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Steklov–Lyapunov type systems [*]

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Abstract

In this paper we describe integrable generalizations of the classical Steklov–Lyapunov systems, which are defined on a certain product $so(m) \times so(m)$, as well as the structure of rank r coadjoint orbits in $so(m) \times so(m)$. We show that the restriction of these systems onto some subvarieties of the orbits written in new matrix variables admits a new $r \times r$ matrix Lax representation in a generalized Gaudin form with a rational spectral parameter.

In the case of rank 2 orbits a corresponding 2×2 La x pair for the reduced systems enables us to perform a separation of variables.

1 Introduction. Gaudin magnets and the hierarchy of the Steklov–Lyapunov systems.

Many finite-dimensional integrable systems, as well as finite-gap reductions of some integrable PDE's, can be regarded as Hamiltonian flows on finite-dimensional coadjoint orbits of the loop algebra $\widetilde{gl}(r)$ described by $r \times r$ Lax equations with a spectral parameter $\lambda \in \mathbb{C}$,

$$\dot{L}(\lambda) = [\, L(\lambda), \mathcal{M}(\lambda)\,], \qquad L = Y + \sum_{i=1}^{n} \frac{\mathcal{N}_i}{\lambda - a_i}, \quad L, \mathcal{M} \in gl(r), \qquad (1.1)$$

where $\mathcal{N}_i$ are $r \times r$ matrix variables, $Y \in gl(r)$ is a constant matrix and $a_1, \ldots, a_n$ are arbitrary distinct constants (see [1, 2]). In particular, $L(\lambda)$ can be taken in form

$$L(\lambda) = Y + G^T (\lambda \mathbf{I}_n - A)^{-1} F \qquad (1.2)$$

where $\mathbf{I}_n$ is the $n \times n$ unit matrix and G, F are $n \times r$ matrices of rank r. Integrable systems described by the corresponding Lax equations are usually referred to as *Gaudin magnets* ([8]) .

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As shown in [1], such systems naturally arise in connection with so called rank r perturbations of the constant matrix $A = \text{diag}(a_1, \ldots, a_n)$, namely

$$A \to \mathcal{L}(\nu) = A + F(Y - \nu I_r)^{-1}G^T, \qquad \nu \in \mathbb{C},$$

where now $\mathbf{I}_r$ is the $r \times r$ unit matrix. The matrices $L(\lambda), \mathcal{L}(\nu)$ are *dual* in the sense that their spectral curves are birationally equivalent and the parameter ν plays the role of the eigenvalue parameter for $L(\lambda)$. The characteristic polynomials of the dual Lax matrices are related by the Weinstein–Aronzjan formula (see [1, 11])

$$\det(\lambda \mathbf{I}_n - A)\det(Y + G^T(\lambda \mathbf{I}_n - A)^{-1}F - \nu \mathbf{I}_r)$$
$$= \det(\nu \mathbf{I}_r - Y)\det(A + F(Y - \nu I_r)^{-1}G^T - \lambda \mathbf{I}_n) \qquad (1.3)$$

On the other hand, there exists a series of integrable systems which are known to admit a Lax pair with an elliptic spectral parameter only. The examples that we consider here are integrable cases of the classical Kirchhoff equations found by Steklov and Lyapunov ([15, 13]).

Recall that the Kirchhoff equations on the Lie coalgebra $e^*(3) = (K, p)$, $K = (K_1, K_2, K_3)^T$, $p = (p_1, p_2, p_3)^T$ are Hamiltonian with respect to the standard Lie–Poisson bracket

$$\{K_\alpha, K_\beta\} = \varepsilon_{\alpha\beta\gamma}K_\gamma, \quad \{K_\alpha, p_\beta\} = \varepsilon_{\alpha\beta\gamma}p_\gamma, \quad \{p_\alpha, p_\beta\} = 0,$$
$$(\alpha, \beta, \gamma) = (1, 2, 3),$$

Here (K, p), (p, p) are Casimir functions of the bracket. The Steklov and Lyapunov systems are described respectively by the Hamiltonians

$$H_S = \frac{1}{2}\sum_{\alpha=1}^3 \left(b_\alpha K_\alpha^2 + 2\nu b_\beta b_\gamma K_\alpha p_\alpha + \nu^2 b_\alpha(b_\beta - b_\gamma)^2 p_\alpha^2\right),$$

$$H_L = \frac{1}{2}\sum_{\alpha=1}^3 \left(K_\alpha^2 - 2\nu b_\alpha K_\alpha p_\alpha + \nu^2(b_\beta - b_\gamma)^2 p_\alpha^2\right), \qquad (1.4)$$

$$b_1, b_2, b_3, \ \nu = \text{const}, \quad (\alpha, \beta, \gamma) = (1, 2, 3),$$

where ν is an arbitrary parameter.

It can be checked that $\{H_S, H_L\} = 0$ with respect to the above Poisson bracket on $e^*(3)$, which implies the integrability of the Steklov and Lyapunov systems.

These systems were explicitly integrated by Kötter [12], who used the change of variables $(K, p) \to (z, p)$:

$$z_\alpha = K_\alpha - \frac{\nu}{2}(b_\beta + b_\gamma)p_\alpha, \qquad \alpha = 1, 2, 3, \quad (\alpha, \beta, \gamma) = (1, 2, 3) \qquad (1.5)$$

and actually represented the equations of motion in a Lax form

$$\dot{L}(s) = [L(s), A(s)], \qquad L(s), A(s) \in so(3), \quad s \in \mathbb{C},$$
$$L_{\alpha\beta}(s) = \varepsilon_{\alpha\beta\gamma}\sqrt{s - b_\gamma}\,(z_\gamma + sp_\gamma), \qquad (1.6)$$

where $\varepsilon_{\alpha\beta\gamma}$ is the Levi-Civita tensor and the matrix $A(s)$ depends on the Hamiltonian of the problem.

The roots in (1.6) are single-valued functions on the elliptic curve $\widehat{\Sigma}$, the 4-sheeted unramified covering of the plane curve $\Sigma = \{w^2 = (s - b_1)(s - b_2)(s - b_3)\}$, which is obtained by doubling of both periods of Σ. This implies that the Lax pair is elliptic. Equivalent $su(2)$ matrix Lax pairs, where the roots are replaced by elliptic functions on Σ are indicated in [3].

According to [5, 9], the Steklov–Lyapunov systems admit multidimensional integrable generalizations defined not on the coalgebra $e^*(n)$, as one might expect, but on a product $so(m) \times so(m)$ with matrix variables $Z, P \in so^*(m)$. The generalized systems admit a Lax pair with a hyperelliptic spectral parameter.

Contents of the paper. In Section 2 we briefly describe m-dimensional Hamiltonian Steklov–Lyapunov systems, the Poisson structure on $so(m) \times so(m)$, and the structure of generic and rank r coadjoint orbits $\mathcal{S}_{c,d}^r$ in $so(m) \times so(m)$, which are characterized by values c, d of the corresponding Casimir functions.

Section 3 shows that the restriction of m-dimensional Steklov–Lyapunov systems onto certain invariant subvarieties $\mathcal{F}_{c,d}^r$ of $\mathcal{S}_{c,d}^r$ admits $r \times r$ matrix Lax representation in a generalized Gaudin form. Namely, the $r \times m$ matrices F, G in (1.2) became linear functions of the spectral parameter λ:

$$G = (\mathcal{X}, \ -\mathcal{Y} - \lambda\mathcal{V}), \quad F = (\mathcal{Y} + \lambda\mathcal{V}, \ \mathcal{X}),$$

where $\mathcal{X}, \mathcal{Y}, \mathcal{V}$ are $(r/2) \times m$ matrices related to the variables $(Z, P) \in so(m) \times so(m)$ as follows

$$\forall s \in \mathbb{R}, \quad Z + sP = \mathcal{X}^T(\mathcal{Y} + s\mathcal{V}) - (\mathcal{Y} + s\mathcal{V})^T\mathcal{X},$$

so that the corresponding $r \times r$ Lax matrix $L(\lambda)$ obtains a linear part in the spectral parameter:

$$L(\lambda) = \begin{pmatrix} \mathcal{X}(\lambda\mathbf{I} - B)^{-1}[\mathcal{Y} + \lambda\mathcal{V}]^T & \mathcal{X}(\lambda\mathbf{I} - B)^{-1}\mathcal{X}^T \\ -(\mathcal{Y} + \lambda\mathcal{V})(\lambda\mathbf{I} - B)^{-1}[\mathcal{Y} + \lambda\mathcal{V}]^T & -[\mathcal{Y} + \lambda\mathcal{V}](\lambda\mathbf{I} - B)^{-1}\mathcal{X}^T \end{pmatrix}$$
$$= L_1\lambda + L_0 + (\mathcal{X} \ -\mathcal{Z})^T(\lambda\mathbf{I}_n - B)^{-1}(\mathcal{Z} \ \mathcal{X}), \tag{1.7}$$

where $B = \mathrm{diag}(b_1, \ldots, b_m)$, $\mathcal{Z} = \mathcal{Y} + B\mathcal{V}$, and L_1, L_0 are certain off-diagonal matrices. This Lax matrix leads to a new rational Lax pair for Steklov–Lyapunov systems on $\mathcal{F}_{c,d}^r$. Note that, apparently, in this case the Weinstein–Aronzjan formula (1.3) is not applicable and the dual Lax matrix of $L(\lambda)$ may not exist.

In Section 4 we consider in detail the motion on rank 2 orbits $\mathcal{S}_{c,d}^2$ and show that it allows a special vestion of the Marsden–Weinstein reduction onto certain symplectic $2(m-1)$-dimensional manifolds $\mathcal{O}_{c,d}^2$. The latter are foliated with $(m-1)$-dimensional Jacobians of hyperelliptic curves, and the reduced systems are just standard algebraic completely integrable Jacobi–Mumford systems (see, e.g., [2, 16]).

Finally, we perform a separation of variables for these systems by indicating the Abel–Jacobi quadratures in terms of ce rtain coordinates on $\mathcal{O}_{c,d}^2$, which are Darboux coordinates with respect to the original Lie–Poisson structure on $so(m) \times so(m)$.

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In the classical case $m = 3$, the orbits $\mathcal{S}^2_{c,d}$ are just coverings of $\mathcal{O}^2_{c,d}$, and the above coordinates, as separating variables, were first introduced by F. Kötter in his short paper [12] without discussing their symplectic nature.

2 Steklov–Lyapunov system on generic and special rank r coadjoint orbits in $so(m) \times so(m)$

Following [5, 9], multidimensional Steklov–Lyapunov systems are defined on a product $so(m) \times so(m)$ with matrix variables $Z, P \in so^*(m)$, which is endowed with the following Poisson bracket

$$
\begin{aligned}
\{f, h\}_1 = \langle Z, [\, d_Z f, d_Z h\,] \rangle + \langle P, [\, d_Z f, d_P h\,] + [\, d_P f, d_Z h\,] \rangle \\
- \langle P, (d_Z f \, B \, d_Z h - d_Z h \, B \, d_Z f) \rangle,
\end{aligned}
\tag{2.1}
$$

$$
\langle X, Y \rangle = -\frac{1}{2} \mathrm{tr}(XY), \quad B = \mathrm{diag}(b_1, \ldots, b_m),
$$

$$
(d_Z f)_{ij} = \partial f / \partial Z_{ij}, \quad (d_P f)_{ij} = \partial f / \partial P_{ij},
$$

where $b_1, \ldots, b_m$ are arbitrary distinct constants. This implies that equations of motion can be written in the Hamiltonian form

$$
\begin{aligned}
\dot{Z} &= \left[Z, \frac{\partial \mathcal{H}}{\partial Z} \right] + B \frac{\partial \mathcal{H}}{\partial Z} P - P \frac{\partial \mathcal{H}}{\partial Z} B + \left[P, \frac{\partial \mathcal{H}}{\partial P} \right], \\
\dot{P} &= \left[P, \frac{\partial \mathcal{H}}{\partial Z} \right].
\end{aligned}
\tag{2.2}
$$

The bracket $\{f, h\}_1$ has exactly $2[m/2]$ independent Casimir functions

$$
\begin{aligned}
\mathcal{P}_k &= -\mathrm{tr}(P^k), \\
\mathcal{Q}_k &= \mathrm{tr}(ZP^{k-1} + P^k B),
\end{aligned}
\quad k = 2, 4, \ldots, 2[m/2]
\tag{2.3}
$$

(here and below, in indices, the symbol $[\ ]$ denotes the integer part of the number).

