Simple examples of dual coupling networks

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Abstract-*Most mechanisms are both underconstrained and overconstrained. The motions attributable to underconstraint can be seen so that they are easily imagined from a drawing whereas actions attributable to overconstraint cannot. Dual coupling networks have the property that the action and motion systems of one are transposed in the other. So, by finding the dual of a mechanism, actions attributable to overconstraint become motions in its dual that can be imagined. Earlier work cited explains the methodology and validates the theory mathematically: this paper provides some simple examples.*

Keywords: action, graph, motion, screw

I. Introduction

The term coupling network is applied to an assemblage of coupled rigid bodies that can be underconstrained, overconstrained, both or neither. Coupling networks include statically indeterminate structures that are overconstrained but not underconstrained, kinematically designed kinematic chains that are underconstrained but not overconstrained, simply stiff (statically determinate) structures that are neither and kinematic chains that are both.

There is no general agreement about the essential features of duality as it applies to TMM. Huang [1] writes of the duals of joints, chains and single circuit mechanisms; forces said to be dual to the motions of bodies in a four-bar linkage are described but not the entirety of the coupling network within which these actions can exist. Shai [2, 3] uses dual graphs to extend the study of duality to mechanisms having more than one circuit again showing that actions in the dual are analogous to the motions of the mechanism. Shai and Pennock [4] also include characteristics of serial and parallel manipulators.

Gosselin and Lallemand [5] review nine earlier papers that refer to the duality of serial and parallel manipulators. They write: "…this duality concerns essentially the velocity kinematics of serial robots and the statics of parallel manipulators." They imply that there is more to duality than the demonstration of an analogy. In [5] they show that simultaneously there is also an analogy between the statics of serial robots and the kinematics of parallel robots. Dual sets of equations are provided. The understanding that Gosselin and Lallemand have of duality appears to correspond closely with the statement below.

Two related mathematical concepts are said to be dual if two properties of each are transposed in the other. The Platonic solids provide examples. The number of faces of one of a dual pair is the number of vertices of the other. Thus the cube has six faces and eight vertices; its dual, the octahedron, has eight faces and six vertices. In [6], and in this paper, the motion and action screw systems of a coupling network are the two properties that are transposed with those of the dual coupling network. In [6] the dual coupling networks N^+ and N^- studied are far from simple. The network N^+, analysed comprehensively in [7], has two circuits and nett degrees of freedom and constraint F_N^+ and C_N^+ that are three and two respectively. The coupling network N^-, dual with N^+, has three circuits and nett degrees of freedom and constraint F_N^- and C_N^- that, because of duality, must be two and three respectively. For this paper there are many simpler examples to choose from but space constraints mean that few can be selected. From these, instructive variations are developed.

II. The gross and nett degrees of freedom and constraint

A coupling between two bodies has a characteristic degree of freedom (*dof*) *f* and a degree of constraint (*doc*) *c*, where $f + c = 6$. A coupling network with *n* bodies and *e* couplings can be said to have a gross *dof* $F = \Sigma f$ and a gross *doc* $C = \Sigma c$. The gross *dof* *F* is also the true (nett) *dof* F_N provided that there are no circuits of couplings. It has been shown [7] that the effect of closing circuits of couplings is to introduce indirect couplings between pairs of directly coupled bodies in parallel with those direct couplings. These indirect couplings can reduce the gross *dof* *F* to a lower value, the nett *dof* F_N.

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Analogous statements can be made about the gross and nett degrees of constraint C and C_N. It is necessary first to introduce the idea of a cutset of couplings. The term *cutset* is borrowed from graph theory. A cutset of couplings [7] is a set of couplings the removal of which creates two disconnected coupling networks each of which could comprise one body and no couplings. Thus the absence of a cutset implies only one body. A single body could nonetheless provide terminals for couplings. Because the body is rigid and forms part of a closed circuit with each coupling, the body can transmit any action that a coupling can transmit. Thus, for a coupling network comprising a single body with couplings, the gross doc C is also the nett doc C_N. A stringed musical instrument is such a body if the parts involved in tensioning strings are immobilised or ignored. The eight strings of a mandolin can be regarded as couplings between points (terminals) on the mandolin. All eight strings can be at different unknown tensions so both C and C_N are eight. The effect of cutting a mandolin in two would be to break all eight circuits that are vital for maintaining string tensions.

