Measures of Growth of Discrete Rational Equations

by

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Abstract

The general scope of this thesis is aimed at investigating certain classes of discrete equations through the analysis of certain characteristics of the solutions of these equations. We construct new methods of analysis based on the growth of these characteristics that let us single out known integrable discrete equations from certain class of equations. These integrable discrete equations are discrete analogues of the famous Painlevé equations.

We investigate the Diophantine integrability of the following class of discrete equations:

$$y_{n+1} + y_{n-1} = \frac{a_n + b_n y_n + c_n y_n^2}{1 - y_n^2},$$

where all the coefficients are rational functions of n and rational numbers and the right hand side of the above equation is irreducible. We constructed a rigorous method to examine the growth of the logarithmic height of the equation solution y_n i.e. $h(y_n)$ where the type of solution that we consider is called admissible (i.e. $h(y_n)$ grows faster than the height of the coefficients). We show that provided the equation has an admissible solution y_n if $c_n \neq 0$ or ± 2 $\forall n$, then our analysis implies that $\sum_{n=r_0}^{r} h(y_n)$ grows exponentially with r where r_0 is sufficiently large integer such that $r \geq r_0$. If $c_n = 0$ $\forall n$, then either $\sum_{n=r_0}^{r} h(y_n)$ is not growing polynomially with r or the equation reduces to a discrete analogue of P_{II} or y_n solves a difference Riccati equation. If $c_n = -2$ or 2 for all n, then $\sum_{n=r_0}^{r} h(y_n)$ grows exponentially with r.

Also, we study another class of equations:

$$(x_{n-1} + x_n)(x_n + x_{n+1}) = \frac{x_n^4 + \alpha_n x_n^3 + \beta_n x_n^2 + \gamma_n x_n + \eta_n}{(x_n - a_n)(x_n - b_n)},$$

where all the coefficients are rational functions of n and the right hand side is irreducible. The analysis here is based on considering x_n for all n as a non-constant rational function of a variable $z \in \mathbb{C}$ external to the equation. We investigate the growth of $\deg_z(x_n)$ with n. We show that if the coefficients of the equation satisfy certain assumptions, then either $\sum_{n=r_0}^{r} \deg_z(x_n)$ grows exponentially with r (for sufficiently large integer r_0, $r \geq r_0$) or we have a case where the equation reduces to a discrete analogue of P_{IV}.

- **Key words:** discrete rational equations, Diophantine integrability, height, rational functions, degree of rational functions, ultra-discrete equations, Max-plus Nevanlinna characteristic function, order.

"If I have ever made any valuable discoveries, it has been owing more to patient attention, than to any other talent".

Issac Newton (1642 - 1727)

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Contents

List of Figures

List of Tables

Chapter 1

Introduction

1.1 Introduction

The general theme that governs this thesis is the quest of *integrability.* Integrability is a very important characteristic of an equation or a map. It has different interpretations in different contexts, for example a Hamiltonian system (continuous or discrete) is integrable if there exist sufficiently many independent integrals of motion [5]. In geometry, integrable systems which describe various classes of surfaces have nice transformation properties such as Bäcklund transformations. Also, existence of Lax representations or soliton solutions of differential or difference equations are other properties which single out integrable equations from others. Investigating equations and maps for integrability is a task that occupied mathematicians' minds for a long time and still does, generating a rich area of research. This intensive research yields many proposed detectors to test equations for their integrability. One reason for this fascination of integrable equations is that when they appear in applications and physical models, their behaviour is predictable, at least asymptotically.

There are evidences (see section 1.3) which indicate that integrability is related to the slow growth of certain characteristics of the equations or maps. The first mathematician who noticed this relation between integrability and the slow growth of some characteristics of the equation is Veselov [82]. Our work here is a step in this direction. We focus here on certain classes of discrete and difference equations. These classes are

$$y_{n+1} + y_{n-1} = \frac{a_n + b_n y_n + c_n y_n^2}{1 - y_n^2},$$

and

$$(x_{n-1} + x_n)(x_n + x_{n+1}) = \frac{x_n^4 + \alpha_n x_n^3 + \beta_n x_n^2 + \gamma_n x_n + \eta_n}{(x_n - a_n)(x_n - b_n)}.$$

The analysis we use to study them is different, but its scope in general is similar. The general framework of the analysis is that we are considering the growth of different characteristics in these two classes. We show that under certain assumptions on the coefficients, these characteristics grow exponentially. It implies that with these assumptions, these classes of equations are non-integrable. Otherwise, we have cases in which these classes of equations reduce to known integrable equations such as discrete analogues of Painlevé equations, or some of their solutions solve difference Riccati equations. We apply the same framework numerically to some ultra-discrete equations (both independent and dependent variables are discrete) in the last chapter. We illustrate there our numerical results and what they suggest in relation to the integrability of these equations.

The above framework, whether it is analytical or numerical, is the backbone of this thesis. The tools we use to execute this framework are taken from different fields which vary from number theory and complex analysis to Nevanlinna theory for rational functions over the max-plus semi-field. These tools are the height of rational numbers, the degree of rational functions over $\mathbb{C} \cup \{\infty\}$ and the max-plus Nevanlinna characteristic of continuous piecewise linear functions of a real variable. Many mathematicians in their quest for integrability used a similar approach to our framework. An essential difference is that we present a classification for the above two classes of equations in terms of the growth of the characteristic under consideration. A constructive novel method is presented in which we use a rigorous analysis examining the growth of the height of the solution of the first class of equations stated above. The basic idea of this method is related to the Diophantine integrability detector by Halburd [32]. Also, we investigate in a systematic way the degree growth of solutions of the second class of equations stated above, in which we consider these solutions as non-constant rational functions of $z \in \mathbb{C} \cup \{\infty\}$. In addition, the numerical results in the last chapter are encouraging for the start of an analytical analysis concerning an integrability detector of ultra-discrete equations.

The rest of this chapter presents a literature review, which has two main parts. The first illustrates a basic overview of the history of Painlevé differential equations along with their discrete and ultra-discrete counterparts. The second part presents four methods which are proposed as integrability detectors to single out integrable discrete systems from classes of discrete equations. We end the chapter with a summary of the rest of this thesis where we present our main results.

1.2 History of Painlevé equations

The *Painlevé equations* are six nonlinear ordinary differential equations denoted traditionally by $P_I, P_{II}, ..., P_{VI}$, and are listed in Table 1.1 below[1]. They were derived in 1895-1910 by French mathematicians Paul Painlevé (1863-1933), Gambier and their colleagues whilst studying a problem posed first by Picard. Picard's original problem is: Given $F(y', y, z)$, where F is polynomial in y', rational in y and analytic in z, what are the second order ordinary differential equations of the form

$$y'' = F(y', y, z) \tag{1.1}$$

with the property that singularities other than poles of any solution of (1.1) depend on the equation itself and not on the constants of integrations?

Painlevé [66] and Gambier [23] proved that there are 50 canonical equations of the form (1.1) that have the so-called *Painlevé property.* Simply, this property means that their movable (i.e. initial condition-dependent) singularities are just poles. Among the 50 equations, 6 (see Table 1.1) are called the Painlevé equations. Of the remaining 44, solutions for 11 could be expressed in terms of the solutions of the 6 Painlevé equations, while the other 33 are solvable in terms of solutions of linear differential equations of second or third order or in terms of elliptic functions.

Many mathematicians studied these equations which contributed greatly to the development of this field. In particular, the work of Boutroux and Garnier inspired and motivated many mathematicians and as a result there were tremendous advances in the field. Relating the integrability of the Painlevé equations to their singularity structure was always a key factor to many developments in their field of study. Note that the relation between singularity structure of differential equations and their integrability was first noticed by Kowalevskaya in her work on the equations of a spinning top in 1889-1890. Following the pioneering work of Ablowitz and Segur [2], it was shown that the Painlevé equations can be linearised in terms of integro-differential equations, using the inverse scattering transform scheme. This confirms their integrability.

Fuchs was the first to relate Painlevé equations to linear systems in 1905. His origional paper was reprinted in 2006 [14]. Also, in 1922-1923, Malmquist recognised a relation between Painlevé equations and polynomial and rational Hamiltonian systems. Later, Okamoto showed the exact forms of these Hamiltonian systems and gave each system a geometric interpretation [63, 64]. Fuchs's work was the basis for an independent

[1]Most of the historic events in the development of the field of Painlevé differential equations and the original references and papers stated here are following the treatment in [31].

point of view of Painlevé differential equations that started to blossom in the 1970s, which interested many mathematicians. It is the isomonodromy method, which was developed for the study of the Painlevé equations. Some of the papers that discussed this issue is a series of three papers by Jimbo, Miwa and their colleagues [44, 42, 43]. They showed that the Painlevé equations are the monodromy-preserving deformation of linear differential equations. This simply means that, given a system of linear differential equations, we could deform it in such a way that the monodromy group stays constant. Painlevé equations are found to be the conditions for the invariance of the monodromy, expressed in terms of the coefficients of the given linear system.

General solutions of the Painlevé equations for generic parameters are called the *Painlevé transcendents* since these equations cannot be in general integrated in terms of elementary functions or solutions of linear differential equations. Painlevé transcendents could be considered as nonlinear special functions. For some special choices of parameters, however, the Painlevé equations have solutions in terms of elementary functions or of transcendental functions such as Airy, Bessel or hypergeometric functions.

One of the reasons that Painlevé equations fascinated mathematicians is that they possess so many special features. One of these features is that, given a solution of a Painlevé equation $(P_{II}, \cdots, P_{VI})$ with choice of some parameter, a special method based on Bäcklund transformations can be used for deriving a new solution with a different value of the parameter, either for the same Painlevé equation or for another in Table 1.1 below. *Symmetry* is a word used frequently to refer to such a mechanism to construct new solutions by transformation and more on symmetry can be found in Noumi's book "Painlevé equations through symmetry" [61].

Another reason that made mathematicians become interested in Painlevé equations is that they appeared in many applications and fields such as hydrodynamics, plasma physics, nonlinear optics and solid state physics. Table 1.1 below shows the six Painlevé equations.

Table 1.1: Six Painlevé equations

$$
\begin{array}{ll}
P_I & : y'' = 6y^2 + t, \\
P_{II} & : y'' = 2y^3 + ty + \alpha, \\
P_{III} & : y'' = \frac{1}{y}(y'')^2 - \frac{1}{t}y' + \frac{1}{t}(\alpha y^2 + \beta) + \gamma y^3 + \frac{\delta}{y}, \\
P_{IV} & : y'' = \frac{1}{2y}(y')^2 + \frac{3}{2}y^3 + 4ty^2 + 2(t^2 - \alpha)y + \frac{\beta}{y}, \\
P_V & : y'' = (\frac{1}{2y} + \frac{1}{y-1})(y')^2 - \frac{1}{t}y' + \frac{(y-1)^2}{t^2}\left(\alpha y + \frac{\beta}{y}\right) + \frac{\gamma}{t}y + \delta\frac{y(y+1)}{y-1}, \\
P_{VI} & : y'' = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)(y')^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)y' \\
& \quad + \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left(\alpha + \beta\frac{t}{y^2} + \gamma\frac{t-1}{(y-1)^2} + \delta\frac{t(t-1)}{(y-t)^2}\right).
\end{array}
$$

In Table 1.1, $y = y(t)$ is the dependent variable and $' = d/dt$ stands for the derivative with respect to the independent variable t. The symbols $\alpha, \beta, \delta, \gamma$ are parameters.

Discrete analogues of the Painlevé equations had been derived using several methods. Some of these are based on orthogonal polynomials [41] or the singularity confinement method [70] (described in sub-section 1.3.1). There are also other methods for deriving integrable non-autonomous discrete equations. These are based on the discrete AKNS approach, on the use of Bäcklund and Schlesinger transforms of the continuous Painlevé equations and on the discrete analogues of the Miura transformations [21].

There are many similarities between discrete and continuous Painlevé equations. In fact, the discrete equations are richer and for each property of the continuous Painlevé equations there appears to exist a discrete analog. One of these properties is the *coalescence cascade*, which is true for both continuous and discrete Painlevé equations. Simply, it means that one could get a lower Painlevé equation from a higher one by taking an appropriate limit on the dependent and independent variables as well as the parameter in the higher equation. Hence, in the discrete case (often denoted by d-Ps), we have $d\text{-}P_{VI} \to d\text{-}P_V \to \{d\text{-}P_{IV}, d\text{-}P_{III}\} \to d\text{-}P_{II} \to d\text{-}P_I$. This particular property is described further in [54, 69, 70].

In the continuous case, there exists just one canonical form for each Painlevé equation, written as $y'' = f(y', y, t)$ with f rational in y', algebraic in y and analytic in t. In the discrete case, however, there is no unique discrete equation known as an analogue for each of the six Painlevé equations. We could get more than one discrete equation for each of the six d-Ps as long as we could obtain the continuous Ps equations through a limiting process and as long as these discrete equations share the same properties with their continuous counterparts. This is true even when we make the restriction to the three-point rational mappings, resulting from the de-autonomisation of a (Quispel, Roberts and Thompson)

QRT form [68]:

$$x_{n+1} = \frac{f_1(x_n) - x_{n-1} f_2(x_n)}{f_4(x_n) - x_{n-1} f_3(x_n)}.$$

Here, $\{f_i\}_{i=1,2,3,4}$ are specific quartic polynomials involving 5 parameters. The first discrete equation in Table 1.2 was shown to be a discrete analogue of P_I in [12]. Also, independently and using a different approach, the second discrete equation in Table 1.2 was shown to be a discrete analogue of P_{II} in [59]. The authors of [60] present in their paper other discrete equations that are analogues of P_I and P_{II}. For historical reasons, the basic forms of the first five *d*-*P*s are:

Table 1.2: First five discrete Painlevé equations

$d\text{-}P_I$	$: x_{n+1} + x_{n-1} + x_n = \frac{z}{x_n} + a,$
$d\text{-}P_{II}$	$: x_{n+1} + x_{n-1} = \frac{z x_n + a}{1 - x_n^2},$
$d\text{-}P_{III}$	$: x_{n+1} x_{n-1} = \frac{ab(x_n - p)(x_n - q)}{(x_n - a)(x_n - b)},$
$d\text{-}P_{IV}$	$: (x_{n+1} + x_n)(x_n + x_{n-1}) = \frac{(x_n^2 - a^2)(x_n^2 - b^2)}{(x_n - z)^2 - c^2},$
$d\text{-}P_V$	$: (x_{n+1} x_n - 1)(x_n x_{n-1} - 1) = \frac{pq(x_n - a)(x_n - 1/a)(x_n - b)(x_n - 1/b)}{(x_n - p)(x_n - q)},$

where $z = \alpha n + \beta$, $p = p_0 \lambda^n$, $q = q_0 \lambda^n$ and a, b, c are constants. Since the discovery of *d*-*P*s, many mathematicians considered them as an extension of the Painlevé differential equations to the discrete world. Recently, Sakai [75] showed that starting from special rational surfaces, the translation part of the corresponding affine Weyl group gives rise to *d*-*P*s. Simply, we could consider *d*-*P*s as independent discrete equations not as extensions of *P*s to the discrete world, although they share with them many similarities, as stated above, and using appropriate limits we could obtain *P*s from *d*-*P*s. Discrete analogues of Painlevé equations appear in many places throughout this thesis.

Integrability is a desirable criterion when we investigate any type of equations. Discrete analogues of Painlevé equations were shown to be integrable where one way of showing their integrability was through the associated isomonodromy method [26, 60]. In 1992, the authors of [45] constructed a discrete isomonodromy deformation problem and from it they deduced discrete versions of the first and second Painlevé equations. A different construction of discrete isomonodromy deformation problems was given in [67]. Also, integrable discrete Painlevé equations are deduced through a similarity reduction process [59]. Searching for discrete analogues for the Painlevé property became a very extensive area of research from the last decade. All the integrability detectors described in the next section are based on examining two main criteria of a discrete equation which are singularity and growth of some characteristic in the discrete equation. Singularity confinement proposed in [27] is related to the disappearance of the singularity of a discrete equation after a few iterations with the perseverance of the initial data. Veselov [82] and Arnold [4] were the first to notice the connection between growth and integrability. Veselov, in his paper, stated that "$\cdots$ *integrability has an essential correlation with the weak growth of certain characteristics*". Many integrability detectors such as algebraic entropy, Diophantine integrability [32] and Nevanlinna theory approach [1, 34, 33] are linked somehow to Veselov's observation. Viallet and his colleagues interpreted Arnold's idea of complexity by proposing a detector of integrability based on algebraic entropy [39, 8].

A blossoming area of research concerning ultra-discrete equations started to gain much attention recently. Ultra-discrete equations are discrete in both independent and dependent variables. We could obtain this kind of equation systematically, starting from a discrete equation through a limiting procedure called ultra-discretisation which was introduced first in [79, 55]. In this procedure, we write a solution x of a discrete equation as $x = \exp\left(\frac{X}{\delta}\right)$ and obtain an ultra-discrete equation in X when we take the limit $\delta \longrightarrow 0^+$, using the ultra-discretisation identity

$$\lim_{\delta \to 0^+} \delta \log\left(\exp\left(\frac{A}{\delta}\right) + \exp\left(\frac{B}{\delta}\right)\right) = \max(A, B).$$

In Chapter 5, we present the algebra associated with this identity. This procedure is used to relate an integrable soliton cellular automata system (known as box-ball system) [78] to an integrable difference equation related to KdV equation [79]. We could obtain ultra-discrete analogues of Painlevé equations from *d-P*s through the ultra-discretisation procedure [25]. These equations share at least some of the same beautiful properties as their continuous and discrete counterparts. One of these is that they have special solutions such as rational and hypergeometric [65].

The integrability of some ultra-discrete analogues of Painlevé equations is shown through the associated isomonodromy problem formulated by Joshi and Ormerod in [48]. In [47], the authors derived a Lax pair for an ultra-discrete analogue of P_{III}. A test for integrability of cellular automata by Joshi and Lafortune was introduced in 2005 [46]. This test is an analogue of the singularity confinement in discrete equations. In [29], the authors analysed this method further and showed that there is a non-integrable system which satisfies this test as well as the integrable ones. Research is still very active in this area.

1.3 Integrability detectors

In this section, we present four methods used as detectors of integrability for discrete equations. The first is the *singularity confinement* method. This method was first presented as a detecter of integrability of discrete mappings in [27] but it was also used to recover many discrete analogues of Painlevé equations [70]. In [39], it was shown that a non-integrable mapping could pass this test too and the authors of [39] suggested that the method could be more effective if augmented with another criterion. They introduced a test based on the degree growth of some initial data under the action of the equation and this test is called *algebraic entropy* which is the second detector presented here. The third and the fourth detectors are *Diophantine integrability* and *Nevanlinna theory approach*, respectively.

1.3.1 Singularity confinement

This method was first introduced in 1991 by Grammaticos, Ramani and Papageorgiou in [27]. They considered singularity confinement as the discrete analogue of the Painlevé property for continuous systems. The basic idea of singularity confinement relies on the observation that for integrable mappings, the singularities that may appear are confined, i.e., they do not propagate indefinitely when one iterates the mapping.

Now let us illustrate how we could apply this method to a discrete mapping by means of the next example, as shown in [27]. Let us start with the nonautonomous mapping (this mapping is the nonautonomous case of the generalised McMillan mapping after rescaling the variable),

$$x_{n+1} + x_{n-1} = \frac{-(x_n^2 + B(n)x_n + C(n))}{x_n(x_n+1)}. \tag{1.2}$$

It is clear that the mapping (1.2) has two singular points, that is when $x_n = 0$ and $x_n = -1$. First, let us consider the first singularity $x_n = 0$; at this point, the mapping is infinite. Therefore, let us assume that all the previous iterations x_is where $i < n$ are finite values. Now let $x_{n-1} = k$, $x_n = \varepsilon$, where k is a finite value and $\varepsilon > 0$ such that $x_n \longrightarrow 0$ when $\varepsilon \longrightarrow 0$. Solving (1.2) for x_{n+1} and iterating the resulting equation as a power series in ε to get x_{n+2} and x_{n+3} (we did some algebraic calculations in Mathematica to perform these iterations) yields

$$x_{n+1} = \frac{-C(n)}{\varepsilon} + (C(n) - B(n) - k) + (B(n) - C(n) - 1)\varepsilon + O(\varepsilon^2), \tag{1.3}$$

$$x_{n+2} = -1 + \frac{(B(n+1) - C(n) - 1)}{C(n)}\varepsilon + O(\varepsilon^2), \tag{1.4}$$

$$x_{n+3} = \frac{\frac{C(n)(B(n+1)-C(n)+C(n+2)-B(n+2))}{(B(n+1)-C(n)-1)}}{\varepsilon} + O(\varepsilon^0). \tag{1.5}$$

Now from (1.3), it is obvious that x_{n+1} is infinite if $\varepsilon \longrightarrow 0$ but in (1.4) x_{n+2} is a finite value. From (1.5), it is clear that in order for x_{n+3} to be a finite value when $\varepsilon \longrightarrow 0$, we should have the following equation

$$\frac{C(n)(B(n+1) - C(n) + C(n+2) - B(n+2))}{(B(n+1) - C(n) - 1)} = 0.$$

This leads to the following condition for singularity confinement,

$$C(n+1) - C(n-1) + B(n) - B(n+1) = 0. \tag{1.6}$$

Let us consider now the second singularity, that is, when $x_n = -1$. Similarly, assume that all the previous iterates x_is where $i < n$ are finite values. Put $x_{n-1} = k_1$ and $x_n = -1+\varepsilon$, where k_1 is a finite value and $\varepsilon > 0$ such that if $\varepsilon \longrightarrow 0$, then $x_n \longrightarrow -1$. Solving (1.2) for x_{n+1} and iterating the resulting equation as a power series in ε to get x_{n+2} and x_{n+3} leads to the following equations,

$$x_{n+1} = \frac{1 - B(n) + C(n)}{\varepsilon} + (C(n) - k_1 - 1) + C(n)\varepsilon + O(\varepsilon^2), \tag{1.7}$$

$$x_{n+2} = \frac{B(n+1) - B(n) + C(n)}{B(n) - C(n) - 1}\varepsilon + O(\varepsilon^2), \tag{1.8}$$

$$x_{n+3} = \frac{\frac{(B(n)-C(n)-1)(B(n+1)-B(n)+C(n)-C(n+2))}{B(n+1)-B(n)+C(n)}}{\varepsilon} + O(\varepsilon^0). \tag{1.9}$$

As before, in (1.7) x_{n+1} is infinite when $\varepsilon \longrightarrow 0$ but from (1.8) it is clear that x_{n+2} is finite. For x_{n+3} in (1.9) to be finite when $\varepsilon \longrightarrow 0$, we require that

$$\frac{(B(n) - C(n) - 1)(B(n+1) - B(n) + C(n) - C(n+2))}{B(n+1) - B(n) + C(n)} = 0.$$

This gives us the second condition for singularity confinement,

$$B(n) - B(n-1) + C(n-1) - C(n+1) = 0. \tag{1.10}$$

Adding (1.6) and (1.10), then multiplying the result by -1, we get

$$B(n+1) - 2B(n) + B(n-1) = 0. \tag{1.11}$$

This implies that

$$B(n) = \lambda n + \mu, \tag{1.12}$$

where λ and μ are arbitrary constants. Substituting (1.12) into (1.6) yields this relation for $C(n)$

$$C(n) = \lambda\frac{n}{2} + \nu + \rho(-1)^n, \tag{1.13}$$

where ν and ρ are arbitrary constants. Having these expressions for $B(n)$ and $C(n)$ (with $\rho = 0$) and by setting $z_n = 2x_n + 1$, equation (1.2) transforms to

$$z_{n+1} + z_{n-1} = \frac{z_n(\alpha n + \beta) + \gamma}{1 - z_n^2}. \tag{1.14}$$

This is precisely the discrete analogue of P_{II} (d-P_{II}) given in Table 1.2. We note that applying this method of singularity confinement allows us to find some conditions on the coefficients such that the singularities are confined, i.e., we were able to cancel them out after a finite number of iterations. This algorithm allows us also to recover a discrete analogue of Painlevé equation II (i.e. d-P_{II}) in this case. Similarly, we could recover a discrete analogue of Painlevé equation I (i.e. $x_{n+1} + x_{n-1} + x_n = b + \frac{\alpha n+\beta}{x_n}$) if we apply this method to the following system

$$x_{n+1} + x_{n-1} = -x_n + B(n) + \frac{C(n)}{x_n}. \tag{1.15}$$

A question to be asked here: Are the singularities really confined by considering only a finite number of iterations? Also, the authors of [40] showed that when (i.e. at which step of the iteration) we impose a singularity condition is crucial. If we did not impose a singularity condition when it first appears and impose another condition at a later iterate, then the system is believed to be not integrable since the degree grow exponentially as shown in [40]. In our analysis in Chapter 3, when we comment on the similarities and differences between our method and singularity confinement, we show that this weakness of the latter does not exist in our method. Although singularity confinement did not hold as a sufficient condition for integrability, it is still a useful method for recovering some of the discrete analogues of Painlevé equations, as we illustrated using examples in this sub-section. In the next sub-section, we present another algorithmic method for testing discrete systems for integrability. This method depends on calculating the algebraic entropy of the system.

1.3.2 Algebraic entropy

In this sub-section, we discuss the integrability of two examples of discrete maps. In these examples, we use a different detector of integrability based on algebraic entropy of a discrete system, as shown in [39]. It was first introduced by Viallet *et al.* [39, 8]. Basically, algebraic entropy is a measure of complexity of a map using the degree growth of its iterates. Usually, integrability is associated with polynomial growth (slow growth) while exponential growth (fast growth) is associated with chaotic systems.

The idea of degree growth interested many mathematicians. In 2000, the authors of [71] studied a class of linearisable mappings which are second-order discrete systems and investigated their degree growth. In [50], the authors presented a method for constructing an integrable Nth-order mapping with degree growth n^N. Also in this paper, they investigated the degree growth of integrable third-order mappings. An approach which is based on group theory and algebra for the study of degree growth is adopted by the authors of [20, 3]. In [38], the authors explored the idea of degree growth and algebraic entropy for a particular class of rational maps, which is monomial maps in the projective space. Their analysis used the algebraic geometry setting. A study of the degree growth of higher order maps is given in [10, 7].

We follow the same steps to present this method, as shown in [39]. Let us start with the first example taken from [39],

$$x_{n+1} + x_{n-1} = x_n + \frac{a}{x_n^2}, \tag{1.16}$$

where a is an arbitrary nonzero constant. In order to study the degree growth of the iterates in (1.16), let us define a map Ω as follows:

$$\Omega : P_n = (x_{n-1}, x_n) \longrightarrow P_{n+1} = (x_n, x_{n+1}). \tag{1.17}$$

We rewrite (1.17) in terms of homogeneous coordinates (y_n, z_n, t_n) by setting

$$P_n = \left(\frac{z_n}{t_n}, \frac{y_n}{t_n}\right). \tag{1.18}$$

Now this means that we are working in the two-dimensional projective space $\mathbb{C}\mathcal{P}^2$. In $\mathbb{C}\mathcal{P}^2$, the points (y, z, t) and $(\lambda y, \lambda z, \lambda t)$ are the same. This implies that the map (1.17) can be written in the following way:

$$\Omega : \begin{pmatrix} y \\ z \\ t \end{pmatrix} \longrightarrow \begin{pmatrix} y^3 + at^3 - y^2 z \\ y^3 \\ ty^2 \end{pmatrix}. \tag{1.19}$$

The map Ω singularity arose if we start with $(0, u, 1)$ in Ω and iterate. After a few iterations we get $(0, 0, 0)$ which is not in $\mathbb{C}\mathcal{P}^2$. To clarify the situation we expand around the singularity and start with the point $(\varepsilon, u, 1)$ in (1.19), then we get the following sequence:

$$\begin{pmatrix} \varepsilon \\ u \\ 1 \end{pmatrix} \longrightarrow \begin{pmatrix} a - u\varepsilon^2 + \varepsilon^3 \\ \varepsilon^3 \\ \varepsilon^2 \end{pmatrix} \longrightarrow \begin{pmatrix} a^3 - 3a^2u\varepsilon^2 + \ldots - u\varepsilon^8 \\ a^3 - 3a^2u\varepsilon^2 + \ldots + \varepsilon^9 \\ a^2\varepsilon^2 + \ldots + \varepsilon^8 \end{pmatrix} \longrightarrow$$

$$\begin{pmatrix} -a^8\varepsilon^3 + \ldots - u^2\varepsilon^{25} \\ a^9 - 9a^8u\varepsilon^2 + \ldots - u^3\varepsilon^{24} \\ a^8\varepsilon^2 + \ldots + u^2\varepsilon^{24} \end{pmatrix} \longrightarrow \begin{pmatrix} a^{24}u\varepsilon^8 + \ldots - u^6\varepsilon^{75} \\ -a^{24}\varepsilon^9 + \ldots - u^6\varepsilon^{75} \\ a^{24}\varepsilon^8 + \ldots + u^6\varepsilon^{74} \end{pmatrix} = \begin{pmatrix} a^{24}u + \ldots - u^6\varepsilon^{67} \\ -a^{24}\varepsilon + \ldots - u^6\varepsilon^{67} \\ a^{24} + \ldots + u^6\varepsilon^{66} \end{pmatrix}.$$

In the last term, we cancelled the factor ε^8, and if we let $\varepsilon \longrightarrow 0$, then the last term is $(a^{24}u, 0, a^{24})$. This shows that the sequence survives the singularity with the initial data u. Without the cancellation, we will not be able to emerge from the singularity, so the cancellation is necessary to survive the singularity. Also, it reduces the growth of the degree of the nth iterate Ω^n of Ω. Note that we mean by the degree of the iterate the highest degree of the term $u^i\varepsilon^j$, where the degree of $u^i\varepsilon^j$ is considered as the sum of both exponents of u^i and ε^j, respectively, $i + j$. If we did not have a cancellation, then the degree of the iterate Ω^n is d^n where d is the degree of Ω. For the map (1.19), the sequence of the degree growth of the iterates is $1, 3, 9, 27, 73, 195, 513, 1347, 3529, \ldots$. Actually, it has been noted in [39] that there is a generating function for the degrees which is

$$\begin{aligned} g &= \frac{1 + 3x^3}{(1-x)(1+x)(x^2 - 3x + 1)} \\ &= 1 + 3x + 9x^2 + 27x^3 + 73x^4 + 195x^5 + 513x^6 + 1347x^7 + \ldots. \end{aligned} \tag{1.20}$$

Note that the coefficient of x^n is the degree of Ω^n denoted by d_n. If α is the smallest modulus of the roots of the denominator of (1.20), then $d_{n+1} \approx \alpha^{-1}d_n$ asymptotically. Here, $\alpha = \frac{3-\sqrt{5}}{2}$ and we define the algebraic entropy of the map (1.19) by

$$\mathcal{E} = \lim_{n\to\infty} \left(\frac{1}{n}\log(d_n)\right) = \log\left(\frac{3+\sqrt{5}}{2}\right).$$

This implies that the map (1.19) has a nonvanishing entropy and, hence, is likely to be non-integrable. In [39], this specific example was used to show that it passed the singularity confinement test although it is showing numerical chaos in the picture of the orbit of its map. In that paper, the authors used the algebraic entropy as a sensitive criterion that possibly could be used as a detector of integrability.