The multidimensional integrable analogs of the Lyapunov and Steklov systems are described by the following quadratic Hamiltonians that generalize (1.4),

$$
\mathcal{H}_L = \langle Z, Z \rangle + 2 \langle Z, (BP + PB) \rangle + \langle P, (B^2 P + BPB + PB^2) \rangle,
$$

$$
\mathcal{H}_S = \langle Z, BZ + ZB \rangle + 2 \langle Z, \{P, B^2\} \rangle + \langle P, \{P, B^3\} \rangle - \mathbf{tr}\, B\, \mathcal{H}_L.
\tag{2.4}
$$

Here and below the bracket $\{X^l, Y^r\}$ (without an index) denotes a homogeneous symmetric matrix polynomial in X and Y of degrees s and r respectively, for example: $\{X, Y^0\} = X$, $\{X, Y\} = XY + YX$, $\{X, Y^2\} = XY^2 + YXY + Y^2 X$, etc.

The corresponding flows admit the following Lax pairs, which generalize (1.6),

$$
\dot{L}(s) = [\, L(s), A(s)\,], \qquad L(s), A(s) \in so(m), \quad s \in \mathbb{C},
\tag{2.5}
$$

$$
L(s)_{ij} = \frac{\sqrt{\Phi(s)}}{\sqrt{(s - b_i)(s - b_j)}} \, (Z + sP)_{ij}, \qquad i, j = 1, \ldots, m,
\tag{2.6}
$$

$$
\Phi(s) = (s - b_1) \cdots (s - b_m), \qquad b_1, \ldots, b_m = \mathrm{const},
$$

where the roots $w_{ij} = \sqrt{(s - b_i)(s - b_j)}$ are assumed to satisfy the relations $w_{ik}w_{kj} = (s - b_k)w_{ij}$. Under this condition, the roots, as well as $\sqrt{\Phi(s)}$, are single-valued functions on an unramified covering of the hyperelliptic curve $\Sigma = \{w^2 = \Phi(s)\}$. In this connection the Lax pair (2.5) is referred to as *hyperelliptic*.

To obtain the generalized Lyapunov and Steklov systems, in (2.5) we put

$$A(s)_{ij} = -\frac{1}{s}\sqrt{(s - b_i)(s - b_j)}\, P_{ij}, \qquad \text{and, respectively,}$$

$$A(s)_{ij} = \sqrt{(s - b_i)(s - b_j)}\,(sP_{ij} + Z_{ij} + (\mathbf{tr}\, B - b_i - b_j)P_{ij})\,.$$

Moreover, as shown in [5, 9], there exists a hierarchy of "higher" Steklov–Lyapunov systems. In particular, putting in (2.5)

$$A = A_{1,\rho}(s) = -S\tilde{A}_{1,\rho}(s)S, \qquad S = \mathrm{diag}(\sqrt{s - b_1}, \ldots, \sqrt{s - b_m}), \quad \rho = 0, 1, 2, \ldots,$$
$$\tilde{A}_{1,\rho}(s) = s^\rho P + s^{\rho-1}\{B, P\} + \cdots + \{B^\rho, P\} + s^{\rho-1}Z + \cdots + \{B^{\rho-1}, Z\},$$

we obtain the following subhierarchy of systems with quadratic right hand sides

$$\dot{Z} = [Z, \{Z, B^\rho\}] + Z\{P, B^\rho\}B - B\{P, B^\rho\}Z\,,$$
$$\dot{P} = [P, \{P, B^{\rho+1}\}] + [P, \{Z, B^\rho\}], \qquad \rho \in \{0, \mathbb{N}\}. \tag{2.7}$$

The matrix $A_{1,0}$ coincides with the above operator defining the multidimensional generalization of the Lyapunov system.

Following [5], apart from the bracket $\{\,,\,\}_1$, on $so(m) \times so(m)$ there is another Poisson bracket $\{\,,\,\}_0$, such that $\{\,,\,\}_1, \{\,,\,\}_0$ form a pencil of consistent (or compatible) Poisson brackets. The coefficients of the spectral curve provide a complete set of first integrals in involution with respect to all the brackets of the pencil, which proves the Liouville integrability of all the systems of the hierarchy.

Remark 1. Under the change of matrix variables

$$(Z, P) \to (M, P): \quad M = Z + \frac{1}{2}(BP + PB), \tag{2.8}$$

which is actually a generalization of Kötter's substitution (1.5), the bracket $\{f, h\}_1$ becomes precisely the Lie–Poisson bracket of the semi-direct product $so(m) \times_s so(m)$ specified by the commutator

$$\text{for } (X, Y) \in so(m) \times_s so(m)\,,$$
$$[(X_1, Y_1), (X_2, Y_2)] = ([X_1, X_2], [X_1, Y_2] - [X_2, Y_1]). \tag{2.9}$$

Indeed, for (M, P) in the dual space to $so(m) \times_s so(m)$, we introduce the natural pairing

$$\overline{\langle (M, P), (X, Y)\rangle} = \langle M, X\rangle + \langle P, Y\rangle.$$

Then, by the definition of a Lie–Poisson bracket and in view of (2.9),

$$\{f(M, P), h(M, P)\}_1 = \langle M, [d_M f, d_M h]\rangle + \langle P, [d_M f, d_P h] + [d_P f, d_M h]\rangle\,,$$

which transforms to (2.1) under the substitution (2.8).

In the classical case $m = 3$, in the vector variables z, p such that

$$Z_{ij} = \varepsilon_{ijk} z_k, \quad P_{ij} = \varepsilon_{ijk} p_k, \tag{2.10}$$

the bracket $\{f, h\}_1$ is just the Lie–Poisson bracket on $e^*(3)$.

According to (2.9), for any $x, y \in SO(m)$, the adjoint action of the semi-direct product $SO(m) \times_{Ad} so(m)$ on the algebra $so(m) \times_s so(m)$ has the form

$$Ad_{(x,y)}(X, Y) = (x^{-1} X x, y^{-1} X y + x^{-1} Y x).$$

Then, from the definition $\overline{\langle (M, P), Ad_{(x,y)}(X, Y) \rangle} = \overline{\langle Ad^*_{(x,y)}(M, P), (X, Y) \rangle}$, the coadjoint action on the dual space is found to be

$$Ad^*_{(x,y)}(M, P) = (x M x^{-1} + y P y^{-1}, x P x^{-1}). \tag{2.11}$$

Alexey, please check the above Remark 1. – Yuri

Although the matrix variables M, P are more convenient than Z, P from the point of view of the Hamiltonian description, for our future purposes we shall continue using both sets of variables.

First integrals and generic coadjoint orbits.   The characteristic polynomial of the Lax matrix (2.6) has the form

$$|L(s) - wI| = w^m + \sum_k w^{m-k} \, \Phi^{k/2-1}(s) \, \widetilde{\mathcal{I}}_k(s, Z, P), \tag{2.12}$$

$$k = 2, \ldots, 2[m/2] \quad (k \text{ is even}),$$

$$\widetilde{\mathcal{I}}_k(s, Z, P) = \sum_I \frac{\Phi(s)}{(s - b_{i_1}) \ldots (s - b_{i_k})} \, |Z + sP|_I^I = \sum_{\mu=0}^m s^\mu H_{k\mu}(Z, P), \tag{2.13}$$

where $|Z + sP|_I^I$ denotes the k-order diagonal minor corresponding to the multi-index $I = \{i_1 \ldots i_k\}$, $i_1 < \cdots < i_k$, which ranges over the set of all such indices. In particular, the two major coefficients

$$H_{km} = \sum_I |P|_I^I,$$

$$H_{k,m-1} = \sum_I (b_{i_1} + \cdots + b_{i_k}) |P|_I^I - (\operatorname{tr} B) \, H_{km}(P) + \operatorname*{Res}_{\varkappa=0} \sum_I |\varkappa^{-1} Z + P|_I^I \tag{2.14}$$

$$\equiv \operatorname*{Res}_{\varkappa=0} \sum_I |\varkappa^{-1} M + P|_I^I - (\operatorname{tr} B) \, H_{km}(P)$$

are annihilators of the bracket (2.1), and they are linear combinations of the Casimir functions (2.3).

The family of quadratic integrals has the form

$$\widetilde{\mathcal{I}}_2(\lambda, Z, P) = \sum_{i<j}^{m} \frac{\Phi(\lambda)}{(\lambda - b_i)(\lambda - b_j)} (Z_{ij} + \lambda P_{ij})^2 = \lambda^m \langle P, P \rangle + H_{2,m-1}(Z,P)\lambda^{m-1}$$
$$+ H_{2,m-2}(Z,P)\lambda^{m-2} + H_{2,m-3}(Z,P)\lambda^{m-3} + \cdots + H_{2,0}(Z,P),$$

$$(2.15)$$

where

$$H_{2,m-1} = \mathfrak{h}_{m-1} - \Delta_1 \mathfrak{h}_m,$$
$$H_{2,m-2} = \mathfrak{h}_{m-2} - \Delta_1 \mathfrak{h}_{m-1} + \Delta_2 \mathfrak{h}_m,$$
$$H_{2,m-3} = \mathfrak{h}_{m-3} - \Delta_1 \mathfrak{h}_{m-2} + \Delta_2 \mathfrak{h}_{m-1} - \Delta_3 \mathfrak{h}_m, \qquad (2.16)$$
$$\cdots \quad \cdots \quad \cdots \quad \cdots$$
$$H_{2,0} = \sum_{s=0}^{m} (-1)^s \Delta_s \mathfrak{h}_s = \det B \langle Z, B^{-1} Z B^{-1} \rangle.$$

and where $\mathfrak{h}_s(Z, P)$ are integrals in a "canonical" form,

$$\mathfrak{h}_m = \langle P, P \rangle, \quad \mathfrak{h}_{m-1} = 2\langle Z, P \rangle + \langle P, BP + PB \rangle,$$
$$\mathfrak{h}_{m-2-\rho} = \langle Z, \{Z, B^\rho\} \rangle + 2\langle Z, \{P, B^{\rho+1}\} \rangle + \langle P, \{P, B^{\rho+2}\} \rangle,$$
$$\rho = 0, 1, \ldots, m-2,$$

Δ_s being elementary symmetric functions of $b_1, \ldots, b_m$ of degree s, e.g., $\Delta_0 = 1$, $\Delta_1 = b_1 + \cdots + b_m$, $\Delta_2 = b_1 b_2 + \cdots + b_{m-1} b_m$, etc.

Notice that $\langle P, P \rangle$ and $H_{2,m-1}$ are quadratic Casimir functions, whereas (up to adding such functions) $H_{2,m-2}$ and $H_{2,m-3}$ coincide with the Lyapunov and Steklov Hamiltonians in (2.4) respectively.

As shown in [5, 9], for odd dimension m, the polynomials

$$H_{k\nu}(Z, P), \qquad k = 2, 4, \ldots \leq m, \quad \nu = 0, 1, \ldots, m$$

form a complete involutive set of $(m+1)[m/2]$ independent first integrals of the systems. The same holds for even dimension m with the only exception: the polynomial $\widetilde{\mathcal{I}}_m(s)$ is the full square of a polynomial $\mathcal{I}'_m(s)$ of degree $m/2$ in Z, P, which is the Pfaffian of $L(s)$. The coefficients of $\mathcal{I}'_m(s)$ are independent of each other and of the integrals $H_{k\nu}$ with $k = 2, \ldots, m-2$. The two major coefficients of $\mathcal{I}'_m(s)$ are again annulators of the bracket (2.1). Thus we again have a complete set of $(m+1)[m/2]$ independent first integrals in involution (see [5, 9]).

