = (h_1, w_1) \in TM|_m \cong Y$, one finds that

$$\mathcal{D}(v_H) = \left\{ (h, w) \in M : \; w(0) = 0, \; \frac{1}{2}\left(w^2(1) + \frac{1}{h_s^2(1)}\right) + \alpha(h(1)-1) = \frac{\mu}{2} \right\}$$

and that Hamilton's equations are given explicitly by

$$h_x = h_s w, \tag{16}$$
$$w_x = \frac{1}{2}\left(w^2 + \frac{1}{h_s^2}\right)_s - \gamma(-s). \tag{17}$$

Observe that Hamilton's equations are reversible; the reverser $S : X \to X$ is defined by $S(h, w) = (h, -w)$.

Proposition 2.2 *Suppose that $(h, w) \in C(I, \mathcal{D}(v_H)) \cap C^1(I, X)$, $I = (x_1, x_2)$ solves Hamilton's equations and let $I' = (x_1', x_2')$ with $x_1 < x_1' < x_2' < x_2$. The functions $\tilde{h}, \tilde{w}$ defined by*

$$\tilde{h}(x, s) = h(x)(s), \qquad \tilde{w}(x, s) = w(x)(s)$$

belong to respectively $H^2(D_{\eta,I'}) \cap C^1(\overline{D_{\eta,I'}})$ and $H^1(D_{\eta,I'}) \cap C(\overline{D_{\eta,I'}})$. These functions satisfy $\tilde{h}_s(x, s) > 0$ in $\overline{D_{\eta,I'}}$ and the equations

$$\tilde{h}_x = \tilde{h}_s \tilde{w}, \qquad \tilde{w}_x = \frac{1}{2}\left(\tilde{w}^2 + \frac{1}{\tilde{h}_s^2}\right)_s + \gamma(-s)$$

in $D_{\eta,I'}$ with boundary conditions

$$\tilde{h}(x, 0) = \tilde{w}(x, 0) = 0, \qquad \frac{1}{2}\left(\tilde{w}^2(x, 1) + \frac{1}{\tilde{h}_s^2(x, 1)}\right) + \alpha(\tilde{h}(x, 1) - 1) = \frac{\mu}{2}.$$

The above proposition is proved using the methods given by Groves & Toland [18]. Eliminating $\tilde{w}$ between the above equations and defining $\tilde{\eta}(x) = \tilde{h}(x, 1)$, we find that $\tilde{h}$ and $\tilde{\eta}$ satisfy equations (11)–(13) and Proposition 2.1 yields a solution of the hydrodynamic problem (7)–(9). Note that the additional regularity $\gamma \in C^{k,\alpha}[0, 1]$ and $(h, w) \in C^{k+2}(I, \mathcal{D}(v_H)) \cap C^{k+3}(I, X)$ for some $\alpha \in (0, 1)$ and some nonnegative integer k implies that $\psi \in C^{k+2,\alpha}(\overline{D_{\eta,I'}})$. In the remainder of this article we take $\gamma \in H^1(0, 1)$ rather than $\gamma \in L^2(0, 1)$ in order to simplify the spectral theory presented in Section 3.

We proceed by seeking solutions $(h, w) \in C(\mathbb{R}, \mathcal{D}(v_H)) \cap C^1(\mathbb{R}, X)$ of Hamilton's equations which satisfy $h(x, 1) \to 1$ as $x \to \pm\infty$. These solutions take the form of perturbations of equilibrium (that is x independent) solutions $(h_0(s), w_0(s))$, where necessarily $h_0(1) = 1$ and $w_0 = 0$ (see equation (14)); our solitary waves therefore 'ride' a horizontal laminar flow (which is in general not uniform). The following lemma, whose proof is given by Groves & Wahlén [19, Lemma 2.3], shows that the value of the Bernoulli constant μ is determined by the requirement that a horizontal laminar flow exists.

Lemma 2.3 *Suppose that*

$$\int_0^1 \frac{1}{\sqrt{2\Gamma(s) - 2\Gamma_{\min}}} \, \mathrm{d}s > 1, \tag{18}$$

where

$$\Gamma_{\min} = \min_{s \in [0,1]} \Gamma(s), \qquad \Gamma_{\max} = \max_{s \in [0,1]} \Gamma(s).$$

There exists a unique value $\mu^\star > -2\Gamma_{\min}$ of μ for which Hamilton's equations (16), (17) admit a solution of the form $(h, w) = (\theta(s), 0)$ with $\theta(1) = 1$ for all $\alpha > 0$. The function $\theta(s)$ is given by the formula

$$\theta(s) = \int_0^s a^{-1}(t) \, \mathrm{d}t, \qquad a(s) = \sqrt{\mu^\star + 2\Gamma(s)}.$$