Now let us consider the next example which is related to *d*-P_I [21]. This example is taken from [39]:

$$x_{n+1} + x_{n-1} = \frac{a}{x_n^2} + \frac{b}{x_n}, \tag{1.21}$$

where a and b are arbitrary constants. If we rewrite (1.21) in $\mathbb{C}\mathcal{P}^2$, then similar to the previous example, we get the following map:

$$\Omega : \begin{pmatrix} y \\ z \\ t \end{pmatrix} \longrightarrow \begin{pmatrix} at^3 + bt^2 y - y^2 z \\ y^3 \\ ty^2 \end{pmatrix}. \tag{1.22}$$

Here again, the map (1.21) has a singularity when we start with $(0, u, 1)$, so if we expand around the singularity and iterate (1.22), we get the following sequence:

$$\begin{pmatrix} \varepsilon \\ u \\ 1 \end{pmatrix} \longrightarrow \begin{pmatrix} a + b\varepsilon - u\varepsilon^2 \\ \varepsilon^3 \\ \varepsilon^2 \end{pmatrix} \longrightarrow \begin{pmatrix} -a^2\varepsilon^3 - ab\varepsilon^4 + \ldots - u^2\varepsilon^7 \\ a^3 + 3a^2 b\varepsilon + \ldots - u^3\varepsilon^6 \\ a^2\varepsilon^2 + 2ab\varepsilon^3 + \ldots + u^2\varepsilon^6 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} a^6 u\varepsilon^8 + \ldots - u^7\varepsilon^{20} \\ -a^6\varepsilon^9 + \ldots - u^6\varepsilon^{21} \\ a^6\varepsilon^8 + \ldots + u^6\varepsilon^{20} \end{pmatrix} = \begin{pmatrix} a^6 u + \ldots - u^7\varepsilon^{12} \\ -a^6\varepsilon + \ldots - u^6\varepsilon^{13} \\ a^6 + \ldots + u^6\varepsilon^{12} \end{pmatrix}.$$

Here, in the last term, we took the factor ε^8 and cancelled it out. If we let $\varepsilon \longrightarrow 0$ in the last term, therefore we have $(a^6 u, 0, a^6)$. Hence, the sequence emerged from the singularity with the initial data u. For the map (1.22), the sequence of the degrees of the iterates is $1, 3, 9, 19, 33, 51, 73, 99, 129, 163, \ldots$. It is noted also in [39] that there is a generating function for this sequence:

$$\begin{aligned} g &= \frac{1 + 3x^2}{(1 - x)^3} \\ &= 1 + 3x + 9x^2 + 19x^3 + 33x^4 + 51x^5 + 73x^6 + 99x^7 + \ldots, \end{aligned} \tag{1.23}$$

where the coefficients of x^ns are the degrees of Ω^ns for n non-negative integer. Note that the degree growth is slower than the degree growth in (1.19). This means that we have more cancellations of the factor ε in many iterations like the above which reduce the degree growth. The growth of the degree is polynomial. Actually, the degree of the map (1.22) grows according to the rule

$$d_n = 2n^2 + 1, \tag{1.24}$$

where d_n is the degree of Ω^n. Let us calculate the algebraic entropy for this map:

$$\mathcal{E} = \lim_{n\to\infty} \left(\frac{1}{n} \log(d_n) \right) \longrightarrow 0.$$

Since the discrete equation is related to d-P_I [21], this suggests that vanishing algebraic entropy is related to integrability.

As we illustrated above, the exponential growth of the degree of the iterates is associated with chaotic systems. This also means nonvanishing algebraic entropy as in map (1.16). On the other hand, the polynomial growth of the degree of the iterates is associated with integrable systems, which yields vanishing algebraic entropy. In the next sub-section, we introduce another sensitive detector of integrability which depends on the height of the iterates. Algebraic entropy and height are strongly related. Both are measures of complexity. In arithmetic geometry, height measures arithmetical complexity of points on varieties, while in dynamical systems, entropy measures the orbit complexity of maps [19].

1.3.3 Diophantine integrability

In this sub-section, we describe one of the recent results in the field of discrete equations and maps, for which Diophantine integrability is a test of integrability. As we described in the previous sub-section, integrability was associated with slow growth of the degree of the iterates under the action of the map. Diophantine integrability is similar in this sense and is based on the slow growth of the height of the iterates of a discrete map. A discrete equation

$$y_{n+1} = f(n, y_n, y_{n-1}),$$

where f is a rational function in the previous iterates y_n, y_{n-1}, is considered Diophantine integrable if the logarithmic height of the iterates $h(y_n) = \log H(y_n)$ (i.e. $H(x) = \max\{|p|, |q|\}$ for $x = \frac{p}{q} \in \mathbb{Q} \setminus \{0\}$ where p and q are coprime) grows no faster than polynomial in n. In [32], the author illustrated his method by a number of examples which show how efficient and quick numerically this method could be. We could check the growth of the height of a large number of iterates in a short time which give you an indication of the Diophantine integrability of a discrete equation. Recall that in sub-section 1.3.2, we stated that in arithmetic geometry height measures arithmetical complexity of points on varieties. Hence, the height $H(x)$ of an element x of a number field K is a measure of the complexity of x. Here, the number field we now consider is the rational number field, so $K = \mathbb{Q}$. For any non-zero $x \in \mathbb{Q}$, its height is $H(x) = \max\{|p|, |q|\}$, where $x = \frac{p}{q}$ and $\gcd(p, q) = 1$. By definition, $H(0) = 1$.

We illustrate the Diophantine integrability method by the next example. Consider the following discrete equation:

$$y_{n+1} + y_{n-1} = \frac{a_n}{y_n} + b_n. \tag{1.25}$$

Note that if $a_n = \lambda n + \mu$ and $b_n = \nu$, where λ, μ and ν are constants, then (1.25) is believed to be an integrable discrete equation related to *d-*P_I. Solving (1.25) for y_{n+1} and starting with initial data $y_0, y_1 \in \mathbb{Q}$ and choosing a_n, b_n to be in $\mathbb{Q}$ for all $n \in \mathbb{Z}$, we get a sequence of iterates $\{y_n\} \subset \mathbb{Q}$. If we take the height and then the logarithmic height for each iterate of this sequence, we get another sequence $\{h(y_n)\}$. By plotting $\log h(y_n)$ versus $\log n$, the resulting graph would be asymptotically a straight line in an integrable case ($a_n = 3, b_n = 5, y_0 = 2/5, y_1 = 3/7$) and asymptotically a nonlinear curve corresponds to a non-integrable case ($a_n = 3, b_n = 5 + n, y_0 = 2/5, y_1 = 3/7$), as shown in Figure 1.1 below.

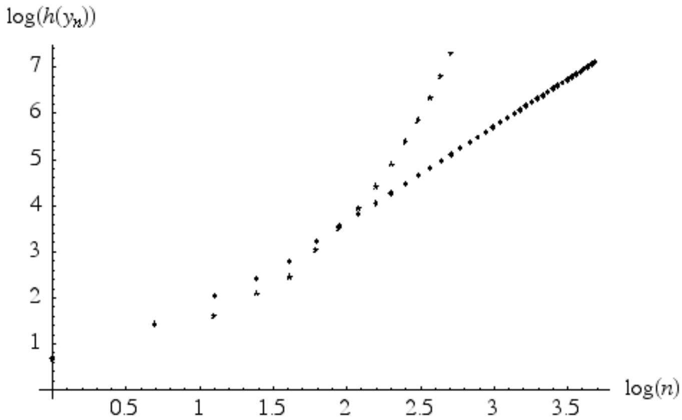


Figure 1.1: Plot of $\log h(y_n)$ versus $\log n$ for equation (1.25)

The above example shows us how quick this method is in giving an indication about the Diophantine integrability of a discrete map or an equation in question. We could write simple calculations in our computer using mathematical packages (e.g. Mathematica or Maple) to calculate the sequence of the logarithmic heights of our iterates and plot the graph of $\log h(y_n)$ versus $\log n$ in a short time, no matter how complicated our discrete equation. Figure 1.1 is plotted using Mathematica. This test proved until now to be a very powerful tool for testing discrete equations for integrability in a sufficient amount of time. We expand more (in an analytical and rigorous way) on the idea of Diophantine integrability in Chapter 3, where we analyse a certain class of discrete equations. Our analysis depends on studying the growth of the height of a particular type of solutions of this class of equations. This implies a classification of the class according to the height growth of its solution. Another test for integrability which recently attracted great attention is Nevanlinna theory approach. We explore this test more in the next sub-section.

1.3.4 Nevanlinna theory approach

The principal idea behind this approach is given first by Ablowitz and collaborators in [1]. They considered discrete equations in the complex domain as difference or delay equations. This enabled them to search for conditions formulated in the complex analysis language under which a difference equation is considered to be integrable. Their analysis implied that integrability of many difference equations is linked to the structure of their solutions at infinity in the complex plane. Since Nevanlinna theory (described in Appendix A) is the study of value distribution of meromorphic functions in the complex plane, it provides all the necessary tools and concepts needed in their investigations. They suggested that the order (i.e. $\sigma = \limsup_{r\to\infty} \frac{\log T(f,r)}{\log r}$) of a meromorphic solution of a difference equation plays a crucial role in their integrability. Their results show that in a certain class of difference equations, known integrable difference equations in this class have finite order meromorphic solutions. In particular, they considered

$$y(z+1) + y(z-1) = R(z; y(z)), \tag{1.26}$$

where R is rational in both of its arguments. They showed that if (1.26) admits at least one non-rational finite order meromorphic solution, then $\deg_y(R) \leq 2$. Note that (1.26) class of equations includes the difference Painlevé II (i.e. $y(z+1)+y(z-1) = \frac{(\lambda z+\mu)y(z)+\nu}{1-y^2(z)}$) and many other equations considered to be non-integrable. The existence of a finite number of finite order meromorphic solutions of (1.26) is not always enough to single out the difference Painlevé II from (1.26).

In [30, 72], the authors used the Ablowitz *et al.* [1] idea and complement it with the singularity confinement method to produce a new test for integrability. They used this test to recover known forms of d-Ps and to show that no new ones may exist within a given parametrisation. A few years later, Halburd and Korhonen [33, 35] explored the Ablowitz *et al.*[1] idea further. They showed that the existence of sufficiently many finite order meromorphic solutions is a good analogue of the Painlevé property for discrete equations, in which the independent variable is taken to be complex. Finally, we end this sub-section with a result proved in [34] by Halburd and Korhonen, who presented this interesting classification theorem. In Chapter 3, we illustrate the analogy between Theorem 1.3.4.1 in Nevanlinna theory and our analysis in Diophantine approximation. In the theorem below, an admissible solution $y(z)$ simply means it is growing faster than any of the coefficients of R in the equation (1.26) in the sense of Nevanlinna theory. Also, $\mathcal{S}(y)$ denotes the field of small functions with respect to $y(z)$ in terms of Nevanlinna theory.

Theorem 1.3.4.1. *If the equation (1.26), where $R(z; y(z))$ is rational in y and meromorphic in z, has an admissible meromorphic solution of finite order, then either y satisfies a difference Riccati equation*

$$y(z+1) = \frac{p(z+1)y(z)+q}{y(z)+p},$$

where $p, q \in \mathcal{S}(y)$, or equation (1.26) can be transformed by a linear change in y to one of the following equations:

$$\begin{aligned}
y(z+1)+y(z)+y(z-1) &= \frac{\pi_1 z+\pi_2}{y(z)}+\kappa_1\\
y(z+1)-y(z)+y(z-1) &= \frac{\pi_1 z+\pi_2}{y(z)}+(-1)^z\kappa_1\\
y(z+1)+y(z-1) &= \frac{\pi_1 z+\pi_3}{y(z)}+\pi_2\\
y(z+1)+y(z-1) &= \frac{\pi_1 z+\kappa_1}{y(z)}+\frac{\pi_2}{y(z)^2}\\
y(z+1)+y(z-1) &= \frac{(\pi_1 z+\kappa_1)y(z)+\pi_2}{(-1)^{-z}-y(z)^2}\\
y(z+1)+y(z-1) &= \frac{(\pi_1 z+\kappa_1)y(z)+\pi_2}{1-y(z)^2}\\
y(z+1)y(z)+y(z)y(z-1) &= p\\
y(z+1)+y(z-1) &= py(z)+q
\end{aligned}$$

where π_k, $\kappa_k \in \mathcal{S}(y)$ are arbitrary finite order periodic functions with period k.

1.4 Main results and structure of thesis

The main result of this thesis concerns the height growth of solutions of the equation

$$y_{n+1}+y_{n-1} = \frac{a_n+b_n y_n+c_n y_n^2}{1-y_n^2}, \tag{1.27}$$

where a_n, b_n and $c_n \in \mathbb{Q}, \forall n$. The result which we state as Theorem 1.4.1 below concerns admissible solutions. A solution of equation (1.27) is called admissible if the height of y_n grows fast compared to the height of the coefficients a_n, b_n and c_n. A formal definition of admissible solution is given in Definition 3.1.2.

Theorem 1.4.1. *Let* $(y_n) \subset \mathbb{Q} \setminus \{-1, 1\}$ *be an admissible solution of (1.27), where* a_n*,* b_n *and* c_n *are rational functions of* n *with coefficients in* $\mathbb{Q}$ *and the right hand side of (1.27) is irreducible. Then either*

1. $a_n = \alpha n + \beta$, $b_n = \gamma$, $c_n = 0$ *for constants* α, β, γ*; or*

2. y_n *is also an admissible solution of the difference Riccati equation*

$$y_{n+1} = \frac{1/2(a_n + \theta b_n - 2\theta) + y_n}{1 - \theta y_n}, \quad \text{where } \theta = -1 \text{ or } 1; \text{ or} \tag{1.28}$$

3. $$\limsup_{r \to \infty} \frac{\log \log \sum_{n=r_0}^{r} h(y_n)}{\log r} \geq 1.$$

Of the three possible outcomes described in Theorem 1.4.1, the first says that equation (1.27) is the discrete analogue of P_{II} given in Table 1.2. The second says that y_n solves a famous linearisable first-order equation and the third implies that $h(y_n)$ does not grow polynomially. If equation (1.27) has more than two one-parameter families of admissible solutions, they cannot all solve difference Riccati equations of the form described by the second conclusion. Hence, the theorem says that if for all admissible solutions $h(y_n)$ grows polynoimally, equation (1.27) is the discrete analogue of the second Painlevé equation.

The idea of Diophantine integrability is a property of all solutions, not just admissible ones. Our methods only allow us to work with one solution at a time. An admissibility-type condition is necessary to avoid counterexamples, some of which can be easily constructed in which the height of the solution grows at approximately the same rate as the heights of the coefficients. This will be discussed in Chapter 3.

The central idea of the proof of Theorem 1.4.1 relies on the fact that there is a simple relationship between the height of a rational number x and a certain sum over all non-trivial absolute values of x. These absolute values consist of the p-adic absolute values (which are non-Archimedean, see Chapter 2) and the usual absolute value (which is Archimedean). In Chapter 3, for each absolute value a sequence (ϵ_n) will be defined in terms of the absolute value of certain combinations of the coefficients of equation (1.27) as given in (3.51). This sequence defines a way of measuring "small" quantities. A key step on our way to proving Theorem 1.4.1 is the following

Theorem 1.4.2. *Let* $(y_n)_{n=k-1}^{k+3} \subset \mathbb{Q} \setminus \{-1, 1\}$ *satisfy*

$$y_{n+1} + y_{n-1} = \frac{a_n + b_n y_n}{1 - y_n^2},$$

where k is sufficiently large and the right hand side of the equation is irreducible. Assume that for a fixed absolute value $|.|_p$ ($\forall p \leq \infty$) we have $|y_{k-1}|_p \leq |1-\theta y_k|_p^{-1/2}$ for $\theta = 1$ or -1. Furthermore, for sufficiently small $\delta > 0$, if $|1-\theta y_k|_p < \epsilon_k$ (where ϵ_k is defined in (3.51)), then

1. $y_{k+1} = \dfrac{a_k + \theta b_k}{2(1-\theta y_k)} + A_k$, *where* $|A_k|_p \leq |1-\theta y_k|_p^{-1/2}$ *for non-Archimedean absolute value and* $|A_k|_p \leq \frac{11}{10} \cdot |1-\theta y_k|_p^{-1/2}$ *for Archimedean absolute value.*

2. $y_{k+2} = -\theta + \left(\dfrac{\theta a_k + b_k - 2b_{k+1}}{a_k + \theta b_k}\right)(1-\theta y_k) + B_k$,
   where $|B_k|_p \leq |1-\theta y_k|_p^{3/2-5\delta}$ *for non-Archimedean absolute value and* $|B_k|_p \leq \frac{1}{2} \cdot |1-\theta y_k|_p^{3/2-5\delta}$ *for Archimedean absolute value.*

3. $y_{k+3} = \dfrac{\big(a_{k+2} - \theta b_{k+2} - \theta(\theta a_k + b_k - 2b_{k+1})\big)}{2(1+\theta y_{k+2})} + C_k$
   where $|C_k|_p \leq |1+\theta y_{k+2}|_p^{-(2/3+2\delta)}$ *for non-Archimedean absolute value and* $|C_k|_p \leq 2|1+\theta y_{k+2}|_p^{-(2/3+2\delta)}$ *for Archimedean absolute value.*

We can think of Theorem 1.4.2 as a way of expressing singularity (non-) confinement in terms of absolute values. It should be stressed, however, that we do not make assumptions about the long term behaviour of solutions or whether they are eventually confined. Theorem 1.4.2 is used to estimate certain quantities measuring how close y_n is to the special values ± 1 and ∞.

The rest of the chapters is structured as follows. Chapter 2 introduces some background material serving as a wide base for our subsequent work. We discuss some of the important properties of height.

Our original work starts in Chapter 3 and ends in Chapter 5. As outlined above Chapter 3 is devoted to the proof of Theorem 1.4.1, where we prove key lemmas and theorems to achieve this. In section 3.1, we study (1.27) and find that the logarithmic height of an admissible solution is not growing polynomially if $c_n \not\equiv 0$ or ± 2. This result is attained through Lemma 3.1.1 and Theorem 3.1.1. In section 3.2, we study a sub-class of (1.27) when $c_n \equiv 0$. We find that in this sub-class either the logarithmic height of an admissible solution is not growing polynomially with n or the equation reduces to a discrete analogue of P_{II} or (y_n) solves a difference Riccati equation. This result is reached through Theorem 1.4.2, Corollary 3.2.1 and the discussion in the section. The last section in this chapter (section 3.3) analyses the remaining sub-class of (1.27) with $c_n \equiv \pm 2$. The treatment and the analysis in this section are similar to those of section 3.2. The results of this section are summarised in Theorem 3.3.1 and Corollary 3.3.1.

In Chapter 4, we study a class of difference equations (namely $(x_{n-1}+x_n)(x_n+x_{n+1}) = \frac{P_n(x_n)}{(x_n-a_n)(x_n-b_n)} = \frac{x_n^4+\alpha_n x_n^3+\beta_n x_n^2+\gamma_n x_n+\eta_n}{(x_n-a_n)(x_n-b_n)}$), where our analysis depends on the degree growth of the solution x_n in terms of an external variable z to the equation rather than its height.

Since we consider x_n $\forall n \in \mathbb{Z}$ to be a rational function in z, we discuss in section 4.1 the degree of rational functions where the independent variable $z \in \mathbb{C} \cup \{\infty\}$. We study the degree growth of non-constant rational function x_n ($\deg_z(x_n)$) in the above class of equations in section 4.2. We show in Theorem 4.2.1 that if $\alpha_n \not\equiv \mu_{-1} + \mu_1 - a_n - b_n$ (where $\mu_i \in \{a_{n+i}, b_{n+i}\}$) for all $n \in \mathbb{Z}$, then $\sum_{n=r_0}^{r} \deg_z(x_n) \geq 2^r K$ for non-zero K. This implies that $\deg_z(x_n)$ grows fast with n. Hence, it suggests that the equation is non-integrable, where we could see the connection with the algebraic entropy approach. We prove in Theorem 4.2.2 that if the equation coefficients satisfy certain assumptions then either $\sum_{n=r_0}^{r} \deg_z(x_n) \geq 2^{\lfloor r/2 \rfloor} K$ or $P_n(x_n)$ have some special forms. Analysing the results of these two theorems leads us to a case where we could reduce the equation to a discrete analogue of P_{IV}.

Chapter 5 is concerned with ultra-discrete equations. In section 5.1, we lay the algebraic setting for this kind of equation, so we introduce the max-plus semi-field. In section 5.2, we present some preliminary numerical results obtained when we tried to extend the Ablowitz *et al.*[1] idea to the ultra-discrete equations. The numerical results suggest that the solution of integrable ultra-discrete equations is of finite order (in Nevanlinna theory sense). We believe that the finite order criterion could be used as a necessary condition for the integrability of ultra-discrete equations. This could serve as a basis for a proposed integrability detector of ultra-discrete equations. Chapter 6 gives a summary of the whole thesis and outlines some future work.

Appendices A and B have topics mentioned in various places in this thesis. We do not consider them as a basis for our work, so we did not include them in the background material chapter (Chapter 2). These appendices are an overview of Nevanlinna theory and the differential and difference Riccati equations.

Chapter 2

Some topics from number theory

A general glance at the topics of this chapter shows that they may not be related. Actually, these topics serve as a wide base with which our analysis in the next chapter is linked. This chapter is split into two main sections concerned with rational points on elliptic curves and p-adic absolute values. The first section considers rational points on elliptic curves, where we highlight some of the main properties of these rational points from geometric and algebraic points of view. A very important property of these points on elliptic curves is the group law, in which we recover a discrete equation when we construct its algebraic formula. This discrete equation is the autonomous version of a well known discrete Painlevé equation. The logarithmic height of these rational points is bounded by a quadratic function. This proves in particular that the equation is Diophantine integrable. Our analysis in Chapter 3 is based on a tool from number theory, namely height. The height of a rational number x is related to a certain sum that involves all the non-trivial absolute values over $\mathbb{Q}$ of x. In the second section we present a type of absolute value on $\mathbb{Q}$, called p-adic absolute values. We give an expression for the height function in terms of these absolute values in Lemma 2.2.1 and give some of the height properties.

2.1 Rational points on elliptic curves

In this section, we explore some of the algebraic and geometric structures of elliptic curves over $\mathbb{Q}$. We describe the beautiful properties of these curves, by showing the group law for rational points on them. In addition, we state a very powerful theorem which shows the algebraic structure of rational points on these curves. This is called *Mordell's theorem*. Then we give a definition of the height function for rational points on elliptic curves. The treatment and structure in this section follow closely the treatment and structure given in [77].

2.1.1 Group law and heights of rational points on elliptic curves

An equation whose coefficients and solutions are integers or rational numbers is called a *Diophantine equation.* We consider the two-variable Diophantine equation

$$f(x, y) = 0,$$

where f is polynomial in x and y. If f is of degree 3, then the above equation is called a cubic equation. Any cubic equation can be mapped to the *Weierstrass normal* form

$$y^2 = f(x) = 4x^3 - g_2x - g_3, \tag{2.1}$$

where g_2 and g_3 are arbitrary constants, or in a more general form

$$y^2 = f(x) = x^3 + ax^2 + bx + c, \tag{2.2}$$

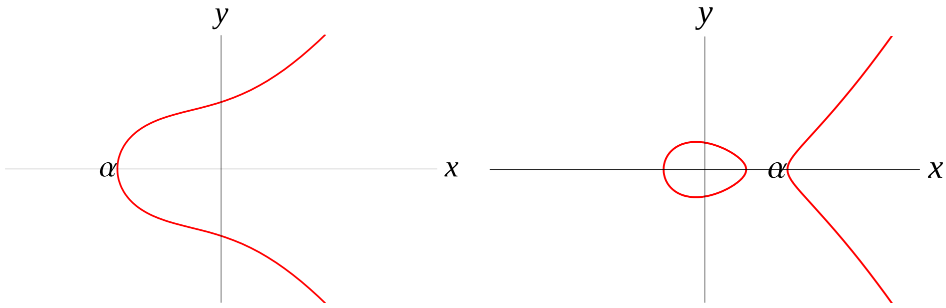
where a, b and c are arbitrary constants. Both equations are called the Weierstrass equations. The graphs of the Weierstrass equations in the xy-plane are called elliptic curves if they are non-singular (i.e. every point on the curve has a well-defined tangent line). It is named elliptic curve because it first arose in studying the problem of how to compute the arc length of an ellipse.

Here we focus on the case where the coefficients and solution (x, y) of the cubic equation are rationals. We call a point (x, y) on an elliptic curve a rational point if both its coordinates x and y are rational numbers. Usually, the graph of the Weierstrass equation has a different shape from that of the original cubic equation. But there is a bijection between rational points on both curves up to a few exceptional points. Therefore, if we are interested in studying rational points on cubic curves in general it suffices to study rational points on elliptic curves in Weierstrass form. Here we analyse non-singular curves. The main reason for not considering singular curves is because they are trivial to analyse as far as rational points go and Mordell's theorem will not hold for them. Actually, rational points on singular cubic curves can be put in one-to-one correspondence with rational points on a line. The group of rational points on them is not finitely generated. More on this can be found in [77].

Now the coefficients g_2 and g_3 in (2.1) are rationals, so they are real. Since the polynomial $f(x)$ is of degree 3, it has at least one real root. The polynomial $f(x)$ could be factored in $\mathbb{R}$ as

$$f(x) = (x - \alpha)(x^2 + \alpha x + \gamma),$$

with α and $\gamma \in \mathbb{R}$. Therefore, the elliptic curve looks like Figure 2.1(a) if $f(x)$ has exactly one real solution. If $f(x)$ has three real roots, then the curve looks like Figure 2.1(b). In this case, the real points form two components. This is true because all the roots of $f(x)$ are distinct since the curve is non-singular. For simplicity, we use the case where $f(x)$ has one real root to illustrate the group structure of rational points on an elliptic curve.



(a) One real solution of equation (2.1) (b) Three real solutions of equation (2.1)

Figure 2.1: Real solutions of (2.1)

We start with the equation in (2.1), $y^2 = 4x^3 - g_2x - g_3$. Thinking in the projective space sense, we associate with the elliptic curve E of the equation a point at infinity called $\mathcal{O}$. This point is located on both ends of the $y-$axis at infinity, so we cannot see it. The point is considered as a rational point and we take it as the zero element when we construct the group of rational points on the elliptic curve. A line is said to pass through the point at infinity $\mathcal{O}$ when this line is vertical, i.e. x = constant. From projective geometry, we know that the line which connects all infinity points is called the line at infinity. This line intersects with the elliptic curve with multiplicity three at $\mathcal{O}$. To make this clearer, let us set $x = \frac{X}{Z}$ and $y = \frac{Y}{Z}$ in (2.1). We get

$$Y^2Z = 4X^3 - g_2XZ^2 - g_3Z^3. \tag{2.3}$$

In projective space terminology, the line $Z = 0$ is called the line at infinity. Substituting $Z = 0$ into equation (2.3) yields $X^3 = 0$, so the root $X = 0$ has multiplicity three. This means that the elliptic curve intersects with the line at infinity at three points, all of them being the same point, namely $\mathcal{O}$. Now we are ready to define rational points on our elliptic curve E. Rational points on the elliptic curve E consist of the ordinary points on the affine $xy-$plane, together with the point at infinity $\mathcal{O}$ that we cannot see,

$$\mathrm{E}(\mathbb{Q}) = \{\mathcal{O}\} \cup \{(x, y) \in \mathbb{Q} \times \mathbb{Q} : y^2 = 4x^3 - g_2x - g_3\}.$$

A vertical line intersects with E at three points (counting multiplicities), two of which are in the $xy-$plane and the third is the point $\mathcal{O}$. A non-vertical line intersects with E at three points (counting multiplicities), all in the $xy-$plane. In general, we may allow x and y to be complex numbers. Now we are ready to describe geometrically the group structure of rational points on E with the addition operation of rational points. Let us draw a non-vertical line through two distinct rational points P and Q on the curve E. This line intersects with E in another point, call it $P*Q$. Then draw another line through $P*Q$ and $\mathcal{O}$ which is just a vertical line through $P*Q$. Since the curve E is symmetric about the $x-$axis, the vertical line through $P*Q$ intersects with E in another point we call $P+Q$. It is obvious from Figure 2.2 that $P+Q$ is just the reflection of $P*Q$ about the $x-$axis. We claim that rational points with the addition operation just described form a group. The process of adding two distinct rational points on the elliptic curve is shown in Figure 2.2.