A generic symplectic leave of the Poisson bracket (2.1),

$$\mathcal{S}_{c,d} = \left\{ Z, P \mid H_{km}(P) = c_k, \ H_{k,m-1}(Z,P) = d_k, \qquad k = 2, 4, \ldots, 2[m/2] \right\}$$

can be regarded as orbits of the coadjoint action (2.11) and has dimension $m(m-1) - 2[m/2]$. As follows from (2.13), this is twice the number of the rest of the involutive integrals $H_{k,m-2}(P, Z), \ldots, H_{k,0}(Z)$. As a result, the dimension of generic invariant tori of the Steklov–Lyapunov systems equals $m(m-1)/2 - [m/2]$.

7

Special rank r orbits. Apart from generic coadjoint orbits there exists a hierarchy of lower-dimensional orbits. As follows from the Hamilton equations (2.2), on each coadjoint orbit the rank of P is constant. We consider coadjoint orbits in $so(m) \times so(m)$ passing through the matrices (Z^*, P^*) of the form

$$P^* = e_1 \wedge e_2 + \cdots + e_{r-1} \wedge e_r, \quad Z^* = \sum_{l=1}^{r} e_l \wedge u_l, \qquad (2.17)$$

where $e_1, \ldots, e_r$ are any mutually orthogonal vectors and $u_1, \ldots, u_r$ are arbitrary generic vectors in $\mathbb{R}^m$. Such orbits will be called *special rank r orbits* $\mathcal{S}_{c,d}^r$. As follows from expressions (2.14), all the higher Casimir functions

$$H_{r+2,m}(P), \ H_{r+2,m-1}(Z, P), \ \ldots, \quad H_{2[m/2],m}(P), \ H_{2[m/2],m-1}(Z, P)$$

equal zero for $(Z, P) = (Z^*, P^*)$ and, therefore, on the whole orbits. Hence, special rank r orbits are parameterized by values of the rest of the Casimir functions $H_{k,m}(P), H_{k,m-1}(Z, P)$ of order $\leq r$.

Remark 2. If all the Casimir functions $H_{2,m}(P), \ldots, H_{2[m/2],m}(P)$ equal zero, one has $P = 0$. As follows from (2.11), in this special case the orbits $\mathcal{S}_{c,d}^r$ reduce to rank r orbits $\mathcal{O}_{so(m)}^r$ of coadjoint action of $SO(m)$ on $so^*(m) = \{Z\}$, and equations (2.2) reduce to the Hamilton equations $\dot{Z} = [Z, \partial \mathcal{H}/\partial Z]$. In the sequel we exclude this case from consideration.

Proposition 2.1 *1). Generic* **????** *orbits $\mathcal{S}_{c,d}^r$ has dimension $2r(m - 1 - r/2)$, which is twice the dimension of rank r orbits $\mathcal{O}_{so(m)}^r$.*

2). On them the invariant polynomials $\widetilde{\mathcal{I}}_{2r+2}(s), \ldots, \widetilde{\mathcal{I}}_{2[m/2]}(s)$ vanish identically.

3). A complete set of independent nonzero first integrals and Casimir functions are given by the coefficients of the polynomials

$$\widetilde{\mathcal{I}}_k(s) = \begin{cases} \displaystyle\sum_I \frac{\Phi(s)}{(s - b_{i_1}) \ldots (s - b_{i_k})} |Z + sP|_I^I, & k = 2, \ldots, r, \\[2em] \displaystyle\sum_I \frac{\Phi(s)}{(s - b_{i_1}) \ldots (s - b_{i_k})} \sum_{j=0}^{g-k} |s^j P^j, Z^{k-j}|_I^I & k = r + 2, \ldots, g, \end{cases}$$

$$\qquad (2.18)$$

$$g = \min\{2r, 2[m/2]\}, \quad \Phi(\lambda) = (\lambda - b_1) \cdots (\lambda - b_m),$$

where k is even, $|s^j P^j, Z^{k-j}|_I^I$ denotes the diagonal minor of order k with $I = \{i_1 \ldots i_k\}$ that contains products of j components of P and $k - j$ components of Z.

4). These polynomials provide $r(m - 1 - r/2)$ independent and involutive first integrals, hence $\mathcal{S}_{c,d}^r$ are foliated with $r(m - 1 - r/2)$-dimensional invariant tori.

Sketch of the proof. 1). According to Remark 1, in the matrix variables (M, P) the orbits $\mathcal{S}^r_{c,d}$ are coadjoint orbits in the dual to the semi-direct product $so(m) \times_s so(m)$. The latter orbits thus pass through the matrices $(M^* = Z^* - \frac{1}{2}(BP^* + P^*B), P^*)$, which have the same structure as (Z^*, P^*) above. Due to the form of the coadjoint action (2.11), under the projection $(M, P) \to P$ the orbit $\mathcal{S}^r_{c,d}$ is mapped onto the coadjoint orbit $\mathcal{O}^r_{so(m)} \subset so^*(m)$, which is the factor variety $SO(m)/(SO(m-r) \times \mathbb{T}^{r/2})$ of dimension $r(m-1-r/2)$. Next, let $St(P^*) \in so(m)$ be the stabilizer of $P^* \in so(m)$ and $\hat{M}^*$ be the projection of M^* onto $St(P^*)$. In the orthogonal basis $\{e_1, \dots, e_r, \dots, e_n\}$ these sets have the following block matrix structure

$$St(P^*) = \begin{pmatrix} \mathbf{T} & \mathbf{0} \\ \mathbf{0}^T & so(m-r) \end{pmatrix}, \quad \hat{M}^* = \begin{pmatrix} \mathbf{T} & \mathbf{0} \\ \mathbf{0}^T & \mathbf{O} \end{pmatrix},$$

where $\mathbf{T}$ ranges over $\mathbb{T}^{r/2}$ and $\mathbf{O}$ denotes a zero $(m-r) \times (m-r)$ matrix. According to the Raís formula (see, e.g., [10]),

$$\dim \mathcal{S}^r_{c,d} = \dim \mathcal{O}^r_{so(m)} + \dim SO(m) - \dim \mathrm{Ann}_{St(P^*)} \hat{M}^*,$$

the last term being the dimension of the annihilator of $\hat{M}^*$ in $St(P^*)$, which equals $\dim so(m-r) + r/2$. T his gives the dimension of $\mathcal{S}^r_{c,d}$ stated by the proposition.

2). Next, we note that the variety $U^r \subset so(m) \times so(m)$ of the matrices Z, P that can be represented in the form (2.17) in an appropriate basis $\{e_1, \dots, e_r, \dots, e_n\}$ has dimension $2r(m-1-r/2) + r$, and therefore the subvariety of U^r consisting of the pairs Z, P with the same Casimir functions is $2r(m-1-r/2)$-dimensional. This shows that actually *all* the points of $\mathcal{S}^r_{c,d}$ admit representation in the form (2.17).

3). Now we evaluate the first integrals given by (2.13) on the matrices Z^*, P^*. First, note that for any s, rank $|Z^* + sP^*| \leq 2r$, hence

$$\widetilde{\mathcal{I}}_{2r+2}(s, Z^*, P^*) = 0, \quad \dots, \quad \widetilde{\mathcal{I}}_{2[m/2]}(s, Z^*, P^*) = 0.$$

Further, one can show that for $r < k \leq g$, the minors $|Z^* + sP^*|^I_I$ must contain at least $2(k-r)$ nonzero components of Z^*, hence the minors have at most degree $2r - k$ in s and in the components of P^*. Finally, for $2 \leq k \leq r$, there are no restrictions on the degree of the polynomials $|Z^* + sP^*|^I_I$, and all their coefficients are generally nonzero. As a result, the integrals given by (2.13) take the form (2.18) on the entire orbit $\mathcal{S}^r_{c,d}$. The latter formula provides $r(m-1-r/2)$ nonzero nontrivial integrals, which is the maximal number of independent integrals in involution on the orbit. The proposition is proved.

Remark 3. The orbits $\mathcal{S}^r_{c,d}$ contain invariant subvarieties

$$\mathcal{F}^r_{c,d} = \{(Z, P) \in \mathcal{S}^r_{c,d} \mid \forall s \in \mathbb{R}, \quad \mathrm{rank} \; |Z + sP| = r\}.$$

On $\mathcal{F}^r_{c,d}$ the higher order invariant polynomials $\widetilde{\mathcal{I}}_{r+2}(s, Z, P), \dots, \widetilde{\mathcal{I}}_g(s, Z, P)$ are identically zero. Then

$$\dim \mathcal{F}^r_{c,d} = \dim \mathcal{S}^r_{c,d} - \text{number of the coefficients of } \widetilde{\mathcal{I}}_{r+2}(s), \dots, \widetilde{\mathcal{I}}_g(s) \text{ in (2.18)},$$

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which equals $r(3m/2 - 3/2 - r/2)$. Let $\bar{\omega}_s$ be the 2-form on $\mathbb{R}^m$ with the components $(Z + sP)_{ij}$. Then, for $r < m$ and a fixed s, the components of the r-form $\bar{\omega}_s^{r/2}$ can be regarded as Plücker coordinates of an r-plane passing through the origin in $\mathbb{R}^m$, whereas the family of such linear spaces parameterized by s is *a pencil* of r-planes $\mathcal{L}$ having a common $r/2$-plane $\mathfrak{P}$, the focus of $\mathcal{L}$.

3 Flows on the matrix triplet variety

Let $\mathcal{W}^r$ be a union of the subvarieties $\mathcal{F}_{c,d}^r$ corresponding to all nonzero Casimir functions given by (2.18). As follows from above,

$$\dim \mathcal{W}^r = \dim \mathcal{F}_{c,d}^r + r = \frac{3}{2}mr - \frac{r}{2} - \frac{r^2}{2}.$$

There exist $r/2$ triples of vectors $x^{(l)}, y^{(l)}, v^{(l)} \in \mathbb{R}^n$, $l = 1, \ldots, r/2$ such that any point of $\mathcal{W}^r$ can be represented in form

$$\forall s \in \mathbb{C}, \quad Z + sP = \sum_{l=1}^{r/2} x^{(l)} \wedge (y^{(l)} + sv^{(l)}) \equiv \mathcal{X}^T(\mathcal{Y} + s\mathcal{V}) - (\mathcal{Y} + s\mathcal{V})^T \mathcal{X}, \quad (3.1)$$

where $\mathcal{X}, \mathcal{Y}, \mathcal{V}$ are $r/2 \times m$ matrices,

$$\mathcal{X}^T = (x^{(1)} \cdots x^{(r/2)}), \quad \mathcal{Y}^T = (y^{(1)} \cdots y^{(r/2)}), \quad \mathcal{V}^T = (v^{(1)} \cdots v^{(r/2)}).$$

(Notice that the linear span of $x^{(1)}, \ldots, x^{(r/2)}$ gives the above $r/2$-dimensional focus $\mathfrak{P}$ of $\mathcal{L}$.) It is seen that for a generic pair Z, P, such vectors are not unique. In particular, under the transformations

$$y^{(l)} \to y^{(l)} + \tau_l x^{(l)}, \quad v^{(l)} \to v^{(l)} + \delta_l x^{(l)}, \qquad \text{for any} \quad \tau_l, \delta_l \in \mathbb{R}$$