In accordance with Lemma 2.3 we take $\mu = \mu^\star$ and seek solutions of Hamilton's equations for (X, Ω, H) of the form $h = \theta + \phi$, where $\phi_s(s) > -a^{-1}(s)$ for $s \in [0, 1]$. Let us write $\alpha = \alpha_0 + \varepsilon$, where α_0 is fixed and ε lies in a neighbourhood Λ of the origin in $\mathbb{R}$, and consider solutions (ϕ, w) which lie in a neighbourhood Z of the origin in Y; here Λ and Z are chosen small enough so that

$$\phi_s(s) > -\frac{1}{2}(\mu^\star + 2\Gamma_{\max})^{-1/2} > -a^{-1}(s)$$

for each $s \in [0, 1]$. This change of variable transforms (X, Ω, H) into $(X, \Omega, H^\varepsilon)$, where $H^\varepsilon \in C^\infty(Z, \mathbb{R})$ is defined by the formula

$$H^\varepsilon(\phi, w) = \int_0^1 \left\{ \frac{1}{2} \left(\frac{(a^{-1}(s) + \phi_s)^2 w^2 - 1}{a^{-1}(s) + \phi_s} \right) + \frac{1}{2} a(s) - \Gamma(s)\phi_s \right\} \mathrm{d}s + \frac{1}{2}(\alpha_0 + \varepsilon)\phi(1)^2 - \frac{1}{2}\mu^\star \phi(1)$$

(a constant term has also been added to the Hamiltonian to ensure that $H^\varepsilon(0) = 0$). Hamilton's equations (16), (17) are transformed into

$$\phi_x = (a^{-1}(s) + \phi_s)w, \tag{19}$$

$$w_x = \frac{1}{2}\left(w^2 + \frac{a^2(s)}{(1+a(s)\phi_s)^2}\right)_s - \gamma(-s), \tag{20}$$

the domain $\mathcal{D}(v_{H^\varepsilon})$ of the Hamiltonian vector field on the right-hand side of this system of equations is the set of elements $(\phi, w) \in Z$ which satisfy

$$\begin{aligned} &w(0) = 0, \\ &\frac{1}{2}\left(w^2(1) + \frac{1}{(a^{-1}(1) + \phi_s(1))^2}\right) + (\alpha_0 + \varepsilon)\phi(1) = \frac{\mu^\star}{2} \end{aligned} \tag{21}$$

and the action of the reverser $S : X \to X$ is given by $S(\phi, w) = (\phi, -w)$. Our task is to find homoclinic solutions of the above equations, that is solutions $(\phi, w) \in C(\mathbb{R}, Z) \cap C^1(\mathbb{R}, X)$ which satisfy $(\phi(x), w(x)) \to (0, 0)$ as $x \to \pm\infty$.

3 Centre-manifold reduction

We find solutions of equations by applying a reduction principle which asserts that $(X, \Omega, H^\varepsilon)$ is locally equivalent to a finite-dimensional Hamiltonian system. The key result is the following theorem, which is a parametrised, Hamiltonian version of a reduction principle for quasilinear evolutionary equations presented by Mielke [23, Theorem 4.1] (see Buffoni, Groves & Toland [7, Theorem 4.1]).

Theorem 3.1 *Consider the differential equation*

$$u_x = \mathcal{L}u + \mathcal{N}(u; \lambda), \tag{22}$$

which represents Hamilton's equations for the reversible Hamiltonian system $(X, \Omega^\lambda, H^\lambda)$. *Here* u *belongs to a Hilbert space* $\mathcal{X}$, $\lambda \in \mathbb{R}^\ell$ *is a parameter and* $\mathcal{L} : \mathcal{D}(\mathcal{L}) \subset \mathcal{X} \to \mathcal{X}$ *is a densely defined, closed linear operator. Regarding* $\mathcal{D}(\mathcal{L})$ *as a Hilbert space equipped with the graph norm, suppose that* 0 *is an equilibrium point of (22) when* $\lambda = 0$ *and that*

(H1) The part of the spectrum $\sigma(\mathcal{L})$ *of* $\mathcal{L}$ *which lies on the imaginary axis consists of a finite number of eigenvalues of finite multiplicity and is separated from the rest of* $\sigma(\mathcal{L})$ *in the sense of Kato, so that* $\mathcal{X}$ *admits the decomposition* $\mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_2$, *where* $\mathcal{X}_1 = \mathcal{P}(\mathcal{X})$, $\mathcal{X}_2 = (I - \mathcal{P})(\mathcal{X})$ *and* $\mathcal{P}$ *is the spectral projection corresponding the purely imaginary part of* $\sigma(\mathcal{L})$.

(H2) The operator $\mathcal{L}_2 = \mathcal{L}|_{\mathcal{X}_2}$ *satisfies the estimate*

$$\|(\mathcal{L}_2 - \mathrm{i}\xi I)^{-1}\|_{\mathcal{X}_2 \to \mathcal{X}_2} \leq \frac{C}{1+|\xi|}, \qquad \xi \in \mathbb{R}$$

for some constant C *that is independent of* ξ.

(H3) There exists a natural number k *and neighbourhoods* $\Lambda \subset \mathbb{R}^\ell$ *of* 0 *and* $U \subset \mathcal{D}(\mathcal{L})$ *of* 0 *such that* $\mathcal{N}$ *is* $(k+1)$ *times continuously differentiable on* $U \times \Lambda$*, its derivatives are bounded and uniformly continuous on* $U \times \Lambda$ *and* $\mathcal{N}(0,0) = 0$, $\mathrm{d}_1\mathcal{N}[0,0] = 0$.