For a coupling network comprising two or more bodies and two or more couplings the gross *dof C* may be reduced to a nett *dof C_N*. The circuit that is vital in order for a coupling **A** to transmit action will now have in it another coupling **B**. Whereas a rigid body can transmit any action, coupling **B** may be incapable of transmitting some or all of the actions that coupling **A** is capable of transmitting.

III. Creating the dual of a coupling network

There are four stages to the process of creating the dual of a coupling graph N^+. It is necessary to find couplings dual to those of N^+; to create G_C^+, the coupling graph of N^+; to construct G_C^-, the coupling graph dual with G_C^+, and finally; to synthesise a coupling network N^-, having G_C^- as its coupling graph and couplings dual with those of N^+.

A. Dual couplings

Two couplings that exhibit the property that the system of motion screws for one is identical, geometrically, to the system of action screws of the other will be referred to as dual couplings. Several pairs of dual couplings have been described [6]. For a given coupling **A** there can be several couplings that meet the requirement that their motion and action systems are the transpose of those of coupling **A**.

B. The coupling graph of a coupling network

A coupling network N^+ has a coupling graph G_C^+ in which each node of G_C^+ represents a rigid body of N^+; and each edge of G_C^+ represents a coupling of N^+.

C. Dual coupling graphs

The dual of G_C^+ is required. Only planar graphs have a dual graph so it is necessary to explain planar graphs before proceeding.

C.1 Planar graphs

A planar graph is a graph that can be embedded onto the surface of a sphere or plane without having any pairs of edges that cross one another. When a planar graph is embedded on the surface of a sphere each area of that spherical surface that is surrounded by edges of the graph is called a region [8] or face [9] of the graph.

C.2 Dual graphs

A planar graph G^+ has a dual planar graph G^- that is created in the following way. Within each region of G^+ there exists one node of G^-. Also, each edge of G^+ is crossed by one edge of G^-. It is convenient to refer to a pair of crossing edges, one from each of a pair of dual graphs, as corresponding edges. For dual coupling graphs G_C^+ and G_C^- corresponding edges represent dual couplings. Dual coupling graphs that appear later are interlinked; G_C^+ is drawn in blue and G_C^- in yellow.

D. The nett dof and doc of dual coupling networks

Because screw systems are transposed in dual coupling networks it is a necessary, but insufficient, condition that $F_N^+ = C_N^-$ and $C_N^+ = F_N^-$.

IV. The dual of a chain of couplings in series

The coupling network N^+ shown in the top left of Fig. 1 comprises three bodies numbered 1-3, coupled by two revolute couplings with parallel axes labelled **A**, **B**. The network is an adaptation of part of a figure provided by Hunt [10], page 66, as a proof, by analogy, of the Kennedy-Aronhold theorem of three centres. One change from Hunt's figure is that the central angular velocity vector is reversed in direction. Now the three vectors are angular velocities of pairs of bodies in cyclic sequence 12, 23 and 31, where ω_{12}, for example, is the angular velocity, relative to member 1, of member 2. The vectors must sum to zero.