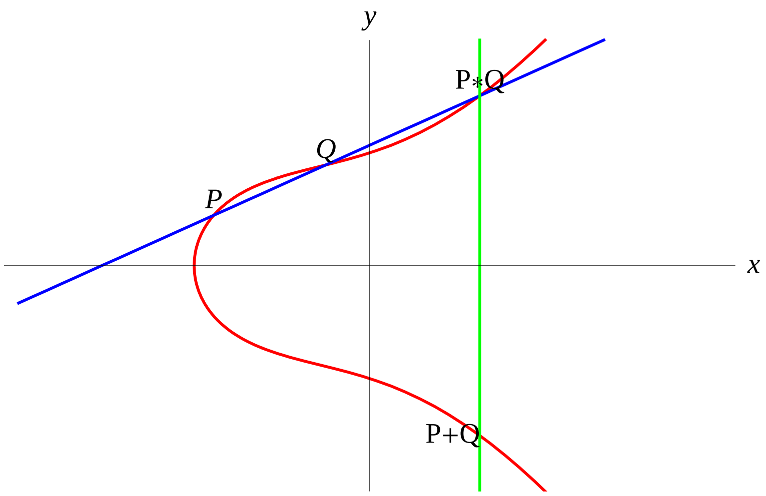


Figure 2.2: Addition of two rational points on a cubic curve

To show that this set of rational points with the addition operation forms a group we check the group axioms for the set. First, it will be apparent from the explicit formulae in (2.5-2.6) below that if P and Q are rational points in $\mathrm{E}(\mathbb{Q})$, then $P+Q$ is also a rational point in $\mathrm{E}(\mathbb{Q})$, hence $\mathrm{E}(\mathbb{Q})$ is a closed set under the addition operation described above. As stated earlier, we set $\mathcal{O}$ to be the zero element for the group, i.e. for any rational point P on E, then $P+\mathcal{O}=P$ by convention. Now the negative of a point $Q=(x,y)$ is $-Q=(x,-y)$. To verify this, we need to show that $Q+(-Q)=\mathcal{O}$. Draw a line through Q and $-Q$; this line is vertical, hence it intersects with the curve in the point $\mathcal{O}$, so $Q*-Q=\mathcal{O}$. To find $Q+(-Q)$, we need to connect $\mathcal{O}$ to itself and take the intersection with the curve. The line which connects $\mathcal{O}$ to $\mathcal{O}$ is the line at infinity and again it meets the curve at $\mathcal{O}$, since it intersects with E with multiplicity 3. To prove associativity is a complicated process geometrically and algebraically, so we will not describe it here. It is described in detail in both [77] and [83]. This shows that rational points on an elliptic curve form a group with the addition operation. Later, we show that it is an abelian group.

The above analysis was from a geometric point of view. Now we give explicit formulae (algebraically) for the addition of two distinct rational points on the elliptic curve in terms of their coordinates. Set

$$P_1 = (x_1, y_1), \quad P_2 = (x_2, y_2), \quad P_1 * P_2 = (x_3, y_3), \quad P_1 + P_2 = (x_4, y_4).$$

Suppose that P_1 and P_2 are given. We show how we compute $P_1 * P_2$. The equation of the line joining P_1 and P_2 is

$$y = \lambda x + \nu, \quad \text{where} \quad \lambda = \frac{y_2 - y_1}{x_2 - x_1} \quad \text{and} \quad \nu = y_1 - \lambda x_1 = y_2 - \lambda x_2. \tag{2.4}$$

This line intersects with the elliptic curve at three points, of which two are P_1 and P_2 and the third is the point $P_1 * P_2$. To get the third point, we substitute (2.4) into equation (2.1) of the elliptic curve. Simplifying the resulting equation and putting everything in one side yields

$$0 = 4x^3 - \lambda^2 x^2 - (g_2 + 2\lambda\nu)x - (g_3 + \nu^2).$$

This is a cubic equation in x that has three roots x_1, x_2 and x_3, so

$$x^3 - \frac{(\lambda^2)}{4}x^2 - \frac{(g_2 + 2\lambda\nu)}{4}x - \frac{(g_3 + \nu^2)}{4} = (x - x_1)(x - x_2)(x - x_3).$$

Equating the coefficients of the term x^2 on both sides yields

$$\frac{-\lambda^2}{4} = -(x_1 + x_2 + x_3).$$

Since x_1 and x_2 are given, we get the coordinates (x_3, y_3) of the point $P_1 * P_2$ by

$$x_3 = \frac{\lambda^2}{4} - x_1 - x_2, \qquad y_3 = \lambda x_3 + \nu. \tag{2.5}$$

The point $P_1 + P_2$ is the reflection of $P_1 * P_2$ about the $x-$axis, therefore

$$(x_4, y_4) = (x_3, -y_3). \tag{2.6}$$

The equations in (2.5) and (2.6) indicate that $P_1 + P_2 = P_2 + P_1$. Hence, the group of rational points on an elliptic curve is abelian. Now we show that we could recover from the formula of the x-coordinate of $P_1 + P_2$ (2.5) a second order autonomous discrete equation. We could parameterise this formula (2.5) using the Weierstrass $\wp$ function which is an elliptic function that has this useful property:

$$(\wp'(z))^2 = 4\wp(z)^3 - g_2\wp(z) - g_3. \tag{2.7}$$

It is clear that (2.7) and (2.1) are the same if $y = \wp'(z)$ and $x = \wp(z)$. The $\wp$ function has the following addition formula:

$$\wp(z+w) = \frac{1}{4}\left(\frac{\wp'(w) - \wp'(z)}{\wp(w) - \wp(z)}\right)^2 - \wp(z) - \wp(w), \tag{2.8}$$

which is the same formula for the x-coordinate of $P_1 + P_2$ given in (2.5). Now let $x_{n+1} = \wp(\xi_0 + nh + h)$, $x_{n-1} = \wp(\xi_0 + nh - h)$ and $x_n = \wp(\xi_0 + nh)$ for some constants ξ_0 and h. Then using (2.8) and (2.7), we have

$$\begin{aligned} x_{n+1} + x_{n-1} &= \frac{1}{2}\Big(\frac{(\wp'(\xi_0+nh))^2 + (\wp'(h))^2}{\wp^2(\xi_0+nh) + \wp^2(h) - 2\wp(\xi_0+nh)\wp(h)}\Big) - 2(\wp(\xi_0+nh) + \wp(h)) \\ &= \frac{4\wp(h)x_n^2 + (4\wp(h)^2 - g_2)x_n - (2g_3 + g_2\wp(h))}{2(\wp^2(\xi_0+nh) + \wp^2(h) - 2\wp(\xi_0+nh)\wp(h))}. \end{aligned}$$

Hence,

$$x_{n+1} + x_{n-1} = \frac{Ax_n^2 + Bx_n + C}{x_n^2 + Dx_n + E}, \tag{2.9}$$

where $A = 2\wp(h)$, $B = 2\wp(h)^2 - g_2/2$, $C = -(g_3 + \frac{g_2\wp(h)}{2})$, $D = -2\wp(h)$ and $E = \wp^2(h)$. The autonomous version of the discrete analogue of P_{II} in Chapter 3 has the same form as (2.9) with different choice of coefficients.

Now we state Mordell's theorem, which Mordell proved in 1923. We omit the proof of the theorem, which is beyond the scope of this thesis. It is however given in many elliptic curves books, in particular [77] and [83].

Mordell's theorem. *Let* E *be a non-singular plane cubic curve given by an equation*

$$\mathrm{E} : y^2 = x^3 + ax^2 + bx,$$

where a and b are integers. Then the group of rational points $\mathrm{E}(\mathbb{Q})$ *is finitely generated.*

Mordell's theorem was generalised by Weil (1928) in his thesis to cover elliptic curves over number fields (i.e. finite extensions of $\mathbb{Q}$) and abelian varieties (i.e. higher-dimensional analogues of elliptic curves) [83].

Now we turn our attention to a very useful tool of number theory, heights of rational points on elliptic curves. The proof of Mordell's theorem uses heights of rational points and their finiteness property. Height of a rational point measures how complicated the point is from the number theory viewpoint. Let $x = \frac{a}{b}$ be a non-zero rational number written in lowest terms. Recall from section 1.3.3 that we defined the height of x by

$$H(x) = H(a/b) = \max\{|a|, |b|\},$$

so $H(x)$ is a positive integer. By convention, $H(0) = 1$. One of the most useful properties of height is the following.

Finiteness property of height. *The set of all rational numbers whose height is less than some fixed number is a finite set.*

The proof is straightforward. Assume that we have for a fixed number c a set $\{\frac{a}{b} : a, b \neq 0 \in \mathbb{Z}$ and $H(\frac{a}{b}) < c\}$. From the height definition, we have $|a| \leq c$ and $|b| \leq c$. Since $a, b \in \mathbb{Z}$, there are finitely many possibilities for a and b and therefore for the rational number $\frac{a}{b}$. □

Now let us consider heights of rational points on an elliptic curve E. Recall the Weierstrass equation in (2.2),

$$y^2 = f(x) = x^3 + ax^2 + bx + c,$$

where it is an equation of a non-singular elliptic curve with integer coefficients a, b, c. If $P = (x, y)$ is a rational point on the curve E, we define the height of P to be the height of its x-coordinate,

$$H(P) = H(x).$$

Also, the logarithmic height or small h is

$$h(P) = \log H(P).$$

So $h(P)$ is always a non-negative real number. For the point at infinity $\mathcal{O}$, we define its height to be $H(\mathcal{O}) = 1$ and hence $h(\mathcal{O}) = 0$.
We end this sub-section by stating a lemma which gives bounds for logarithmic heights of rational points on elliptic curves. We omit the proof since it is not essential to the scope of this thesis and an interested reader could find it in [77], in addition to more results.

Lemma 2.1.1.1. *There is a constant t, depending on a, b, c, so that $h(2P) \geq 4h(P) - t$ for all $P \in \mathrm{E}(\mathbb{Q})$.*

Actually, there is a stronger result

$$m^2h(P) - \kappa \leq h(mP) \leq m^2h(P) + \kappa,$$

where $m \in \mathbb{Z}$, $P \in \mathrm{E}(\mathbb{Q})$ and $\kappa > 0$ is a constant depending on E and m. The proof of this result and more about it and other results related to the height are found in [76]. The above result shows that the logarithmic height of the x-coordinate of mP is less than or equals a quadratic function in m. This implies the polynomial growth of $h(mP)$.

2.2 Heights of rational numbers and p-adic absolute values

In Chapter 3 we explore Diophantine integrability of a certain class of discrete equations. The main tool used is the height of rational numbers. Since the height of a rational number is related to a sum over all absolute values on $\mathbb{Q}$ for this rational number, we introduce in this section p-adic absolute values on the field of rational numbers. Since the absolute value is used to measure the distance between elements of the field, p-adic absolute values give a different way to measure distance between rational numbers. Therefore, this introduces a new geometry and topology on the field of rational numbers, in which all triangles are isosceles.

First we start by stating the definition of an absolute value. The following definition of absolute value is valid for any field $\mathbb{K}$, but here we consider the field of rational numbers $\mathbb{Q}$. For any numbers $x, y \in \mathbb{Q}$, an absolute value $|.|$ on $\mathbb{Q}$ is a non-negative function such that

1. $|x| \geq 0$ with $|x| = 0$ if and only if $x = 0$,
2. $|xy| = |x| \cdot |y|$,
3. $|x + y| \leq |x| + |y|$.

Axiom 3 is called the triangle inequality. In geometry, the triangle inequality means that no side of a triangle is greater in length than the sum of lengths of the other two sides. If we replace axiom 3 by the stronger inequality

$$|x + y| \leq \max\{|x|, |y|\}, \tag{2.10}$$

then this absolute value is called *non-Archimedean*, otherwise *Archimedean*. The inequality in (2.10) is called the isosceles inequality. Geometrically, it implies that for any triangle, two of its sides are equal in length.

We fix a prime p, then for any non-zero rational number c we write c in terms of its p factorisation as

$$c = p^v \frac{m}{n}, \tag{2.11}$$

where $v, m \in \mathbb{Z}$, $n \in \mathbb{N}$ and $p \nmid mn$. We define the p-adic absolute value of c by

$$|c|_p = p^{-v}. \tag{2.12}$$

By convention, if $c = 0$, then $|0|_p = 0$. This absolute value is non-Archimedean for any prime p. This type of absolute value was first introduced by Hensel (1904), where he chose $p = 2$ [15]. Equivalently, the p-adic absolute value is

$$|q|_p = \begin{cases} 1 & \text{for prime } q \neq p, \\ \frac{1}{p} & \text{if } q = p. \end{cases}$$

The usual (non-trivial) Archimedean absolute value on $\mathbb{Q}$ is sometimes called the absolute value at infinity ($|.|_\infty$). Any two absolute values on a field (specifically here $\mathbb{Q}$) are said to be equivalent if they induce the same topology on the field [22]. A theorem by Ostrowski classifies the absolute values on $\mathbb{Q}$. The proof of this theorem is given in [24] and the statement of the theorem is as follows.

Theorem (Ostrowski). *Every non-trivial absolute value on $\mathbb{Q}$ is equivalent to one of the absolute values $|.|_p$, where either p is a prime number or $p = \infty$.*

Hence, in the field of rational numbers $\mathbb{Q}$, the only Archimedean absolute value is the ordinary absolute value $|.|$ (denoted by $|.|_\infty$). The non-Archimedean absolute values on $\mathbb{Q}$ are equivalent to the p-adic absolute values. Furthermore, we have the product formula:

$$\prod_{p \leq \infty} |x|_p = 1, \tag{2.13}$$

for any $x \in \mathbb{Q} \setminus \{0\}$. The proof is straightforward and given in [24].

From the previous sub-section, the logarithmic height $h(x)$ for any non-zero rational number $x = \frac{a}{b}$ is defined as

$$h(x) = \log H(x) = \log(\max\{|a|_\infty, |b|_\infty\}), \tag{2.14}$$

where a and $b \neq 0$ are coprime. There is another equivalent expression for the logarithmic height $h(x)$ which involves the p-adic absolute values given by the next lemma.

Lemma 2.2.1. *For a non-zero rational number x,*

$$h(x) = \sum_{p \leq \infty} \log^+ |x|_p, \tag{2.15}$$

where $\log^+ y = \max\{\log y, 0\}$ *for any* $y \in \mathbb{Q}^+$.

Proof Now we show that the two expressions (2.14) and (2.15) are equivalent. For any $x \in \mathbb{Q} \setminus \{0\}$, let $x = \frac{a}{b} = \frac{t_1^{k_1} \cdot t_2^{k_2} \ldots t_n^{k_n}}{q_1^{l_1} \cdot q_2^{l_2} \ldots q_m^{l_m}}$. Note that $a, b \in \mathbb{Z}$ are coprime, $b \neq 0$ and k_i, l_j are non-negative integers. Since a and b are coprime, it implies that $t_i \neq q_j \ \forall i, j \in \mathbb{N}$, where $\prod_{i=1}^n t_i^{k_i}, \prod_{j=1}^m q_j^{l_j}$ are the prime factorisation of a and b, respectively. For a fixed prime t_i, where $i \in \{1, \ldots, n\}$, the t_i-adic absolute value of x is

$$|x|_{t_i} = t_i^{-k_i}. \tag{2.16}$$

It implies $\log^+ |x|_{t_i} = 0$. Also for a fixed prime q_j, where $j \in \{1, \ldots, m\}$, the q_j-adic absolute value of x is

$$|x|_{q_j} = q_j^{l_j}. \tag{2.17}$$

Hence, $\log^+ |x|_{q_j} = \log q_j^{l_j}$. Consequently, $\sum_{j=1}^m \log^+ |x|_{q_j} = \log \prod_{j=1}^m q_j^{l_j}$. For all primes $p \leq \infty$,

$$\sum_{p \leq \infty} \log^+ |x|_p = \sum_{i=1}^n \log^+ |x|_{t_i} + \sum_{j=1}^m \log^+ |x|_{q_j} + \log^+ \left| \frac{a}{b} \right|_\infty. \tag{2.18}$$

If $|\frac{a}{b}|_\infty > 1$ ($|a|_\infty > |b|_\infty$), then (2.18) is

$$\begin{aligned} \sum_{p \leq \infty} \log^+ |x|_p &= \log \left(\prod_{i=1}^m q_i^{l_i} \right) + \log \left| \frac{\prod_{i=1}^n t_i^{k_i}}{\prod_{i=1}^m q_i^{l_i}} \right|_\infty, \\ &= \log \left| \prod_{i=1}^n t_i^{k_i} \right|_\infty = \log |a|_\infty. \end{aligned} \tag{2.19}$$

If $|\frac{a}{b}|_\infty < 1$ ($|b|_\infty > |a|_\infty$), then (2.18) is

$$\begin{aligned} \sum_{p \leq \infty} \log^+ |x|_p &= \log \prod_{i=1}^m q_i^{l_i} \\ &= \log \left| \prod_{i=1}^m q_i^{l_i} \right|_\infty = \log |b|_\infty. \end{aligned} \tag{2.20}$$

From (2.19) and (2.20), it is clear that

$$\sum_{p\leq\infty} \log^+ |x|_p = \log(\max\{|a|_\infty, |b|_\infty\}) = h(x). \qquad \square$$

The $\log^+$ function has similar properties to the usual logarithmic function log. We state here some of its properties that we often use in the next chapter. Let x, $y \in \mathbb{Q}^+$, then

- $\log^+(x \cdot y) \leq \log^+ x + \log^+ y$,
- $\log^+ x^y = y \log^+ x$,
- $\log^+ \left(\sum_{i=1}^n x_i\right) \leq \log n + \sum_{i=1}^n \log^+ x_i$.

There are more properties of the $\log^+$ function given in [51].

Also, we prove a property of logarithmic height that we use frequently in the next chapter. This property is given in the next lemma [9].

Lemma 2.2.2. *If $x \in \mathbb{Q} \setminus \{0\}$ and $\lambda \in \mathbb{Z}$, then $h(x^\lambda) = |\lambda|_\infty \cdot h(x)$. In particular, $h(\frac{1}{x}) = h(x)$.*

Proof If $\lambda \geq 0$, then the result is clear from (2.14) and (2.15). If $\lambda < 0$, then we could write it as $\lambda = -1 \cdot (-\lambda)$ where $-\lambda > 0$. Hence, it is enough to consider only $\lambda = -1$. Let $\lambda = -1$ and x is a non-zero rational number that equals $\frac{a}{b}$, where $b \neq 0$. Using (2.14) we have

$$h(x) = \log H(x) = \log\left(\max\{|a|_\infty, |b|_\infty\}\right) = \log H\left(\frac{1}{x}\right) = h\left(\frac{1}{x}\right),$$

which proves the lemma. $\square$

We end this section by proving another property of logarithmic heights of rational numbers. A generalised result is proved in [9] for an affine space of dimension n over $\bar{\mathbb{Q}}$ (which is an algebraic closure of $\mathbb{Q}$).

Lemma 2.2.3. *If $\{x_1, x_2, \ldots, x_n\} \subset \mathbb{Q} \setminus \{0\}$ where $n \in \mathbb{N}$, then*

$$h(x_1 + x_2 + \cdots + x_n) \leq h(x_1) + h(x_2) + \cdots + h(x_n) + \log n.$$

Proof Using Lemma 2.2.1 we have the following chain of inequalities,

$$
\begin{aligned}
h(x_1 + x_2 + \cdots + x_n) &= \sum_{p\leq\infty} \log^+ |x_1 + \cdots + x_n|_p \\
&\leq \sum_{p<\infty} \log^+ (\max\{|x_1|_p, \ldots, |x_n|_p\}) + \log^+ \left(\sum_{i=1}^{n} |x_i|_\infty\right), \\
&\leq \sum_{p<\infty} \max(\log^+ |x_1|_p, \ldots, \log^+ |x_n|_p) + \sum_{i=1}^{n} \log^+ |x_i|_\infty + \log n, \\
&\leq \sum_{p<\infty} (\log^+ |x_1|_p + \cdots + \log^+ |x_n|_p) + \sum_{i=1}^{n} \log^+ |x_i|_\infty + \log n, \\
&= \sum_{p\leq\infty} \log^+ |x_1|_p + \cdots + \sum_{p\leq\infty} \log^+ |x_n|_p + \log n, \\
&= h(x_1) + h(x_2) + \cdots + h(x_n) + \log n.
\end{aligned}
$$

In the above chain of inequalities we used the isosceles and triangle inequalities, properties of $\log^+$ and the fact that the sum of non-negative elements in a set is greater than or equals the maximum element of that set. This proves the lemma.

□

Chapter 3

Diophantine integrability

The main purpose of this chapter is to prove Theorem 1.4.1, which is the most important result of this thesis. It provides further evidence that there is a strong relationship between the integrability of a discrete equation and the growth in height of its solutions — justifying the term *Diophantine integrability* for equations with solutions having logarithmic heights that grow polynomially. Recall that a solution y_n is said to be admissible if the logarithmic heights of the coefficients are small compared to the logarithmic height of y_n. Suppose that y_n is an admissible solution of (1.27) with polynomial height growth Theorem 1.4.1 says either y_n is also a solution of the Riccati (1.28) or (1.27) is the d-P_{II} (given in Table 1.2).

Ideally, we would have liked to have a theorem in which we assume that the heights of all solutions grow no faster than polynomials and conclude that the equation must be d-P_{II}. Our methods do not allow us to use properties of different solutions at once so we only assume the existence of a single solution of this type. It is very simple to construct non-integrable equations with at least one explicit slow height growth solution. It is because of this that we need an admissibility-type assumption. Starting from a particular choice for y_n, say $y_n \equiv n$, it is easy to construct an equation such as

$$y_{n+1} + y_{n-1} = \frac{-2n^3 + 2y_n}{1 - y_n^2},$$

that y_n satisfies. Due to the absence of significant cancellation in general, and in the example just mentioned in particular, the height of the coefficients of the resulting equation will be comparable to the height of the solution. In other words, the solution is inadmissible.

The absolute values $|\cdot|_p$ on $\mathbb{Q}$, where $p \leq \infty$, play a central role in our analysis. Recall that for $x \in \mathbb{Q}$,

$$h(x) = \sum_{p \leq \infty} \log^+ |x|_p.$$

Some of our calculations, especially those in Theorem 1.4.2, are essentially a refinement of calculations from singularity confinement re-expressed in terms of these absolute values. There is also an analogy between some of the methods used in our proof and methods used in the proof of similar classification problems in Nevanlinna theory by Halburd and Korhonen (see Theorem 1.3.4.1). This analogy further supports the philosophy underlying Vojta's dictionary.

This entire chapter is dedicated to the proof of Theorem 1.4.1. The main objective of section 3.1 is to prove Theorem 3.1.1, which shows that if $c_n \not\equiv 0$ or ± 2, then the height of an admissible solution is not growing polynomially. The cases $c_n \equiv 0$ and ± 2 are analysed further in the next two sections.

3.1 Diophantine integrability

The motivation of our work in this chapter arose from the work of Halburd and Korhonen in [34] (Theorem 1.3.4.1). In 2007, they proved the following: let $w(z)$ be an admissible (in Nevanlinna theory sense) finite-order meromorphic solution of the second order difference equation

$$w(z+1) + w(z-1) = R(z, w(z)), \tag{3.1}$$

where R is rational in $w(z)$ with coefficients meromorphic in z. Then either $w(z)$ satisfies a difference linear or Riccati equation or else equation (3.1) can be transformed to one of a list of canonical difference equations. This list consists of all known difference Painlevé equations of the form (3.1), together with their autonomous versions. Their work implies that the existence of finite-order meromorphic solutions is a good indicator of integrability for difference equations. Recall that the order (σ) of a meromorphic function f, involves the Nevanlinna characteristic function $T(f, r)$ in its definition, i.e. $\sigma = \limsup_{r\to\infty} \frac{\log T(f,r)}{\log r}$ (see Appendix A for more details on Nevanlinna functions). A relation between the order of meromorphic solutions of a rational equation and integrability of the equation was noticed and discussed first by Ablowitz, Halburd and Herbst in [1]. In [1], the authors showed that in order for an admissible solution $w(z)$ of (3.1) to be of finite order, the degree of R must be ≤ 2. Halburd and Korhonen explored the case when the degree of $R \leq 2$ further and proved Theorem 1.3.4.1.

There is a relation between Nevanlinna theory and Diophantine approximation. This relation was observed first by Osgood [9]. In 1986, Vojta from his PhD thesis created a dictionary (called the Vojta dictionary) giving an analogy between Diophantine approximation and Nevanlinna theory [9]. The analogy that concerns us in our work is given in Table 3.1 [74]:

Table 3.1: Examples from Vojta dictionary

infinite sequence (x) in $\mathbb{Q}$	$\longleftrightarrow$ non-constant meromorphic function f
$h(x)$	$\longleftrightarrow T(r, f)$

Since we have a correspondence between Nevanlinna theory and Diophantine approximation, we expect to have a result similar to the Halburd and Korhonen work in Diophantine approximation (using the height function as a tool) for the class of equations

$$y_{n+1} + y_{n-1} = R(n, y_n). \tag{3.2}$$

Here, R is rational in y_n with coefficients that are rational functions in n and rational numbers; also the degree of $R \leq 2$. Halburd and Morgan [36] studied in detail a particular case of the class of equations (3.2)(namely $y_{n+1} + y_{n-1} = \frac{\alpha_n+\beta_n y_n+\gamma_n y_n^2}{y_n^2}$). They used an analysis based on the height growth of the solution $h(y_n)$. Here we analyse a different case of the same class of equations (3.2). However, there are essential differences and difficulties which distinguish between the two cases and consequently the analysis used to treat each of them. In the equation considered by Morgan in his PhD thesis [36], there is one singularity at $y_n = 0$ of multiplicity 2. In equation (3.3) we are considering, we have 2 distinct singularities, each of multiplicity 1 at $y_n = 1$ and $y_n = -1$. Unlike the equation considered by Morgan, we have a major technical difficulty that arose in our case, in which for certain forms of the equation coefficients, y_n solves a difference Riccati equation (see Appendix B).

In this chapter, we explore the idea of Diophantine integrability of the following discrete equation:

$$y_{n+1} + y_{n-1} = \frac{a_n + b_n y_n + c_n y_n^2}{1 - y_n^2}, \tag{3.3}$$

where a_n, b_n and c_n are rational functions in n and rational numbers and the right hand side of (3.3) is irreducible. This chapter gives a rigorous proof of Theorem 1.4.1. We show in this section that the summed logarithmic height of an admissible solution of equation (3.3) is greater than an increasing exponential function, provided that $c_n \neq 0$ or ± 2 $\forall n$. This implies the exponential growth of the summed logarithmic height of an admissible solution unless c_n is identically 0 or ± 2 for all n. We give in this section formal definitions of the summed logarithmic height, an admissible solution and a Diophantine integrable equation. In section 3.2, we show that for an admissible solution y_n of (3.3) when $c_n \equiv 0$, either the summed logarithmic height of the solution $h_r(y_n)$ grows fast with $r \to \infty$ or the solution y_n solves a difference Riccati equation (i.e. $y_{n+1} = \frac{\frac{1}{2\theta}(\theta a_n + b_n - 2) + y_n}{1 - \theta y_n}$, for $\theta = 1$

or -1) or equation (3.3) reduces to a discrete analogue of the second Painlevé equation $d\text{-}P_{II}$ (i.e. $y_{n+1}+y_{n-1}=\frac{(\alpha n+\beta)y_n+\gamma}{1-y_n^2}$). In section 3.3, we show, using a similar analysis to section 3.2, that when $c_n \equiv \pm 2$ the summed logarithmic height of an admissible solution $h_r(y_n)$ grows exponentially with $r\to\infty$.

Before we proceed to state and prove the lemma and the theorem of this section, we need to define formally some terms stated above such as summed logarithmic height, admissible solution and Diophantine integrable equation.

Definition 3.1.1. *Let $(y_n)\subset\mathbb{Q}$ be a solution for some discrete equation. We define the summed logarithmic height $h_r(y_n)$ by*

$$h_r(y_n)=\sum_{n=r_0}^{r} h(y_n)=\sum_{n=r_0}^{r}\sum_{p\leq\infty}\log^+|y_n|_p,$$

for some integer r_0.

For the purpose of our work we define an admissible solution as follows:

Definition 3.1.2. *Let $(y_n)\subset\mathbb{Q}$ be a solution of a rational discrete equation with coefficients $a_i(n)$ that are rational functions in n and $a_i(n)\in\mathbb{Q}\ \forall i$. Then (y_n) is said to be an admissible solution if for some r_0,*

$$\max\{r,h_r(a_i)\}=o(h_r(y_n))\quad \forall i \text{ and } r\geq r_0.$$

Now we define formally when the discrete equation is called Diophantine integrable.

Definition 3.1.3. *For a discrete equation, if every solution (y_n) satisfies*

$$h(y_n)=O(n^m),$$

for some non-negative integer m, then this equation is called Diophantine integrable.

Note that in (3.3) we consider coefficients a_n, b_n and c_n that are all rational functions of n and rational numbers at every n. Any rational function (except the zero function) has a finite number of zeros and poles. Also, any linear combination of rational functions is a rational function too. This property of rational functions made us consider equation (3.3) with coefficients that are rational functions in n rather than any other type of coefficients with a finite number of zeros and poles. It also helps in avoiding many technical details that would occur if the coefficients were taken to lie in some other class of functions. There exists an integer K such that it is greater than all the real zeros and poles of the rational functions b_n, c_n, $a_n\pm b_n+c_n$ and $c_n\pm 2$ (if there are any). In case these functions have no real poles and no real zeros, then we could choose K to be any integer. For the rest of this section we assume that $c_n \neq 0$ or ± 2 $\forall n$ and the right hand side of (3.3) is irreducible for all n, which implies that $a_n \pm b_n + c_n \neq 0$ $\forall n$. We set a notation used throughout this chapter that whenever we write an expression $A \pm B$, both terms $A+B$ and $A-B$ are included.

The analogy between our analysis and singularity confinement is recognised first through Lemma 3.1.1. We introduce a quantity ϵ_n, that serves as our measure for the size of the iterate y_n in equation (3.3) and to measure its distance from the singularities ± 1. For a fixed sufficiently small $\delta > 0$ we define ϵ_n ($\forall n > K$) as follows:

$$\begin{aligned}\epsilon_n^{-\delta} = \kappa_p \max\Big\{&|2|_p^{-1}, |a_n \pm b_n + c_n|_p^{-1}, |b_n|_p, |c_n|_p, \left|\frac{1}{2}\right|_p \cdot |a_{n+1} \pm b_{n+1} + c_{n+1}|_p,\\ &\left|\frac{1}{2}\right|_p \cdot |a_{n-1} \pm b_{n-1} + c_{n-1}|_p, |c_{n-1}|_p^{-1}, |c_{n+1}|_p^{-1},\\ &|c_{n+1} \pm 2|_p^{-1}, |c_{n-1} \pm 2|_p^{-1}\Big\}\end{aligned} \tag{3.4}$$

where $\kappa_p = 1$ ($\forall p < \infty$), $\kappa_p = 10$ when $p = \infty$. We keep the same expression for ϵ_n throughout this section but in the next sections we have different expressions for ϵ_n. The equation in (3.4) implies the following inequality for non-Archimedean absolute value: $|c_n|_p \leq \epsilon_n^{-\delta} \Rightarrow \epsilon_n^{\delta} \leq |c_n|_p^{-1}$. For the Archimedean absolute value, we have $10|c_n|_\infty \leq \epsilon_n^{-\delta} \Rightarrow \epsilon_n^{\delta} \leq \frac{1}{10}|c_n|_\infty^{-1}$.