Under these hypotheses there exist neighbourhoods $\tilde{\Lambda} \subset \Lambda$ *of* 0 *and* $\tilde{U}_1 \subset U \cap \mathcal{X}_1$, $\tilde{U}_2 \subset U \cap \mathcal{X}_2$ *of* 0 *and a reduction function* $r : \tilde{U}_1 \times \tilde{\Lambda} \to \tilde{U}_2$ *with the following properties. The reduction function* r *is* k *times continuously differentiable on* $\tilde{U}_1 \times \tilde{\Lambda}$*, its derivatives are bounded and uniformly continuous on* $\tilde{U}_1 \times \tilde{\Lambda}$ *and* $r(0;0) = 0$, $\mathrm{d}_1 r[0;0] = 0$. *The graph* $\tilde{X}^\lambda = \{u_1 + r(u_1;\lambda) \in \tilde{U}_1 \times \tilde{U}_2 : u_1 \in \tilde{U}_1\}$ *is a Hamiltonian centre manifold for (22), so that*

(i) $\tilde{X}^\lambda$ *is a locally invariant manifold of (22): through every point in* $\tilde{X}^\lambda$ *there passes a unique solution of (22) that remains on* $\tilde{X}^\lambda$ *as long as it remains in* $\tilde{U}_1 \times \tilde{U}_2$.

(ii) Every small bounded solution $u(x)$, $x \in \mathbb{R}$ *of (22) satisfying* $(u_1(x), u_2(x)) \in \tilde{U}_1 \times \tilde{U}_2$ *lies completely in* $\tilde{X}^\lambda$.

(iii) Every solution $u_1 : (x_1, x_2) \to \tilde{U}_1$ *of the reduced equation*

$$u_{1x} = \mathcal{L}u_1 + \mathcal{P}\mathcal{N}(u_1 + r(u_1;\lambda);\lambda) \tag{23}$$

generates a solution

$$u(x) = u_1(x) + r(u_1(x);\lambda) \tag{24}$$

of the full equation (22).

(iv) $\tilde{X}^\lambda$ *is a symplectic submanifold of* X *and the flow determined by the Hamiltonian system* $(\tilde{X}^\lambda, \tilde{\Omega}^\lambda, \tilde{H}^\lambda)$*, where the tilde denotes restriction to* $\tilde{X}^\lambda$*, coincides with the flow on* $\tilde{X}^\lambda$ *determined by* $(X, \Omega^\lambda, H^\lambda)$. *The reduced equation (23) is reversible and represents Hamilton's equations for* $(\tilde{X}^\lambda, \tilde{\Omega}^\lambda, \tilde{H}^\lambda)$.

Theorem 3.1 cannot be applied directly to equations (19), (20) because of the nonlinear boundary condition (21). We overcome this difficulty using the following change of variable, which leads to an equivalent problem in a linear space. Define $F : Z \to H^1(0,1)$ by the formula

$$F(\phi, w) = -\frac{1}{2}\left(w^2 + \frac{1}{(a^{-1}(s) + \phi_s)^2}\right) - a^3(s)\phi_s + \frac{1}{2}a^2(s),$$

so that the boundary condition (21) is equivalent to

$$(\alpha_0+\varepsilon)\phi(1)-a^3(1)\phi_s(1)=F(\phi,w)|_{s=1}.$$

Consider the function $G_1 : Z \to H^2(0,1)\times H^1(0,1)$ given by $G_1(\phi,w)=(\zeta,w)$, where

$$\zeta=\phi-a^{-3}(1)s\int_s^1 F(\phi,w)(t)\,\mathrm{d}t.$$

Because it is a near-identity transformation the mapping G_1 is a smooth diffeomorphism from Z onto a neighbourhood $\tilde{Z}$ of the origin in Y, and moreover

$$(\alpha_0+\varepsilon)\zeta(1)-a^3(1)\zeta_s(1)=(\alpha_0+\varepsilon)\phi(1)-a^3(1)\phi_s(1)-F(\phi,w)|_{s=1},$$

so that the boundary condition (21) is transformed into

$$(\alpha_0+\varepsilon)\zeta(1)-a^3(1)\zeta_s(1)=0. \tag{25}$$

The next step is to consider the linear function $G_2^\varepsilon : Y \to Y$ given by $G_2^\varepsilon(\zeta,w)=(\xi,w)$, where

$$\xi=\zeta+\varepsilon a^{-3}(1)s\int_s^1 \zeta(t)\,\mathrm{d}t.$$

A straightforward modification of the proof of Lemma 4(ii) in Groves & Mielke [17] shows that $G_1^\varepsilon : Y \to Y$ is an isomorphism for each $\varepsilon\in\Lambda$, and furthermore

$$\alpha_0\xi(1)-a^3(1)\xi_s(1)=(\alpha_0+\varepsilon)\zeta(1)-a^3(1)\zeta_s(1),$$

so that the boundary condition (25) is transformed into

$$\alpha_0\xi(1)-a^3(1)\xi_s(1)=0.$$

The following lemma confirms that $G^\varepsilon=G_2^\varepsilon\circ G_1$ defines a valid change of variable; part (ii) is proved using the method explained by Groves & Mielke [17, Lemma 4(ii)]).