Any coupling that is capable of transmitting a force, but no more than a single force, is dual with a revolute coupling. Thus, any dual of N^+ has two couplings that each transmits a force. N^+ has a coupling graph G_C^+, with three nodes and two edges and its dual G_C^- has one node and two edges. Both graphs are shown at the top right of Fig. 1. Both edges of G_C^- are called *loops* because their ends terminate at the same node. Two coupling networks, either of which can be N^-, a dual of N^+, are shown at the bottom of Fig 1. For the version on the left two couplings are provided by elastic bands in tension. In the other, bolts are screwed through tapped holes in the main body until the hemispherical ends of the bolts make contact with a surface of the same body and the bolts are in compression. The bolts are then welded to the main body otherwise there would be three bodies in N^-, not one. Note that $F_N^+ = C_N^- = 2$ and $C_N^+ = F_N^- = 0$. Because $C_N^- = 2$ the equilibrant force transmitted by the web of the I-shaped body has a line of action parallel with, and anywhere between, the other lines of action. Dually, in N^+, the axis of angular velocity ω_{31} must lie between the other two.
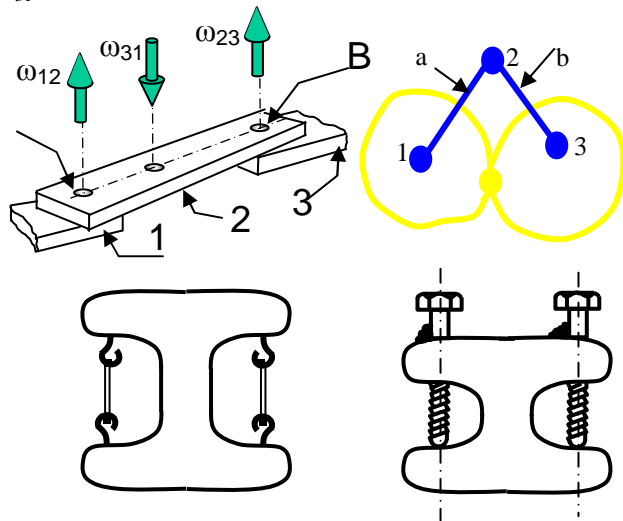

Fig 1. Bodies coupled in series shown top left, alternative duals below and their dual coupling graphs top right

Generally the dual of a coupling network comprising bodies and couplings in series is a single body providing ports [11] for the same number of couplings in parallel. So the metal strap of a wristwatch with nine links and eight pins and a mandolin with eight strings, and with the means of tensioning immobilised, can be dual coupling networks. They are dual only if some more conditions are met. The watchstrap must not be pinned to the watch otherwise a closed circuit is created the dual of which would have two bodies not one. Adjacent links of the strap must be in relative motion. Finally, at the instant that the duality is valid, the pins of the watchstrap must be coplanar.

V. Duals of some four bar linkages

Figure 2a shows a planar four-bar linkage N^+ with links of equal length instantaneously in the configuration whereby the centrelines of the (R) couplings cut any plane perpendicular to them at points that form the vertices of a square.
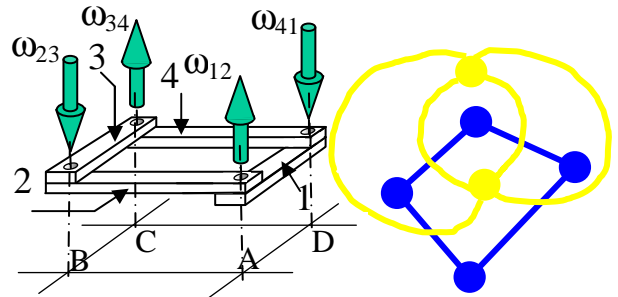

Fig. 2a. A planar 4-bar and associated dual graphs

Also shown are the four angular velocity vectors of equal magnitude representing the instantaneous relative motions of contiguous links when progressing in a consistent direction around the circuit. A magnitude ω_{ij} is the angular velocity, relative to body i, of body j. The dual couplings must therefore be capable of transmitting equal forces. On the right of Fig. 2a are the coupling graphs: G_C^+ with four nodes, G_C^- with two nodes.

A. Variations on the theme of an unstable bar stool

There are several kinds of coupling that can transmit a force. Figure 2b shows two images of a miniature barstool. Integral with the base are two posts that have hemispheric upper ends; each end makes a single point contact with the flat underside of the seat. The same forces are transmitted if legs integral with the seat replace the posts integral with the base, but that arrangement is less stable.