Lemma 3.1.1. *Let $(y_n) \subset \mathbb{Q} \setminus \{-1, 1\}$ be a solution to the equation (3.3):*

$$y_{n+1} + y_{n-1} = \frac{a_n + b_n y_n + c_n y_n^2}{1 - y_n^2},$$

where c_n is a rational function not identically 0 or ± 2. Furthermore, assume that the numerator and the denominator of (3.3) are coprime and the coefficients a_n, b_n and c_n are all rational functions of n and rational numbers for all n. For a fixed prime $p \leq \infty$ and $\forall k \in \mathbb{Z}$ and $k > K$, let ϵ_k be as defined in (3.4). If $|1 - \theta y_k|_p < \epsilon_k$ for $\theta = 1$ or -1, then either

$$|y_{k+1}|_p \geq \frac{1}{|1 - \theta y_k|_p^{1-\delta}} \quad \text{and} \quad |1 \pm \theta y_{k+2}|_p \geq \epsilon_{k+2},$$

or

$$|y_{k-1}|_p \geq \frac{1}{|1 - \theta y_k|_p^{1-\delta}} \quad \text{and} \quad |1 \pm \theta y_{k-2}|_p \geq \epsilon_{k-2}.$$

Proof Since the coefficients a_n, b_n and c_n are rational functions of n, then $a_n + b_n + c_n$ and $a_n - b_n + c_n$ are rational functions of n. If the rational function $a_n + b_n + c_n$ or $a_n - b_n + c_n$ is identically zero for all n, then this implies that 1 or -1 is a root of the numerator but ± 1 are roots of the denominator. Since the numerator and the denominator are coprime then we have a contradiction. Hence, $a_n \pm b_n + c_n$ are rational functions not identically zero for all n.

Before we proceed to prove the main statement of the lemma, we show first that $\epsilon_n < 1$ when $p = \infty$ and $p = 2$, while $\epsilon_n \leq 1$ when $p < \infty$ and $p \neq 2$. For the Archimedean absolute value ($p = \infty$), we have from the definition of $\epsilon_n^{-\delta}$ (3.4) this inequality $1 < 10 \cdot |2|_\infty^{-1} \leq \epsilon_n^{-\delta}$. Consequently, $\epsilon_n^{\delta} < 1 \Rightarrow \epsilon_n < 1 \; \forall n \in \mathbb{Z}$. Also from (3.4) for the 2-adic absolute value, we have $1 < 2 = |2|_2^{-1} \leq \epsilon_n^{-\delta} \Rightarrow \epsilon_n < 1$. Similarly, from (3.4) for the non-Archimedean absolute values ($p < \infty$ and $p \neq 2$), we have $\epsilon_n \leq 1 \; \forall n \in \mathbb{Z}$. Also note that $\theta = 1$ or -1, which means $\theta^2 = 1$ and $|\theta|_p = 1$ $\forall p \leq \infty$.

Now we prove the main statement of the lemma starting with the non-Archimedean absolute values (p-adic absolute values ($\forall p < \infty$)) and then we prove it for the Archimedean absolute value ($p = \infty$).

For a fixed prime ($p < \infty$), let $|1 - \theta y_k|_p < \epsilon_k$ for some integer $k > K$, where $\theta = 1$ or -1. The definition of $\epsilon_k^{-\delta}$ is given in (3.4) where $\kappa_p = 1$. Now the equation (3.3) can be rewritten as follows:

$$\begin{aligned} y_{k+1} + y_{k-1} &= \frac{a_k + b_k y_k + c_k y_k^2}{1 - y_k^2} \\ &= \frac{a_k + \theta b_k + c_k - \theta \cdot (b_k + \theta c_k)(1 - \theta y_k) - \theta c_k y_k (1 - \theta y_k)}{(1 - \theta y_k)(1 + \theta y_k)}. \end{aligned}$$

If we multiply the above equation by $(1 + \theta y_k)$ and simplify, then we get

$$(y_{k+1} + y_{k-1})(1 + \theta y_k) = \frac{a_k + \theta b_k + c_k}{1 - \theta y_k} - \theta b_k - c_k(1 + \theta y_k). \tag{3.5}$$

Note that

$$\begin{aligned} |1 - \theta y_k|_p^{-(1-\delta)} &= \frac{|1 - \theta y_k|_p^{\delta}}{|1 - \theta y_k|_p} < \frac{\epsilon_k^{\delta}}{|1 - \theta y_k|_p}, \quad (\text{from } |1 - \theta y_k|_p < \epsilon_k) \\ &\leq \frac{|a_k + \theta b_k + c_k|_p}{|1 - \theta y_k|_p}, \quad (\text{from equation (3.4)}) \\ &\leq |(y_{k+1} + y_{k-1})(1 + \theta y_k) + \theta b_k + c_k(1 + \theta y_k)|_p, \\ &\quad (\text{from equation (3.5)}) \\ &\leq \max\{|y_{k+1} + y_{k-1}|_p \cdot |1 + \theta y_k|_p, |b_k|_p, |c_k|_p \cdot |1 + \theta y_k|_p\}. \end{aligned} \tag{3.6}$$

In the last step of the above inequalities we used the isosceles inequality. To find the maximum in the set in the above inequality (3.6), we find relations between elements of the set and $|1-\theta y_k|_p^{-(1-\delta)}$. First, from $\epsilon_k^{-\delta}$ definition in (3.4) we have

$$|b_k|_p \leq \epsilon_k^{-\delta} \leq \epsilon_k^{-(1-\delta)} < |1-\theta y_k|_p^{-(1-\delta)},$$

where we have used the facts $\epsilon_k \leq 1$ (from (3.4)) and sufficiently small $(0 < \delta)$. Therefore, the inequality in (3.6) reduces to

$$\begin{aligned} |1-\theta y_k|_p^{-(1-\delta)} &\leq \max\{|y_{k+1}+y_{k-1}|_p \cdot |1+\theta y_k|_p, |c_k|_p \cdot |1+\theta y_k|_p\} \\ &= |1+\theta y_k|_p . \max\{|y_{k+1}+y_{k-1}|_p, |c_k|_p\}. \end{aligned} \tag{3.7}$$

Also,

$$|1+\theta y_k|_p = |2-(1-\theta y_k)|_p \leq \max\{|2|_p, |1-\theta y_k|_p\} \leq \max\{1, \epsilon_k\} = 1, \tag{3.8}$$

since $\epsilon_k \leq 1$, and since the absolute value is non-Archimedean, so that $|2|_p \leq 1$. Using the inequality (3.8) in (3.7), it follows that

$$|1-\theta y_k|_p^{-(1-\delta)} \leq \max\{|y_{k+1}+y_{k-1}|_p, |c_k|_p\}. \tag{3.9}$$

From (3.4) we have the following relation $|c_k|_p \leq \epsilon_k^{-\delta} \leq \epsilon_k^{-(1-\delta)} < |1-\theta y_k|_p^{-(1-\delta)}$. Hence, the above inequality (3.9) is

$$|1-\theta y_k|_p^{-(1-\delta)} \leq |y_{k+1}+y_{k-1}|_p \leq \max\{|y_{k+1}|_p, |y_{k-1}|_p\}. \tag{3.10}$$

Therefore, either the maximum is $|y_{k+1}|_p$ where we would have the proof completed for $|y_{k+1}|_p \geq \frac{1}{|1-\theta y_k|_p^{(1-\delta)}}$ or the maximum is $|y_{k-1}|_p$ where in this case we would complete the proof of $|y_{k-1}|_p \geq \frac{1}{|1-\theta y_k|_p^{(1-\delta)}}$. Without loss of generality, we choose the maximum to be $|y_{k+1}|_p$ and for the rest of the proof we use $|y_{k+1}|_p \geq |1-\theta y_k|_p^{-(1-\delta)}$.

Heuristically speaking, the assumption $|1-\theta y_k|_p < \epsilon_k$ means that we start with a rational number y_k close (in the p-adic sense) to 1 or -1 if $\theta = 1$ or -1. Hence, $|1-\theta y_k|_p$ is a small quantity ($< \epsilon_k$) and from the above proof we showed that the next iterate of equation (3.3) is a big quantity $|y_{k+1}|_p \geq \frac{1}{|1-\theta y_k|_p^{(1-\delta)}}$. Now we want to show that y_{k+2} is not close to either -1 or 1 ($|1 \pm \theta y_{k+2}|_p \geq \epsilon_{k+2}$). When we use the terms close, big and small, we mean with respect to the absolute value under consideration when p is fixed and with respect to ϵ_k. To identify the similarity with singularity confinement we iterate equation (3.3) with $y_k = \theta - \theta(1-\theta y_k)$ and y_{k-1}

any rational number. Treating each iterate as a power series of $(1-\theta y_k)$, we get

$$y_{k+1} = \frac{a_k + \theta b_k + c_k}{2(1-\theta y_k)} + \cdots,$$
$$y_{k+2} = -\theta - c_{k+1} + \frac{\theta(a_k + \theta b_k - 2\theta b_{k+1} + c_k)(1-\theta y_k)}{(a_k + \theta b_k + c_k)} + \cdots,$$

where c_{k+1} is not identically $0, \pm 2$ by assumption. Hence, the resulting iterates are similar to singularity confinement structure where $(1-\theta y_k)$ plays the role of ε in singularity confinement method, as illustrated in Chapter 1. We show that after iterating (3.3) twice, y_{k+2} is naturally away from the singularities 1 and -1 when $c_n \neq 0$ or ± 2.

In order to show that $|1 \pm \theta y_{k+2}|_p \geq \epsilon_{k+2}$ in our analysis, we need first to show that $|y_{k+2} + \theta + c_{k+1}|_p$ is a small quantity bounded from above by a power of ϵ_k. Since c_{k+1} is not identically $0, \pm 2$, then if $|y_{k+2} + \theta + c_{k+1}|_p$ is a small quantity ($\leq$ a power of ϵ_k), this implies that y_{k+2} is not close to ± 1. We show this as follows: from equation (3.3) we have

$$\begin{aligned} y_{k+2} + y_k &= \frac{a_{k+1} + b_{k+1}y_{k+1} + c_{k+1}y_{k+1}^2}{1-y_{k+1}^2}, \\ &= \frac{a_{k+1} + b_{k+1}y_{k+1} - c_{k+1}(1-y_{k+1}^2) + c_{k+1}}{1-y_{k+1}^2}. \end{aligned}$$

Adding $\theta + c_{k+1} - y_k$ to both sides of this equation, we get the following:

$$y_{k+2} + c_{k+1} + \theta = \frac{a_{k+1} + b_{k+1}y_{k+1} + c_{k+1}}{1-y_{k+1}^2} + \theta(1-\theta y_k). \tag{3.11}$$

We rewrite the fraction in the right hand side of equation (3.11) using partial fractions which yields

$$y_{k+2} + c_{k+1} + \theta = \frac{\frac{1}{2}(a_{k+1} + b_{k+1} + c_{k+1})}{1-y_{k+1}} + \frac{\frac{1}{2}(a_{k+1} - b_{k+1} + c_{k+1})}{1+y_{k+1}} + \theta(1-\theta y_k). \tag{3.12}$$

Note that we have from the assumption $|1-\theta y_k|_p < \epsilon_k$ and from the relation $|y_{k+1}|_p \geq \frac{1}{|1-\theta y_k|_p^{1-\delta}}$, the following chain of inequalities:

$$\frac{1}{\epsilon_k^{1-\delta}} < \frac{1}{|1-\theta y_k|_p^{1-\delta}} \leq |y_{k+1}|_p = |1-(1-y_{k+1})|_p \leq \max\{1, |1-y_{k+1}|_p\}.$$

Now if 1 is the maximum, then we have $\frac{1}{\epsilon_k^{1-\delta}} < 1$. Since $\epsilon_k \leq 1$, which means $\epsilon_k^{-1} \geq 1$ and consequently $\frac{1}{\epsilon_k^{1-\delta}} \geq 1$, we have a contradiction. Therefore, the maximum is

$|1 - y_{k+1}|_p$ and

$$\frac{1}{\epsilon_k^{1-\delta}} < |y_{k+1}|_p \leq |1 - y_{k+1}|_p. \tag{3.13}$$

Consequently,

$$\frac{1}{|1 - y_{k+1}|_p} < \epsilon_k^{1-\delta}. \tag{3.14}$$

Similarly, we get

$$\frac{1}{|1 + y_{k+1}|_p} < \epsilon_k^{1-\delta}. \tag{3.15}$$

By using the isosceles inequality in equation (3.12) and using (3.4), we get

$$\begin{aligned} |y_{k+2} + c_{k+1} + \theta|_p \leq \max \Bigg\{ & \frac{|\frac{1}{2}|_p.|a_{k+1} + b_{k+1} + c_{k+1}|_p}{|1 - y_{k+1}|_p}, \\ & \frac{|\frac{1}{2}|_p.|a_{k+1} - b_{k+1} + c_{k+1}|_p}{|1 + y_{k+1}|_p}, |1 - \theta y_k|_p \Bigg\} \\ & < \max\{\epsilon_k^{1-2\delta}, \epsilon_k^{1-2\delta}, \epsilon_k\} = \epsilon_k^{1-2\delta}. \end{aligned} \tag{3.16}$$

The maximum is $\epsilon_k^{1-2\delta}$, since $\epsilon_k \leq 1$ and $0 < \delta < \frac{1}{2}$. From (3.4) we have the following:

$$\begin{aligned} \epsilon_k^{\delta} \leq |c_{k+1}|_p &= |(y_{k+2} + c_{k+1} + \theta) - \theta(1 + \theta y_{k+2})|_p \\ &\leq \max\{|y_{k+2} + c_{k+1} + \theta|_p, |1 + \theta y_{k+2}|_p\}. \end{aligned} \tag{3.17}$$

If the maximum in the above inequality is $|y_{k+2} + c_{k+1} + \theta|_p$, then

$$\epsilon_k^{\delta} \leq |c_{k+1}|_p = |y_{k+2} + c_{k+1} + \theta|_p < \epsilon_k^{1-2\delta}.$$

The above inequality gives a contradiction, since $\epsilon_k^{1-2\delta} \leq \epsilon_k^{\delta}$ (from $\epsilon_k \leq 1$ and sufficiently small δ). Hence, $|1 + \theta y_{k+2}|_p > |y_{k+2} + c_{k+1} + \theta|_p$ and the maximum in (3.17) is $|1 + \theta y_{k+2}|_p$ which yields

$$|c_{k+1}|_p \leq |1 + \theta y_{k+2}|_p. \tag{3.18}$$

From the definition of $\epsilon_{k+2}^{-\delta}$ in (3.4) and using the above inequality, we get

$$\epsilon_{k+2} \leq \epsilon_{k+2}^{\delta} \leq |c_{k+1}|_p \leq |1 + \theta y_{k+2}|_p.$$

Hence, we proved that $|1 + \theta y_{k+2}|_p \geq \epsilon_{k+2}$.

Now we prove that $|1 - \theta y_{k+2}|_p \geq \epsilon_{k+2}$, starting with the following inequality where we have used (3.4)

$$\epsilon_k^{\delta} \leq |c_{k+1} + 2\theta|_p = |(y_{k+2} + c_{k+1} + \theta) + \theta(1 - \theta y_{k+2})|_p$$
$$\leq \max\{|y_{k+2} + c_{k+1} + \theta|_p, |1 - \theta y_{k+2}|_p\}. \tag{3.19}$$

Similarly to the above argument, if we have $|y_{k+2} + c_{k+1} + \theta|_p \geq |1 - \theta y_{k+2}|_p$, then $|y_{k+2} + c_{k+1} + \theta|_p$ is the maximum in (3.19). Therefore, (3.19) is

$$\epsilon_k^{\delta} \leq |c_{k+1} + 2\theta|_p \leq \max\{|y_{k+2} + c_{k+1} + \theta|_p, |1 - \theta y_{k+2}|_p\}$$
$$= |y_{k+2} + c_{k+1} + \theta|_p < \epsilon_k^{1-2\delta}.$$

The above inequality gives a contradiction as before, therefore $|1 - \theta y_{k+2}|_p > |y_{k+2} + c_{k+1} + \theta|_p$ and the maximum in (3.19) is $|1 - \theta y_{k+2}|_p$. Hence, the inequality in (3.19) is

$$|c_{k+1} + 2\theta|_p \leq |1 - \theta y_{k+2}|_p. \tag{3.20}$$

The $\epsilon_{k+2}^{-\delta}$ definition in (3.4) and the above inequality (3.20) yield

$$\epsilon_{k+2} \leq \epsilon_{k+2}^{\delta} \leq |c_{k+1} + 2\theta|_p \leq |1 - \theta y_{k+2}|_p,$$

Therefore, $|1 - \theta y_{k+2}|_p \geq \epsilon_{k+2}$, which proves the lemma for the non-Archimedean absolute values ($\forall p < \infty$). By symmetry, had we considered the case $|y_{k-1}|_p \geq |1 - \theta y_k|_p^{-(1-\delta)}$, then we would have obtained $|1 \pm \theta y_{k-2}|_p \geq \epsilon_{k-2}$.

Now we prove the lemma for the Archimedean absolute value ($p = \infty$). Recall that in the definition of $\epsilon_k^{-\delta}$ in (3.4) $\kappa_\infty = 10$. Assume that $|1 - \theta y_k|_\infty < \epsilon_k$. As in the non-Archimedean absolute value case, we start from (3.5). Note that

$$\begin{aligned}
10|1-\theta y_k|_\infty^{-(1-\delta)} &= \frac{10|1-\theta y_k|_\infty^{\delta}}{|1-\theta y_k|_\infty},\\
&< \frac{10\epsilon_k^{\delta}}{|1-\theta y_k|_\infty} \quad \text{(from assumption } |1-\theta y_k|_\infty<\epsilon_k),\\
&\le \frac{|a_k+\theta b_k+c_k|_\infty}{|1-\theta y_k|_\infty} \quad \text{(from (3.4))},\\
&\le |(y_{k+1}+y_{k-1})(1+\theta y_k)+\theta b_k+c_k(1+\theta y_k)|_\infty\\
&\qquad \text{(from equation (3.5))},\\
&\le |y_{k+1}+y_{k-1}|_\infty\cdot|1+\theta y_k|_\infty+|b_k|_\infty+|c_k|_\infty\cdot|1+\theta y_k|_\infty\\
&\qquad \text{(triangle inequality)}.
\end{aligned} \tag{3.21}$$

As before, we need to find a relation between $|1+\theta y_k|_\infty$ and $|1-\theta y_k|_\infty^{-(1-\delta)}$. We start by $|1+\theta y_k|_\infty = |2-(1-\theta y_k)|_\infty$, then using the triangle inequality, we have

$$|1+\theta y_k|_\infty \le |1-\theta y_k|_\infty + |2|_\infty < \epsilon_k+2 < 1+2 = 3.$$

Also we have from (3.4) the following relations:

$$|b_k|_\infty \le \frac{1}{10}\epsilon_k^{-\delta} \le \frac{1}{10}\epsilon_k^{-(1-\delta)} \le \epsilon_k^{-(1-\delta)} < |1-\theta y_k|_\infty^{-(1-\delta)}. \tag{3.22}$$

Similarly,

$$|c_k|_\infty < |1-\theta y_k|_\infty^{-(1-\delta)}. \tag{3.23}$$

In the above inequalities, since $\epsilon_k<1$ and δ is sufficiently small, then the following relation is true $\epsilon_k < \epsilon_k^{-\delta} < \epsilon_k^{-(1-\delta)}$. Using (3.22), (3.23) and $|1+\theta y_k|_\infty < 3$ in the (3.21) inequality, we get

$$\begin{aligned}
10|1-\theta y_k|_\infty^{-(1-\delta)} &< 3|y_{k+1}+y_{k-1}|_\infty + |b_k|_\infty + 3|c_k|_\infty,\\
&< 3|y_{k+1}+y_{k-1}|_\infty + |1-\theta y_k|_\infty^{-(1-\delta)} + 3|1-\theta y_k|_\infty^{-(1-\delta)},\\
&\le 3|y_{k+1}+y_{k-1}|_\infty + 4|1-\theta y_k|_\infty^{-(1-\delta)}.
\end{aligned} \tag{3.24}$$

Therefore,

$$2|1-\theta y_k|_\infty^{-(1-\delta)} \le |y_{k+1}+y_{k-1}|_\infty \le |y_{k+1}|_\infty + |y_{k-1}|_\infty.$$

Hence, either $|y_{k+1}|_\infty \geq |1-\theta y_k|_\infty^{-(1-\delta)}$ or $|y_{k-1}|_\infty \geq |1-\theta y_k|_\infty^{-(1-\delta)}$, which proves the first assertion of the lemma. Here, without any loss of generality, we let $|y_{k+1}|_\infty \geq |1-\theta y_k|_\infty^{-(1-\delta)}$.

Now we prove that $|1 \pm \theta y_{k+2}|_\infty \geq \epsilon_{k+2}$ for the Archimedean case and we start with our assumption $|1-\theta y_k|_\infty < \epsilon_k$. Similar to the non-Archimedean case, we need first to show that the term $|y_{k+2}+c_{k+1}+\theta|_\infty$ is bounded from above by a power of ϵ_k. As before, to get the term $y_{k+2}+c_{k+1}+\theta$, we rewrite equation (3.3) in the following form using partial fractions:

$$\begin{aligned} y_{k+2}+c_{k+1}+\theta = & \frac{\frac{1}{2}(a_{k+1}+b_{k+1}+c_{k+1})}{1-y_{k+1}} \\ & + \frac{\frac{1}{2}(a_{k+1}-b_{k+1}+c_{k+1})}{1+y_{k+1}} + \theta(1-\theta y_k). \end{aligned} \tag{3.25}$$

To get a bound on the term $|y_{k+2}+c_{k+1}+\theta|_\infty$ we need to find bounds on the terms $|1-y_{k+1}|_\infty^{-1}$ and $|1+y_{k+1}|_\infty^{-1}$. Using the assumption and the first part of the lemma that we just proved, we get the following relation:

$$\begin{aligned} 10\epsilon_k^{-(1-\delta)} & < 10|1-\theta y_k|_\infty^{-(1-\delta)} \leq 10|y_{k+1}|_\infty = 10|1-(1-y_{k+1})|_\infty, \\ & \leq 10|1-y_{k+1}|_\infty + 10|1|_\infty, \\ & \quad \text{(using the triangle inequality)} \\ & < 10|1-y_{k+1}|_\infty + 2\epsilon_k^{-(1-\delta)} \\ & \quad \text{(using (3.4), } 10 = 2\cdot 10 \cdot |2|_\infty^{-1} \leq 2\epsilon_k^{-\delta} < 2\epsilon_k^{-(1-\delta)}). \end{aligned}$$

So,

$$\frac{1}{|1-y_{k+1}|_\infty} < \frac{5}{4}\epsilon_k^{(1-\delta)}. \tag{3.26}$$

Similarly, we get the following inequality:

$$\frac{1}{|1+y_{k+1}|_\infty} < \frac{5}{4}\epsilon_k^{(1-\delta)}. \tag{3.27}$$

Applying the Archimedean absolute value on equation (3.25) and using the triangle inequality, we have

$$
\begin{aligned}
|y_{k+2}+c_{k+1}+\theta|_\infty &\leq \frac{|\frac{1}{2}|_\infty.|a_{k+1}+b_{k+1}+c_{k+1}|_\infty}{|1-y_{k+1}|_\infty} \\
&+ \frac{|\frac{1}{2}|_\infty.|a_{k+1}-b_{k+1}+c_{k+1}|_\infty}{|1+y_{k+1}|_\infty} + |1-\theta y_k|_\infty, \\
&< \frac{1}{8}\epsilon_k^{1-2\delta} + \frac{1}{8}\epsilon_k^{1-2\delta} + \epsilon_k, \quad (\star_3) \\
&< 3\epsilon_k^\delta + 3\epsilon_k^\delta + 3\epsilon_k^\delta = 9\epsilon_k^\delta \quad (\star_4).
\end{aligned} \tag{3.28}
$$

In the above chain of inequalities, we used (3.26), (3.27), (3.4) and the assumption in step $(\star_3)$. In step $(\star_4)$ we used the fact that $\epsilon_k < \epsilon_k^{1-2\delta} < \epsilon_k^\delta$ since $\epsilon_k < 1$ and sufficiently small δ.

Note that we have from (3.4) the following inequalities:

$$
\begin{aligned}
10\epsilon_k^\delta \leq |c_{k+1}|_\infty &= |(y_{k+2}+c_{k+1}+\theta) - \theta(1+\theta y_{k+2})|_\infty, \\
&\leq |y_{k+2}+c_{k+1}+\theta|_\infty + |1+\theta y_{k+2}|_\infty, \\
&< 9\epsilon_k^\delta + |1+\theta y_{k+2}|_\infty.
\end{aligned}
$$

In the above inequalities, we used (3.28) and the triangle inequality. So $\epsilon_k^\delta < |1+\theta y_{k+2}|_\infty$ which implies

$$\epsilon_k < \epsilon_k^\delta < |1+\theta y_{k+2}|_\infty. \tag{3.29}$$

Now we use contradiction to prove $|1+\theta y_{k+2}|_\infty \geq \epsilon_{k+2}$. Assume that $|1+\theta y_{k+2}|_\infty < \epsilon_{k+2}$, then from (3.29) and the assumption in the beginning of the proof (Archimedean case), we have

$$|1-\theta y_k|_\infty < \epsilon_k < |1+\theta y_{k+2}|_\infty < \epsilon_{k+2}. \tag{3.30}$$

Therefore, we have

$$
\begin{aligned}
10\epsilon_{k+2} &< 10\epsilon_{k+2}^\delta, \quad \text{(since } \epsilon_{k+2} < 1 \text{ and } \delta \text{ is sufficiently small)} \\
&\leq |c_{k+1}|_\infty, \quad \text{(from } \epsilon_{k+2}^{-\delta} \text{ definition in (3.4))} \\
&= |(y_{k+2}+c_{k+1}+\theta) - \theta(1+\theta y_{k+2})|_\infty, \\
&\leq |y_{k+2}+c_{k+1}+\theta|_\infty + |1+\theta y_{k+2}|_\infty, \\
&< 9\epsilon_k^\delta + \epsilon_{k+2}, \quad \text{(from (3.28) and (3.30))} \\
&< 9\epsilon_{k+2}^\delta + \epsilon_{k+2}^\delta = 10\epsilon_{k+2}^\delta. \\
&\qquad \text{(from (3.30) and } \epsilon_{k+2} < 1,\ 0 < \delta).
\end{aligned} \tag{3.31}
$$

Note that in the above inequalities we have $10\epsilon_{k+2}^{\delta} < 10\epsilon_{k+2}^{\delta}$, which is a contradiction, hence $|1+\theta y_{k+2}|_\infty \geq \epsilon_{k+2}$.

Now we show that $|1-\theta y_{k+2}|_\infty \geq \epsilon_{k+2}$, starting from the same assumption $|1-\theta y_k|_\infty < \epsilon_k$. From (3.4), we have the following chain of inequalities:

$$\begin{aligned} 10\epsilon_k^{\delta} &\leq |c_{k+1}+2\theta|_\infty = |(y_{k+2}+c_{k+1}+\theta)+\theta(1-\theta y_{k+2})|_\infty, \\ &\leq |y_{k+2}+c_{k+1}+\theta|_\infty + |1-\theta y_{k+2}|_\infty, \quad \text{(triangle inequality)} \\ &< 9\epsilon_k^{\delta} + |1-\theta y_{k+2}|_\infty \quad \text{(from (3.28))}. \end{aligned}$$

Subtracting $9\epsilon_k^{\delta}$ from both sides of the above inequality yields

$$\epsilon_k < \epsilon_k^{\delta} < |1-\theta y_{k+2}|_\infty. \tag{3.32}$$

To prove $|1-\theta y_{k+2}|_\infty \geq \epsilon_{k+2}$, we use contradiction. Assume that $|1-\theta y_{k+2}|_\infty < \epsilon_{k+2}$, then we have

$$|1-\theta y_k|_\infty < \epsilon_k < |1-\theta y_{k+2}|_\infty < \epsilon_{k+2}. \tag{3.33}$$

From the inequalities in (3.33) and the definition of $\epsilon_{k+2}^{-\delta}$ in (3.4), we have

$$\begin{aligned} 10\epsilon_{k+2} &< 10\epsilon_{k+2}^{\delta}, \\ &\leq |c_{k+1}+2\theta|_\infty = |(y_{k+2}+c_{k+1}+\theta)+\theta(1-\theta y_{k+2})|_\infty, \\ &\leq |y_{k+2}+c_{k+1}+\theta|_\infty + |1-\theta y_{k+2}|_\infty, \\ &< 9\epsilon_k^{\delta} + \epsilon_{k+2}, \\ &< 9\epsilon_{k+2}^{\delta} + \epsilon_{k+2}^{\delta} = 10\epsilon_{k+2}^{\delta}. \end{aligned}$$

In the above inequalities we have a contradiction, since it shows that $10\epsilon_{k+2}^{\delta} < 10\epsilon_{k+2}^{\delta}$. Hence, $|1-\theta y_{k+2}|_\infty \geq \epsilon_{k+2}$. By symmetry, had we considered the case $|y_{k-1}|_\infty \geq |1-\theta y_k|_\infty^{-(1-\delta)}$, then we would have obtained $|1 \pm \theta y_{k-2}|_\infty \geq \epsilon_{k-2}$. Therefore, the proof is completed for this lemma. □

We explored in the above lemma and its proof how far or close the iterates of equation (3.3) are to the singularities ± 1 of the equation, given that $c_n \not\equiv 0, \pm 2$. Note that the terms used to describe the distance here (far, close) are with respect to the fixed absolute value under consideration and with respect to ϵ_n. Now we focus on the main result of this section which is a consequence of Theorem 3.1.1 below. The main result of this section is to show that if c_n is not identically 0 or ± 2, then equation (3.3) is not a candidate for Diophantine integrability given that it has an admissible solution. We state and prove Theorem 3.1.1 next where we use Lemma 3.1.1 in the proof.