Lemma 3.2

(i) *For each $\varepsilon\in\Lambda$, the mapping G^ε is a smooth diffeomorphism from Z onto a neighbourhood $\tilde{Z}$ of the origin in Y. The mappings G^ε and $(G^\varepsilon)^{-1}$ and their derivatives depend smoothly upon ε.*

(ii) *For each $(\phi,w,\varepsilon)\in Z\times\Lambda$ the operator $\mathrm{d}G^\varepsilon[\phi,w] : Y \to Y$ extends to an isomorphism $\widehat{\mathrm{d}G}^\varepsilon[\phi,w] : X \to X$. The operators $\widehat{\mathrm{d}G}^\varepsilon[\phi,w]$ and $\widehat{\mathrm{d}G}^\varepsilon[\phi,w]^{-1}$ depend smoothly on $(\phi,w,\varepsilon)\in Z\times\Lambda$.*

The change of variable G^ε transforms $(X, \Omega, H^\varepsilon)$ into (X, Φ, K^ε), where Φ^ε and K^ε are defined on the manifold domain $\tilde{Z}$ of X by

$$\Phi^\varepsilon|_z(v_1, v_2) = \Omega(\widehat{\mathrm{d}G}^\varepsilon[(G^\varepsilon)^{-1}(z)]^{-1}(v_1), \widehat{\mathrm{d}G}^\varepsilon[(G^\varepsilon)^{-1}(z)]^{-1}(v_2))$$

for $v_1, v_2 \in TX|_z$ and

$$K^\varepsilon(z) = H^\varepsilon((G^\varepsilon)^{-1}(z)).$$

The corresponding Hamiltonian vector field is given by

$$v_{K^\varepsilon}(u) = \widehat{\mathrm{d}G}^\varepsilon[(G^\varepsilon)^{-1}(u)](v_{H^\varepsilon}((G^\varepsilon)^{-1}(u))),$$

where

$$\mathcal{D}(v_{K^\varepsilon}) = \{(\xi, w) \in \tilde{Z} : w(0) = 0,\ \alpha_0\xi(1) - a^3(1)\xi_s(1) = 0\}.$$

The next step is to verify that Hamilton's equations for $(X, \Phi^\varepsilon, K^\varepsilon)$ satisfy the hypotheses of Theorem 3.1. We write these equations as

$$u_x = Lu + N^\varepsilon(u),$$

in which the linear operator $L : \mathcal{D}(L) \subset X \to X$ with

$$\mathcal{D}(L) = \{(\xi, w) \in Y : w(0) = 0,\ \alpha_0\xi(1) - a^3(1)\xi_s(1) = 0\}$$

is given by

$$L\begin{pmatrix} \xi \\ w \end{pmatrix} = \begin{pmatrix} a^{-1}(s)w \\ -(a^3(s)\xi_s)_s \end{pmatrix}$$

(the linearisation of the Hamiltonian vector field v_{K^ε} at $\varepsilon = 0$). It follows from the following lemma, whose proof is a straightforward modification of those of Proposition 3.3 and Lemma 3.4 in Groves & Wahlén [19], that L satisfies hypotheses (H1) and (H2); hypothesis (H3) is clearly satisfied for an arbitrary value of k.

Lemma 3.3

(i) The spectrum of the operator $L : \mathcal{D}(L) \subset X \to X$ consists of isolated, geometrically simple eigenvalues of finite algebraic multiplicity.

(ii) There exist real constants C, $\xi_0 > 0$ such that each solution $u \in Y$ of the resolvent equation

$$(L - \mathrm{i}\xi I)u = f^\star, \tag{26}$$

in which $f^\star$ belongs to X and ξ is a real number with $|\xi| > \xi_0$, satisfies the estimates

$$\|v\|_Y \leq C\|f^\star\|_X, \qquad \|v\|_X \leq \frac{C}{|\xi|}\|f^\star\|_X.$$

Let us now examine the spectrum of L in more detail, in particular the qualitative dependence of its eigenvalues upon α_0. Eliminating w, we find that the eigenvalue problem $Lu = \kappa u$ is equivalent to

$$
\begin{aligned}
-a^{-1}(s)(a^3(s)\xi_s)_s &= \kappa^2\xi, \qquad 0 < s < 1,\\
\alpha_0\xi(1) - a^3(1)\xi_s(1) &= 0,\\
\xi(0) &= 0.
\end{aligned}
$$

The change of variable

$$y = \int_0^s a^{-1}(t)\,\mathrm{d}t, \qquad v(y) = a(s)\xi(s)$$

transforms the above equations into the equivalent self-adjoint Sturm-Liouville problem

$$-v_{yy} + Q(y)v = \nu v, \tag{27}$$
$$v_y(1) = \hat{\alpha}v(1), \tag{28}$$
$$v(0) = 0, \tag{29}$$

where $\nu = \kappa^2$, $Q(y) = -\gamma'(-s)$ and $\hat{\alpha} = a'(1) + a^{-2}(1)\alpha_0$.