Fig. 2b Duals of the planar 4-bar linkage

A.1 Assuming weightlessness

On the left of Fig. 2b there are also two couplings provided by elastic bands between the seat and base. The

locations of the four force vectors transmitted by the posts and bands correspond with those of the angular velocity vectors of the four-bar linkage. The four forces would be equal if the seat could be assumed to have zero mass. If that assumption is made then the barstool is a dual coupling network N^- of the 4-bar linkage. It is well known that the 4-bar linkage N^+ has a nett dof $F_N^+ = 1$. Duality requires that, for N^-, $C_N^- = 1$. This means that, provided that the elastic bands are in tension, if the magnitude of any one of the four forces is known then the forces in the other three couplings can be found. Obviously, for this example, all four forces are equal.

Less obviously, but of greater significance, $F_N^- = 3$. Because the hemispherical ends of the posts make single point contact with the underside of the seat, the seat can translate in any horizontal direction or rotate about a vertical axis through an infinitesimal displacement. The tension in the elastic bands is unaffected by infinitesimal displacements. These motions are all transitory motions that belong to the system classified by Hunt [10] as the fifth special 3-system of motion screws. They are the motions of planar kinematics. Duality requires that $C_N^+ = 3$; furthermore the actions that can exist within N^+, the 4-bar linkage, are geometrically identical to the motions that the bodies of N^- can have. It follows that, within the four-bar linkage, all bodies and couplings are capable of transmitting a torque in any direction perpendicular to the revolute axes and a force parallel with those axes.

A.2 with a gravitational coupling

There is a simple way of avoiding the need to assume that the seat is weightless. One of the two elastic bands can be removed and the centre of mass of the seat shifted so that it lies above the point where the missing elastic band was attached. The weight of the seat can replace the force in the elastic band that is removed provided that the weight is identical to the force exerted by the remaining band. An obvious way of shifting the location of the mass centre of the seat without modifying the seat is to have someone sit on the seat. On the right of Fig. 2b this is demonstrated. Such an unstable stool cannot be recommended but Barbie can.

B. Changes resulting in a stable bar stool

In the planar four-bar linkage shown in Fig. 2a, suppose that a spherical (S) coupling replaces the (R) coupling at **A** and a Hooke coupling replaces the (R) coupling opposite **A** at **C** to create a new coupling network N^+. The orientation of the Hooke coupling is important. Whereas, for the 4R linkage, **ABCD** were points in any one of the planes perpendicular to the coupling axes, for the new linkage, let **ABCD** occupy the only one of these

planes in which the centre of the (S) coupling lies. The point **C** is now the centre of the Hooke coupling; the plane of the rotation axes of the Hooke coupling is perpendicular to the line **AC** and the two (R) axes of the Hooke coupling are both at 45 degrees to the plane **ABCD**. For this new coupling network N^+, F_N^+ remains one but C_N^+ drops to zero. The dual coupling graphs are unaffected by this change but two couplings of the dual coupling network N^- are altered.

The (S) coupling is self-dual. Self-duality is explained in greater detail in [6] and in Section **VII** *C* of this paper. It means that, in the barstool, an (S) coupling, or a coupling that can transmit the same actions as an (S) coupling, must replace the single point contacts at **A**. A trihedral depression can be formed under the seat, or a part fixed to the underside that has this depression, as shown in Fig. 3. The resultant of the three forces transmitted between the three faces of the depression and the hemispherical end of the post that contacts these faces is along the centreline of the post as it is in the dual of the planar 4R linkage.
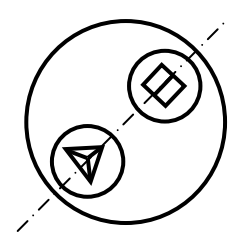

Fig. 3. Modifications to the underside of the barstool seat to reduce the nett dof from three to zero

The dual of the Hooke coupling can be a two-point contact between the hemispheric end of the second post and two faces of a depression made in the seat, or in a part added to the seat. To be the correct dual these faces should be perpendicular to one another, both being at 45 degrees to the plane of the underside of the seat. Furthermore, the plane of intersection of these faces must pass through the centre of the trihedral depression as shown in Fig. 3. The common normals at the two contacts correspond to the locations for the intersecting axes of the Hooke coupling. With these changes C_N^- remains one but F_N^- is reduced from three to zero. The seat can no longer move without the disengagement of a coupling. Disengagement requires the temporary use of an additional active coupling, for example one provided by human intervention.