Theorem 3.1.1. *Let $(y_n) \subset \mathbb{Q} \setminus \{-1, 1\}$ be an admissible solution of the equation (3.3):*

$$y_{n+1} + y_{n-1} = \frac{a_n + b_n y_n + c_n y_n^2}{1 - y_n^2},$$

where a_n, b_n and c_n are rational functions of n with $c_n \neq 0$ or ± 2 $\forall n$ and the right hand side of (3.3) is irreducible $\forall n$. There exists an integer r_0 such that $\forall r \geq r_0$ and for any $1 < F < 2$ and $D > 0$, then $h_r(y_n) \geq F^r D$ for sufficiently large r unless c_n is identically 0 or ± 2 $\forall n$.

Proof We defined $\epsilon_n^{-\delta}$ for all $n > K$ to be as in (3.4) and $\theta = 1$ or -1, where we choose an integer $r_0 \gg K$.

Since we are concerned with the behaviour of the solution y_n near the singularities ± 1, we start by finding an upper bound for the expression $\sum_{k=r_0}^{r} (\log^+ \frac{1}{|1-y_k|_p} + \log^+ \frac{1}{|1+y_k|_p})$ for a fixed absolute value. Once we get this upper bound, we could sum over all the p-adic and Archimedean absolute values $\forall p \leq \infty$ for both sides of the inequality. Then in the left hand side of the inequality $h_r(y_n)$ appears and we attain our result given in the statement of the theorem when $r \to \infty$.
First we define four sets of points (for a fixed absolute value $|.|_p$) by

$$\begin{aligned} A_1(r) &= \{n | r_0 \leq n \leq r \text{ and } |1 - y_n|_p < \epsilon_n\}, \\ A_2(r) &= \{n | r_0 \leq n \leq r \text{ and } |1 - y_n|_p \geq \epsilon_n\}, \\ A_3(r) &= \{n | r_0 \leq n \leq r \text{ and } |1 + y_n|_p < \epsilon_n\}, \\ A_4(r) &= \{n | r_0 \leq n \leq r \text{ and } |1 + y_n|_p \geq \epsilon_n\}. \end{aligned} \tag{3.34}$$

We estimate the above expression on the four sets of points defined in (3.34). Therefore,

$$\begin{aligned} &\sum_{k=r_0}^{r} \left(\log^+ \frac{1}{|1 - y_k|_p} + \log^+ \frac{1}{|1 + y_k|_p} \right) \\ &= \sum_{k=r_0}^{r} \log^+ \frac{1}{|1 - y_k|_p} + \sum_{k=r_0}^{r} \log^+ \frac{1}{|1 + y_k|_p}, \\ &= \sum_{k \in A_1(r)} \log^+ \frac{1}{|1 - y_k|_p} + \sum_{k \in A_2(r)} \log^+ \frac{1}{|1 - y_k|_p} \\ &+ \sum_{k \in A_3(r)} \log^+ \frac{1}{|1 + y_k|_p} + \sum_{k \in A_4(r)} \log^+ \frac{1}{|1 + y_k|_p}. \end{aligned} \tag{3.35}$$

For the two sets $A_2(r)$ and $A_4(r)$, since $\log^+ \frac{1}{|1\pm y_n|_p} = 0$ for any $|1 \pm y_n|_p \geq 1$, the only terms left to be considered in these sets are when $\epsilon_n \leq |1 \pm y_n|_p \leq 1$. Hence,

$$\begin{aligned}
&\sum_{k\in A_2(r)} \log^+ \frac{1}{|1-y_k|_p} = \sum_{k\in A_2(r)} \log^+ |1-y_k|_p^{-1}, \\
&\leq \sum_{k\in A_2(r)} \log^+ \epsilon_k^{-1} = \sum_{k\in A_2(r)} \log^+ (\epsilon_k^{-\delta})^{\frac{1}{\delta}}, \\
&= \frac{1}{\delta} \sum_{k\in A_2(r)} \log^+ \Big(\kappa_p \max\{|2|_p^{-1}, |a_k \pm b_k + c_k|_p^{-1}, |b_k|_p, |c_k|_p, \\
&\left|\frac{1}{2}\right|_p .|a_{k+1} \pm b_{k+1} + c_{k+1}|_p, \left|\frac{1}{2}\right|_p .|a_{k-1} \pm b_{k-1} + c_{k-1}|_p, \\
&|c_{k-1}|_p^{-1}, |c_{k+1}|_p^{-1}, |c_{k+1} \pm 2|_p^{-1}, |c_{k-1} \pm 2|_p^{-1}\}\Big), \\
&\leq \frac{1}{\delta} \sum_{k=r_0}^{r} \log^+ (\kappa_p \max\{|2|_p^{-1}, \ldots, |c_{k-1} \pm 2|_p^{-1}\}) \\
&\leq \frac{1}{\delta}(r - r_0 + 1) \log^+ \kappa_p \\
&+ \frac{1}{\delta} \sum_{k=r_0}^{r} \max\{\log^+ |2|_p^{-1}, \log^+ |a_k \pm b_k + c_k|_p^{-1}, \\
&\ldots, \log^+ |c_{k-1} \pm 2|_p^{-1}\}, \\
&\leq \frac{1}{\delta}(r - r_0 + 1) \log^+ \kappa_p + \frac{1}{\delta}(r - r_0 + 1) \log^+ |2|_p^{-1} \\
&+ \frac{2}{\delta}(r - r_0 + 1) \log^+ |1/2|_p + \frac{1}{\delta} \sum_{k=r_0}^{r} \Big(\log^+ |a_k \pm b_k + c_k|_p^{-1} \\
&+ \cdots + \log^+ |c_{k-1} \pm 2|_p^{-1}\Big), \\
&\leq \frac{1}{\delta}(r - r_0 + 1) \log^+ \kappa_p + \frac{1}{\delta}(r - r_0 + 1) \log^+ |2|_p^{-1} \\
&+ \frac{2}{\delta}(r - r_0 + 1) \log^+ |1/2|_p + \frac{1}{\delta} \sum_{k=r_0-1}^{r+1} \Big(\log^+ |a_k \pm b_k + c_k|_p^{-1} \\
&+ \cdots + \log^+ |c_{k-1} \pm 2|_p^{-1}\Big).
\end{aligned} \tag{3.36}$$

Here we have used (3.4) and the fact that the maximum of a set of non-negative elements is less than or equal to the sum of all the elements in that set. Similarly, we have the following inequality:

$$\sum_{k\in A_4(r)} \log^+ \frac{1}{|1+y_k|_p} \le \frac{1}{\delta}(r-r_0+1)\log^+ \kappa_p + \frac{1}{\delta}(r-r_0+1)\log^+ |2|_p^{-1}$$
$$+\frac{2}{\delta}(r-r_0+1)\log^+ |1/2|_p + \frac{1}{\delta}\sum_{k=r_0-1}^{r+1} (\log^+ |a_k \pm b_k + c_k|_p^{-1}$$
$$+\cdots+\log^+ |c_{k-1} \pm 2|_p^{-1}). \tag{3.37}$$

It is crucial in the proof of this theorem to prove that $A_1(r) \cap A_3(r) = \emptyset$. We show this for the absolute value $|.|_p$ where $p \le \infty$. Given $|1-y_n|_p < \epsilon_n$ and since $2 = (1-y_n) + (1+y_n)$, then for the Archimedean absolute value ($p = \infty$) we have

$$|2|_\infty \le |1-y_n|_\infty + |1+y_n|_\infty < \epsilon_n + |1+y_n|_\infty.$$

Subtracting ϵ_n from both sides of the inequality above we have $2-\epsilon_n < |1+y_n|_\infty$. Since we have $\epsilon_n < 1$ from (3.4) then we have $2\epsilon_n < 2$ which yields

$$2\epsilon_n - \epsilon_n < 2 - \epsilon_n < |1+y_n|_\infty.$$

Hence, $\epsilon_n < |1+y_n|_\infty$. Therefore, for any $n \in A_1(r)$ we have $n \notin A_3(r)$ in the Archimedean case, so we avoid double counting of points n in the inequality (3.35). Now for the non-Archimedean case where ($p < \infty$) we show that we do not have double counting of points, basically for the following reason. If $|1-y_n|_p < \epsilon_n$ where $p < \infty$, then using (3.4) we have

$$\epsilon_n^\delta \le |2|_p \le \max\{|1-y_n|_p, |1+y_n|_p\}.$$

If we have $|1-y_n|_p \ge |1+y_n|_p$, then this implies that the above inequality is

$$\epsilon_n^\delta \le \max\{|1-y_n|_p, |1+y_n|_p\} = |1-y_n|_p < \epsilon_n. \tag{3.38}$$

This is a contradiction since $\epsilon_n \le \epsilon_n^\delta$. Therefore, $|1+y_n|_p > |1-y_n|_p$ and the maximum is $|1+y_n|_p$. Hence, $\epsilon_n \le \epsilon_n^\delta \le |1+y_n|_p$. This implies that we avoid double counting in the inequality (3.35). Consequently, $\forall p \le \infty$ we do not have points in both sets $A_1(r)$ and $A_3(r)$ at the same time, i.e. $A_1(r) \cap A_3(r) = \emptyset$. Define $\sigma_n = -1$ or 1, depending on the location of n in the set $A_1(r)$ such that if $|1-y_n|_p < \epsilon_n$, then $|y_{n+\sigma_n}|_p \geq \frac{1}{|1-y_n|_p^{1-\delta}}$, (proved in Lemma 3.1.1). Also define $\widehat{\sigma}_m = -1$ or 1, depending on the location of m in the set $A_3(r)$, so that if $|1+y_m|_p < \epsilon_m$, then $|y_{m+\widehat{\sigma}_m}|_p \geq \frac{1}{|1+y_m|_p^{1-\delta}}$ also from Lemma 3.1.1. Note that for each $n \in A_1(r)$ we have $n + \sigma_n \neq m + \widehat{\sigma}_m$ $\forall m \in A_3(r)$ since the distance between any small terms $|1-\theta y_n|_p < \epsilon_n$ (for $\theta = -1$ or 1) is more than 2 steps (Lemma 3.1.1). This implies $\{n+\sigma_n | n \in A_1(r)\} \cap \{m + \widehat{\sigma}_m | m \in A_3(r)\} = \emptyset$. Therefore, we will not have double counting of points in the expression defined in (3.35). Now we have

$$\begin{aligned} \sum_{k\in A_1(r)} \log^+ \frac{1}{|1-y_k|_p} &= \sum_{k\in A_1(r)} \log^+(|1-y_k|_p^{-(1-\delta)})^{\frac{1}{1-\delta}}, \\ = \frac{1}{1-\delta} \sum_{k\in A_1(r)} \log^+ |1-y_k|_p^{-(1-\delta)} &\leq \frac{1}{1-\delta} \sum_{k\in A_1(r)} \log^+ |y_{k+\sigma_k}|_p. \end{aligned} \tag{3.39}$$

We used in the inequality above the result from Lemma 3.1.1. Similarly, we have

$$\begin{aligned} \sum_{k\in A_3(r)} \log^+ \frac{1}{|1+y_k|_p} &= \sum_{k\in A_3(r)} \log^+(|1+y_k|_p^{-(1-\delta)})^{\frac{1}{1-\delta}}, \\ = \frac{1}{1-\delta} \sum_{k\in A_3(r)} \log^+ |1+y_k|_p^{-(1-\delta)} &\leq \frac{1}{1-\delta} \sum_{k\in A_3(r)} \log^+ |y_{k+\widehat{\sigma}_k}|_p. \end{aligned} \tag{3.40}$$

Therefore, adding the inequalities in (3.39) and (3.40), we get the relation below:

$$\begin{aligned} &\sum_{k\in A_1(r)} \log^+ \frac{1}{|1-y_k|_p} + \sum_{k\in A_3(r)} \log^+ \frac{1}{|1+y_k|_p} \\ &\leq \frac{1}{1-\delta}\Big(\sum_{k\in A_1(r)} \log^+ |y_{k+\sigma_k}|_p + \sum_{k\in A_3(r)} \log^+ |y_{k+\widehat{\sigma}_k}|_p \Big), \\ &\leq \frac{1}{1-\delta} \sum_{k=r_0-1}^{r+1} \log^+ |y_k|_p, \end{aligned} \tag{3.41}$$

since we do not have double counting of points. Now we have from (3.35) and the previous calculations in (3.36),(3.37) and (3.41) the following inequality. For a fixed absolute value $|.|_p$ ($p \leq \infty$), we have

$$\begin{aligned}
&\sum_{k=r_0}^{r}\left(\log^+ \frac{1}{|1-y_k|_p} + \log^+ \frac{1}{|1+y_k|_p}\right) \leq \frac{2}{\delta}(r-r_0+1)\log^+ \kappa_p \\
&+ \frac{2}{\delta}(r-r_0+1)\log^+ |2|_p^{-1} + \frac{4}{\delta}(r-r_0+1)\log^+ |1/2|_p \\
&+ \frac{2}{\delta}\sum_{k=r_0-1}^{r+1}\Big(\log^+ |a_k \pm b_k + c_k|_p + \cdots + 2\log^+ |c_k \pm 2|_p^{-1}\Big) \\
&+ \frac{1}{1-\delta}\sum_{k=r_0-1}^{r+1}\log^+ |y_k|_p. \qquad (3.42)
\end{aligned}$$

To get the height, we sum over all the primes ($p \leq \infty$) in (3.42) which yields

$$\begin{aligned}
&\sum_{k=r_0}^{r} h\left(\frac{1}{1-y_k}\right) + \sum_{k=r_0}^{r} h\left(\frac{1}{1+y_k}\right) \leq \frac{2}{\delta}(r-r_0+1)\log 10 \\
&+ \frac{2}{\delta}(r-r_0+1)\log 2 + \frac{4}{\delta}(r-r_0+1)\log 2 \\
&+ \frac{2}{\delta}\sum_{k=r_0-1}^{r+1}(2h(a_k \pm b_k + c_k) + \cdots + 2h(c_k \pm 2)) + \frac{1}{1-\delta}\sum_{k=r_0-1}^{r+1} h(y_k), \\
&= \frac{6}{\delta}(r-r_0+1)\log 2 + \frac{2}{\delta}(r-r_0+1)\log 10 \\
&+ \frac{2}{\delta}\sum_{k=r_0-1}^{r+1}(2h(a_k \pm b_k + c_k) + \cdots + 2h(c_k \pm 2)) + \frac{1}{1-\delta}\sum_{k=r_0-1}^{r+1} h(y_k). \qquad (3.43)
\end{aligned}$$

Note that we have from the height properties (given in Chapter 2) the following inequalities:

$$h\left(\frac{1}{1-\theta y_k}\right) = h(1-\theta y_k) \leq h(1) + h(\theta y_k) + \log 2 = h(y_k) + \log 2,$$

and

$$\begin{aligned}
h(y_k) &= h(\theta y_k) = h(1-(1-\theta y_k)) \leq h(1) + h(1-\theta y_k) + \log 2, \\
&= h(1-\theta y_k) + \log 2 = h\left(\frac{1}{1-\theta y_k}\right) + \log 2.
\end{aligned}$$

The above inequalities imply

$$\left| h\left(\frac{1}{1-\theta y_k}\right) - h(y_k)\right|_\infty \leq \log 2, \qquad (3.44)$$

where $\theta = 1$ or -1. Using (3.44) result and using the notation for the summed logarithmic height $\sum_{k=r_0}^{r} h(x_k) = h_r(x_k)$ in (3.43), we get

$$\sum_{k=r_0}^{r}(h(y_k) - \log 2) + \sum_{k=r_0}^{r}(h(y_k) - \log 2) \le \frac{6}{\delta}(r - r_0 + 1)\log 2$$
$$+ \frac{2}{\delta}(r - r_0 + 1)\log 10 + \frac{2}{\delta}\Big(2h_{r+1}(a_k \pm b_k + c_k) + \cdots + 2h_{r+1}(c_k \pm 2)\Big)$$
$$+ \frac{2}{\delta}\Big(2h(a_{r_0-1} \pm b_{r_0-1} + c_{r_0-1}) + \cdots + 2h(c_{r_0-1} \pm 2)\Big)$$
$$+ \frac{1}{1-\delta}h(y_{r_0-1}) + \frac{1}{1-\delta}h_{r+1}(y_k).$$

So

$$2h_r(y_k) - 2(r - r_0 + 1)\log 2 \le \frac{6}{\delta}(r - r_0 + 1)\log 2 + \frac{2}{\delta}(r - r_0 + 1)\log 10$$
$$+ \frac{2}{\delta}\Big(2h_{r+1}(a_k \pm b_k + c_k) + \cdots + 2h_{r+1}(c_k \pm 2)\Big)$$
$$+ \frac{2}{\delta}\Big(2h(a_{r_0-1} \pm b_{r_0-1} + c_{r_0-1}) + \cdots + 2h(c_{r_0-1} \pm 2)\Big)$$
$$+ \frac{1}{1-\delta}h(y_{r_0-1}) + \frac{1}{1-\delta}h_{r+1}(y_k). \tag{3.45}$$

Simplifying the inequality in (3.45), we have

$$h_r(y_k) \le \left(1 + \frac{3}{\delta}\right)(r - r_0 + 1)\log 2 + \frac{1}{\delta}(r - r_0 + 1)\log 10$$
$$+ \frac{1}{\delta}\Big(2h_{r+1}(a_k \pm b_k + c_k) + \cdots + 2h_{r+1}(c_k \pm 2)\Big)$$
$$+ \frac{1}{\delta}\Big(2h(a_{r_0-1} \pm b_{r_0-1} + c_{r_0-1}) + \cdots + 2h(c_{r_0-1} \pm 2)\Big)$$
$$+ \frac{1}{2(1-\delta)}h(y_{r_0-1}) + \frac{1}{2(1-\delta)}h_{r+1}(y_k). \tag{3.46}$$

For an admissible solution y_k of the equation (3.3), where $\max\{h_r(a_k), h_r(b_k), h_r(c_k), r\} = o(h_r(y_k))$, the inequality in (3.46) becomes

$$h_{r+1}(y_k) \ge 2(1-\delta)h_r(y_k) + o(h_{r+1}(y_k)).$$

Using the shift $r \to r - 1$ in the above inequality, we have

$$h_r(y_k) \ge 2(1-\delta)h_{r-1}(y_k) + o(h_r(y_k)). \tag{3.47}$$

Any function $R_r = o(h_r(y_k))$ satisfies the following inequality for any fixed $\nu > 0$, there exists r_0 such that $|R_r|_\infty < \nu \cdot |h_r(y_k)|_\infty$ holds for all $r > r_0$. Using this fact in (3.47) and applying this recurrence relation repeatedly, we get

$$h_r(y_k) \geq \left(\frac{2(1-\delta)}{1+\nu}\right)^r D. \tag{3.48}$$

Here $D > 0$, and for sufficiently small δ, ν, $1 < F = \frac{2(1-\delta)}{1+\nu} < 2$. Consequently, the summed logarithmic height of an admissible solution for the equation (3.3) is bounded from below by an increasing exponential function. This implies that it is increasing exponentially unless c_k is identically 0 or ± 2, which proves the theorem.

□

The above theorem implies that equation

$$y_{n+1} + y_{n-1} = \frac{a_n + b_n y_n + c_n y_n^2}{1 - y_n^2},$$

is not a candidate for Diophantine integrability if it has at least one admissible solution unless c_n is identically 0 or ± 2. Hence, we have 3 cases to examine to investigate the Diophantine integrability of the above equation. In the next sections, we consider these 3 cases where $c_n = 0$ or $c_n = \pm 2$ for all n. We find sub-cases that reduce our equation to a discrete analogue of Painlevé II equation or where y_n solves a difference Riccati equation or $h_r(y_n)$ grows fast with r.

3.2 Diophantine integrability analysis of equation $y_{n+1} + y_{n-1} = \frac{a_n + b_n y_n}{1 - y_n^2}$

Now we consider the case in which c_n vanishes identically i.e.

$$y_{n+1} + y_{n-1} = \frac{a_n + b_n y_n}{1 - y_n^2}, \tag{3.49}$$

where the right hand side of (3.49) is irreducible for all n. The strategy in this section is to show that there is a number $\tau < 2$ such that for each absolute value $|.|_p$ ($\forall p \leq \infty$) on $\mathbb{Q}$ and for all $r \geq r_0$,

$$\sum_{n=r_0}^{r} \left(\log^+ \frac{1}{|1 - y_n|_p} + \log^+ \frac{1}{|1 + y_n|_p}\right) \leq \tau \sum_{n=r_0}^{r+1} \log^+ |y_n|_p. \tag{3.50}$$

We can then sum this inequality over all absolute values to show that the summed logarithmic height grows exponentially. Before we proceed to apply our strategy we set the assumptions and definitions that are used throughout this section. Note that the functions $a_n \pm b_n$ are not equal to zero $\forall n$, otherwise the right hand side of (3.49) is reducible, contradicting our assumption. We define an integer K such that it is greater than all the zeros and poles of a_n, b_n and their linear combinations $a_n \pm b_n$, $\pm a_n + b_n - 2b_{n+1}$, $\pm a_n + b_n - 2b_{n-1}$ and $a_n \pm b_n \pm (\pm a_{n-2} + b_{n-2} - 2b_{n-1})$ (if there are any). If these rational functions have no zeros and no poles, then K could be any integer. We introduce a quantity ϵ_n that is used to measure the distance (in terms of a fixed absolute value $|.|_p$ where $p \leq \infty$) between the iterate y_n and the singularities ± 1 and also to measure the size of the iterate y_n. We introduce a definition for ϵ_n where we take the maximum of a set of elements when they are all finite values for all n. If any of these elements is ∞ for all n, then we remove it from the set and take the maximum of all the remaining finite elements. For a fixed absolute value, we define $\epsilon_n^{-\delta}$ for all $n > K$ where δ is a sufficiently small real number ($0 < \delta$) as follows:

$$\begin{aligned}\epsilon_n^{-\delta} = \kappa_p \max\Big\{ & |2|_p^{-1}, |1/2|_p \cdot |a_n \pm b_n|_p, |1/2|_p^{-1} \cdot |a_n \pm b_n|_p^{-1}, |a_{n+1}|_p, |b_{n+1}|_p, \\ & |a_{n-1}|_p, |b_{n-1}|_p, |1/2|_p \cdot |a_{n+2} \pm b_{n+2}|_p, |1/2|_p \cdot |a_{n-2} \pm b_{n-2}|_p \\ & |\pm a_n + b_n - 2b_{n+1}|_p, |\pm a_n + b_n - 2b_{n-1}|_p, |a_n \pm b_n|_p^{-1}, \\ & |\pm a_n + b_n - 2b_{n+1}|_p^{-1}, |\pm a_n + b_n - 2b_{n-1}|_p^{-1}, |a_n \pm b_n|_p, \\ & |1/2|_p^{-1} \cdot |a_n \mp b_n \mp (\pm a_{n-2} + b_{n-2} - 2b_{n-1})|_p^{-1} \Big\}. \end{aligned} \tag{3.51}$$

For the non-Archimedean absolute value, $\kappa_p = 1$, and for the Archimedean absolute value, $\kappa_p = 10$. It is evident from the definition that $\epsilon_n \leq 1$ when $p < \infty$ and $p \neq 2$, while $\epsilon_n < 1$ when $p = \infty$ or $p = 2$. Now we are ready to execute our strategy. For a fixed absolute value $|.|_p$ and $r \geq r_0$ where $r_0 \gg K$, define the sets

$$\begin{aligned} A_r^{\pm} &= \{n : r_0 \leq n \leq r \text{ and } |1 \mp y_n|_p < \epsilon_n\} \quad \text{and} \\ B_r^{\pm} &= \{n : r_0 \leq n \leq r \text{ and } \epsilon_n \leq |1 \mp y_n|_p < 1\}. \end{aligned}$$

The points of A_r^+ will be called 1 points (since y_n is close to 1 with respect to the absolute value) and the points of A_r^- will be called -1 points. Recall from the proof of Theorem 3.1.1 (where in this part of the proof we did not use the assumption $c_n \not\equiv 0$, so it is still valid here) that if $|1-\theta y_n|_p < \epsilon_n$ for $\theta = 1$ or -1 then $|1+\theta y_n|_p \geq \epsilon_n$. Hence $A_r^+ \cap A_r^- = \emptyset$. Clearly

$$\sum_{n=r_0}^{r} \left(\log^+ \frac{1}{|1-y_n|_p} + \log^+ \frac{1}{|1+y_n|_p} \right) = \sum_{n \in A_r^+} \log^+ \frac{1}{|1-y_n|_p} + \sum_{n \in A_r^-} \log^+ \frac{1}{|1+y_n|_p} + \Phi_r,$$

where

$$\Phi_r = \sum_{n \in B_r^+} \log^+ \frac{1}{|1-y_n|_p} + \sum_{n \in B_r^-} \log^+ \frac{1}{|1+y_n|_p}.$$

Using the same argument as in the previous section, we see that the admissibility of y_n implies that $\sum_{p \leq \infty} \Phi_r$ is bounded from above by $o(h_{r+1}(y_n))$.

In order to bound

$$\sum_{n \in A_r^+} \log^+ \frac{1}{|1-y_n|_p} + \sum_{n \in A_r^-} \log^+ \frac{1}{|1+y_n|_p}$$

by a multiple of $\sum_{n=r_0}^{r+1} \log^+ |y_n|_p$, we construct a number of disjoint subintervals containing only 1 points, -1 points and points where y_n is sufficiently large to make a significant contribution to the right hand side of the inequality (3.50). These subintervals are called oscillating sequences.

Definition 3.2.1. *Suppose that $|1-\theta y_k|_p < \epsilon_k$, for some $k \in \mathbb{Z}$ and $\theta = 1$ or $\theta = -1$. Then the oscillating sequence S containing k is the longest interval in $\mathbb{Z}$ (possibly unbounded) satisfying the following conditions.*

1. *If $k + 2l \in S$ then $|1-(-1)^l \theta y_{k+2l}|_p < \epsilon_{k+2l}$;*
2. *If $\{k+2l-1, k+2l\} \in S$, then $|y_{k+2l-1}|_p \geq |1-(-1)^l \theta y_{k+2l}|_p^{-(1-\delta)}$; and*
3. *If $\{k+2l, k+2l+1\} \in S$, then $|y_{k+2l+1}|_p \geq |1-(-1)^l \theta y_{k+2l}|_p^{-(1-\delta)}$.*

Recall from the proof of Lemma 3.1.1 that if $|1-\theta y_n|_p < \epsilon_n$ then either $|y_{n+1}|_p \geq |1-\theta y_n|_p^{-(1-\delta)}$ or $|y_{n-1}|_p \geq |1-\theta y_n|_p^{-(1-\delta)}$ (where for this part of the proof we did not use the assumption $c_n \not\equiv 0$, so it is still valid here). This implies that every ± 1 point lies in an oscillating sequence containing at least two elements. For a fixed oscillating sequence S and $r \geq r_0$, we will now obtain a suitable upper bound for

$$\sum_{n \in S \cap A_r^+} \log^+ \frac{1}{|1-y_n|_p} + \sum_{n \in S \cap A_r^-} \log^+ \frac{1}{|1+y_n|_p}. \tag{3.52}$$

Case 1: Let $m+1$ be the total number of 1 points and -1 points in $S \cap [r_0, r]$ and assume that $m \geq 2$. Let I be the shortest subinterval of $S \cap [r_0, r]$ containing these ± 1 points. Let k be the first term in I, so that $|1-\theta y_k|_p < \epsilon_k$ for some choice of $\theta = -1$ or 1. Then $I = \{k, k+1, \ldots, k+2m\}$ and contains exactly m points on which y_n is big in the following sense:

$$|y_{k+1}|_p \geq |1-\theta y_k|_p^{-(1-\delta)}, \quad |y_{k+2m-1}|_p \geq |1-(-1)^m \theta y_{k+2m}|_p^{-(1-\delta)}$$

and

$$|y_{k+2l+1}|_p \geq \max\{|1-(-1)^l\theta y_{k+2l}|_p^{-(1-\delta)}, |1-(-1)^{l+1}\theta y_{k+2l+2}|_p^{-(1-\delta)}\}, \qquad l = 1, \ldots, m-2.$$

Hence

$$\begin{aligned}
&\sum_{n\in S\cap A_r^+} \log^+ \frac{1}{|1-y_n|_p} + \sum_{n\in S\cap A_r^-} \log^+ \frac{1}{|1+y_n|_p} \\
&= \sum_{l=0}^{m} \log^+ \frac{1}{|1-(-1)^l\theta y_{k+2l}|_p} \\
&= \sum_{l=1}^{m} \frac{l}{m} \log^+ \frac{1}{|1-(-1)^l\theta y_{k+2l}|_p} + \sum_{l=0}^{m-1} \frac{m-l}{m} \log^+ \frac{1}{|1-(-1)^l\theta y_{k+2l}|_p} \\
&= \sum_{l=0}^{m-1} \frac{l+1}{m} \log^+ \frac{1}{|1-(-1)^{l+1}\theta y_{k+2l+2}|_p} + \sum_{l=0}^{m-1} \frac{m-l}{m} \log^+ \frac{1}{|1-(-1)^l\theta y_{k+2l}|_p} \\
&\leq \frac{1}{1-\delta} \sum_{l=0}^{m-1} \left(\frac{l+1}{m} + \frac{m-l}{m}\right) \log^+ |y_{k+2l+1}|_p \\
&= \frac{m+1}{(1-\delta)m} \sum_{l=0}^{m-1} \log^+ |y_{k+2l+1}|_p = \frac{m+1}{(1-\delta)m} \sum_{n\in S\cap[r_0,r]} \log^+ |y_n|_p \\
&\leq \frac{3}{2(1-\delta)} \sum_{n\in S\cap[r_0,r]} \log^+ |y_n|_p,
\end{aligned}$$

where the last inequality follows from $m \geq 2$.