The Sturm-Liouville problem (27)–(29) has a countable number of simple eigenvalues $\nu_0 < \nu_1 < \nu_2 < \ldots$ with $\nu_n \to \infty$ as $n \to \infty$. These eigenvalues correspond to the intersections in the (ν, s) plane of the line $s = \hat{\alpha}$ and the curve $s = B(\nu)$, where $B(\nu) = v_y(1;\nu)/v(1;\nu)$ and $v(y;\nu)$ solves the initial-value problem

$$-v_{yy} + Q(y)v = \nu v, \qquad v(0;\nu) = 0.$$

The function $B(\nu)$ has poles exactly at the *Dirichlet eigenvalues* ν_n^{D} (the positive eigenvalues of the problem in which (28) is replaced by $v(1) = 0$); it is strictly decreasing from $+\infty$ to $-\infty$ in each interval $(-\infty, \nu_0^{\mathrm{D}})$ and $(\nu_{n-1}^{\mathrm{D}}, \nu_n^{\mathrm{D}})$, $n \in \mathbb{N}$. It follows that ν_0 lies in the interval $(-\infty, \nu_0^{\mathrm{D}})$ while ν_n lies in the interval $(\nu_{n-1}^{\mathrm{D}}, \nu_n^{\mathrm{D}})$, $n \in \mathbb{N}$ (see Figure 2).

The eigenvalues κ of L are recovered from the above analysis by the formula $\nu = \kappa^2$, so that in particular they occur in plus-minus pairs. Clearly L has precisely one simple real eigenvalue in each interval $((\nu_{n-1}^{\mathrm{D}})^{1/2}, (\nu_n^{\mathrm{D}})^{1/2})$ and $(-(\nu_n^{\mathrm{D}})^{1/2}, -(\nu_{n-1}^{\mathrm{D}})^{1/2})$, $n \in \mathbb{N}$ and there are two additional eigenvalues (counted according to algebraic multiplicity). Figure 2 shows that the nature of the two additional eigenvalues depends upon the sign of ν_0: they are real (with magnitude less than $(\nu_0^{\mathrm{D}})^{1/2}$) for $\nu_0 > 0$ and purely imaginary for $\nu_0 < 0$. The remaining case $\nu_0 = 0$ leads to a zero eigenvalue of L, the algebraic

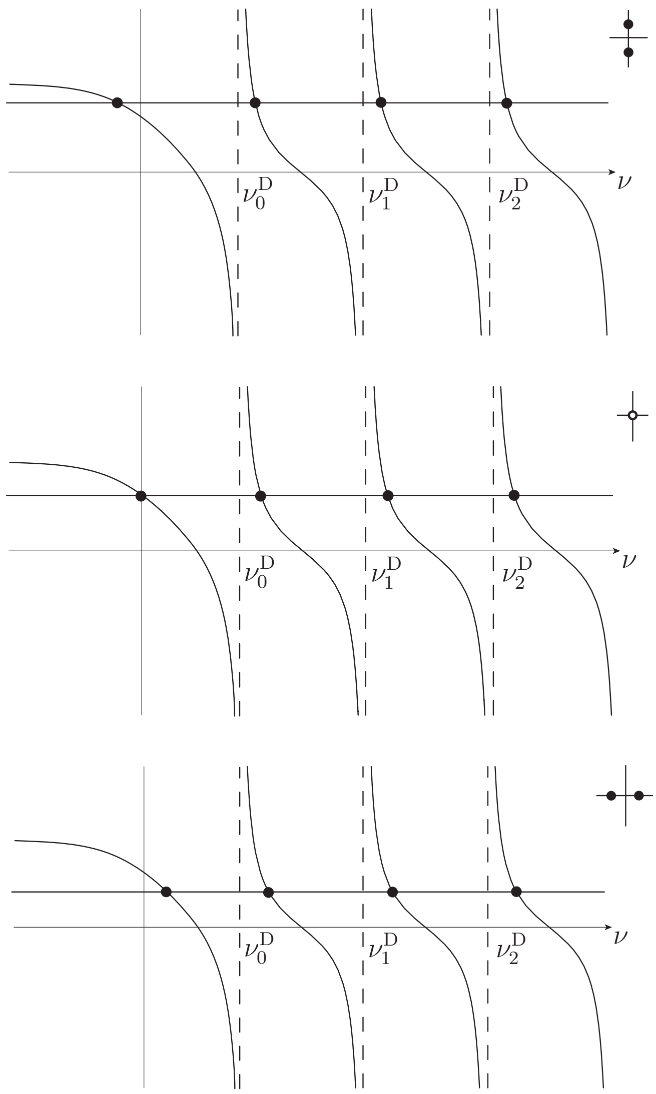

Fig. 2. Geometric characterisation of the eigenvalues ν_n of (27)–(29) as the points of intersection of the curve $s = B(\nu)$ with the straight line $s = \hat{\alpha}$. The eigenvalues ν_n, $n \in \mathbb{N}$ are positive, while ν_0 can be negative (top), zero (centre) or positive (bottom). The insets show the eigenvalues κ of L which satisfy $|\mathrm{Re}\,\kappa| < (\nu_0^{\mathrm{D}})^{1/2}$; solid and hollow dots denote respectively algebraically simple and double eigenvalues.