Z, P remain unchanged. To get rid of the ambiguity, we introduce constraint submanifold

$$\mathcal{T}^r = \{\mathcal{X}, \mathcal{Y}, \mathcal{V} \,|\, \mathcal{X}\mathcal{X}^T = \mathbf{I}, \quad \mathcal{V}\mathcal{X}^T = 0, \quad \mathcal{X}[\mathcal{Y} + B\mathcal{V}]^T + [\mathcal{Y} + B\mathcal{V}]\mathcal{X}^T = 0\}, \quad (3.2)$$

which is defined by $\frac{r}{2} + \frac{r^2}{2}$ scalar constraint equations in $\mathbb{R}^{3mr/2}$ and therefore has the same dimension as $\mathcal{W}^r$. (We shall refer to it as the *matrix triplet variety*.) Then a complete preimage of a generic point of $\mathcal{W}^r$ in $\mathcal{T}^r$ is a discrete orbit of the group $\mathfrak{R}$ generated by reflections

$$(x^{(l)}, y^{(l)}, v^{(l)}) \to (-x^{(l)}, -y^{(l)}, -v^{(l)}), \qquad l = 1, \ldots, r/2.$$

The main observation of this section is that *the restriction of the Steklov–Lyapunov systems on $\mathcal{W}^r$ can be described as dynamical systems on $\mathcal{T}^r$, which admit $r \times r$ matrix Lax pairs with a rational parameter.*

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As model systems, we take equations (2.7), which are described by the quadratic Hamiltonians

$$\frac{1}{2}\mathfrak{h}_{m-2-\rho} = \frac{1}{2}\langle Z, \{Z, B^\rho\}\rangle + \langle Z, \{P, B^{\rho+1}\}\rangle + \frac{1}{2}\langle P, \{P, B^{\rho+2}\}\rangle, \qquad \rho = 0, 1, \ldots,$$

and can be represented in form

$$\begin{aligned} \dot{Z} &= [\, Z, \Omega_\rho \,] + PB^{\rho+1}Z - ZB^{\rho+1}P\,, \\ \dot{P} &= [\, P, \Omega_\rho \,]\,, \\ \Omega_\rho &= \frac{1}{2}\frac{\partial \mathfrak{h}_{m-2-\rho}}{\partial Z} = \{Z, B^\rho\} + \{P, B^{\rho+1}\} \in so(m). \end{aligned} \tag{3.3}$$

On the other hand, consider the following dynamical system on the variety $\mathcal{T}^r$

$$\begin{aligned} \dot{\mathcal{X}}^T &= -\Omega_\rho \mathcal{X}^T + P\mathcal{X}^T\,\mathcal{X}B^{\rho+1}\mathcal{X}^T \equiv -\Omega_\rho \mathcal{X}^T - \mathcal{V}^T \mathcal{X}B^{\rho+1}\,mathcalX^T, \\ \dot{\mathcal{V}}^T &= -\Omega_\rho \mathcal{V}^T + P\mathcal{V}^T\,\mathcal{X}B^{\rho+1}\mathcal{X}^T \equiv -\Omega_\rho \mathcal{V}^T + \mathcal{X}^T\,\mathcal{V}\mathcal{V}^T \mathcal{X}B^{\rho+1}\mathcal{X}^T, \\ \dot{\mathcal{Y}}^T &= -\Omega_\rho \mathcal{Y}^T + \mathcal{Y}^T\mathcal{X}B^{\rho+1}\mathcal{V}^T + PB^{\rho+1}\mathcal{Y}^T + \mathcal{X}^T\Xi_\rho \\ &\equiv -\Omega_\rho \mathcal{Y}^T + \mathcal{Y}^T\mathcal{X}B^{\rho+1}\mathcal{V}^T - \mathcal{V}^T\mathcal{X}B^{\rho+1}\mathcal{Y}^T + \mathcal{X}^T\mathcal{V}B^{\rho+1}\mathcal{Y}^T + \mathcal{X}^T\Xi_\rho, \end{aligned} \tag{3.4}$$

where Ξ_ρ is the $r/2 \times r/2$ symmetric matrix

$$\Xi_\rho = \mathcal{V}B^{\rho+1}\mathcal{Y}^T + \mathcal{Y}B^{\rho+1}\mathcal{V}^T - \mathcal{V}B^{\rho+2}\mathcal{V}^T + \frac{1}{2}(\Lambda + \Lambda^T),$$

$$\Lambda = \mathcal{X}B^{\rho+1}\mathcal{X}^T[\mathcal{V}\mathcal{Y}^T + \mathcal{Y}\mathcal{V}^T + \mathcal{V}B\mathcal{V}^T] + \mathcal{X}B^{\rho+2}\mathcal{X}^T\mathcal{V}\mathcal{V}^T - \mathcal{X}B\mathcal{X}^T\mathcal{V}\mathcal{V}^T\mathcal{X}B^{\rho+1}\mathcal{X}^T,$$

and where one must substitute the above expression for Ω_ρ and then the expressions (3.1).

The matrices Ξ_ρ are chosen in such a way that equations (3.4) preserve the constraints (4.4) and therefore indeed describe a flow on $\mathcal{T}^r$.

Theorem 3.1 *1). Under the substitution (3.1) solutions of the system (3.4) pass to rank r solutions of the multidimensional Steklov system (3.3).*

2). Up to the action of the discrete group generated by reflections $(\mathcal{X}, \mathcal{Y}, \mathcal{V}) \to (-\mathcal{X}, -\mathcal{Y}, -\mathcal{V})$ th e system (3.4) is described by the following Lax pair with $r \times r$ matrices and rational parameter λ

$$\dot{L}(\lambda) = [L(\lambda), A_\rho(\lambda)], \tag{3.5}$$

$$\begin{aligned} L(\lambda) &= \begin{pmatrix} \mathcal{X}(\lambda\mathbf{I} - B)^{-1}[\mathcal{Y} + \lambda\mathcal{V}]^T & \mathcal{X}(\lambda\mathbf{I} - B)^{-1}\mathcal{X}^T \\ -(\mathcal{Y} + \lambda\mathcal{V})(\lambda\mathbf{I} - B)^{-1}[\mathcal{Y} + \lambda\mathcal{V}]^T & -[\mathcal{Y} + \lambda\mathcal{V}](\lambda\mathbf{I} - B)^{-1}\mathcal{X}^T \end{pmatrix} \\ &= L_1\lambda + L_0 + \begin{pmatrix} \mathcal{X}(\lambda\mathbf{I} - B)^{-1}\mathcal{Z}^T & \mathcal{X}(\lambda\mathbf{I} - B)^{-1}\mathcal{X}^T \\ -\mathcal{Z}(\lambda\mathbf{I} - B)^{-1}\mathcal{Z}^T & -\mathcal{Z}(\lambda\mathbf{I} - B)^{-1}\mathcal{X}^T \end{pmatrix}, \end{aligned} \tag{3.6}$$

where $\mathcal{Z} = \mathcal{Y} + B\mathcal{V}$,

$$L_1 = \begin{pmatrix} 0 & 0 \\ -\mathcal{V}\mathcal{V}^T & 0 \end{pmatrix}, \quad L_0 = \begin{pmatrix} \mathcal{X}\mathcal{V}^T & 0 \\ -\mathcal{V}B\mathcal{V}^T - \mathcal{V}\mathcal{Y}^T - \mathcal{Y}\mathcal{V}^T & -\mathcal{V}\mathcal{X}^T \end{pmatrix},$$

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and

$$A(\lambda)_\rho = \begin{pmatrix} \mathcal{X}\,\mathcal{B}(\lambda)\,[\mathcal{Y}+\mathcal{V}B]^T & \mathcal{X}\,\mathcal{B}(\lambda)\,\mathcal{X}^T \\ -[\mathcal{Y}+\lambda\mathcal{V}]\,\mathcal{B}(\lambda)\,[\mathcal{Y}+\lambda\mathcal{V}]^T+??? & -[\mathcal{Y}+\mathcal{V}B]\,\mathcal{B}(\lambda)\,\mathcal{X}^T \end{pmatrix},$$
$$\mathcal{B}(\lambda) = \lambda^\rho\mathbf{I} + \lambda^{\rho-1}B + \cdots + B^\rho.$$

3). *The coefficients of the characteristic polynomial* $|\Phi(\lambda)\,L(\lambda)-w\mathbf{I}|$ *are functions of the right hand sides of* (3.1), *and they can be expressed only in terms of* Z_{ij}, P_{ij} *as follows*

$$|w\mathbf{I} - \Phi(\lambda)\,L(\lambda)| = w^r + \sum_l w^{r-l}\Phi^{l-1}(\lambda)\,\tilde{\mathcal{I}}_l(\lambda, Z, P), \qquad l = 2, 4, \ldots, r,$$

$$\Phi(\lambda) = (\lambda - b_1)\cdots(\lambda - b_m),$$

thus giving all nonzero invariant polynomials $\tilde{\mathcal{I}}_2(\lambda), \ldots, \tilde{\mathcal{I}}_r(\lambda)$ *on* $\mathcal{W}^r$.

In view of Theorem 3.1 one can say that (3.5) *is a Lax representation with a rational parameter for multidimensional Steklov–Lyapunov systems restricted onto* $\mathcal{W}^r \subset so(m) \times so(m)$. Notice that, according to item 3, for $Z, P \in \mathcal{W}^r$, the spectral curve of the hyperelliptic Lax pair (2.5) is birationally equivalent to that of the rational Lax pair (3.5).

Proof of Theorem 3.1. 1). Differentiating left and right hand sides of (3.1) by virtue of equations (3.3) and (3.4) respectively, we find that both derivatives coincide under the substitution (3.1).

2). We differentiate $L(\lambda)$ along the flow of the system (3.4). In view of matrix relations in (3.2) and the identity $(\lambda\mathbf{I}-B)^{-1}B = \lambda(\lambda\mathbf{I}-B)^{-1}-\mathbf{I}$, the result coincides with the commutator in (3.5).

3). First, notice that $L(\lambda) \in sp\,(r/2)$, hence all the odd-order diagonal minors of $L(\lambda)$ equal zero. The sum of all the diagonal minors of even order k of $\Phi(\lambda)L(\lambda)$ can be represented in the form

$$\Phi^k(\lambda) \sum_I \frac{1}{(\lambda - b_{i_1})\ldots(\lambda - b_{i_k})} \left(\sum \mathcal{M}_{i_1 i_2}\cdots\mathcal{M}_{i_{k-1}i_k}\right)^2,$$

where

$$\mathcal{M}_{ij} = \sum_{s=1}^{r/2} \left(x_i^{(s)}(y_j^{(s)} + \lambda v_j^{(s)}) - x_j^{(s)}(y_i^{(s)} + \lambda v_i^{(s)})\right),$$

$$\{i_1 < \cdots < i_k\} \subset \{1, \ldots, m\},$$

which, in view of (3.1) and (2.13), coincides with the polynomial $\Phi^{k-1}(\lambda)\,\tilde{\mathcal{I}}_k(\lambda)$. This establishes item 3 of Theorem 3.1. □

4   Reductions in the rank 2 case

Now we consider in detail the simplest case of the motion on rank 2 coadjoint orbits $\mathcal{S}_{c,d}^2 \subset so(m) \times s(m)$, which nevertheless are generic in the classical problem $m = 3$. As follows from Proposition 2.1, these orbits are $4(m-2)$-dimensional and on them all the invariant polynomials $\widetilde{\mathcal{I}}_6(s, Z, P), \ldots, \widetilde{\mathcal{I}}_g(s, Z, P)$ and two leading coefficients of $\widetilde{\mathcal{I}}_4(s, Z, P)$ (Casimir functions) are identically zero. According to (2.18), the set of $2(m-2)$ nonzero independent integrals and two quadratic Casimir functions is given by the coefficients of the polynomials

$$
\begin{aligned}
\widetilde{\mathcal{I}}_2(s, Z, P) &= \sum_{1 \le i < j \le m} \frac{\Phi(s)}{(s - b_i)(s - b_j)} \, (Z + sP)^2 = \sum_{\mu=0}^{m} s^\mu H_{2\mu}(Z, P), \\
\widetilde{\mathcal{I}}_4(s, Z) &= \sum_{I} \frac{\Phi(s)}{(s - b_{i_1}) \ldots (s - b_{i_4})} \, |Z|_I^I = \sum_{\mu=0}^{m-4} s^\mu H_{4\mu}(Z),
\end{aligned}
\tag{4.1}
$$

where now $I = \{i_1 \cdots i_4\}$, $i_1 < \cdots < i_4$. The subvariety

$$
\mathcal{F}_{c,d}^2 = \{(Z, P) \in \mathcal{S}_{c,d}^2 \mid \forall s \in \mathbb{R}, \ \operatorname{rank}|Z + sP| = 2\}
$$

is obtained by fixing to zero $m - 3$ quartic Hamiltonians $H_{4,0}, \ldots, H_{4,m-4}$. Thus $\mathcal{F}_{c,d}^2$ has dimension $3m - 5$. Equivalently, $\mathcal{F}_{c,d}^2$ can be defined as the intersection of the orbit $\mathcal{S}_{c,d}^2$ with the quadrics

$$
\left\{ Pf(|Z|_I^I) \equiv Z_{i_1 i_2} Z_{i_3 i_4} - Z_{i_1 i_3} Z_{i_4 i_2} + Z_{i_2 i_3} Z_{i_1 i_4} = 0 \right\},
$$

$Pf(|Z|_I^I)$ being the Pfaffian of the 4×4 determinant $|Z|_I^I$.