C. A 4H kinematic chain with parallel axes and its dual

The zero pitch motion screws of the planar four bar linkage shown in Fig. 2a belong to the 5th special 3-system of screws. The screws of this special system must be all of the same pitch but not necessarily of zero pitch. Thus, four helical (H) couplings of the same pitch can

replace the (R) couplings. For this 4H kinematic chain $F_N^+ = 1$ and $C_N^+ = 3$, like the 4R linkage. The coupling graphs are unaltered but the four couplings of N^- must all be capable of transmitting wrenches on screws of the same pitch. In Fig. 2b posts provide two forces on the seat of the barstool. Elastic bands attached to the upper surface of the seats can replace these forces. Fig.4 shows such an arrangement turned on its side. Replacing the seat in this figure is a square plate suspended within a box. By suspending the plate from the top of the box its weight can be ignored. Imagine the four elastic bands being replaced by twisted cords of the kind used to tension a bow saw. All the cords must be twisted in the same sense, as well as providing the same torque/force ratio, the pitch of the screw.
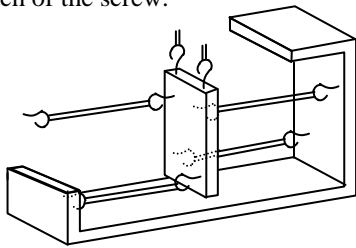

Fig. 4. Another dual of the planar 4-bar linkage

VI The dual of a conventional barstool

This barstool has three vertical legs terminating in hemispherical feet in contact with a horizontal floor at points **ABC** that form an equilateral triangle. There is stability so the alternative design, using vertical posts instead of legs, is not needed. The weight W of the barstool has a line of action through **D** at the centroid of the triangle **ABC**. Obviously the force on the seat through each leg is $W/3$ upwards.

Consider now the planar four-bar linkage that is the dual of this barstool. The four centrelines linking bearing axes form a re-entrant quadrilateral with vertices at **ABCD** such that **D** is at the centre of an equilateral triangle **ABC**. Duality makes conventional kinematic analysis unnecessary. Moving around the circuit in a consistent direction, if the instantaneous relative angular velocity of the two members directly coupled by the bearing at **D** is ω, then the relative angular velocity of each of the other three pairs of directly coupled members is $-\omega/3$.

If N^+ is the linkage and N^- the barstool as before then $F_N^+ = C_N^- = 1$ and $C_N^+ = F_N^- = 3$ again. One change is that the three degrees of freedom of N^- are expressed now by the unrestricted freedom of the entire barstool to slide on the floor instead of the transitory freedom of the seat to slide on the posts.

VII. Self-dual coupling networks

It is possible to anticipate one property of self-dual coupling networks even before an example is found. Because $F_N^+ = C_N^-$ and $C_N^+ = F_N^-$ for dual coupling networks N^+ and N^-, it follows that $F_N = C_N$ for a self-dual coupling network N. Obviously the + and – superscripts are unnecessary for a self-dual coupling network. Both self-dual couplings and self-dual graphs are needed.

A. Self-dual couplings

A coupling will be said to be self-dual if the screw system describing all possible motions that the coupling allows is identical to the screw system that describes all possible actions that the coupling can transmit. Among the surface contact couplings, often called lower kinematic pairs, the spherical (S) coupling and its variant, the planar or ebene (E) coupling, are self-dual.

The screw system associated with both motions and actions of a spherical coupling is categorised by Hunt [10] as the 2nd special 3-system of screws. This special 3-system comprises screws of the same pitch and the instantaneous screw axes (ISAs) form a star of ∞^2 lines in all directions but having one point in common, the centre of the sphere. For the (S) coupling the pitches of the screws are all zero. Thus the motions that the two bodies directly coupled by an (S) coupling are capable of are all angular velocities about axes through the sphere centre *provided that* any other couplings that exist between those bodies, in parallel with the (S) coupling, do not inhibit those motions. Furthermore, the actions that can be transmitted by an (S) coupling are all forces with lines of action that pass through the sphere centre *provided that* the (S) coupling belongs to a closed circuit of bodies and couplings all of which are capable of transmitting those actions.