multiplicity of which is readily determined by studying the equation $Lu = 0$ directly. A straightforward calculation shows that zero is an eigenvalue of L if and only if $\alpha_0 = \alpha_\star$, where

$$\alpha_\star = \left(\int_0^1 a^{-3}(s)\,\mathrm{d}s \right)^{-1}.$$

The eigenvalue has algebraic multiplicity 2; the generalised eigenvectors u_1, u_2, where $Lu_1 = 0$, $Lu_2 = u_1$, are given by

$$u_1 = \begin{pmatrix} \int_0^s a^{-3}(t)\,\mathrm{d}t \\ 0 \end{pmatrix}, \qquad u_2 = \begin{pmatrix} 0 \\ a(s)\int_0^s a^{-3}(t)\,\mathrm{d}t \end{pmatrix}. \tag{30}$$

Examining Figure 2, we conclude that the three cases $\nu_0 > 0$, $\nu_0 = 0$ and $\nu_0 < 0$ correspond to respectively $\alpha_0 > \alpha_\star$, $\alpha_0 = \alpha_\star$ and $\alpha_0 < \alpha_\star$; Theorem 3.1 therefore yields a two-dimensional centre manifold $\tilde{X}^\varepsilon$ whenever $\alpha_0 \geq \alpha_\star$. The centre manifold is equipped with the single coordinate chart $\tilde{U}_1 \subset \mathcal{X}_1$ and coordinate map $\chi : \tilde{X}^\varepsilon \to \tilde{U}_1$ defined by $\chi^{-1}(u_1) = u_1 + r(u_1;\varepsilon)$. It is however more convenient to use an alternative coordinate map for calculations.

Define $\hat{r} : \tilde{U}_1 \times \tilde{\Lambda} \to Z$ by

$$\hat{u}_1 + \hat{r}(\hat{u}_1;\varepsilon) = (G^\varepsilon)^{-1}(u_1 + r(u_1;\varepsilon)),$$

where $\hat{r}(0;0) = 0$, $\mathrm{d}_1\hat{r}[0;0] = 0$, and equip $\tilde{X}^\varepsilon$ with the coordinate map $\hat{\chi} : \tilde{X}^\varepsilon \to \tilde{U}_1$ given by $\hat{\chi}^{-1}(\hat{u}_1) = \hat{u}_1 + \hat{r}(\hat{u}_1;\varepsilon)$. In this coordinate system the reduced 2-form $\tilde{\Omega}^\varepsilon$ is given by the formula

$$\begin{aligned}\tilde{\Omega}^\varepsilon|_{\hat{u}_1}(v^1,v^2) &= \Omega|_{\hat{u}_1+\hat{r}(\hat{u}_1;\varepsilon)}(v^1 + \mathrm{d}_1\hat{r}[\hat{u}_1;\varepsilon](v^1), v^2 + \mathrm{d}_1\hat{r}[\hat{u}_1;\varepsilon](v^2)) \\ &= \Omega(v^1,v^2) + O(|(\hat{u}_1,\varepsilon)||v^1||v^2|).\end{aligned}$$

According to the parameter-dependent version of Darboux's theorem presented by Buffoni & Groves [6, Theorem 4] there exists a near-identity change of variable $\hat{u}_1 = \tilde{u}_1 + \Theta(\tilde{u}_1;\varepsilon)$ of class C^{k-1} which transforms $\tilde{\Omega}^\varepsilon$ into Ψ, where

$$\Psi(v^1,v^2) = \Omega(v^1,v^2).$$

We accordingly introduce $\tilde{r} : \tilde{U}_1 \times \tilde{\Lambda} \to Z$ by the formula

$$\tilde{u}_1 + \tilde{r}(\tilde{u}_1;\varepsilon) = \tilde{u}_1 + \Theta(\tilde{u}_1;\varepsilon) + \hat{r}(\tilde{u}_1 + \Theta(\tilde{u}_1;\varepsilon);\varepsilon),$$

where $\tilde{r}(0;0) = 0$, $\mathrm{d}_1\tilde{r}[0;0] = 0$, and equip $\tilde{X}^\varepsilon$ with the coordinate map $\tilde{\chi} : \tilde{X}^\varepsilon \to \tilde{U}_1$ given by $\tilde{\chi}^{-1}(\tilde{u}_1) = \tilde{u}_1 + \tilde{r}(\tilde{u}_1;\varepsilon)$. It is always possible to choose a basis for the two-dimensional central subspace X_1 of X so that Ψ is the canonical symplectic 2-form Υ in this coordinate system (a 'symplectic basis'). Choosing $\tilde{\chi}$ as coordinate map and identifying $\tilde{U}_1$ with a neighbourhood $\mathcal{M}$ of the origin in $\mathbb{R}^2$, one therefore obtains a two-dimensional reduced Hamiltonian system $(\mathcal{M}, \Upsilon, \tilde{K}^\varepsilon)$ for $\alpha_0 \geq \alpha_\star$, in which

$$\tilde{K}^\varepsilon(\tilde{u}_1) = H^\varepsilon(\tilde{u}_1 + \tilde{r}(\tilde{u}_1;\varepsilon)).$$