On $\mathcal{F}_{c,d}^2$ and on $\mathcal{W}^2$ the flows generated by the quartic Hamiltonians $H_{4,0}, \ldots, H_{4,m-4}$ are zero. Instead, we consider the flows of the *quadratic* Hamiltonians $\mathcal{H}_I = Pf(|Z|_I^I)$, which, in view of equations (2.2), have the simple matrix form

$$
Z' = B\hat{Z}_I P - P\hat{Z}_I B, \quad P' = P\hat{Z}_I - \hat{Z}_I P, \qquad (\hat{Z}_I)_{ij} = \frac{\partial Pf\left(|Z|_I^I\right)}{\partial Z_{ij}}.
\tag{4.2}
$$

One can show by hand that for any 4-indices I, J, the Poisson bracket $\{\mathcal{H}_I, \mathcal{H}_J\}_1$ is a linear combination of the functions $\mathcal{H}_I$, hence on $\mathcal{W}^2$ they commute with each other. As we shall see later (item 3 of Theorem 4.6), all $\mathcal{H}_I$ also commute with the coefficients of $\widetilde{\mathcal{I}}_2(s, Z, P)$.

Notice that the corresponding flows (4.2) do not commute even on $\mathcal{W}^2$! In the sequel we denote these flows by $\mathcal{P}_I$.

Special Poisson Reduction. Below we are going to make a kind of reduction with respect to the flows $\mathcal{P}_I$, which is similar to the classical Marsden–Weinstein reduction by an action of a finite-dimensional Lie group. However, in our case there is no action and, moreover, the integrals of the system into consideration ($\mathcal{H}_I$) are not general, but partial. That is why we want now to describe briefly our reduction procedure from a more abstract point of view.

Theorem 4.1 *Suppose we have a Hamiltonian system $\dot{x} = X_H(x)$ on a symplectic manifold $(\mathcal{M}, \omega)$ and there are k functions satisfying the following properties:*

1) *the common level $L = \{f_1 = 0, \ldots, f_k = 0\}$ is a smooth submanifold of codimension k; in particular, the differentials of these functions are linearly independent on L,*

2) *the Hamiltonian vector fields $X_{f_1}, \ldots, X_{f_k}$ are all tangent to L or, which is the same, $\{f_i, f_j\} = 0$ on L,*

3) *$\{f_i, H\} = 0$ on L for every $i = 1, \ldots, k$.*

Then

1) *The distribution on L generated by $X_{f_1}, \ldots, X_{f_k}$ is integrable and, therefore, it generates a foliation ρ on L of dimension k.*

2) *For the case of compact leaves of the foliation, the quotient space L/ρ obtained by identifying each leaf into a point has a natural symplectic structure and the initial Hamiltonian system $\dot{x} = X_H(x)$ can naturally be reduced onto L/ρ.*

The precise description of the symplectic structure on L/ρ is given in terms of a reduced Poisson bracket as follows. Let g, h be two arbitrary smooth functions on L/ρ. These functions are naturally identified with functions $\tilde{g}, \tilde{h}$ on L which are constant on the leaves of ρ. To define the bracket $\{g, h\}$ we simply want to take the bracket of $\tilde{g}$ and $\tilde{h}$. But to do so we need to extend $\tilde{g}$ and $\tilde{h}$ from L to the whole $\mathcal{M}$ because on L there is no natural Poisson structure. Let $\hat{g}, \hat{h}$ be any smooth functions on $\mathcal{M}$ such that $\tilde{g} = \hat{g}|_L$, $\tilde{h} = \hat{h}|_L$.

Proposition 4.2 *1) The restriction of $\{\hat{g}, \hat{h}\}$ onto L does not depend on the choice of $\hat{g}$ and $\hat{h}$;*

2) $\{\hat{g}, \hat{h}\}|_L$ is a first integral of the Hamiltonian flows $X_{f_1}, \ldots, X_{f_k}$, i.e., it is constant on the leaves of the foliation ρ and, therefore, can be considered as a function on the reduced space L/ρ.

The function so obtained is, by definition, the (reduced) Poisson bracket $\{g, h\}_{\mathrm{red}}$ on L/ρ. It is easy to see that this structure is non-degenerate, so L/ρ obtains a natural symplectic structure. Since the original Hamiltonian H is invariant with respect to X_{f_i}, the reduced Hamiltonian on L/ρ and the corresponding reduced Hamiltonian system are correctly defined.

Proof of Theorem 4.1. Since $\{f_i, f_j\} \equiv 0$ on L, the differential of the bracket $\{f_i, f_j\}$ considered as a function on $\mathcal{M}$ is a linear combination of $df_1(x), \ldots, df_k(x)$ at each point $x \in L$. Hence, for $x \in L$ we have

$$[X_{f_i}, X_{f_j}](x) = -X_{\{f_i, f_j\}}(x) = -\omega^{-1}(d\{f_i, f_j\}(x))$$
$$= -\omega^{-1}\left(\sum c_l(x) df_l(x)\right) = \sum c_l(x) X_{f_l}(x).$$

Notice that this relation takes place only on L and nowhere else in general. Thus the Frobenius integrability condition holds, which establishes item 1). Item 2) follows from Proposition 4.2.

Proof of Proposition 4.2. 1) Let $\hat{g}$, $\hat{g}'$ be two different functions both satisfying $\hat{g}|_L = \tilde{g}$, $\hat{g}'|_L = \tilde{g}$. To show that $\{\hat{g}, \hat{h}\}|_L = \{\hat{g}', \hat{h}\}|_L$ it suffices to verify that $\{\hat{g} - \hat{g}', \hat{h}\}|_L = 0$. We now use the fact that the function $\hat{g} - \hat{g}'$ is identically zero on L. This implies that at each point $x \in L$, $d(\hat{g} - \hat{g}')$ is a linear combination of $df_1, \ldots, df_k$. Hence,

$$\{\hat{g} - \hat{g}', \hat{h}\}(x) = -\langle d(\hat{g} - \hat{g}')(x), X_{\hat{h}}(x)\rangle = -\left\langle \sum_{l=1}^{k} c_l(x) df_l(x), X_{\hat{h}}(x)\right\rangle =$$

$$= -\sum_{l=1}^{k} c_l(x)\langle df_l(x), X_{\hat{h}}(x)\rangle = -\sum c_l(x)\{f_l, \hat{h}\}(x),$$

$c_l(x)$ being certain functions. Now, since $\{f_l, \hat{h}\}|_L = 0$ for any $1 \le l \le k$, we obtain the required result.

2) It remains to show that the function $\{\hat{g}, \hat{h}\}|_L$ is invariant under the flows $X_{f_1}, \ldots, X_{f_k}$. This is equivalent to conditions $\{f_i, \{\hat{g}, \hat{h}\}\}|_L \equiv 0$. We have

$$\{f_i, \{\hat{g}, \hat{h}\}\} = -\{\hat{g}, \{\hat{h}, f_i\}\} + \{\hat{h}, \{\hat{g}, f_i\}\}.$$

Since $\{\hat{h}, f_i\}|_L \equiv 0$ and $\{\hat{g}, f_i\}|_L \equiv 0$, we arrive at item 2).

In the above construction we assumed the functions $f_1, \ldots, f_k$ to be independent on L. However, everything can be repeated under the weaker assumption that the submanifold L is coisotropic or, which is the same, $\operatorname{codim} L = \operatorname{corank}(\omega|_{TL})$.

Below we apply this construction in our case. As the symplectic manifold $\mathcal{M}$ and its submanifold L we shall consider the rank 2 orbit $S^2_{c,d}$ and the common level surface of the Pfaffians $Pf(|Z|)^I_I$ respectively.

Steklov–Lyapunov flows and the flows $\mathcal{P}_I$ on $\mathcal{T}^2$. In the rank 2 case, the matrices $\mathcal{X}^T, \mathcal{Y}^T, \mathcal{V}^T$ in relations (3.1) become just vectors x, y, v, whereas the relations themselves take the form

$$Z = x \wedge y, \quad P = x \wedge v, \qquad x, y, v \in \mathbb{R}^m. \tag{4.3}$$

The constraint submanifold $\mathcal{T}^2 \in \mathbb{R}^{3m}$ is defined by three conditions

$$(x, x) = 1, \quad (x, v) = 0, \quad (x, y + Bv) = 0. \tag{4.4}$$

Notice that in view of (4.3), $x_1, \ldots, x_m$ become homogeneous coordinates of the focus of pencil of lines $\mathcal{L} = \{Z + sP\}$ in $\mathbb{P}^n$ ($n = m - 1$).

The formulas (4.3) can be inverted to give a pair of points on $\mathcal{T}^2$ in view of the following proposition.

Proposition 4.3 *Let* $\mathfrak{w}_{123} \subset \mathcal{W}^2$ *be a domain defined by the conditions* $Z_{\alpha\beta} = P_{\alpha\beta} = 0$, $\alpha, \beta = 1, 2, 3$. *Then the redundant coordinates* x, y, v *can be expressed in terms of* Z, P *on the open subset* $\mathcal{W}^2 \setminus \mathfrak{w}_{123}$ *as follows*

$$x = \pm\bar{x}^{(123)} / \left| \underline{a}rx^{(123)} \right|, \quad v = -Px, \quad y = (-Z + (x, BPx)) x, \qquad (4.5)$$

where

$$\begin{aligned}
\bar{x}_1^{(123)} &= Z_{12}P_{13} - Z_{13}P_{12}\,, \\
\bar{x}_2^{(123)} &= Z_{23}P_{21} - Z_{21}P_{23}\,, \\
\bar{x}_3^{(123)} &= Z_{31}P_{32} - Z_{32}P_{31}\,, \\
\bar{x}_j^{(123)} &= -(Z_{12}P_{3j} - Z_{13}P_{2j} + Z_{23}P_{1j})\,, \qquad j = 4, \ldots, m\,.
\end{aligned} \qquad (4.6)$$

Expressions on other open subsets $\mathcal{W}^2 \setminus \mathfrak{w}_{\alpha\beta\gamma}$ *are obtained from (4.6) by the corresponding permutation of indices.*

Note that in the classical case $m = 3$, in the vector variables (2.10) the above expressions take the form

$$\begin{aligned}
x &= \frac{1}{\gamma} z \times p, \quad v = x \times p = \frac{1}{\gamma}[(p,z)p - (p,p)z], \\
y &= x \times z - (Bx, x \times p)x = \frac{1}{\gamma}[(z,z)p - (z,p)z] \qquad\qquad\qquad (4.7) \\
&\quad - \frac{1}{\gamma^3}\left(B(z \times p), (p,z)p - (p,p)z\right) z \times p, \qquad \gamma = |z \times p|.
\end{aligned}$$

Relations (4.3) and (4.5), (4.6) establish a two-to-one correspondence between $\mathcal{T}^2$ and $\mathcal{W}^2$: the triples x, y, v and $-x, -y, -v$ are mapped to the same pair Z, P.