Case 2: There are exactly two ± 1 points in $S \cap [r_0, r]$. Define k such that these points are k and $k+2$. That is, for some choice of $\theta = \pm 1$, we have $|1-\theta y_k|_p < \epsilon_k$ and $|1+\theta y_{k+2}|_p < \epsilon_{k+2}$. We now use the following corollary of Theorem 1.4.2. We will prove Theorem 1.4.2 and this corollary at the end of this section.

Corollary 3.2.1. *For a fixed absolute value $|.|_p$ ($\forall p \leq \infty$) let $k-1 > K$ be such that $|1-\theta y_k|_p < \epsilon_k$, $|y_{k-1}|_p \leq |1-\theta y_k|_p^{-1/2}$ and $|1+\theta y_{k+2}|_p < \epsilon_{k+2}$ where $\theta = 1$ or -1. Assume that $|a_k - \theta b_k - \theta(\theta a_{k-2} + b_{k-2} - 2b_{k-1})|_p \not\equiv 0$, then $|y_{k+3}|_p > |1+\theta y_{k+2}|_p^{-1/2}$.*

Hence if $|a_k - \theta b_k - \theta(\theta a_{k-2} + b_{k-2} - 2b_{k-1})|_p \not\equiv 0$, then either $|y_{k-1}|_p > |1-\theta y_k|_p^{-1/2}$ or $|y_{k+3}|_p > |1+\theta y_{k+2}|_p^{-1/2}$. This says that, even if neither $k-1$ nor $k+3$ is in S, at least one of y_{k-1} or y_{k+3} has to be moderately large. Without loss of generality, we assume that $|y_{k-1}|_p > |1-\theta y_k|_p^{-1/2}$.

We have

$$
\begin{aligned}
&\sum_{n\in S\cap A_r^+} \log^+ \frac{1}{|1-y_n|_p} + \sum_{n\in S\cap A_r^-} \log^+ \frac{1}{|1+y_n|_p}, \\
&= \log^+ \frac{1}{|1-\theta y_k|_p} + \log^+ \frac{1}{|1+\theta y_{k+2}|_p}, \\
&= \eta \log^+ \frac{1}{|1-\theta y_k|_p} + (1-\eta)\log^+ \frac{1}{|1-\theta y_k|_p} + \log^+ \frac{1}{|1+\theta y_{k+2}|_p}, \\
&\leq 2\eta \log^+ |y_{k-1}|_p + \frac{1-\eta}{1-\delta}\log^+ |y_{k+1}|_p + \frac{1}{1-\delta}\log^+ |y_{k+1}|_p, \\
&= 2\eta \log^+ |y_{k-1}|_p + \frac{2-\eta}{1-\delta}\log^+ |y_{k+1}|_p. \qquad (3.53)
\end{aligned}
$$

This calculation shows that for $\eta > 0$, we can reduce the coefficient of $\log^+ |y_{k+1}|_p$ by introducing a contribution from y_{k-1}. If $k-1 \in S$, this is not problematic and we would have an upper bound for (3.53) of the form

$$\max\left(\frac{2-\eta}{1-\delta}, 2\eta\right) \sum_{n\in S} \log^+ |y_n|_p.$$

However, if $k-1 \notin S$ then we need to be careful because we will later sum our estimates for (3.52) over all oscillating sequences. When we do this we might need to "share" the term $k-1$ with another oscillating sequence, in which case it will appear twice in the upper bound and we will need to sum the contributions. Note that the term $k-1$ here cannot be part of a subinterval I of the type considered in case 1 above as such subintervals of oscillating sequences have only ± 1 points as endpoints. There could, however, be two adjacent oscillating sequences S_1 and S_2 both of the type considered in the present case (case 2) that need to share the contribution from y_{k-1}. If so, then summing over the contributions for both oscillating sequences would give the upper bound

$$\frac{2-\eta}{1-\delta}\log^+ |y_{k-3}|_p + 4\eta \log^+ |y_{k-1}|_p + \frac{2-\eta}{1-\delta}\log^+ |y_{k+1}|_p$$

which, in turn, is bounded from above by

$$\max\left(\frac{2-\eta}{1-\delta}, 4\eta\right) \sum_{n=k-3}^{k+1} \log^+ |y_n|_p.$$

Note that $k-1$ could also be part of an oscillating sequence of the type we are about to consider in case 3.

Case 3: There is exactly one $k_1 \in S \cap [r_0, r]$ such that $|1-\theta y_{k_1}|_p < \epsilon_{k_1}$ for $\theta = -1$

or 1. Since S has at least two points, we know that either $|y_{k_1-1}|_p \geq |1-\theta y_{k_1}|_p^{-(1-\delta)}$ or $|y_{k_1+1}|_p \geq |1-\theta y_{k_1}|_p^{-(1-\delta)}$. Without loss of generality, we assume the latter. Note that since $k_1 \in S \cap [r_0, r]$, then $k_1+1 \in S \cap [r_0, r+1]$.

Now

$$\sum_{n \in S \cap A_r^+} \log^+ \frac{1}{|1-y_n|_p} + \sum_{n \in S \cap A_r^-} \log^+ \frac{1}{|1+y_n|_p} = \log^+ \frac{1}{|1-\theta y_{k_1}|_p} \leq \frac{1}{1-\delta} \log^+ |y_{k_1+1}|_p.$$

It is conceivable that k_1+1 is adjacent to, or part of, a sequence of the type considered in case 2 in such a way that it plays the role of $k-1$ in the analysis above of that case. In other words, summing over the contributions of these two oscillating sequences in the left side of (3.50) leads to a term of the form

$$\left(\frac{1}{1-\delta} + 2\eta\right) \log^+ |y_{k_1+1}|_p$$

on the right hand side.

We have now considered all possible oscillating sequences under the assumption that both $|a_k - \ b_k - (a_{k-2} + b_{k-2} - 2b_{k-1})|_p \neq 0$ and $|a_k + b_k + (-a_{k-2} + b_{k-2} - 2b_{k-1})|_p \neq 0$ for all $k > K$. Combining our results, we have

$$\sum_{n=r_0}^{r} \left(\log^+ \frac{1}{|1-y_n|_p} + \log^+ \frac{1}{|1+y_n|_p}\right) \leq \tau \sum_{n=r_0}^{r+1} \log^+ |y_n|_p + \Phi_r,$$

where

$$\tau = \max\left(\frac{3}{2(1-\delta)}, \frac{2-\eta}{1-\delta}, 2\eta, 4\eta, \frac{1}{1-\delta} + 2\eta\right).$$

In particular, choosing $\eta = 3/8$ and δ sufficiently small, we have

$$\tau = \frac{3}{4} + \frac{1}{1-\delta} < 2.$$

Since $\sum_{p \leq \infty} \Phi_r$ is bounded from above by a small expression (i.e. $o(h_{r+1}(y_n))$), we see that $h_r(y)$ grows exponentially with r. Hence,

$$h_r(y_n) \leq \frac{\tau}{2} h_{r+1}(y_n) + o(h_{r+1}(y_n)), \tag{3.54}$$

If at least one of the above assumptions do not hold, then we could have oscillating sequences which are called special oscillating sequences. We have two forms of these sequences depending on which quantity vanishes.

Definition 3.2.2. *The special oscillating sequence S_p starting with k is $S_p = \{k, k+1, k+2\}$. It is an oscillating sequence of length 3 starting with k in $\mathbb{Z}$ such that $|1-\theta y_k|_p < \epsilon_k$,*

$|y_{k+1}|_p \geq \max\left\{|1-\theta y_k|_p^{-(1-\delta)}, |1+\theta y_{k+2}|_p^{-(1-\delta)}\right\}$ *and* $|1+\theta y_{k+2}|_p < \epsilon_{k+2}$. *Also, we have* $|y_{k-1}|_p \leq |1-\theta y_k|_p^{-1/2}$ *and* $|y_{k+3}|_p \leq |1+\theta y_{k+2}|_p^{-1/2}$.

We need to understand from where the definition of special oscillating sequences emerged. It means that we need to examine closely the size of the first three iterates $\{y_{k+1}, y_{k+2}, y_{k+3}\}$ of equation (3.49). The term size is used here with respect to the quantity ϵ_n (3.51) we are using and to the absolute value under consideration. Recall Theorem 1.4.2 where this examination is illustrated and in the discussion which follows the theorem.

Theorem 1.4.2. *Let* $(y_n)_{n=k-1}^{k+3} \subset \mathbb{Q} \setminus \{-1, 1\}$ *satisfy*

$$y_{n+1} + y_{n-1} = \frac{a_n + b_n y_n}{1 - y_n^2},$$

where k is sufficiently large and the right hand side of (3.49) is irreducible. Assume that for a fixed absolute value $|.|_p$ ($\forall p \leq \infty$) we have $|y_{k-1}|_p \leq |1-\theta y_k|_p^{-1/2}$ for $\theta = 1$ or -1. Furthermore, for sufficiently small δ, if $|1-\theta y_k|_p < \epsilon_k$ with $\epsilon_k^{-\delta}$ defined by (3.51), then

1. $y_{k+1} = \dfrac{a_k + \theta b_k}{2(1-\theta y_k)} + A_k$, *where* $|A_k|_p \leq |1-\theta y_k|_p^{-1/2}$ *for non-Archimedean absolute value and* $|A_k|_p \leq \frac{11}{10} \cdot |1-\theta y_k|_p^{-1/2}$ *for Archimedean absolute value.*
2. $y_{k+2} = -\theta + \left(\dfrac{\theta a_k + b_k - 2b_{k+1}}{a_k + \theta b_k}\right)(1-\theta y_k) + B_k$, *where* $|B_k|_p \leq |1-\theta y_k|_p^{3/2-5\delta}$ *for non-Archimedean absolute value and* $|B_k|_p \leq \frac{1}{2} \cdot |1-\theta y_k|_p^{3/2-5\delta}$ *for Archimedean absolute value.*
3. $y_{k+3} = \dfrac{\left(a_{k+2} - \theta b_{k+2} - \theta(\theta a_k + b_k - 2b_{k+1})\right)}{2(1+\theta y_{k+2})} + C_k$ *where* $|C_k|_p \leq |1+\theta y_{k+2}|_p^{-(2/3+2\delta)}$ *for non-Archimedean absolute value and* $|C_k|_p \leq 2|1+\theta y_{k+2}|_p^{-(2/3+2\delta)}$ *for Archimedean absolute value.*

Intuitively speaking, Theorem 1.4.2 shows that for a fixed absolute value $|.|_p$ ($\forall p \leq \infty$), if we start with an iterate close to θ, $|1-\theta y_k|_p < \epsilon_k$ where $|y_{k-1}|_p \leq |1-\theta y_k|_p^{-1/2}$. Then the first iterate is big $|y_{k+1}|_p$ and the second is small (close to $-\theta$). The terms close, small and big are used with respect to the absolute value under consideration and the definition of $\epsilon_k^{-\delta}$ we are using. The size of the third iterate $|y_{k+3}|_p$ depends on if $|a_n - \theta b_n - \theta(\theta a_{n-2} + b_{n-2} - 2b_{n-1})|_p$ is equivalent to zero or not. In Corollary 3.2.1 above, we showed that if $|a_n - \theta b_n - \theta(\theta a_{n-2} + b_{n-2} - 2b_{n-1})|_p \not\equiv 0$, then $|y_{k+3}|_p > |1+\theta y_{k+2}|_p^{-1/2}$, hence $|y_{k+3}|_p$ is moderately big. The converse of this argument is if $|y_{k+3}|_p \leq |1+\theta y_{k+2}|_p^{-1/2}$, then $|a_n - \theta b_n - \theta(\theta a_{n-2} + b_{n-2} - 2b_{n-1})|_p \equiv 0$. This leads us to a special oscillating sequence definition, defined formally earlier.

Recall that the inequality in (3.54) implies the exponential growth of the summed logarithmic height of an admissible solution y_k that has only oscillating sequences that are not special. A necessary condition to avoid this exponential growth is by having an admissible solution y_k that has infinitely many special oscillating sequences. Theorem 1.4.2 and Corollary 3.2.1 imply that to get a special oscillating sequence we need the following relation between the coefficients of equation (3.49) to hold for all $k > K$, $|a_k - \theta b_k - \theta(\theta a_{k-2} + b_{k-2} - 2b_{k-1})|_p \equiv 0$, for $\theta = -1$ or 1. Consequently, we have two forms of special oscillating sequences, depending on the value of θ. If we have infinitely many special oscillating sequences of both forms, then this reduces equation (3.49) to a discrete analogue of the second Painlevé equation $d\text{-}P_{II}$,

$$y_{k+1} + y_{k-1} = \frac{(\alpha k + \beta)y_k + \lambda}{1 - y_k^2},$$

where α, β and λ are constants. Equation $d\text{-}P_{II}$ is a known integrable equation as given in the references in section 1.2.

If y_k has infinitely many special oscillating sequences of one of the forms rather than the other, then there is a possibility that the admissible solution y_k of equation (3.49) solves a difference Riccati equation. The rest of this section is devoted to the discussion of this case.

For a fixed value of θ, we assume that we have infinitely many special oscillating sequences of the form: $|1 - \theta y_n|_p < \epsilon_n$, $|y_{n+1}|_p \geq \max\{|1 - \theta y_n|_p^{-(1-\delta)}, |1 + \theta y_{n+2}|_p^{-(1-\delta)}\}$ and $|1 + \theta y_{n+2}|_p < \epsilon_{n+2}$ $\forall p \leq \infty$ and $n > K$. Simply it means we have infinitely many special oscillating sequences for all $n > K$ such that an admissible solution of (3.49) is close to θ, ∞ and $-\theta$ respectively, where the term close is with respect to the absolute value $|.|_p$ $\forall p \leq \infty$ under consideration. Also, we assume that there is no special oscillating sequences such that the solution is close to $-\theta$, ∞ and θ respectively. For the rest of this chapter when we refer to special oscillating sequences we mean those sequences of the form: θ, ∞, $-\theta$. We define a rational function f_n in n by

$$f_n = (1 - \theta y_n)y_{n+1} - y_n, \tag{3.55}$$

where $\theta = -1$ or 1. If we solve (3.55) for y_{n+1} or make a shift $n \to n-1$, then solve (3.55) for y_{n-1}, we get

$$y_{n+1} = \frac{f_n + y_n}{1 - \theta y_n}, \qquad y_{n-1} = \frac{y_n - f_{n-1}}{1 + \theta y_n}.$$

Using the above expressions for y_{n-1} and y_{n+1} in (3.49) and simplifying, we get

$$y_{n+1} + y_{n-1} = \frac{(f_n - f_{n-1}) + (2 + \theta f_n + \theta f_{n-1})y_n}{1 - y_n^2} = \frac{a_n + b_n y_n}{1 - y_n^2}.$$

Hence

$$(b_n - 2 - \theta f_n - \theta f_{n-1})y_n = f_n - f_{n-1} - a_n. \tag{3.56}$$

If $b_n - 2 - \theta f_n - \theta f_{n-1} = 0$, $\forall n$, then this implies $f_n - f_{n-1} - a_n = 0$, $\forall n$. Solving these two equations together to get f_n we have

$$f_n = \frac{1}{2\theta}(\theta a_n + b_n - 2).$$

Since the expression of f_n is in terms of the coefficients of equation (3.49), the summed logarithmic height $h_r(f_n)$ is a slow growing function with respect to $h_r(y_n)$ for an admissible solution y_n of (3.49). This leads us to a difference Riccati equation of the form

$$y_{n+1} = \frac{\frac{1}{2\theta}(\theta a_n + b_n - 2) + y_n}{1 - \theta y_n}, \tag{3.57}$$

where y_n solves both (3.49) and the above difference Riccati equation.
Now if $b_n - 2 - \theta f_n - \theta f_{n-1} \neq 0$, $\forall n$, then from (3.56) we have

$$y_n = \frac{f_n - f_{n-1} - a_n}{b_n - 2 - \theta f_n - \theta f_{n-1}}. \tag{3.58}$$

Note that we take n to be greater than all the zeros of the rational function $b_n - 2 - \theta f_n - \theta f_{n-1}$. Taking the height of both sides of (3.58) and using some of the height properties yields

$$\begin{aligned} h(y_n) &= h\left(\frac{f_n - f_{n-1} - a_n}{b_n - 2 - \theta f_n - \theta f_{n-1}}\right), \\ &\leq h(f_n - f_{n-1} - a_n) + h\left(\frac{1}{b_n - 2 - \theta f_n - \theta f_{n-1}}\right), \\ &= h(f_n - f_{n-1} - a_n) + h(b_n - 2 - \theta f_n - \theta f_{n-1}), \\ &\leq 2h(f_n) + 2h(f_{n-1}) + h(a_n) + h(b_n) + \log 24. \end{aligned}$$

Taking $\sum_{n=r_0}^{r}$ for both sides of the inequality above and using the fact that $h(f_n)$ is a non-decreasing function of n, we have

$$h_r(y_n) \leq 4h_{r+1}(f_n) + h_r(a_n) + h_r(b_n) + (r - r_0 + 1)\log 24. \tag{3.59}$$

The inequality in (3.59) shows that $h_r(y_n)$ is bounded from above by $h_{r+1}(f_n)$ for an admissible solution y_n. Recall that for an admissible solution y_n of (3.49), $h_r(a_n)$ and $h_r(b_n)$ are growing slower than $h_r(y_n)$. If $h_{r+1}(f_n)$ grows much slower than $h_{r+2}(y_n)$, then we show it leads to a fast growth of $h_r(y_n)$ with r. Before we proceed to investigate the relation between $h_{r+2}(y_n)$ and $h_{r+1}(f_n)$, we need to prove Lemma 3.2.1 that we use later in our argument. We use this lemma to show the fast growth of $h_r(y_n)$ with r later. Clearly if a non-decreasing sequence of positive numbers (w_n) satisfies $w_{n+s} \geq \alpha w_n \ \forall n$, where $s > 0$ and $\alpha > 1$, then w_n grows exponentially. Lemma 3.2.1 says that if (w_n) satisfies the inequality on a sufficiently large set (which has infinite discrete logarithmic measure), then w_n still grows very fast.

Lemma 3.2.1. *Let $(w_n)_{n \geq n_0}$ $(n_0 > 0)$ be a non-decreasing sequence of positive numbers. For a fixed real number $\alpha > 1$ and a fixed positive integer s we define*

$$F = \{n \geq n_0 : \ w_{n+s} \geq \alpha w_n\}. \tag{3.60}$$

If F has infinite discrete logarithmic measure, i.e.

$$\sum_{n \in F} \frac{1}{n} = \infty,$$

then

$$\limsup_{r \to \infty} \frac{\log \log w_r}{\log r} \geq 1. \tag{3.61}$$

Proof Define a sequence (r_n) using induction as follows. Let $r_0 = \min(F)$ and for all $n > 0$, define $r_n = \min(F \cap [r_{n-1} + s, \infty))$. Hence, $r_{n+1} \geq r_n + s$ and

$$F \subseteq \cup_{n=0}^{\infty} [r_n, r_n + s].$$

This yields $w_{r_{n+1}} \geq w_{r_n + s} \geq \alpha w_{r_n}$ for all $n \geq 0$. Iterating this relation recursively yields

$$w_{r_n} \geq \alpha^n w_{r_0}. \tag{3.62}$$

We use the notation $\lfloor x \rfloor$ to denote the integer part of x in the following chain of inequalities. Assume that there is a constant $\varepsilon > 0$ and an integer $m > 1$ such that

$r_n \geq n^{1+\varepsilon}$ for all $n > m$. Then there is a constant E such that

$$\begin{aligned}
\sum_{j\in F}\frac{1}{j} &\leq E + \sum_{n=m}^{\infty}\sum_{k=\lfloor n^{1+\varepsilon}\rfloor}^{\lfloor n^{1+\varepsilon}\rfloor+s}\frac{1}{k}, \text{ (bounded from above by the Riemann integral)}\\
&\leq E + \sum_{n=m}^{\infty}\int_{\lfloor n^{1+\varepsilon}\rfloor-1}^{\lfloor n^{1+\varepsilon}\rfloor+s}\frac{\mathrm{d}t}{t}, \text{ (since } \tfrac{1}{t} \text{ is a decreasing function in } (0,\infty))\\
&\leq E + \sum_{n=m}^{\infty}\int_{n^{1+\varepsilon}-2}^{n^{1+\varepsilon}+s}\frac{\mathrm{d}t}{t},\\
&= E + \sum_{n=m}^{\infty}\log t\Big|_{n^{1+\varepsilon}-2}^{n^{1+\varepsilon}+s},\\
&= E + \sum_{n=m}^{\infty}\log\left(\frac{n^{1+\varepsilon}+s}{n^{1+\varepsilon}-2}\right),\\
&\leq E + \sum_{n=m}^{\infty}\left((s+2)n^{-(1+\varepsilon)} + O(n^{-2(1+\varepsilon)})\right) < \infty.
\end{aligned}$$

But this is a contradiction to our assumption that F has infinite discrete logarithmic measure. Therefore, there exists a subsequence (r_{n_k}) such that $r_{n_k} < n_k^{1+\varepsilon}$ for all $k \geq 0$. From (3.62) we have

$$w_{r_{n_k}} \geq \alpha^{n_k} w_{r_0}.$$

Hence,

$$\begin{aligned}
\limsup_{r\to\infty}\frac{\log\log w_r}{\log r} &\geq \lim_{k\to\infty}\sup\frac{\log\log w_{r_{n_k}}}{\log r_{n_k}},\\
&\geq \lim_{k\to\infty}\sup\frac{\log\log \alpha^{n_k} w_{r_0}}{\log n_k^{1+\varepsilon}},\\
&= \lim_{k\to\infty}\sup\frac{\log\left(n_k\log\alpha + \log w_{r_0}\right)}{(1+\varepsilon)\log n_k},\\
&\geq \frac{1}{1+\varepsilon}.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary small number, this proves the lemma. □

Now we investigate the relation between $h_{r+1}(f_n)$ and $h_{r+2}(y_n)$ where we use the definition of f_n in (3.55). Adding and subtracting θ to the right hand side of (3.55) and simplifying yields

$$f_n + \theta = \theta(1-\theta y_n)(1+\theta y_{n+1}). \tag{3.63}$$

For every prime $p \leq \infty$, we define a set $C_p \subset \mathbb{Z}$ such that it consists of all the big terms in special oscillating sequences i.e. the terms ∞s in the form: θ, ∞, $-\theta$. For a fixed prime

p and for some $r_0 \gg 0$ where $r \geq r_0$ we have

$$\sum_{n=r_0}^{r} \log^+ \frac{1}{|f_n+\theta|_p} = \sum_{\substack{n=r_0 \\ n\in C_p}}^{r} \log^+ \frac{1}{|f_n+\theta|_p} + \sum_{\substack{n=r_0 \\ n+1\in C_p}}^{r} \log^+ \frac{1}{|f_n+\theta|_p} + \sum_{\substack{n=r_0 \\ n\notin C_p \text{ and } n+1\notin C_p}}^{r} \log^+ \frac{1}{|f_n+\theta|_p}. \tag{3.64}$$

In the above inequality we simply split the interval $[r_0, r]$ into points that are in special oscillating sequences (where n, $n+1 \in C_p$) and points in any other oscillating sequence that is not special. Note that for $n \in C_p$, we have $|1+\theta y_{n+1}|_p^{-(1-\delta)} \leq |y_n|_p$. Therefore, for $n \in C_p$ we have

$$\begin{aligned}\log^+ \frac{1}{|f_n+\theta|_p} &= \log^+ \frac{1}{|1-\theta y_n|_p \cdot |1+\theta y_{n+1}|_p} \leq \log^+ |1-\theta y_n|_p^{-1} \cdot |y_n|_p^{\frac{1}{1-\delta}}, \\ &= \log^+ |1-\theta y_n|_p^{-1} \cdot |y_n|_p^{\frac{\delta+1-\delta}{1-\delta}} = \log^+ |1-\theta y_n|_p^{-1} \cdot |y_n|_p \cdot |y_n|_p^{\frac{\delta}{1-\delta}}, \\ &\leq \frac{\delta}{1-\delta} \log^+ |y_n|_p + \log^+ \left|\frac{y_n}{1-\theta y_n}\right|_p.\end{aligned}$$

Since $|y_n|_p$ is big, it is away from θ and $-\theta$. If $p < \infty$, then $|y_n|_p = |\theta - \theta(1-\theta y_n)|_p \leq \max\{1, |1-\theta y_n|_p\} = |1-\theta y_n|_p$, since $|y_n|_p > 1$. Hence, the term $\log^+ \left|\frac{y_n}{1-\theta y_n}\right|_p$ vanishes. For $p = \infty$ we have the following relation $\epsilon_{n+1}^{-\delta} < \epsilon_{n+1}^{-(1-\delta)} < |1+\theta y_{n+1}|_\infty^{-(1-\delta)} \leq |y_n|_\infty \leq 1 + |1-\theta y_n|_\infty$ which yields $\epsilon_{n+1}^{-\delta} - 1 \leq |1-\theta y_n|_\infty$. Consequently, $\frac{1}{|1-\theta y_n|_\infty} \leq \frac{1}{\epsilon_{n+1}^{-\delta}-1}$. Starting with $|y_n|_\infty \leq 1 + |1-\theta y_n|_\infty$ then dividing both sides by $|1-\theta y_n|_\infty$ implies $\frac{|y_n|_\infty}{|1-\theta y_n|_\infty} \leq \frac{1}{|1-\theta y_n|_\infty} + 1 \leq \frac{1}{\epsilon_{n+1}^{-\delta}-1} + 1$. Therefore, $\left|\frac{y_n}{1-\theta y_n}\right|_\infty \leq \frac{1}{4} + 1 = \frac{5}{4}$ since $5 \leq \epsilon_{n+1}^{-\delta}$. This implies

$$\sum_{p\leq\infty} \sum_{\substack{n=r_0 \\ n\in C_p}}^{r} \log^+ \frac{1}{|f_n+\theta|_p} \leq \frac{\delta}{1-\delta} h_r(y_n) + V(r-r_0+1), \tag{3.65}$$

where $V = \log \frac{5}{4}$. Similarly,

$$\sum_{p\leq\infty} \sum_{\substack{n=r_0 \\ n+1\in C_p}}^{r} \log^+ \frac{1}{|f_n+\theta|_p} \leq \frac{\delta}{1-\delta} h_{r+1}(y_n) + V(r-r_0+1). \tag{3.66}$$

Summing over all $p \leq \infty$ in (3.64) and using (3.63), (3.65) and (3.66) yields

$$h_r(f_n) - (r - r_0 + 1)\log 2 \leq h_r\left(\frac{1}{f_n + \theta}\right) \leq \frac{2\delta}{1-\delta} h_{r+1}(y_n) + 2V(r - r_0 + 1) + \sum_{p\leq\infty}\left\{\sum_{\substack{n=r_0 \\ n+1\notin C_p}}^{r} \log^+ \frac{1}{|1-\theta y_n|_p} + \sum_{\substack{n=r_0 \\ n\notin C_p}}^{r} \log^+ \frac{1}{|1+\theta y_{n+1}|_p}\right\}.$$

Therefore,

$$h_r(f_n) \leq \frac{2\delta}{1-\delta} h_{r+1}(y_n) + (r - r_0 + 1)(\log 2 + 2V) + B_{r+1}, \tag{3.67}$$

where

$$B_r = \sum_{p\leq\infty}\left\{\sum_{\substack{n=r_0 \\ n+1\notin C_p}}^{r} \log^+ \frac{1}{|1-\theta y_n|_p} + \sum_{\substack{n=r_0 \\ n-1\notin C_p}}^{r} \log^+ \frac{1}{|1+\theta y_n|_p}\right\}. \tag{3.68}$$

Recall in our previous analysis of oscillating sequences that are not special we have the following result implied by (3.54),

$$B_r \leq \tau \sum_{p\leq\infty} \sum_{\substack{n=r_0 \\ n\notin C_p}}^{r+1} \log^+ |y_n|_p + R_r. \tag{3.69}$$

Recall that $\tau < 2$ and R_r is an expression that involves the summed logarithmic heights of the coefficients a_n and b_n. Applying the shift $r \to r+1$ in (3.67) and (3.69), then using the result in (3.59) yields

$$h_r(y_n) \leq \frac{8\delta}{1-\delta} h_{r+2}(y_n) + 4\tau \sum_{p\leq\infty} \sum_{\substack{n=r_0 \\ n\notin C_p}}^{r+3} \log^+ |y_n|_p + \widehat{R}_{r+2}, \tag{3.70}$$

where $\widehat{R}_{r+2} = o(h_{r+2}(y_n))$. If we could compare the expression $\sum_{p\leq\infty} \sum_{\substack{n=r_0 \\ n\notin C_p}}^{r+1} \log^+ |y_n|_p$ to $h_{r+1}(y_n)$ such that it is smaller than a suitable constant c times $h_{r+1}(y_n)$ in a big set of positive integers, then we show that $h_r(y_n)$ grows very fast with r. We construct another inequality that has B_r and consequently, the expression $\sum_{p\leq\infty} \sum_{\substack{n=r_0 \\ n\notin C_p}}^{r+1} \log^+ |y_n|_p$. If in this constructed inequality, $\sum_{p\leq\infty}\sum_{\substack{n=r_0\\ n\notin C_p}}^{r+1}\log^+|y_n|_p$ is bounded from below by c times $h_{r+1}(y_n)$ in a big set of positive integers, then we show that this also leads to a very fast growth of $h_r(y_n)$ with r. We use Lemma 3.2.1 to show the above result. Now we consider the following inequality

$$\sum_{p\leq\infty}\sum_{n=r_0}^{r}\left\{\log^+\frac{1}{|1-y_n|_p}+\log^+\frac{1}{|1+y_n|_p}\right\}$$
$$\leq\sum_{p\leq\infty}\left\{\sum_{\substack{n=r_0\\ n+1\in C_p}}^{r}\log^+\frac{1}{|1-\theta y_n|_p}+\sum_{\substack{n=r_0\\ n-1\in C_p}}^{r}\log^+\frac{1}{|1+\theta y_n|_p}\right\}+B_r.$$

Recall that if $n+1\in C_p$ (or $n-1\in C_p$), then $|y_{n+1}|_p\geq|1-\theta y_n|_p^{-(1-\delta)}$ (or $|y_{n-1}|_p\geq|1+\theta y_n|_p^{-(1-\delta)}$). Using this fact and (3.69) we have

$$\sum_{p\leq\infty}\sum_{n=r_0}^{r}\left\{\log^+\frac{1}{|1-y_n|_p}+\log^+\frac{1}{|1+y_n|_p}\right\}$$
$$\leq\frac{2}{1-\delta}\sum_{p\leq\infty}\sum_{\substack{n=r_0\\ n\in C_p}}^{r+1}\log^+|y_n|_p+\tau\sum_{p\leq\infty}\sum_{\substack{n=r_0\\ n\notin C_p}}^{r+1}\log^+|y_n|_p+R_r$$
$$=\frac{2}{1-\delta}\sum_{p\leq\infty}\sum_{\substack{n=r_0\\ n\in C_p}}^{r+1}\log^+|y_n|_p+\frac{2}{1-\delta}\sum_{p\leq\infty}\sum_{\substack{n=r_0\\ n\notin C_p}}^{r+1}\log^+|y_n|_p$$
$$-\frac{2}{1-\delta}\sum_{p\leq\infty}\sum_{\substack{n=r_0\\ n\notin C_p}}^{r+1}\log^+|y_n|_p+\tau\sum_{p\leq\infty}\sum_{\substack{n=r_0\\ n\notin C_p}}^{r+1}\log^+|y_n|_p+R_r$$
$$=\frac{2}{1-\delta}h_{r+1}(y_n)-\left(\frac{2}{1-\delta}-\tau\right)\sum_{p\leq\infty}\sum_{\substack{n=r_0\\ n\notin C_p}}^{r+1}\log^+|y_n|_p+R_r. \tag{3.71}$$

This implies that

$$2h_r(y_n)\leq\frac{2}{1-\delta}h_{r+1}(y_n)-\left(\frac{2}{1-\delta}-\tau\right)\sum_{p\leq\infty}\sum_{\substack{n=r_0\\ n\notin C_p}}^{r+1}\log^+|y_n|_p+\widetilde{R_r}, \tag{3.72}$$

where $\widetilde{R_r}=o(h_r(y_n))$. Considering the two inequalities in (3.70) and (3.72), we have two cases to consider depending on whether the expression $\sum_{p\leq\infty}\sum_{\substack{n=r_0\\ n\notin C_p}}^{r+1}\log^+|y_n|_p$ is very small compared to $h_{r+1}(y_n)$ on a large set or not. We use Lemma 3.2.1 to prove that in both cases $h_r(y_n)$ grows very fast with r.