4 Homoclinic bifurcation

In this section we complete our existence theory for solitary waves by showing that the reduced Hamiltonian system on the centre manifold admits homoclinic solutions. A *Hamiltonian* 0^2 *resonance* takes place at $\alpha = \alpha_\star$: two real eigenvalues become purely imaginary by colliding at the origin and forming a Jordan chain of length 2. This resonance is associated with the bifurcation of a branch of homoclinic solutions into the parameter region with real eigenvalues; we correspondingly choose $\alpha_0 = \alpha_\star$ and seek homoclinic solutions for $\varepsilon < 0$.

Formulae for the generalised eigenvectors u_1, u_2, where $Lu_1 = 0$, $Lu_2 = u_1$, are given in equation (30), and one finds that

$$\Omega(u_1, u_2) = d_1^2, \qquad d_1^2 = \int_0^1 a(s) \left(\int_0^s a^{-3}(t)\,\mathrm{d}t \right)^2 \mathrm{d}s.$$

It follows that $\{e, f\}$, where $e = d_1^{-1} u_1$, $f = d_1^{-1} u_2$, is a symplectic basis for X_1. The coordinates q, p in the e and f directions are canonical coordinates for X_1 and the action of the reverser S on this space is given by

$$S(q,p) = (q, -p).$$

Choosing a coordinate system for $\tilde{X}^\varepsilon$ according to the recipe given at the end of Section 3 we obtain the two-dimensional canonical Hamiltonian system $(\mathcal{M}, \Upsilon, \tilde{K}^\varepsilon)$, where $\mathcal{M}$ is a neighbourhood of the origin in $\mathbb{R}^2$,

$$\Upsilon((q^1,p^1),(q^2,p^2)) = q^1 p^2 - p^1 q^2$$

and

$$\tilde{K}^\varepsilon(q,p) = H^\varepsilon(\tilde{u}_1 + \tilde{r}(\tilde{u}_1;\varepsilon)), \qquad \tilde{u}_1 = qe + pf.$$

A direct calculation shows that

$$\tilde{K}_2^0(q,p) = H_2^0[\tilde{u}_1, \tilde{u}_1] = \frac{1}{2} p^2,$$

where $\varepsilon^i \tilde{K}_j^i(\tilde{u}_1)$ denotes the part of the Taylor expansion of $\tilde{K}^\varepsilon(\tilde{u}_1)$ which is homogeneous of order i in ε and j in $\tilde{u}_1 \cong (q,p)$ and H_j^i denotes the symmetric, k-linear operator $X_1^k \to \mathbb{R}$ which defines the corresponding coefficient in the Taylor expansion of H^ε. Anticipating the scaling $q \sim |\varepsilon| Q$, $p \sim |\varepsilon|^{3/2} P$, we write

$$\tilde{K}^\varepsilon(q,p) = \frac{1}{2} p^2 + c_1 \varepsilon q^2 + c_2 q^3 + O(|p||(q,p)||(\varepsilon,q,p)|) + O(|(q,p)|^2 |(\varepsilon,q,p)|^2),$$

so that the first three terms on the right-hand side of the above equation are $O(|\varepsilon|^3)$ and the remainder is of higher order.

The coefficients c_1 and c_2 are obtained from the calculations

$$c_1 = H_2^1[e,e] + 2H_2^0[e,\tilde{r}_{10}^1], \qquad c_2 = H_3^0[e,e,e] + 2H_2^0[e,\tilde{r}_{20}^0],$$

in which $\tilde{r}_{jk}^i$ denotes the coefficient of $\varepsilon^i q^j p^k$ in the Taylor expansion of $\tilde{r}$. To calculate these coefficients we make use of the fact that

$$\mathrm{d}H^\varepsilon[u](v) = \Omega(v_{H^\varepsilon}(u), v) - T^\varepsilon(u)\tilde{\phi}|_{s=1},$$

where $u = (\phi, w)$, $v = (\tilde{\phi}, \tilde{w})$ and $T^\varepsilon : Z \to \mathbb{R}$ is defined by

$$T^\varepsilon(u) = \frac{1}{2}\left(w^2(1) + \frac{1}{(a^{-1}(1) + \phi_s(1))^2}\right) + (\alpha_\star + \varepsilon)\phi(1) - \frac{\mu^\star}{2}.$$