Proof of Proposition 4.3. The formulas (4.6) can be checked by direct calculations. Their geometric proof is the following. Let $(X_1 : \cdots : X_m)$ be homogeneous coordinates in the projective space $\mathbb{P}^{m-1}$ and $Y_2 = X_2/X_1, \ldots, Y_m = X_m/X_1$ be Cartesian coordinates in $\mathbb{C}^{m-1} = \mathbb{P}^{m-1}\setminus\{X_1 = 0\}$. Now let $\ell_1, \ell_2 \subset \mathbb{P}^{m-1}$ be lines with Plücker coordinates Z_{ij}, P_{ij} respectively. Then their affine parts in $\mathbb{C}^{m-1}$ can be described in parametric form

$$\left\{ Y_i(\tau) = Z_{i1}\tau + \sum_{k=2}^{m} Z_{ik}Z_{k1} \,\middle|\, \tau \in \mathbb{C} \right\}, \quad mboxrespectively$$

$$\left\{ Y_i(\tau') = P_{i1}\tau' + \sum_{k=2}^{m} P_{ik}P_{k1} \,\middle|\, \tau' \in \mathbb{C} \right\}, \quad i = 2, \ldots, m. \qquad (4.8)$$

Without loss of generality, here we assume that $\sum_{i=2}^{m} Z_{1i}^2 = \sum_{i=2}^{m} P_{1i}^2 = 1$. According to the condition rank $|Z + sP| = 2$, the two lines intersect at a point (the focus of the pencil $\mathcal{L}$), whose homogeneous coordinates in $\mathbb{P}^{m-1}$ give the components of x up to a common factor. Matching the right hand sides of the expressions in (4.8) and

using the above normalization conditions, we find the values of τ, τ' corresponding to the intersection point, and, after some calculations, the expressions (4.6).

The formulas (4.5) are then obtained by applying the second and third conditions in (4.4). The proposition is proved. $\square$

It appears that the flows $\mathcal{P}_I$ on $\mathcal{W}^2$ generated by the quadratic Hamiltonians $\mathcal{H}_I = Pf(|Z|_I^I)$ do not change the focus of the pencil of lines $\mathcal{L}$.

Proposition 4.4 *In vector variables x, y, v on $\mathcal{T}^2$ the flows (4.2) have the form*

$$x' = 0, \quad v' = -\hat{Z}_I v, \quad y' = B\hat{Z}_I v, \qquad (\hat{Z}_I)_{ij} = \frac{\partial Pf(|Z|_I^I)}{\partial Z_{ij}}, \qquad (4.9)$$

where one must substitute $Z = x \wedge y$.

One can check that these flows preserve the constraints (4.4) and therefore are indeed flows on $\mathcal{T}^2$.

Sketch of a proof of Proposition 4.4. First, note that the condition rank $|Z + sP| = 2$ for any $s \in \mathbb{R}$ implies

$$\underset{\varkappa=0}{Res} \, Pf(|Z + \varkappa^{-1}P|_I^I) \equiv Z_{i_1 i_2}P_{i_3 i_4} - Z_{i_1 i_3}P_{i_4 i_2} + Z_{i_2 i_3}P_{i_1 i_4}$$

$$+ P_{i_1 i_2}Z_{i_3 i_4} - P_{i_1 i_3}Z_{i_4 i_2} + P_{i_2 i_3}Z_{i_1 i_4} = 0 \qquad (4.10)$$

for $i_1 < i_2 < i_3 < i_4$. Calculating the derivatives of the homogeneous coordinates $\bar{x}_i$ in (4.6) with respect to any of the flows given by (4.2) and using the conditions $\mathcal{H}_I = 0$ and (4.10), we find that the vector $\bar{x}'$ is a linear combination of alternative expressions for $\bar{x}$ obtained from the right hand sides of (4.6) by various permutations of indices. In particular,

$$\left\{\bar{x}_i^{(123)}, \mathcal{H}_{1234}\right\}_1 = (Z_{12} + b_3 P_{12})\bar{x}_i^{(124)} + (Z_{13} + b_2 P_{13})\bar{x}_i^{(134)} + (Z_{23} + b_1 P_{23})\bar{x}_i^{(234)},$$

$$i = 1, \ldots, m.$$

This implies that $\bar{x}'$ is collinear to $\bar{x}$, hence the normalized vector x is constant.

Next, we substitute expressions (4.3) into the Hamilton equations (4.2) and take into account $x' = 0$. As a result, comparing coefficients at different components of x_i, we arrive at two last equations in (4.9), which proves the proposition. $\square$

Theorem 4.5 *The variables x_i commute with respect to the Poisson bracket (2.1), i.e., $\{x_i, x_j\}_1 = 0$.*

Proof. Since $\{x_i, \mathcal{H}_I\}_1$ for any i, from the Jacobi identity we have

$$\{\{x_i, x_j\}_1, \mathcal{H}_I\}_1 = -\{\{x_j, \mathcal{H}_I\}_1, x_i\} - \{\{\mathcal{H}_I, x_i\}_1, x_j\}_j = 0.$$

For $m = 3$, when the flows $\mathcal{P}_I$ do not exist, the proof is direct. Namely, from the vector expressions (4.7) we have

$$\frac{\partial x_i}{\partial z_\alpha} = -\frac{x_\alpha}{\gamma}[p \times x]_i, \quad \frac{\partial x_i}{\partial p_\alpha} = \frac{x_\alpha}{\gamma}[z \times x]_i, \qquad \alpha = 1, 2, 3. \qquad (4.11)$$

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Substituting this into the vector analog of Hamiltonian equations (2.2) with $\mathcal{H} = x_i$, we obtain

$$z' = -\frac{[p \times x]_i}{\gamma}(z - Bp) \times x + \frac{[z \times x]_i}{\gamma}p \times x, \quad p' = -\frac{[p \times x]_i}{\gamma}p \times x,$$

where now prime denotes the derivative with respect to the flow with the Hamiltonian x_i. Hence, in view of (4.11),

$$\{x_i, x_j\}_1 = \left(\frac{\partial x_j}{\partial z}, z'\right) + \left(\frac{\partial x_j}{\partial p}, p'\right) = 0.$$

The theorem is proved.

The systems (3.4) on $\mathcal{T}^2$ take the form

$$\dot{x} = -\Omega_\rho x + (x, B^{\rho+1}x)\, Px,$$
$$\dot{v} = -\Omega_\rho v + (x, B^{\rho+1}x)\, Pv, \qquad (4.12)$$
$$\dot{y} = -\Omega_\rho y + (x, B^{\rho+1}v)y - (x, B^{\rho+1}y)v + (y, B^{\rho+1}v)x + \chi_\rho x,$$

where

$$\Omega_\rho = \{Z, B^\rho\} + \{P, B^{\rho+1}\}, \qquad P = x \wedge v, \quad Z = x \wedge y, \quad \rho \in \{0 \cup \mathbb{N}\},$$
$$\chi_\rho = 2(y, v)(x, B^{\rho+1}x) - 2(y, B^{\rho+1}v)(x, x) - (x, x)(v, B^{\rho+2}v)$$
$$+ (v, v)(x, B^{\rho+2}x) - (x, B^{\rho+1}x)[(v, v)(x, Bx) - (x, x)(v, Bv)],$$

and they admit 2×2 matrix Lax representation, which comes from (3.5),

$$\dot{L}(\lambda) = [L(\lambda), A_\rho(\lambda)], \qquad \lambda \in \mathbb{C}, \qquad (4.13)$$

$$L(\lambda) = \sum_{i=1}^{m} \frac{1}{\lambda - b_i}\begin{pmatrix} x_i(y_i + \lambda v_i) & x_i^2 \\ -(y_i + \lambda v_i)^2 & -x_i(y_i + \lambda v_i) \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ -(v, v) & 0 \end{pmatrix}\lambda + \begin{pmatrix} 0 & 0 \\ -(v, Bv) - 2(v, y) & 0 \end{pmatrix}$$

$$+ \sum_{i=1}^{m} \frac{1}{\lambda - b_i}\begin{pmatrix} x_i(y_i + b_i v_i) & x_i^2 \\ -(y_i + b_i v_i)^2 & -x_i(y_i + b_i v_i) \end{pmatrix}, \qquad (4.14)$$

$$A_\rho(\lambda) = \begin{pmatrix} (x, \mathcal{B}(\lambda)(y + Bv)) & (x, \mathcal{B}(\lambda)x) \\ -Q_\rho(\lambda) & -(x, \mathcal{B}(\lambda)(y + Bv)) \end{pmatrix}, \qquad (4.15)$$

where, as above, $\mathcal{B}(\lambda) = \lambda^\rho\, bfI + \lambda^{\rho-1}B + \cdots + B^\rho$ and $Q_\rho(\lambda)$ is a polynomial of degree $\rho+2$, whose coefficients are chosen uniquely from the condition $\frac{d}{dt}(x, y+Bv) = 0$.

In particular, in view of the constraints (4.4),

$$A_0(\lambda) = \begin{pmatrix} 0 & 1 \\ -Q_0(\lambda) & 0 \end{pmatrix} \qquad A_1(\lambda) = \begin{pmatrix} (Bx, (y + Bv)) & \lambda + (x, Bx) \\ -Q_1(\lambda) & -(Bx, (y + Bv)) \end{pmatrix},$$

$$Q_0 = (v, v)\lambda^2 + [2(v, y) + (v, Bv) - (v, v)(x, Bx)]\lambda + (y, y)$$
$$+ \Delta(v, Bv) - (x, Bx)[2(v, y) + (v, Bv) - (v, v)(x, Bx)]$$
$$- (v, v)[\Delta_2 + (x, B^2x)],$$

$$Q_1 = (v, v)\lambda^3 + [2(v, y) + (v, Bv)]\lambda^2 + [(y, y) - 2\Delta(v, y) + 2(y, Bv) - (v, v)]\lambda$$

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The spectral curve $\mathcal{C} = \{|\Phi(\lambda)\,L(\lambda) - wI| = 0\}$ is now an even order hyperelliptic curve of genus $g = m - 1$, and under the substitution (4.3) it reads

$$w^2 = -\sum_{i<j}^{m} \frac{\Phi^2(\lambda)}{(\lambda - b_i)(\lambda - b_j)} \left[x_i(y_j + \lambda v_j) - x_j(y_i + \lambda v_i)\right]^2$$

$$= -\Phi(\lambda)\,\widetilde{\mathcal{I}}_2(\lambda, Z, P), \qquad\qquad (4.16)$$

thus giving all the quadratic first integrals (2.16) of the Steklov–Lyapunov systems on $\mathcal{S}^2_{c,d}$ and on $\mathcal{W}^2$.