- **Case 1**: Assume that there is a sufficiently small constant $c>0$ such that

$$\sum_{p\leq\infty}\sum_{\substack{n=r_0\\ n\notin C_p}}^{r+1}\log^+|y_n|_p\leq ch_{r+1}(y_n),$$

on a set of infinite discrete logarithmic measure. Considering (3.70) with the above assumption implies

$$h_r(y_n)\leq\left(\frac{8\delta}{1-\delta}+4\tau c\right)h_{r+3}(y_n)+\widehat{R}_{r+2},$$

on a set of infinite discrete logarithmic measure. Hence, Lemma 3.2.1 implies that $h_r(y_n)$ grows very fast with r.

- **Case 2**: Assume that

$$\sum_{p\leq\infty}\sum_{\substack{n=r_0\\ n\notin C_p}}^{r+1}\log^+|y_n|_p> ch_{r+1}(y_n),$$

on a set of infinite discrete logarithmic measure. Using this inequality in (3.72) yields

$$2h_r(y_n)\leq\left[\frac{2}{1-\delta}-\left(\frac{2}{1-\delta}-\tau\right)c\right]h_{r+1}(y_n)+\widetilde{R}_r.$$

Since $\left[\frac{2}{1-\delta}-(\frac{2}{1-\delta}-\tau)c\right]<2$ for sufficiently small δ, then using Lemma 3.2.1 the above inequality implies that $h_r(y_n)$ grows very fast with r.

Finally, we achieved the main goal of this section. We showed that for

$$y_{n+1}+y_{n-1}=\frac{a_n+b_ny_n}{1-y_n^2},$$

we have 3 scenarios. Either $h_r(y_n)$ grows exponentially with r if the admissible solution y_n has infinitely many oscillating sequences that are not special rather than special oscillating sequences. Or if y_n has infinitely many special oscillating sequences of both forms then the above equation reduces to a discrete analogue of P_{II}. Or if y_n has infinitely many special oscillating sequences of one form rather than the other, then y_n solves a difference Riccati equation given in (3.57) or $h_r(y_n)$ grows very fast with r. Therefore, either $h(y_n)$ is not growing polynomially with r, or the above equation reduces to a discrete analogue of P_{II}, or y_n solves a difference Riccati equation. Hence, this proves Theorem 1.4.1 in the case $c_n \equiv 0$. We end this section by giving the proofs of Theorem 1.4.2 and Corollary 3.2.1.

Proof of Theorem 1.4.2 Let $k-1 > K$ and for a fixed absolute value $|.|_p$ ($\forall p \leq \infty$), assume that $|1-\theta y_k|_p < \epsilon_k$ where $\theta = -1$ or 1, and where ϵ_k is as defined above in (3.51). Also, let $|y_{k-1}|_p \leq |1-\theta y_k|_p^{-1/2}$.

Now we start proving the theorem first for the non-Archimedean absolute values ($p < \infty$) and then for the Archimedean absolute value ($p = \infty$). We start with the first part of the theorem. First we rewrite equation (3.49) as follows, using partial fraction:

$$y_{k+1} + y_{k-1} = \frac{1/2(a_k + \theta b_k)}{1-\theta y_k} + \frac{1/2(a_k - \theta b_k)}{1+\theta y_k}. \tag{3.73}$$

From the first part of the theorem, we have $A_k = y_{k+1} - \dfrac{1/2(a_k + \theta b_k)}{1-\theta y_k}$ and using (3.73) we have $A_k = \dfrac{1/2(a_k - \theta b_k)}{1+\theta y_k} - y_{k-1}$. Hence, for non-Archimedean absolute value $|.|_p$ ($p < \infty$), we have

$$|A_k|_p \leq \max\left\{\frac{|1/2|_p \cdot |a_k - \theta b_k|_p}{|1+\theta y_k|_p}, |y_{k-1}|_p\right\}. \tag{3.74}$$

Here we need to find an estimate for the terms in the set above to prove the estimate for $|A_k|_p$ in the theorem. For $p < \infty$, we have

$$\epsilon_k^\delta \leq |2|_p \leq \max\{|1-\theta y_k|_p, |1+\theta y_k|_p\}.$$

If $|1-\theta y_k|_p > |1+\theta y_k|_p$, then we get the following relation:

$$\epsilon_k^\delta \leq \max\{|1-\theta y_k|_p, |1+\theta y_k|_p\} = |1-\theta y_k|_p < \epsilon_k.$$

This is a contradiction since $\epsilon_k \leq \epsilon_k^\delta$. Therefore, $|1+\theta y_k|_p \geq |1-\theta y_k|_p$ and

$$\epsilon_k^\delta \leq \max\{|1-\theta y_k|_p, |1+\theta y_k|_p\} = |1+\theta y_k|_p.$$

This implies $\dfrac{1}{|1+\theta y_k|_p} \leq \epsilon_k^{-\delta} < |1-\theta y_k|_p^{-\delta}$. Using this relation and (3.51) in (3.74) yields

$$\begin{aligned}|A_k|_p &\leq \max\left\{\frac{|1/2|_p \cdot |a_k - \theta b_n|_p}{|1+\theta y_k|_p}, |y_{k-1}|_p\right\}\\ &\leq \max\left\{|1-\theta y_k|_p^{-2\delta}, |1-\theta y_k|_p^{-1/2}\right\} = |1-\theta y_k|_p^{-1/2}.\end{aligned}$$

The above result is valid for sufficiently small δ. This proves the first part of the theorem for the non-Archimedean absolute value ($p < \infty$).

To prove the second part, we start by writing the expression for B_k as follows:

$$B_k = (y_{k+2} + \theta) - \left(\theta - \frac{2b_{k+1}}{a_k + \theta b_k}\right)(1 - \theta y_k).$$

Using equation (3.49) to get an expression for $y_{k+2} + \theta$ and simplifying the above equation using the first part yields

$$B_k = \frac{a_{k+1} + b_{k+1} y_{k+1}}{(1 - y_{k+1})(1 + y_{k+1})} + \frac{b_{k+1}}{y_{k+1} - A_k}.$$

Combining the two terms together in the above equation and simplifying, we get

$$\begin{aligned}B_k = &\frac{a_{k+1}}{(1-y_{k+1})(1+y_{k+1})} + \frac{b_{k+1}}{(1-y_{k+1})(1+y_{k+1})(y_{k+1}-A_k)}\\ &- \frac{b_{k+1} A_k y_{k+1}}{(1-y_{k+1})(1+y_{k+1})(y_{k+1}-A_k)}.\end{aligned} \tag{3.75}$$

In order to get an estimate for $|B_k|_p$, we need an estimate of the three terms on the right hand side of equation (3.75). We start first with the non-Archimedean absolute value ($p < \infty$) where we have the following chain of inequalities:

$$\begin{aligned}|1-\theta y_k|_p^{-(1-\delta)} &= \frac{|1-\theta y_k|_p^{\delta}}{|1-\theta y_k|_p} < \frac{\epsilon_k^{\delta}}{|1-\theta y_k|_p} \leq \frac{|1/2|_p \cdot |a_k + \theta b_k|_p}{|1-\theta y_k|_p},\\ &= |y_{k+1} - A_k|_p \leq \max\{|y_{k+1}|_p, |A_k|_p\},\\ &\leq \max\{|y_{k+1}|_p, |1-\theta y_k|_p^{-1/2}\}.\end{aligned}$$

Here we have used the assumption $|1-\theta y_k|_p < \epsilon_k$ and part one of the theorem. Since $|1-\theta y_k|_p^{-1/2} \leq |1-\theta y_k|_p^{-(1-\delta)}$ for sufficiently small δ, then the maximum of the above set is $|y_{k+1}|_p$. Hence,

$$|1-\theta y_k|_p^{-(1-\delta)} \leq |y_{k+1}|_p = |1-(1-y_{k+1})|_p \leq \max\{1, |1-y_{k+1}|_p\}. \tag{3.76}$$

If $1 \geq |1-y_{k+1}|_p$, then we have $\epsilon_k^{-(1-\delta)} < |1-\theta y_k|_p^{-(1-\delta)} \leq 1$ which is a contradiction, since $\epsilon_k \leq 1$. Therefore, $|1-y_{k+1}|_p \geq 1$ and the maximum in (3.76) is $|1-y_{k+1}|_p$. This implies

$$\frac{1}{|1-y_{k+1}|_p} \leq |1-\theta y_k|_p^{1-\delta}. \tag{3.77}$$

Similarly,

$$\frac{1}{|1+y_{k+1}|_p} \leq |1-\theta y_k|_p^{1-\delta}. \tag{3.78}$$

Also we have

$$\begin{aligned} |1-\theta y_k|_p^{-(1-\delta)} = \frac{|1-\theta y_k|_p^{\delta}}{|1-\theta y_k|_p} < \frac{\epsilon_k^{\delta}}{|1-\theta y_k|_p} &\leq \frac{|1/2|_p \cdot |a_k+\theta b_k|_p}{|1-\theta y_k|_p} \\ &= |y_{k+1}-A_k|_p. \end{aligned}$$

This implies

$$\frac{1}{|y_{k+1}-A_k|_p} \leq |1-\theta y_k|_p^{1-\delta}. \tag{3.79}$$

Moreover, we have from the first part of the theorem

$$\begin{aligned} |y_{k+1}|_p &\leq \max\left\{\frac{|1/2|_p \cdot |a_k+\theta b_k|_p}{|1-\theta y_k|_p}, |A_k|_p\right\} \\ &\leq \max\{|1-\theta y_k|_p^{-(1+\delta)}, |1-\theta y_k|_p^{-1/2}\} = |1-\theta y_k|_p^{-(1+\delta)}, \end{aligned} \tag{3.80}$$

for sufficiently small δ. If we apply the non-Archimedean absolute value ($p < \infty$) to the equation (3.75), we get

$$\begin{aligned} |B_k|_p \leq \max\Bigg\{ &\frac{|a_{k+1}|_p}{|1-y_{k+1}|_p \cdot |1+y_{k+1}|_p}, \frac{|b_{k+1}|_p}{|1-y_{k+1}|_p \cdot |1+y_{k+1}|_p \cdot |y_{k+1}-A_k|_p}, \\ &\frac{|b_{k+1}|_p \cdot |A_k|_p \cdot |y_{k+1}|_p}{|1-y_{k+1}|_p \cdot |1+y_{k+1}|_p \cdot |y_{k+1}-A_k|_p}\Bigg\} \\ \leq{} &\max\left\{|1-\theta y_k|_p^{2-3\delta}, |1-\theta y_k|_p^{3-4\delta}, |1-\theta y_k|_p^{3/2-5\delta}\right\} \\ ={} &|1-\theta y_k|_p^{3/2-5\delta}. \end{aligned}$$

Here we used the results in (3.77), (3.78), (3.79) and (3.80). Also, we used the result from the first part of the theorem and $\epsilon_k^{-\delta}$ definition in (3.51). Hence, we proved the second part of the theorem for the non-Archimedean absolute value ($p < \infty$).

We now prove the third part of the theorem for the non-Archimedean absolute value

$(p < \infty)$. We start by writing

$$C_k = y_{k+3} - \frac{(a_{k+2} - \theta b_{k+2} - \theta(\theta a_k + b_k - 2b_{k+1}))}{2(1 + \theta y_{k+2})}.$$

We use equation (3.49) to get an expression for y_{k+3} and we use partial fraction to express this expression. Simplifying the resulting equation yields

$$C_k = \frac{1/2(a_{k+2} + \theta b_{k+2})}{1 - \theta y_{k+2}} - \frac{a_k + \theta b_k}{2(1 - \theta y_k)} - A_k + \frac{\theta(\theta a_k + b_k - 2b_{k+1})}{2(1 + \theta y_{k+2})}.$$

Combining the two terms $\frac{a_k + \theta b_k}{2(1 - \theta y_k)}$ and $\frac{\theta(\theta a_k + b_k - 2b_{k+1})}{2(1 + \theta y_{k+2})}$, then using the equation in the second part of the theorem to simplify the resulting expression results the following equation:

$$C_k = \frac{1/2(a_{k+2} + \theta b_{k+2})}{1 - \theta y_{k+2}} - \frac{B_k(a_k + \theta b_k)}{2\theta(1 + \theta y_{k+2})(1 - \theta y_k)} - A_k. \tag{3.81}$$

In order to find an upper bound for $|C_k|_p$, we need to find upper bounds for each term in the right hand side of the equation in (3.81). We start with the first term $\frac{1/2(a_{k+2} + \theta b_{k+2})}{1 - \theta y_{k+2}}$. We need to find an upper bound for $\frac{1}{|1 - \theta y_{k+2}|_p}$. From the second part of the theorem, we have

$$\begin{aligned} |1 + \theta y_{k+2}|_p &\leq \max\left\{\frac{|\theta a_k + b_k - 2b_{k+1}|_p}{|a_k + \theta b_k|_p}|1 - \theta y_k|_p, |B_k|_p\right\} \\ &\leq \max\{|1 - \theta y_k|_p^{1-2\delta}, |1 - \theta y_k|_p^{3/2-5\delta}\} \\ &< \max\{\epsilon_k^{1-2\delta}, \epsilon_k^{3/2-5\delta}\} = \epsilon_k^{1-2\delta}. \end{aligned} \tag{3.82}$$

In the above chain of inequalities for sufficiently small δ, we used (3.51) and the assumption $|1 - \theta y_k|_p < \epsilon_k$. Note that if $|\theta a_k + b_k - 2b_{k+1}|_p \equiv 0$, then $|1 + \theta y_{k+2}|_p = |B_k|_p \leq |1 - \theta y_k|_p^{3/2-5\delta} < \epsilon_k^{3/2-5\delta} \leq \epsilon_k^{1-2\delta}$. Also, we have for $p < \infty$ the following relation:

$$\epsilon_k^{\delta} \leq |2|_p \leq \max\{|1 + \theta y_{k+2}|_p, |1 - \theta y_{k+2}|_p\}.$$

If we have $|1 + \theta y_{k+2}|_p \geq |1 - \theta y_{k+2}|_p$, then $\epsilon_k^{\delta} \leq \max\{|1 + \theta y_{k+2}|_p, |1 - \theta y_{k+2}|_p\} = |1 + \theta y_{k+2}|_p < \epsilon_k^{1-2\delta}$. This is a contradiction, since $\epsilon_k^{1-2\delta} \leq \epsilon_k^{\delta}$ for sufficiently small δ. Therefore, $|1 - \theta y_{k+2}|_p > |1 + \theta y_{k+2}|_p$. Hence, we have the following relation:

$$\epsilon_k^{\delta} \leq |2|_p \leq \max\{|1 + \theta y_{k+2}|_p, |1 - \theta y_{k+2}|_p\} = |1 - \theta y_{k+2}|.$$

The above inequality yields

$$\frac{1}{|1-\theta y_{k+2}|_p} \leq \epsilon_k^{-\delta} < |1-\theta y_k|_p^{-\delta}. \tag{3.83}$$

The second term in (3.81) has the following upper bound if $|\theta a_k + b_k - 2b_{k+1}|_p \not\equiv 0$:

$$\left|\frac{B_k(a_k+\theta b_k)}{2\theta(1+\theta y_{k+2})(1-\theta y_k)}\right|_p \leq |1-\theta y_k|_p^{1/2-6\delta} \cdot |1+\theta y_{k+2}|_p^{-1} \leq |1-\theta y_k|_p^{-1/2-8\delta}, \tag{3.84}$$

where we have used (3.51) and the relation $|1-\theta y_k|_p^{1+2\delta} < |1+\theta y_{k+2}|_p$. We get this relation from the second part of the theorem if $|\theta a_k + b_k - 2b_{k+1}|_p \not\equiv 0$, as follows:

$$\begin{aligned}
|1-\theta y_k|_p^{1+2\delta} &= |1-\theta y_k|_p \cdot |1-\theta y_k|_p^{2\delta} < |1-\theta y_k|_p \epsilon_k^{2\delta}, \\
&\leq \frac{|\theta a_k + b_k - 2b_{k+1}|_p}{|a_k+\theta b_k|_p} |1-\theta y_k|_p, \\
&= |(1+\theta y_{k+2}) - B_k|_p, \\
&\leq \max\{|1+\theta y_{k+2}|_p, |B_k|_p\}, \\
&\leq \max\left\{|1+\theta y_{k+2}|_p, |1-\theta y_k|_p^{3/2-5\delta}\right\}.
\end{aligned}$$

Since for sufficiently small δ and $|1-\theta y_k|_p < \epsilon_k \leq 1$, we have $|1-\theta y_k|_p^{3/2-5\delta} \leq |1-\theta y_k|_p^{1+2\delta}$. Therefore, $|1+\theta y_{k+2}|_p$ is the maximum, which implies $|1+\theta y_{k+2}|_p^{-1} < |1-\theta y_k|_p^{-(1+2\delta)}$. Now we could find an upper bound for $|C_k|_p$ by applying the non-Archimedean absolute value ($p < \infty$) for both sides of equation (3.81). Using the isosceles inequality, we have

$$\begin{aligned}
|C_k|_p &\leq \max\left\{\frac{|1/2|_p \cdot |a_{k+2}+\theta b_{k+2}|_p}{|1-\theta y_{k+2}|_p}, \left|\frac{B_k(a_k+\theta b_k)}{2\theta(1+\theta y_{k+2})(1-\theta y_k)}\right|_p, |A_k|_p\right\}, \\
&\leq \max\left\{|1-\theta y_k|_p^{-2\delta}, |1-\theta y_k|_p^{-1/2-8\delta}, |1-\theta y_k|_p^{-1/2}\right\}, \\
&= |1-\theta y_k|_p^{-1/2-8\delta}.
\end{aligned} \tag{3.85}$$

In the above inequality, we used (3.83), (3.84) and the first part of the theorem. We need to find an upper bound with respect to $|1+\theta y_{k+2}|_p$. From (3.82) we have

$$|1+\theta y_{k+2}|_p \leq |1-\theta y_k|_p^{1-2\delta}.$$

Taking the reciprocal of the above inequality and raising both sides to the power $\frac{(1/2+8\delta)}{1-2\delta}$, we get

$$|1-\theta y_k|_p^{-(1/2+8\delta)} \leq |1+\theta y_{k+2}|_p^{\frac{-(1/2+8\delta)}{(1-2\delta)}}. \tag{3.86}$$

To simplify the power in (3.86), we use Taylor series expansion as follows:

$$\frac{(1/2+8\delta)}{1-2\delta} = 1/2 + 9\delta + O(\delta^2) \le 1/2 + 10\delta,$$

for sufficiently small δ. The inequality above yields $-\dfrac{(1/2+8\delta)}{1-2\delta} \ge -1/2 - 10\delta$. Therefore, we have

$$\begin{aligned} |C_k|_p &\le |1-\theta y_k|_p^{-(1/2+8\delta)} \le |1+\theta y_{k+2}|_p^{\frac{-(1/2+8\delta)}{(1-2\delta)}}, \\ &\le |1+\theta y_{k+2}|_p^{-1/2-10\delta}, \\ &\le |1+\theta y_{k+2}|_p^{-2/3-2\delta}, \end{aligned}$$

for sufficiently small δ. Now if $|\theta a_k + b_k - 2b_{k+1}|_p \equiv 0$, then the upper bound on the second term in (3.81) is as follows

$$\begin{aligned} \left|\frac{B_k(a_k+\theta b_k)}{2\theta(1+\theta y_{k+2})(1-\theta y_k)}\right|_p &= \frac{|1+\theta y_{k+2}|_p \cdot |a_k+\theta b_k|_p}{|2|_p \cdot |1+\theta y_{k+2}|_p \cdot |1-\theta y_k|_p} \\ &\le |1-\theta y_k|_p^{-(1+\delta)}. \end{aligned} \tag{3.87}$$

Consequently, we find an upper bound for $|C_k|_p$ as follows

$$\begin{aligned} |C_k|_p &\le \max\left\{\frac{|1/2|_p \cdot |a_{k+2}+\theta b_{k+2}|_p}{|1-\theta y_{k+2}|_p}, \left|\frac{B_k(a_k+\theta b_k)}{2\theta(1+\theta y_{k+2})(1-\theta y_k)}\right|_p, |A_k|_p\right\}, \\ &\le \max\{|1-\theta y_k|_p^{-2\delta}, |1-\theta y_k|_p^{-(1+\delta)}, |1-\theta y_k|_p^{-1/2}\}, \\ &= |1-\theta y_k|_p^{-(1+\delta)}. \end{aligned}$$

Since $|1+\theta y_{k+2}|_p = |B_k|_p \le |1-\theta y_k|_p^{3/2-5\delta}$, it yields that $|1-\theta y_k|_p^{-(1+\delta)} \le |1+\theta y_{k+2}|_p^{\frac{-(1+\delta)}{3/2-5\delta}} \le |1+\theta y_{k+2}|_p^{-2/3-2\delta}$. Hence, $|C_k|_p \le |1+\theta y_{k+2}|_p^{-2/3-2\delta}$ which proves the last part of the theorem for the non-Archimedean absolute value ($p < \infty$).

Now let us prove the theorem for the Archimedean absolute value. As in the non-Archimedean absolute value case, we could rewrite A_k as $A_k = \dfrac{a_k - \theta b_k}{2(1+\theta y_k)} - y_{k-1}$. Applying the Archimedean absolute value for both sides of the equation and using the triangle inequality, we get

$$|A_k|_p \le \frac{|1/2|_p \cdot |a_k - \theta b_k|_p}{|1+\theta y_k|_p} + |y_{k-1}|_p. \tag{3.88}$$

Also, we have the following chain of inequalities:

$$\begin{aligned} 2 = |2|_p &\leq |1-\theta y_k|_p + |1+\theta y_k|_p, \\ &< \epsilon_k + |1+\theta y_k|_p, \\ &< 1 + |1+\theta y_k|_p. \end{aligned}$$

Subtracting 1 from both sides of the inequality yields $1 < |1+\theta y_k|_p$. This implies

$$\frac{1}{|1+\theta y_k|_p} < 1. \tag{3.89}$$

Hence, the inequality in (3.88) is

$$\begin{aligned} |A_k|_p &\leq \frac{|1/2|_p \cdot |a_k - \theta b_k|_p}{|1+\theta y_k|_p} + |y_{k-1}|_p, \\ &\leq \frac{1}{10}\epsilon_k^{-\delta} \cdot 1 + |1-\theta y_k|_p^{-1/2}, \\ &\leq \frac{1}{10}|1-\theta y_k|_p^{-\delta} + |1-\theta y_k|_p^{-1/2}, \\ &\leq \frac{1}{10}|1-\theta y_k|_p^{-1/2} + |1-\theta y_k|_p^{-1/2}, \\ &= \frac{11}{10}|1-\theta y_k|_p^{-1/2}, \end{aligned}$$

for sufficiently small δ, which proves the first part of the theorem for Archimedean absolute value.

Now for the second part of the theorem, we have from (3.75) the following result:

$$\begin{aligned} B_k = &\frac{a_{k+1}}{(1-y_{k+1})(1+y_{k+1})} + \frac{b_{k+1}}{(1-y_{k+1})(1+y_{k+1})(y_{k+1}-A_k)} \\ &- \frac{b_{k+1}A_k y_{k+1}}{(1-y_{k+1})(1+y_{k+1})(y_{k+1}-A_k)}. \end{aligned}$$

We need to find upper bounds for the Archimedean absolute value of each term of the above equation in order to find an upper bound for $|B_k|_p$. We have the following chain of inequalities where we have used the result from the first part of the theorem and the triangle inequality:

$$\begin{aligned}
10|1-\theta y_k|_p^{-(1-\delta)} &= \frac{10|1-\theta y_k|_p^{\delta}}{|1-\theta y_k|_p} < \frac{10\epsilon_k^{\delta}}{|1-\theta y_k|_p} \leq \frac{|1/2|_p \cdot |a_k+\theta b_k|_p}{|1-\theta y_k|_p}, \\
&= |y_{k+1}-A_k|_p \leq |y_{k+1}|_p + |A_k|_p, \\
&\leq |y_{k+1}|_p + \frac{11}{10}|1-\theta y_k|_p^{-1/2}, \\
&\leq |y_{k+1}|_p + \frac{11}{10}|1-\theta y_k|_p^{-(1-\delta)}, \\
&\leq |y_{k+1}|_p + 9|1-\theta y_k|_p^{-(1-\delta)}.
\end{aligned}$$

This implies

$$|1-\theta y_k|_p^{-(1-\delta)} \leq |y_{k+1}|_p. \tag{3.90}$$

Therefore we have the following relations:

$$\begin{aligned}
10|1-\theta y_k|_p^{-(1-\delta)} &\leq 10|y_{k+1}|_p = 10|1-(1-y_{k+1})|_p, \\
&\leq 10|1|_p + 10|1-y_{k+1}|_p, \\
&\leq 2\epsilon_k^{-\delta} + 10|1-y_{k+1}|_p, \\
&< 2|1-\theta y_k|_p^{-\delta} + 10|1-y_{k+1}|_p, \\
&\leq 2|1-\theta y_k|_p^{-(1-\delta)} + 10|1-y_{k+1}|_p.
\end{aligned}$$

Subtracting $2|1-\theta y_k|_p^{-(1-\delta)}$ from both sides of the inequality and multiplying by $\frac{1}{10}$ implies

$$\frac{8}{10}|1-\theta y_k|_p^{-(1-\delta)} < |1-y_{k+1}|_p.$$

Taking the reciprocal of both sides yields

$$\frac{1}{|1-y_{k+1}|_p} < \frac{5}{4}|1-\theta y_k|_p^{1-\delta}. \tag{3.91}$$

Similarly,

$$\frac{1}{|1+y_{k+1}|_p} < \frac{5}{4}|1-\theta y_k|_p^{1-\delta}. \tag{3.92}$$

Also, we get the relation below:

$$\begin{aligned}
10|1-\theta y_k|_p^{-(1-\delta)} &= \frac{10|1-\theta y_k|_p^{\delta}}{|1-\theta y_k|_p}, \\
&< \frac{10\epsilon_k^{\delta}}{|1-\theta y_k|_p} \leq \frac{|1/2|_p \cdot |a_k+\theta b_k|_p}{|1-\theta y_k|_p}, \\
&= |y_{k+1}-A_k|_p.
\end{aligned}$$

Hence,

$$\frac{1}{|y_{k+1} - A_k|_p} < \frac{1}{10}|1 - \theta y_k|_p^{1-\delta}. \tag{3.93}$$

Moreover, we have this chain of inequalities:

$$\begin{aligned} |y_{k+1}|_p &\leq \frac{|1/2|_p \cdot |a_k + \theta b_k|_p}{|1 - \theta y_k|_p} + |A_k|_p, \\ &\leq \frac{1}{10}\epsilon_k^{-(1+\delta)} + \frac{11}{10}|1 - \theta y_k|_p^{-1/2}, \\ &< \frac{1}{10}|1 - \theta y_k|_p^{-(1+\delta)} + \frac{11}{10}|1 - \theta y_k|_p^{-(1+\delta)}, \\ &= \frac{6}{5}|1 - \theta y_k|_p^{-(1+\delta)}. \end{aligned} \tag{3.94}$$

Now we are ready to find an upper bound for $|B_k|_p$. Applying the Archimedean absolute value for both sides in (3.75) and using the triangle inequality, we get

$$\begin{aligned} |B_k|_p \leq &\frac{|a_{k+1}|_p}{|1 - y_{k+1}|_p \cdot |1 + y_{k+1}|_p} + \frac{|b_{k+1}|_p}{|1 - y_{k+1}|_p \cdot |1 + y_{k+1}|_p \cdot |y_{k+1} - A_k|_p} \\ &+ \frac{|b_{k+1}|_p \cdot |A_k|_p \cdot |y_{k+1}|_p}{|1 - y_{k+1}|_p \cdot |1 + y_{k+1}|_p \cdot |y_{k+1} - A_k|_p}, \\ &< \frac{5}{32}|1 - \theta y_k|_p^{2-3\delta} + \frac{1}{64}|1 - \theta y_k|_p^{3-4\delta} + \frac{33}{1600}|1 - \theta y_k|_p^{3/2-5\delta}, \\ &< \frac{1}{4}|1 - \theta y_k|_p^{3/2-5\delta} + \frac{1}{8}|1 - \theta y_k|_p^{3/2-5\delta} + \frac{1}{8}|1 - \theta y_k|_p^{3/2-5\delta}, \\ &= \frac{1}{2}|1 - \theta y_k|_p^{3/2-5\delta}. \end{aligned} \tag{3.95}$$

In the above inequalities, we have used (3.91), (3.92), (3.93) and (3.51). Also, since $|1-\theta y_k|_p < \epsilon_k < 1$ and for sufficiently small δ, we had the above result which proves the second part of the theorem for Archimedean absolute value.