The screw system associated with both motions and actions of an ebene (E) coupling are categorised by Hunt [10] as the 5th special 3-system of screws. This special 3-system comprises screws of infinite pitch anywhere in space that all have a direction parallel to the same plane together with all screws of the same pitch with ISAs perpendicular to that plane. For the (E) coupling this pitch is zero. Thus, subject to the provisos explained above, the motions of the two bodies directly coupled by an (E) coupling are translational velocities in any direction parallel with the plane of a contact surface of the coupling and angular velocities about any axis perpendicular to the same plane. Furthermore, the actions that can be transmitted by an (E) coupling are torques in any direction parallel with the plane of a contact surface of the coupling and forces along any line of action perpendicular to the same plane.

B. Self-dual graphs

A graph *G* that has a dual graph that is isomorphic with *G* is a self-dual graph. Two examples of self-dual graphs are provided in Fig.5 below. As before, each graph and its dual are interlinked.
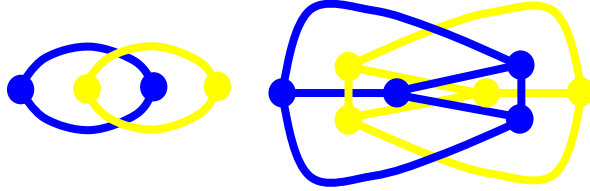

Fig. 5a and b. Pairs of self-dual graphs

Fig.5a, on the left, comprises two nodes and two parallel edges between them. Fig. 5b is the complete graph with four nodes. A graph is described as complete if every pair of nodes is connected by an edge.

C. Examples of self-dual coupling networks

Suppose that the two edges of either of the self-dual graphs shown in Fig. 5a both represent (S) couplings and the graph is the coupling graph G_C of a coupling network. That network could comprise a door, a doorframe, and the two (S) couplings from which the door is hung. The only motion both couplings permit is rotation about an axis through the sphere centres and the only possible action both couplings can transmit is a force along the same line. Thus $F_N = C_N = 1$ and, being overconstrained, it cannot be recommended.
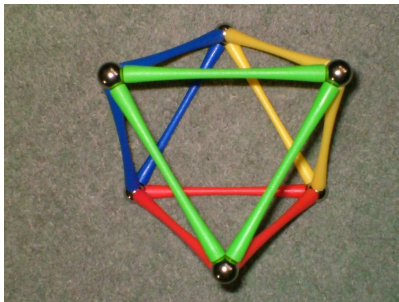

Fig. 6. By integrating rods of the same colour this becomes a self-dual simply stiff structure

A construction is shown in Fig.6 that resembles a regular octahedron. It is made from 12 identical rods with magnets embedded in their concave ends that adhere to steel balls thereby creating couplings that behave like (S) couplings. Because each rod is free to rotate about its central axis F_N is 12. If, however, each set of three rods of the same colour were to be made integral with one another these freedoms would be lost. Then there would be four bodies and six (S) couplings. The coupling network would be self-dual and have a coupling graph

G_C like either of the two self-dual graphs shown in Fig.5b. For this self-dual coupling network $F_N = C_N = 0$.

VIII. Conclusions

A contribution is made to the debate about what constitutes duality and, in particular, duality within TMM. The proposition is that duality requires the transposition of properties; analogy alone is insufficient. For TMM these properties are the action and motion screw systems of couplings in coupling networks. Mechanisms form an important subset of these networks.

Because motions are easier to imagine than internal actions, the creation of the dual of any overconstrained coupling network can enable the internal actions to be recognised. Those actions are transformed into geometrically identical motions within the dual coupling network. Often these motions are easily imagined even from a drawing.

IX Acknowledgement

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