In view of the constraints (4.4), the polynomial Lax matrix $\hat{L}(\lambda) = \Phi(\lambda)L(\lambda)$ has the following structure

$$\hat{L}(\lambda) = \begin{pmatrix} V(\lambda) & U(\lambda) \\ W(\lambda) & -V(\lambda) \end{pmatrix}, \qquad\qquad (4.17)$$

$$U(\lambda) = \lambda^g + U_1\lambda^{g-1} + \cdots + U_g, \quad V(\lambda) = V_1\lambda^{g-1} + \cdots + V_{g+1},$$

$$W(\lambda) = -(v,v)\lambda^{g+2} - W_{-1}\lambda^{g+1} - W_0\lambda^g - \cdots - W_g, \qquad g = m - 1.$$

The set of all such complex matrices forms a $3m$-dimensional li near space $\mathcal{E}_g$ spanned by the coefficients of the polynomials U, V, W. Following [14] (see also [2, 16]), $\mathcal{E}_g$ can be completed to the fiber bundle $\bar{\mathcal{E}}_g$ over the $(2g+2)$-dimensional base space spanned by the coefficients of the characteristic polynomial $R(\lambda) = -\det \hat{L}(\lambda) \equiv U(\lambda)W(\lambda) + V^2(\lambda)$ and parameterizing the corresponding genus g hyperelliptic curves $\mathcal{C}$, with fibers being the Jacobian varieties of the curves.

As follows from (4.14), the Lax matrices constructed of the vectors x, y, v form a $2m$-dimensional subvariety $\mathcal{N}_m \subset \mathcal{E}_g$ specified by conditions $R(b_i) = 0$, $i = 1, \ldots, m$. In this case the two leading coefficients of $R(\lambda)$ are linear combinations of the quadratic Casimir functions $H_{2m}, H_{2,m-1}$ of the bracket (2.1).

It is seen that for $m > 3$, the dimension of $\mathcal{W}^2$ is bigger than that of $\mathcal{N}_m$, hence, in this case, the Lax pair (4.13) is not equivalent to equations (4.12).

Proposition 4.6   *1). The components of the Lax matrix $L(\lambda|x, y, v)$ in (4.14) are invariant with respect to the flows (4.9). Generic orbits of these flows in $\mathcal{T}^2$ are $(m-3)$-dimensional compact real algebraic varieties.*

2). $\mathcal{N}_m$ is the factor variety of $\mathcal{T}^2$ by the action of the group generated by the flows and by the action of the discrete group $\mathfrak{R}$ generated by reflections $(x_i, y_i, v_i) \to (-x_i, -y_i, -v_i)$, $i = 1, \ldots, m$.

3). On $\mathcal{F}^2_{c,d}$ and $\mathcal{W}^2$ the Pfaffians $Pf(|Z|)^I_I$ commute with the quadratic first integrals in (4.1).

4). Generic orbits of the flows $\mathcal{P}_I$ in $\mathcal{F}^2_{c,d}$ are $(m-3)$-dimensional real compact algebraic varieties.

Proof. First, notice that the flows (4.9) do not change the vectors $y + Bv$, which form the Laurent part of $L(\lambda)$ in (4.14). Next, we have $(v, v)' = 0$ and

$$(v, Bv)' + 2(v, y)' = -2(Bv, \hat{Z}_I v) - 2(y, \hat{Z}_I v) + 2(v, B\hat{Z}_I v) \equiv 2\langle y \wedge v, \widehat{(x \wedge y)}_I \rangle,$$

which is zero due to the definition of $\hat{Z}_I$ in (4.2).

Hence, the components of $L(\lambda)$ provide $2m$ *independent* algebraic first integrals of the flows, and, therefore, their orbits are algebraic varieties of dimension $\dim \mathcal{T}^2 - 2m = m - 3$.

Further, from equations (4.9) and the constraints (**??**) we find that for each fixed orbit, the vector v lies on the sphere S^{m-2} in $\mathbb{R}^{m-1} = \{v \mid (v, x) = 0\}$. On the other hand, since on each orbit $y + Bv = d$, $d =$ const and $(v, Bv) + 2(v, y) =$ const, the same vector belongs to the quadric $2(d, v) + (v, Bv) =$ const. As a result, each orbit is diffeomorphic to a connected component of the intersection of two $(m-2)$-dimensional quadrics in $\mathbb{R}^{m-1}$, which is a compact variety. This implies items 1.

Next, the components of $L(\lambda)$ are invariant with respect to reflections of $\mathfrak{R}$, which yields item 2.

Since the above flows preserve $L(\lambda)$, the corresponding flows $\mathcal{P}_I$ on $\mathcal{W}^2$ preserve the quadratic integrals (4.1). Thus, these integrals and $Pf(|Z|)_I^I$ commute on $\mathcal{W}^2$.

Item 4 is a reformulation of item 1 in terms of the coordinates Z, P on $\mathcal{W}^2$. $\square$

Now let $\mathcal{O}_{c,d}^2$ be a $2(m-1)$-dimensional subvariety of $\mathcal{N}_m$ obtained by fixing the two leading coefficients in the polynomial (4.16), i.e., by fixing the two quadratic Casimir functions on $\mathcal{W}^2$. In view of item 3 of the above proposition, $\mathcal{O}_{c,d}^2$ can also be regarded as the factor variety of $\mathcal{F}_{c,d}^2$ by the action of the Ab elian group generated by the flows $\mathcal{P}_I$ and by the action of the discrete group $\mathfrak{R}'$ induced by $\mathfrak{R}$ on $\mathcal{F}_{c,d}^2$.

Combining Propositions 4.4, 4.6, as well as Theorem 4.1, we arrive at the following theorem.

Theorem 4.7 *The manifold $\mathcal{O}_{c,d}^2$ is symplectic and can be regarded as a special Poisson (Marsden–Weinstein) reduction of rank 2 coadjoint orbits $\mathcal{S}_{c,d}^2$ obtained by fixing the Hamiltonians $\mathcal{H}_I = Pf(|Z|_I^I)$ to zero and factorizing by the action of the Hamiltonian flows (4.2) and by the group $\mathfrak{R}'$ action.*

To get a global view on the above manifolds, we represent them in the following commutative diagram where arrows denote the corresponding maps (embeddings or factorizations), and the map $\Lambda : \mathcal{W}^2 \to \mathcal{N}_m$ is given by the composition of the formulas of Proposition 4.3 and (4.14).

$$
\begin{array}{ccccc}
so(m) \times so(m) & \xleftarrow{\tilde{\mathcal{I}}_4(s) = \cdots = \tilde{\mathcal{I}}_g(s) = 0} & \mathcal{W}^2 & \xrightarrow{\Lambda} & \mathcal{N}_m \\
\cup \uparrow & & \cup \uparrow & & \cup \uparrow \\
\mathcal{S}_{c,d}^2 & \xleftarrow{H_{4,\mu}(Z) = 0} & \mathcal{F}_{c,d}^2 & \xrightarrow{/\mathcal{P}_I \, /\mathfrak{R}'} & \mathcal{O}_{c,d}^2.
\end{array}
$$

In the classical case $m = 3$ the above diagram simplifies: the 6-dimensional variety $\mathcal{W}^2$ coincides with the product $so(3) \times so(3)$ itself, and 4-dimensional orbits $\mathcal{S}^2_{c,d}$ are coverings of $\mathcal{O}^2_{c,d}$. They are foliated with 2-dimensional tori, whose complexifications are coverings of the Jacobians of genus 2 hyperelliptic curves $\mathcal{C}$.

Note that another 2×2 matrix Lax pair for the classical Steklov system written in different coordinates related to an integrable geodesic flow on $SO(4)$ was found in [6].

5 Linearization of flows and separation of variables in the rank 2 case

Let $P_1 = (\lambda_1, w_1), \ldots, P_g = (\lambda_g, w_g)$ be a divisor of $g = m - 1$ points on the spectral curve $\mathcal{C}$, whose coordinates satisfy equations

$$U(\lambda_k) = 0, \quad w_k = V(\lambda_k).$$

Since $U(\lambda)$ and $V(\lambda)$ are polynomial of degree g and $g - 1$ respectively, then

$$U = (\lambda - \lambda_1) \cdots (\lambda - \lambda_g), \quad V = \sum_{k=1}^{g} w_k \frac{\prod_{l \neq k} (\lambda - \lambda_l)}{\prod_{l \neq k} (\lambda_k - \lambda_l)} . \tag{5.1}$$

Now, taking residue of the Lax matrix (4.13) at $\lambda = b_i$, we obtain

$$x_i^2 = \frac{(b_i - \lambda_1) \cdots (b_i - \lambda_{m-1})}{\prod_{j \neq i} (b_i - b_j)} ,$$

$$y_i + b_i v_i = x_i \sum_{k=1}^{g} \frac{w_k}{(b_i - \lambda_k) \prod_{s \neq k} (\lambda_k - \lambda_s)} ,$$

$$i = 1, \ldots, m, \tag{5.2}$$

The first set of these expressions implies that $\lambda_1, \ldots, \lambda_g$ are spheroconic coordinates on the unit sphere $\{(x, x) = 1\}$,

Now let us fix constants of motion by setting

$$\widetilde{\mathcal{I}}_2(\lambda, Z, P) = \psi(\lambda), \quad \psi(\lambda) = h_m \lambda^m + \cdots + h_1 \lambda + h_0, \qquad h_0, h_1, \ldots, h_m = \text{const},$$

so that, due to (4.16), $w_k = \sqrt{-\Phi(\lambda_k) \psi(\lambda_k)}$.

Theorem 5.1 *Let $Z(t), P(t)$ be a solution of the Steklov–Lyapunov system on $\mathcal{W}^2$ with the quadratic Hamiltonian*

$$H_f = \frac{1}{2} \left(f_m H_{2,m}(P) + f_{m-1} H_{2,m-1}(P, Z) + \cdots + f_0 H_{20}(Z, P) \right) , \tag{5.3}$$

$$f_0, \ldots, f_{m-2} = \text{const}$$

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and constants of motion $H_{2,m}(P) = h_m, \ldots, H_{20}(Z, P) = h_0$. *Then the evolution of the points* (λ_k, w_k) *is given by the following standard Abel–Jacobi equations involving* g *holomorphic differentials on the curve* $\mathcal{C}$,

$$\sum_{k=1}^{m-1} \frac{\lambda_k^r \, d\lambda_k}{2\sqrt{-\Phi(\lambda_k)\psi(\lambda_k)}} = d\phi_r, \qquad r = 0, 1, \ldots, m-2, \qquad (5.4)$$

where $d\phi_r = f_r \, dt$.

Recall that $H_{2,m}(P)$, $H_{2,m-1}(P, Z)$ are Casimir functions of the bracket $\{\,,\,\}_1$ and notice the corresponding constants f_{m-1}, f_m do not appear in the right hand sides of (5.4).

In particular, for the generalized Steklov and Lyapunov systems described by the Hamiltonians (2.4) the above equations take the form respectively

$$\begin{cases} \displaystyle\sum_{k=1}^{g} \frac{d\lambda_k}{2\sqrt{-\Phi(\lambda_k)\psi(\lambda_k)}} = 0, \\ \qquad \cdots \quad \cdots \quad \cdots \\ \displaystyle\sum_{k=1}^{g} \frac{\lambda_k^{g-2} d\lambda_k}{2\sqrt{-\Phi(\lambda_k)\psi(\lambda_k)}} = dt, \\ \displaystyle\sum_{k=1}^{g} \frac{\lambda_k^{g-1} d\lambda_k}{2\sqrt{-\Phi(\lambda_k)\psi(\lambda_k)}} = 0, \end{cases} \qquad \begin{cases} \displaystyle\sum_{k=1}^{g} \frac{d\lambda_k}{2\sqrt{-\Phi(\lambda_k)\psi(\lambda_k)}} = 0, \\ \qquad \cdots \quad \cdots \quad \cdots \\ \displaystyle\sum_{k=1}^{g} \frac{\lambda_k^{g-2} d\lambda_k}{2\sqrt{-\Phi(\lambda_k)\psi(\lambda_k)}} = 0, \\ \displaystyle\sum_{k=1}^{g} \frac{\lambda_k^{g-1} d\lambda_k}{2\sqrt{-\Phi(\lambda_k)\psi(\lambda_k)}} = dt. \end{cases}$$

Note that, for the classical case $m = 3$, the variables λ_1, λ_2 were first introduced and the quadratures (5) were obtained by F. Kötter in [12].