In particular, we find that

$$2H_2^0[u,v] = \Omega(L(u), v) - T_1^0(u)\tilde{\phi}|_{s=1},$$

where T_j^i is defined in the same way as H_j^i, so that

$$H_2^0[e,v] = 0.$$

It follows that

$$c_1 = H_2^1[e,e] = \frac{1}{2\alpha_\star^2 d_1^2}, \qquad c_2 = H_3^0[e,e,e] = \frac{c_0}{2\alpha_\star^3 d_1^3},$$

where

$$c_0 = \alpha_\star^3 \int_0^1 a^{-5}(s)\,\mathrm{d}s.$$

Hamilton's equations for $(\mathcal{M}, \Upsilon, \tilde{K}^\varepsilon)$ are

$$q_x = p + \mathcal{R}_1(q,p,\varepsilon), \tag{31}$$

$$p_x = -\frac{\varepsilon}{\alpha_\star^2 d_1^2} q - \frac{3c_0}{2\alpha_\star^3 d_1^3} q^2 + \mathcal{R}_2(q,p,\varepsilon), \tag{32}$$

where $\mathcal{R}_1$, $\mathcal{R}_2$ are respectively odd and even in their second arguments and

$$\mathcal{R}_1 = O(|(q,p)||(\varepsilon,q,p)|), \qquad \mathcal{R}_2 = O(|p||(\varepsilon,q,p)|) + O(|(q,p)||(\varepsilon,q,p)|^2).$$

Introducing the scaled variables

$$X = |\varepsilon|^{1/2}\alpha_\star^{-1} d_1^{-1} x, \quad q(x) = c_0^{-1}|\varepsilon|\alpha_\star d_1 Q(X), \quad p(x) = c_0^{-1}|\varepsilon|^{3/2} P(X),$$

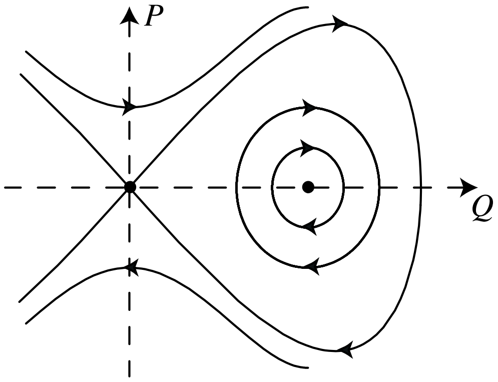

Fig. 3. Phase portrait of the scaled reduced system of equations.

one finds from (31), (32) that

$$Q_X = P + \mathcal{R}_3(Q, P, \varepsilon), \tag{33}$$
$$P_X = -\mathrm{sgn}(\varepsilon)Q - \frac{3}{2}Q^2 + \mathcal{R}_4(Q, P, \varepsilon), \tag{34}$$

where the remainder terms $\mathcal{R}_3$ and $\mathcal{R}_4$ are $O(|\varepsilon|^{1/2})$ and respectively odd and even in their second arguments. In the limit $\varepsilon \uparrow 0$ equations (33), (34) are equivalent to

$$Q_X = P,$$
$$P_X = Q - \frac{3}{2}Q^2,$$

whose phase portrait is easily calculated by elementary methods and is depicted in Figure 3. Notice in particular that it has a nonzero equilibrium $(2/3, 0)$, surrounded by the symmetric homoclinic orbit

$$Q(X) = \mathrm{sech}^2\left(\frac{X}{2}\right), \qquad P(X) = -\mathrm{sech}^2\left(\frac{X}{2}\right)\tanh\left(\frac{X}{2}\right).$$

A familiar argument based upon the reversibility of (33), (34) shows that its phase portrait is qualitatively the same as that shown in Figure 3 for small negative values of ε (e.g. see Groves & Wahlén [19, Section 4.1]); in particular its phase portrait has a reversible homoclinic orbit in the right half-plane for sufficiently small negative values of ε. Tracing back the various changes of variable, one finds that the surface profile of the water corresponding to this homoclinic orbit is given by

$$\eta(x) = 1 + c_0^{-1}|\varepsilon|\mathrm{sech}^2\left(\frac{|\varepsilon|^{1/2}x}{2d_1\alpha_\star}\right) + O(|\varepsilon|^{3/2}).$$

We therefore obtain a symmetric solitary wave of elevation which decays exponentially and monotonically to a horizontal laminar flow as $x \to \pm\infty$ and is sketched in Figure 1(b).

Remark 4.1

(i) Similar arguments show that the phase portrait of (33), (34) for small positive values of ε consists of a family of periodic orbits surrounding the zero equilibrium. The corresponding free-surface flows are symmetric Stokes waves with large periods.

(ii) Choosing $\alpha_0 > \alpha_\star$ and $\varepsilon = 0$, one obtains a one-degree-of-freedom reduced Hamiltonian system with a pair of purely imaginary eigenvalues $\pm \mathrm{i}\omega(\alpha_0)$, where $\omega(\alpha_0)$ is a strictly increasing function of α_0 with $\omega(\alpha_0) \to \infty$ as $\alpha_0 \to \infty$ and $\omega(\alpha_0) \downarrow 0$ as $\alpha_0 \downarrow \alpha_\star$. It follows from the Lyapunov centre theorem that the phase portrait of this reduced system consists of a family of periodic orbits surrounding the zero equilibrium; the corresponding free-surface flows are symmetric Stokes waves with period near $2\pi/\omega_0$.

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