Proof of Theorem 5.1. As follows from the Lax equations (4.13) and expressions for $\hat{L}(\lambda)$ in (4.17), for the system with the Hamiltonian $\mathfrak{h}_{m-2-\rho}$,

$$\dot{U}(\lambda) = 2V(\lambda)[\lambda^\rho + \lambda^{\rho-1}(x, Bx) + \cdots + (x, B^\rho x)] - 2U(\lambda)(x, \mathcal{B}(\lambda)(y + Bv)),$$

Setting here $\lambda = \lambda_k$ and taking into account (5.1), we obtain

$$\dot{\lambda}_k \prod_{s\neq k}(\lambda_k - \lambda_s) = 2w_k[\lambda_k^\rho + \lambda_k^{\rho-1}(x, Bx) + \cdots + (x, B^\rho x)].$$

Then, according to relations (2.16), for the motion with the quadratic Hamiltonian

$$H_{2,m-2-\rho}(Z, P) = \sum_{s=0}^{\rho}(-1)^s \Delta_s \mathfrak{h}_{m-2-\rho+s},$$

we have

$$\frac{\dot{\lambda}_k}{2w_k} \prod_{s\neq k}(\lambda_k - \lambda_s) = \lambda_k^\rho + \lambda_k^{\rho-1}[(x, Bx) - \Delta_1] + \lambda_k^{\rho-2}[(x, B^2 x) - \Delta_1(x, Bx) + \Delta_2]$$

$$+ \cdots + \lambda_k^0[(x, B^\rho x) - \Delta_1(x, B^{\rho-1} x) + \cdots + (-1)^\rho \Delta_\rho]. \qquad (5.5)$$

Now applying relations (5.2) and the known Jacobi identities, we represent the right hand side in form

$$\lambda_k^\rho - \sigma_1\lambda_k^{\rho-1} + \cdots + (-1)^\rho\sigma_\rho\lambda_k^0,$$

where $\sigma_s = (-1)^s U_s$ is the elementary symmetric polynomial of $\lambda_1, \ldots, \lambda_g$ of degree s and, as above, U_s is the coefficients of $U(\lambda)$. Again, in view of the Jacobi identities, for $0 \le r \le g - 1 = m - 2$ we have

$$\sum_{k=1}^g \lambda_k^r \frac{\lambda_k^\rho - \sigma_1\lambda_k^{\rho-1} + \cdots + (-1)^\rho\sigma_\rho\lambda_k^0}{\prod_{s\ne k}(\lambda_k - \lambda_s)} = \delta_{m-2-\rho,r}.$$

This, together with (5.5), implies that for the system with the Hamiltonian $H_{2,m-2-\rho}(Z,P)$ the evolution of λ-coordinates is given by equations

$$\sum_{k=1}^g \frac{\lambda_k^r \, d\lambda_k}{2w_k} = \delta_{m-2-\rho,r} \, dt, \qquad r = 0, \ldots, g-1.$$

By linearity, we conclude that for the motion with the generic Hamiltonian (5.3) this evolution is described by the system (5.4). $\square$

Now introduce variables

$$\mu_k = \frac{w_k}{\Phi(\lambda_k)} = \frac{\sqrt{\lambda^m \langle P, P \rangle + H_{2,m-1}\lambda^{m-1} + \cdots + H_{2,0}}}{\sqrt{(\lambda - \lambda_1) \cdots (\lambda - \lambda_g)}} \,. \qquad (5.6)$$

Theorem 5.2 *On the $2g$-dimensional manifold $\mathcal{O}_{c,d}^2$ the variables $(\lambda_1, \mu_1), \ldots, (\lambda_g, \mu_g)$ form a complete set Darboux coordinates with respect to the Lie–Poisson bracket (2.1) on $so(m) \times so(m)$, i.e.,*

$$\{\lambda_k, \lambda_s\}_1 = \{\mu_k, \mu_s\}_1 = 0, \quad \{\lambda_k, \mu_s\}_1 = \delta_{ks}, \qquad k, s = 1, \ldots, g.$$

As a corollary, we find that for $m = 3$, the Kötter variables $\lambda_1, \mu_1, \lambda_2, \mu_2$ are Darboux coordinates on the orbits $S_{c,d}^2 = \mathcal{O}_{c,d}^2$ with respect to the standard Lie–Poisson bracket on $e^*(3)$.

Proof of Theorem 5.2. As follows from Theorem 5.1,

$$\{\phi_\rho, H_{2,r}\}_1 = \delta_{\rho r}, \qquad \rho, r = 0, 1, \ldots, m-2,$$

where ϕ_ρ are angle type variables defined in a neighborhood of a generic invariant torus. Also, $\{H_{2,\rho}, H_{2,r}\}_1 = 0$. Hence, the reduction of the corresponding symplectic structure on the orbit $\mathcal{S}_{c,d}^2$ onto $\mathcal{O}_{c,d}^2$ can locally be represented as

$$\omega = \sum_{r=0}^{m-2} d\phi_r \wedge dh_r + \sum_{0\le\rho<r\le m-2} C_{\rho r} \, d\phi_\rho \wedge d\phi_r$$

with some coefficients $C_{\rho r}$. Next, due to (5.4) and (5.6),

$$d\phi_r = \sum_{k=1}^g \frac{\partial\mu_k(\lambda_k, h)}{\partial h_r} \, d\lambda_k,$$

which implies

$$\omega = \sum_{k=1}^{g} d\lambda_k \wedge \left[\sum_{r=0}^{m-2} \frac{\partial \mu_k(\lambda_k, h)}{\partial h_r} \, dh_r \right] + \sum_{0 \le \rho < r \le m-2} C_{\rho r} \, d\phi_\rho \wedge d\phi_r$$

$$\equiv \sum_{k=1}^{g} d\lambda_k \wedge d\mu_k + \sum_{1 \le k < s \le g} \tilde{C}_{ks} \, d\lambda_k \wedge d\lambda_s$$

with some coefficients $\tilde{C}_{ks}$. On the o ther hand, Theorem 4.5 says that $\{x_i, x_j\}_1 = 0$, which, together with the first relations in (5.2), implies $\{\lambda_k, \lambda_s\}_1 = 0$. As a result, in the expression for ω we have $\tilde{C}_{ks} = 0$, which proves the theorem.

6 Conclusion

In this paper we considered integrable Steklov–Lyapunov systems on rank r coadjoint orbits $\mathcal{S}_{c,d}^r$ in $so(m) \times so(m)$ and on their invariant subvarieties $\mathcal{F}_{c,d}^r$. We showed that the latter systems, written in terms of matrix triplets $\mathcal{X}, \mathcal{V}, \mathcal{Y}$, admit $r \times r$ matrix Lax representation in a generalized Gaudin form.

It would be interesting to find an appropriate generalization of the Weinstein–Aronzjan formula (1.3) to the case of Lax matrices (1.7) .

In the rank 2 case we described a Marsden–Weinstein reduction of $\mathcal{S}_{c,d}^2$ onto symplectic $2(m-1)$-dimensional manifolds $\mathcal{O}_{c,d}^2$, which is foliated with $(m-1)$-dimensional Jacobians of hyperelliptic spectral curves, and indicated Darboux coordinates with respect to the original Lie–Poisson structure on $so(m) \times so(m)$. For $m = 3$, these coordinates coincide with the mysterious separating variables used by Kötter in order to reduce the systems on $e^*(3)$ to Abel–Jacobi quadratures. They can be used to construct action-angle variables for the classical systems.

The properties of analogous reduction for arbitrary rank r are still not understood completely.

On the other hand, it appears that adding to the Lax matrix $L(\lambda)$ in (3.5) a constant $r \times r$ matrix Y allows a similar description of other generalizations of the Steklov–Lyapunov systems. For example, consider the following matrix "hybrid" system on the phase space $(Z, P, e^{(1)}, \ldots, e^{(k)})$, $Z, P \in so^*(m)$, $e^{(1)}, \ldots, e^{(k)} \in \mathbb{R}^m$, $k \le m$ (see also [9])

$$\dot{Z} = ZPB - BPZ + [\Gamma, B],$$
$$\dot{P} = [P, PB + BP] + [P, Z],$$
$$\dot{\Gamma} = [\Gamma, Z] + \Gamma PB - BP\Gamma,$$
$$\Gamma = \varepsilon(e^{(1)} \otimes e^{(1)} + \cdots + e^{(k)} \otimes e^{(k)}), \quad B = \mathrm{diag}(b_1, \ldots, b_m),$$

$$(6.1)$$

which for $\varepsilon \to 0$ is reduced to the generalized Lyapunov system (2.7) with $\rho = 0$, whereas for $P \to 0$ it becomes the simplest system of the Clebsch–Perelomov-Bogoyavlensky hierarchy on the dual to the semi-direct product Lie algebra $so(m) \times_s$

$\underbrace{(\mathbb{R}^m \times \cdots \times \mathbb{R}^m)}_{k \text{ times}}$ ([4]), i.e.,

$$\dot{Z} = [\,\Gamma, B\,], \qquad \dot{\Gamma} = [\,\Gamma, Z\,].$$

This describes the motion of a spherically symmetrical top with the angular velocity Z in the field of the quadratic potential $\frac{1}{2}\left(e^{(1)}, Be^{(1)}\right) + \cdots + \frac{1}{2}\left(e^{(k)}, Be^{(k)}\right)$.

We mention without a proof that, for an even number r, $2k \le r \le m$, the system (6.1) has invariant manifolds $\tilde{\mathcal{W}}^r$ given by the conditions

$$\forall s \in \mathbb{R}, \quad \text{rank } |Z + sP| = r, \quad \text{rank}\begin{pmatrix} Z & e^{(1)} & \cdots & e^{(r)} \\ -(e^{(1)})^T & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ -(e^{(r)})^T & 0 & \cdots & 0 \end{pmatrix} = r.$$

Then, similarly to (3.1), on $\tilde{\mathcal{W}}^r$ the variables $Z, P, e^{(1)}, \ldots, e^{(k)}$ can be represented in terms of $r/2 \times m$ matrices $\mathcal{X}, \mathcal{Y}, \mathcal{V}$ as follows

$$Z = \mathcal{X}^T\mathcal{Y} - \mathcal{Y}^T\mathcal{X}, \quad P = \mathcal{X}^T\mathcal{V} - \mathcal{V}^T\mathcal{X}, \quad e^{(1)} = x^{(1)}, \ldots, e^{(r)} = x^{(k)},$$

and the restriction of equations (6.1) onto $\tilde{\mathcal{W}}^r$ admits $r \times r$ matrix La x representation

$$\dot{L}(\lambda) = [L(\lambda), A(\lambda)],$$

$$L(\lambda) = Y + \begin{pmatrix} \mathcal{X}(\lambda\mathbf{I} - B)^{-1}[\mathcal{Y} + \lambda\,\mathcal{V}]^T & \mathcal{X}(\lambda\mathbf{I} - B)^{-1}\mathcal{X}^T \\ -(\mathcal{Y} + \lambda\mathcal{V})(\lambda\mathbf{I} - B)^{-1}[\mathcal{Y} + \lambda\mathcal{V}]^T & -[\mathcal{Y} + \lambda\mathcal{V}](\lambda\mathbf{I} - B)^{-1}\mathcal{X}^T \end{pmatrix}, \quad (6.2)$$

with certain polynomial matrix $A(\lambda)$ and the constant matrix Y of the following structure

$$Y = \begin{pmatrix} 0 & 0 \\ \mathbf{I}_k & 0 \end{pmatrix}, \qquad \mathbf{I}_k = \text{diag}(\underbrace{1, \ldots, 1}_{k \text{ units}}, 0, \ldots, 0).$$

A detailed description of a natural Poisson structure on the space of Lax matrices (6.2) and its relation to symplectic properties of various Steklov–Lyapunov type systems, as well as their integrable discretizations, are left for a future publication.

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