To prove the third part of the theorem for Archimedean absolute value, recall from (3.81) the following expression for C_k

$$C_k = \frac{1/2(a_{k+2} + \theta b_{k+2})}{1 - \theta y_{k+2}} - \frac{B_k(a_k + \theta b_k)}{2\theta(1 + \theta y_{k+2})(1 - \theta y_k)} - A_k.$$

We need to find an upper bound for each term in the right hand side of the above equation to get an upper bound for $|C_k|_p$. From the second part of the theorem, we have if $|\theta a_k + b_k - 2b_{k+1}|_p \not\equiv 0$ the following result

$$\begin{aligned}
|1+\theta y_{k+2}|_p &\leq \frac{|\theta a_k + b_k - 2b_{k+1}|_p}{|a_k+\theta b_k|_p}|1-\theta y_k|_p + |B_k|_p, \\
&\leq \frac{1}{10^2}|1-\theta y_k|_p^{1-2\delta} + \frac{1}{2}|1-\theta y_k|_p^{3/2-5\delta}, \\
&< \frac{1}{2}|1-\theta y_k|_p^{1-2\delta} + \frac{1}{2}|1-\theta y_k|_p^{1-2\delta}, \\
&= |1-\theta y_k|_p^{1-2\delta} < \epsilon_k^{1-2\delta}. \qquad (3.96)
\end{aligned}$$

If $|\theta a_k+b_k-2b_{k+1}|_p \equiv 0$, then $|1+\theta y_{k+2}|_p = |B_k|_p \leq |1-\theta y_k|_p^{3/2-5\delta} < \epsilon_k^{3/2-5\delta} \leq \epsilon_k^{1-2\delta}$. Also, we have the following relation:

$$\begin{aligned}
2 = |2|_p &\leq |1+\theta y_{k+2}|_p + |1-\theta y_{k+2}|_p, \\
&< \epsilon_k^{1-2\delta} + |1-\theta y_{k+2}|_p, \\
&< 1 + |1-\theta y_{k+2}|_p.
\end{aligned}$$

Subtracting 1 from both sides yields $1 < |1-\theta y_{k+2}|_p$. Hence,

$$\frac{1}{|1-\theta y_{k+2}|_p} < 1. \qquad (3.97)$$

We have another relation derived from the second part of the theorem given that $|\theta a_k + b_k - 2b_{k+1}|_p \not\equiv 0$:

$$\begin{aligned}
10^2|1-\theta y_k|_p^{1+2\delta} &= 10^2|1-\theta y_k|_p \cdot |1-\theta y_k|_p^{2\delta}, \\
&< |1-\theta y_k|_p 10^2 \epsilon_k^{2\delta}, \\
&\leq \frac{|\theta a_k + b_k - 2b_{k+1}|_p}{|a_k+\theta b_k|_p}|1-\theta y_k|_p, \\
&= |(1+\theta y_{k+2}) - B_k|_p, \\
&\leq |1+\theta y_{k+2}|_p + |B_k|_p, \\
&\leq |1+\theta y_{k+2}|_p + \frac{1}{2}|1-\theta y_k|_p^{3/2-5\delta}, \\
&< |1+\theta y_{k+2}|_p + 99|1-\theta y_k|_p^{1+2\delta}.
\end{aligned}$$

In the above relation, we used (3.51) for sufficiently small δ. The above inequalities imply $|1-\theta y_k|_p^{1+2\delta} < |1+\theta y_{k+2}|_p$. Therefore,

$$|1+\theta y_{k+2}|_p^{-1} < |1-\theta y_k|_p^{-(1+2\delta)}. \qquad (3.98)$$

Using (3.98), (3.95) and (3.51), we get the following inequality:

$$\left|\frac{B_k(a_k+\theta b_k)}{2\theta(1+\theta y_{k+2})(1-\theta y_k)}\right|_p \leq \frac{1}{20}|1-\theta y_k|_p^{1/2-6\delta}\cdot|1+\theta y_{k+2}|_p^{-1},$$
$$< \frac{1}{20}|1-\theta y_k|_p^{-1/2-8\delta}. \quad (3.99)$$

Applying the Archimedean absolute value on (3.81) and using the triangle inequality, we have

$$\begin{aligned}|C_k|_p &\leq \frac{|1/2|_p\cdot|(a_{k+2}+\theta b_{k+2})|_p}{|1-\theta y_{k+2}|_p} + \left|\frac{B_k(a_k+\theta b_k)}{2\theta(1+\theta y_{k+2})(1-\theta y_k)}\right|_p + |A_k|_p,\\ &\leq \frac{1}{10}|1-\theta y_k|_p^{-\delta} + \frac{1}{20}|1-\theta y_k|_p^{-1/2-8\delta} + \frac{11}{10}|1-\theta y_k|_p^{-1/2},\\ &< \frac{1}{4}|1-\theta y_k|_p^{-1/2-8\delta} + \frac{1}{4}|1-\theta y_k|_p^{-1/2-8\delta} + \frac{3}{2}|1-\theta y_k|_p^{-1/2-8\delta},\\ &= 2|1-\theta y_k|_p^{-1/2-8\delta}.\end{aligned} \quad (3.100)$$

We have used in (3.100) the results derived in (3.97), (3.99), the first part of the theorem for Archimedean absolute value and (3.51).
We need an upper bound in terms of $|1+\theta y_{k+2}|_p$ which means that we need to find a relation between $|1+\theta y_{k+2}|_p$ and $|1-\theta y_k|_p$. This relation is derived from the second part of the theorem, as shown in (3.96). Hence,

$$|1+\theta y_{k+2}|_p \leq |1-\theta y_k|_p^{1-2\delta}.$$

We used the above relation to get the inequality:

$$|1-\theta y_k|_p^{-(1/2+8\delta)} \leq |1+\theta y_{k+2}|_p^{\frac{-(1/2+8\delta)}{1-2\delta}}.$$

To simplify the power, we use Taylor series expansion (as in the non-Archimedean case), therefore

$$\begin{aligned}|C_k|_p &\leq 2|1-\theta y_k|_p^{-1/2-8\delta},\\ &\leq 2|1+\theta y_{k+2}|_p^{\frac{-(1/2+8\delta)}{1-2\delta}},\\ &\leq 2|1+\theta y_{k+2}|_p^{-1/2-10\delta} \leq 2|1+\theta y_{k+2}|_p^{-2/3-2\delta},\end{aligned}$$

for sufficiently small δ. If $|\theta a_k + b_k - 2b_{k+1}|_p \equiv 0$, then the upper bound on $|C_k|_p$ is

$$\begin{aligned}
|C_k|_p &\leq \frac{|1/2|_p \cdot |(a_{k+2}+\theta b_{k+2})|_p}{|1-\theta y_{k+2}|_p} + \left|\frac{B_k(a_k+\theta b_k)}{2\theta(1+\theta y_{k+2})(1-\theta y_k)}\right|_p + |A_k|_p, \\
&\leq \frac{1}{10}|1-\theta y_k|_p^{-\delta} + \frac{1}{10}|1-\theta y_k|_p^{-(1+\delta)} + \frac{11}{10}|1-\theta y_k|_p^{-1/2}, \\
&\leq \frac{1}{4}|1-\theta y_k|_p^{-(1+\delta)} + \frac{1}{4}|1-\theta y_k|_p^{-(1+\delta)} + \frac{3}{2}|1-\theta y_k|_p^{-(1+\delta)}, \\
&= 2|1-\theta y_k|_p^{-(1+\delta)}.
\end{aligned}$$

Since $|1+\theta y_{k+2}|_p = |B_k|_p \leq \frac{1}{2}|1-\theta y_k|_p^{3/2-5\delta} < |1-\theta y_k|_p^{3/2-5\delta}$, it implies that $|1-\theta y_k|_p^{-(1+\delta)} \leq |1+\theta y_{k+2}|_p^{-(2/3+2\delta)}$. Hence, $|C_k|_p \leq 2|1+\theta y_{k+2}|_p^{-(2/3+2\delta)}$. This proves the theorem for Archimedean absolute value and the proof is completed for Theorem 1.4.2. □

Proof of Corollary 3.2.1 In this proof, we use (3.51) for $\epsilon_n^{-\delta}$. The proof consists of 2 parts depending on which absolute value $|.|_p$ is under consideration when $(p < \infty)$ or $p = \infty$.

Since $k - 1 > K$, therefore k is greater than all the zeros and poles of a_k, b_k and some of their linear combination, in particular $a_k - \theta b_k - \theta(\theta a_{k-2} + b_{k-2} - 2b_{k-1})$ for $\theta = 1$ or -1. Recall from Theorem 1.4.2 the following expression for y_{k+3}:

$$y_{k+3} = \frac{(a_{k+2} - \theta b_{k+2} - \theta(\theta a_k + b_k - 2b_{k+1}))}{2(1+\theta y_{k+2})} + C_k. \tag{3.101}$$

First, for the non-Archimedean absolute value $|.|_p$ $(p < \infty)$, we have from $\epsilon_{k+2}^{-\delta}$ definition in (3.51) the following relation:

$$\begin{aligned}
|1+\theta y_{k+2}|_p^{-(1-\delta)} &= \frac{|1+\theta y_{k+2}|_p^{\delta}}{|1+\theta y_{k+2}|_p} < \frac{\epsilon_{k+2}^{\delta}}{|1+\theta y_{k+2}|_p}, \\
&\leq \frac{|a_{k+2} - \theta b_{k+2} - \theta(\theta a_k + b_k - 2b_{k+1})|_p}{|2|_p \cdot |1+\theta y_{k+2}|_p}, \\
&= |y_{k+3} - C_k|_p \leq \max\{|y_{k+3}|_p, |C_k|_p\}.
\end{aligned} \tag{3.102}$$

Since in Theorem 1.4.2, $|C_k|_p \leq |1+\theta y_{k+2}|_p^{-2/3-2\delta}$, then for sufficiently small δ, $|C_k|_p \leq |1+\theta y_{k+2}|_p^{-2/3-2\delta} \leq |1+\theta y_{k+2}|_p^{-(1-\delta)}$. Therefore, the inequality in (3.102) reduces to $|y_{k+3}|_p > |1+\theta y_{k+2}|_p^{-(1-\delta)} \geq |1+\theta y_{k+2}|_p^{-1/2}$. This proves the corollary for the non-Archimedean absolute $|.|_p$ (where $p < \infty$).

The last part of the proof is for the Archimedean absolute value where $\kappa_\infty = 10$ in (3.51). Note that we have the following relation where we had used (3.101) and the triangle inequality:

$$
\begin{aligned}
10|1+\theta y_{k+2}|_p^{-(1-\delta)} &= \frac{10|1+\theta y_{k+2}|_p^{\delta}}{|1+\theta y_{k+2}|_p} < \frac{10\epsilon_{k+2}^{\delta}}{|1+\theta y_{k+2}|_p}, \\
&\leq \frac{|a_{k+2}-\theta b_{k+2}-\theta(\theta a_k + b_k - 2b_{k+1})|_p}{|2|_p \cdot |1+\theta y_{k+2}|_p}, \\
&= |y_{k+3}-C_k|_p \leq |y_{k+3}|_p + |C_k|_p, \\
&\leq |y_{k+3}|_p + 2|1+\theta y_{k+2}|_p^{-2/3-2\delta}, \\
&\leq |y_{k+3}|_p + 9|1+\theta y_{k+2}|_p^{-(1-\delta)}.
\end{aligned} \tag{3.103}
$$

In the chain of inequalities in (3.103), we used Theorem 1.4.2 and the fact that $|1+\theta y_{k+2}|_p^{-2/3-2\delta} \leq |1+\theta y_{k+2}|_p^{-(1-\delta)}$ for sufficiently small δ. The inequality in (3.103) implies $|y_{k+3}|_p > |1+\theta y_{k+2}|_p^{-(1-\delta)} \geq |1+\theta y_{k+2}|_p^{-1/2}$ which proves the corollary for the Archimedean absolute value and with this last part the proof of the corollary is completed. □

3.3 Diophantine integrability analysis of equation $y_{n+1} + y_{n-1} = \frac{a_n+b_ny_n+2\xi y_n^2}{1-y_n^2}$

In this section, we study the remaining case of equation (3.3) with $c_n \equiv 2\xi$, where $\xi = -1$ or 1. This equation is

$$y_{n+1} + y_{n-1} = \frac{a_n + b_n y_n + 2\xi y_n^2}{1 - y_n^2}.$$

Without loss of generality, we consider the equation

$$y_{n+1} + y_{n-1} = \frac{a_n + b_n y_n + 2y_n^2}{1 - y_n^2}, \tag{3.104}$$

where the right hand side is irreducible $\forall n$. We could derive the other equation with $c_n \equiv -2$ if we start with (3.104) and substitute $y_n = -Y_n$. Intuitively speaking, for a fixed absolute value ($\forall p \leq \infty$) and a quantity ϵ_n defined later, we remark that in the sequence of the solution (y_n) of (3.104) the solution is alternating between being close to the singularity -1 and being a big term. Or in the sequence (y_n) the solution is alternating in the following pattern: y_n is close to 1, y_{n+1} is a big term, y_{n+2} is close to -3, y_{n+3} is a big term, y_{n+4} is close to 1 again and the pattern repeats itself.

We use the same strategy as in the previous section and we get a slightly different result. For (3.104), we find that the summed logarithmic height of an admissible solution

$h_r(y_n)$ that has oscillating sequences that are not special grows exponentially with r (as $r \to \infty$). Later, we define formally oscillating sequences and special oscillating sequences, although they are in concept similar to their definitions in the previous section. If there is an admissible solution y_n of (3.104) that has infinitely many special oscillating sequences, then the coefficients in (3.104) satisfy a certain relation $a_{n-2}+a_n+4-b_{n-2}-b_n+2b_{n-1} \equiv 0$. The difference with the previous section result is that (y_n) does not solve a difference Riccati equation as well and (3.104) is not reduced to any discrete analogue of P_{II}.

Since the analysis and calculation used in this section are similar to the previous section we omit them here. We state and define oscillating sequence S, special oscillating sequence S_p and $\epsilon_n^{-\delta}$ since their expressions are slightly different from the previous section, although they have the same concept and meaning in our analysis in both sections. We assume that the right hand side of (3.104) is irreducible for all n. Also, $a_n \pm b_n + 2$ are not the zero function, otherwise the right hand side of (3.104) is reducible, which is a contradiction to our assumption. We define an integer K such that it is greater than all the zeros and poles of the coefficients b_n, $a_n + 2$ and their linear combinations $a_n + 2 \pm b_n$, $a_n + 2 - b_n + 2b_{n-1}$, $a_n + 2 - b_n + 2b_{n+1}$ and $a_{n-2} + a_n + 4 - b_{n-2} - b_n + 2b_{n-1}$. Now we give a definition for $\epsilon_n^{-\delta}$ for all $n > K$ and for sufficiently small δ. For a fixed absolute value $|.|_p$, $\forall p \leq \infty$, we define $\epsilon_n^{-\delta}$ as follows:

$$\begin{aligned}\epsilon_n^{-\delta} = \kappa_p \max \Big\{ & |2|_p^{-1}, |1/2|_p \cdot |a_n+2+b_n|_p, |1/2|_p^{-1} \cdot |a_n+2-b_n|_p^{-1}, \\ & |b_{n-1}|_p, |b_{n+1}|_p, |a_{n-1}+2|_p, |a_{n+1}+2|_p, |a_n-b_n+2|_p^{-1}, |a_n-b_n+2|_p, \\ & |a_n+2-b_n+2b_{n-1}|_p, |a_n+2-b_n+2b_{n+1}|_p, |a_n+2-b_n+2b_{n-1}|_p^{-1} \\ & |a_n+2-b_n+2b_{n+1}|_p^{-1}, |1/2|_p \cdot |a_{n-2}+2+b_{n-2}|_p, \\ & |1/2|_p \cdot |a_{n+2}+2+b_{n+2}|_p, |1/2|_p \cdot |a_{n-2}+a_n+4-b_{n-2}-b_n+2b_{n-1}|_p \Big\}\end{aligned} \tag{3.105}$$

As in the previous section in (3.105) the maximum is taken over all the finite values in the above set and if any element in the set is infinite then we remove it from the set and take the maximum of the remaining terms. For the non-Archimedean, $\kappa_p = 1$, and for the Archimedean absolute value, $\kappa_p = 10$.

Now we are ready to give formal definitions of an oscillating sequence and a special oscillating sequence. In our definitions we let $k > K$ for a fixed absolute value $|.|_p$ ($\forall p \leq \infty$) and $\theta = -1$ or 1.

Definition 3.3.1. *Suppose that $|1 - \theta y_k|_p < \epsilon_k$, for some $k \in \mathbb{Z}$ and some $\theta = 1$ or $\theta = -1$. Then the* <u>*oscillating sequence*</u> *S containing k is the longest interval in $\mathbb{Z}$ (possibly unbounded) satisfying the following conditions.*

1. *If $k + 2l \in S$ then $|1 - \theta y_{k+2l}|_p < \epsilon_{k+2l}$;*

2. *If* $\{k+2l-1, k+2l\} \in S$, *then* $|y_{k+2l-1}|_p \geq |1-\theta y_{k+2l}|_p^{-(1-\delta)}$; *and*

3. *If* $\{k+2l, k+2l+1\} \in S$, *then* $|y_{k+2l+1}|_p \geq |1-\theta y_{k+2l}|_p^{-(1-\delta)}$.

Definition 3.3.2. *A special oscillating sequence* $S_p = \{k, k+1, k+2\}$ *is an oscillating sequence of length 3 starting with k in* $\mathbb{Z}$ *such that* $|1+y_k|_p < \epsilon_k$, $|y_{k+1}|_p \geq \max\Big\{|1+y_k|_p^{-(1-\delta)}, |1+y_{k+2}|_p^{-(1-\delta)}\Big\}$ *and* $|1+y_{k+2}|_p < \epsilon_{k+2}$. *Also, we have* $|y_{k-1}|_p \leq |1+y_k|_p^{-1/2}$ *and* $|y_{k+3}|_p \leq |1+y_{k+2}|_p^{-1/2}$.

To understand the origin of S_p definition, we need to analyse the iterations of the solution y_n of (3.104) when y_n is close to -1. This analysis is given through the statement of Theorem 3.3.1 which is analogue to Theorem 1.4.2 in the last section. For a fixed absolute value $|.|_p$, $\forall p \leq \infty$, we have the statement of Theorem 3.3.1.

Theorem 3.3.1. *Let* $(y_n)_{n=k-1}^{k+3} \subset \mathbb{Q} \setminus \{-1, 1\}$ *satisfy (3.104)*

$$y_{n+1} + y_{n-1} = \frac{a_n + b_n y_n + 2y_n^2}{1 - y_n^2},$$

where k is sufficiently large and the right hand side of (3.104) is irreducible. Assume that $|y_{k-1}|_p \leq |1+y_k|_p^{-1/2}$. *Furthermore, for sufficiently small* δ *and for* ϵ_k *as defined in (3.105), if* $|1+y_k|_p < \epsilon_k$, *then*

1. $y_{k+1} = \dfrac{a_k - b_k + 2}{2(1+y_k)} + A_k$, *where* $|A_k|_p \leq |1+y_k|_p^{-1/2}$ *for non-Archimedean absolute value and* $|A_k|_p \leq \frac{5}{2}|1+y_k|_p^{-1/2}$ *for Archimedean absolute value.*

2. $y_{k+2} = -1 + \left(\dfrac{a_k - b_k + 2 + 2b_{k+1}}{a_k - b_k + 2}\right)(1+y_k) + B_k$,
where $|B_k|_p \leq |1+y_k|_p^{3/2-5\delta}$ *for non-Archimedean absolute value and* $|B_k|_p \leq \frac{1}{2}|1+y_k|_p^{3/2-5\delta}$ *for Archimedean absolute value.*

3. $y_{k+3} = \dfrac{(a_{k+2} - b_{k+2} + a_k - b_k + 4 + 2b_{k+1})}{2(1+y_{k+2})} + C_k$,
where $|C_k|_p \leq |1+y_{k+2}|_p^{-(2/3+2\delta)}$ *for non-Archimedean absolute value and* $|C_k|_p \leq \frac{15}{4}|1+y_{k+2}|_p^{-(2/3+2\delta)}$ *for Archimedean absolute value.*

Theorem 3.3.1 simply means that for a fixed absolute value $|.|_p$ ($\forall p \leq \infty$), if we start with a small term (close to -1) $|1+y_k|_p < \epsilon_k$ where $|y_{k-1}|_p \leq |1+y_k|_p^{-1/2}$, then the first iterate is big $|y_{k+1}|_p \geq |1+y_k|_p^{-(1-\delta)}$. The second iterate $|y_{k+2}|_p$ is small (close to -1) and the size of the third iterate depends on the relation $|a_{k-2}+a_k+4-b_{k-2}-b_k+2b_{k-1}|_p$ if it is identically zero or not. If $|a_{k-2}+a_k+4-b_{k-2}-b_k+2b_{k-1}|_p \not\equiv 0$, then $|y_{k+3}|_p > |1+y_{k+2}|_p^{-1/2}$ as shown in Corollary 3.3.1 below. Note that, if $|y_{k+3}|_p \leq |1+y_{k+2}|_p^{-1/2}$, which implies the special oscillating sequence definition given before, then $|a_{k-2}+a_k+4-b_{k-2}-b_k+2b_{k-1}|_p \equiv 0$.

Corollary 3.3.1. *For a fixed absolute value $|.|_p$ ($\forall p \leq \infty$) let $k-1 > K$ such that $|1+y_k|_p < \epsilon_k$, $|y_{k-1}|_p \leq |1+y_k|_p^{-1/2}$ and $|1+y_{k+2}|_p < \epsilon_{k+2}$. Assume that $|a_{k-2}+a_k+4-b_{k-2}-b_k+2b_{k-1}|_p \not\equiv 0$, then $|y_{k+3}|_p > |1+y_{k+2}|_p^{-1/2}$.*

Recall that the summed logarithmic height of an admissible solution $h_r(y_n)$ grows exponentially with $r \to \infty$ if y_n has oscillating sequences that are not special. Now if y_n has infinitely many special oscillating sequences, then Theorem 3.3.1 and Corollary 3.3.1 imply that the following relation holds between the coefficients of equation (3.104) for all $n > K$: $a_{n-2}+a_n+4-b_{n-2}-b_n+2b_{n-1} \equiv 0$. The analogue of the outlined procedure in section 3.2 is when we have infinitely many special oscillating sequences of only one of the forms and we define a rational function of n, f_n similar to (3.55)

$$f_n = x_n + x_{n+1}(1+x_n). \tag{3.106}$$

Here, x_n solves a difference Riccati equation in (3.106). Note that when x_n takes the value -1 at certain n, then the next iterates are ∞ and -1, respectively. We want to show if we could get $x_{n-1}+x_{n+1} = \frac{A_n+B_nx_n+2x_n^2}{1-x_n^2}$ (for some rational functions in n, A_n and B_n) that is in the same class as (3.104), then it means that x_n solves a difference Riccati equation which is given in (3.106) as well as solving (3.104). Using the shift $n \to n-1$ in (3.106) and solving for x_{n-1}, we get

$$x_{n-1} = \frac{f_{n-1}-x_n}{(1+x_n)}. \tag{3.107}$$

From (3.106) and (3.107) we have

$$x_{n+1}+x_{n-1} = \frac{f_n+f_{n-1}-2x_n}{1+x_n} = \frac{(f_n+f_{n-1})-(2+f_n+f_{n-1})x_n+2x_n^2}{1-x_n^2}.$$

This shows that x_n solves a difference Riccati equation (3.106) but $x_{n+1}+x_{n-1}$ do not give us an equation of the class of (3.104) as shown above since the right hand side is reducible. Unlike equation (3.49) in the previous section, there is no solution y_n of (3.104) which solves a difference Riccati equation as well. This proves Theorem 1.4.1 for $c_n \equiv 2$. A similar result is obtained if we exchange y_n by $-Y_n$ in (3.104) for $c_n \equiv -2$ and apply our strategy using the new equation.

Chapter 4

Degree growth of difference equations

Since the remark of Veselov [82] and the idea of Viallet *et al.* [39, 8] where they noticed a correlation between integrability and slow growth of some characteristics of a mapping, many studies which are related to the degree growth were presented. Some of these studies led to the introduction of integrability detectors such as algebraic entropy (discussed in Chapter 1). The Viallet *et al.* idea is based on analysing the degree growth of the iterates of some initial data under the action of the mapping. We could notice the connection with our work in this chapter (described below) where both algebraic entropy and our analysis here consider the degree growth of solutions of difference equations in the complex plane setting. In our analysis, however, we consider the iterate x_n for every n as a rational function of an external variable to the mapping $z \in \mathbb{C} \cup \{\infty\}$. We consider the degree growth of the iterate x_n in terms of this variable z, i.e. $\deg_z(x_n)$.

In this chapter, we study the following class of difference equations:

$$(x_{n-1} + x_n)(x_n + x_{n+1}) = \frac{P_n(x_n)}{(x_n - a_n)(x_n - b_n)} = \frac{x_n^4 + \alpha_n x_n^3 + \beta_n x_n^2 + \gamma_n x_n + \eta_n}{(x_n - a_n)(x_n - b_n)},$$

where $\alpha_n, \beta_n, \gamma_n$, η_n, a_n and b_n are rational functions of n independent of z. It is worth noting that z is not a function of n. We investigate in this chapter the degree growth of x_n ($\deg_z(x_n)$) for a sequence of iterations $(x_n)_{n\in\mathbb{Z}}$ of the above difference equation. This chapter consists of 2 sections. The first defines a rational map and discusses the degree of rational functions where the independent variable of this map is $z \in \mathbb{C} \cup \{\infty\}$. The main results of section 4.2 are presented in Theorem 4.2.1 and Theorem 4.2.2 and their proofs, where we study the above class of difference equations. We show in Theorem 4.2.1 that if $\alpha_n \not\equiv \omega_{-1} + \omega_1 - a_n - b_n$ (where $\omega_i \in \{a_{n+i}, b_{n+i}\}$) for all $n \in \mathbb{Z}$, then $\sum_{n=r_0}^{r} \deg_z(x_n) \geq K(\frac{2}{1+\mu})^r$ for all $r \geq r_0$, where μ is an arbitrary constant and μ, r_0 and K are all positive constants. This implies that $\deg_z(x_n)$ grows fast with n (as $n \to \infty$). This suggests that the above class of difference equations with the above assumptions on its coefficients is not integrable. In Theorem 4.2.2 we show that for the above class of difference equations either $\sum_{n=r_0}^{r} \deg_z(x_n)$ grows exponentially with r or $P_n(x_n)$ has some special forms. Analysing the results of the two theorems implies that for certain assumptions on the coefficients and the roots of the denominator the above class of difference equations reduces to a discrete analogue of P_{IV}, namely $(x_{n+1} + x_n)(x_n + x_{n-1}) = \frac{(x_n^2 - p^2)(x_n^2 - q^2)}{(x_n - (\psi n + \xi))^2 - u^2}$.

4.1 Degree of rational functions

A rational map is a function of the form

$$R(z) = \frac{S(z)}{Q(z)} = \frac{a_0 + a_1 z + \cdots + a_s z^s}{b_0 + b_1 z + \cdots + b_q z^q}, \tag{4.1}$$

where $S(z)$ and $Q(z)$ are coprime polynomials in z. Also, z and $R(z) \in \mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$, where we call $\mathbb{C}_\infty$ the extended complex plane. If S is the zero polynomial, then the rational function $R(z)$ is the zero function. If Q is the zero polynomial, then R is a constant function ∞. If $Q(z) = 0$ for some $z \in \mathbb{C}$ and S is not the zero polynomial, then $R(z)$ is defined to be ∞. Note that R has a pole at this particular z. Also, we define $R(\infty)$ as the limit of $R(z)$ as $z \longrightarrow \infty$ [6].

Suppose that both S and Q are not zero polynomials and $a_s, b_q \neq 0$. Then, we define the degree of the rational function $R(z)$ by

$$\deg_z(R) = \max\{\deg_z(S), \deg_z(Q)\},$$

where in this case $\deg_z(S) = s$ and $\deg_z(Q) = q$. For example, if $R(z) = m$ is a constant map, then $\deg_z(R) = 0$ for any value $m \in \mathbb{C}_\infty$. It is important to note that for a rational function R of a positive degree d (i.e. R is a non-constant rational function), R is a d-fold map of $\mathbb{C}_\infty$ onto itself. This means for any $w \in \mathbb{C}_\infty$, the equation

$$R(z) = w \tag{4.2}$$

has exactly d solutions in z counting multiplicities. The number of solutions of (4.2) equals the number of preimages of w under R in $\mathbb{C}_\infty$ counting multiplicities. Adopting this terminology of preimages, we show that equation (4.2) has precisely d solutions as follows:

$$\text{number of preimages of } w \text{ under } R \text{ in } \mathbb{C}_\infty = \\ (\text{number of preimages in } \mathbb{C}\ ) + (\text{multiplicity of } w \text{ (if any) at } z = \infty). \tag{4.3}$$

Denote the number of preimages of w under R in $\mathbb{C}_\infty$ (counting multiplicities) by $\mathrm{P}(R, w)$. Using (4.3) to calculate the number of preimages when $w = 0$, we get the following:

$$\mathrm{P}(R, 0) = s + \max(q - s, 0) = \max(q, s) = d.$$

Also, when $w = \infty$ we get the same result as follows:

$$\mathrm{P}(R, \infty) = q + \max(s - q, 0) = \max(s, q) = d.$$

Now we choose $w \in \mathbb{C} \setminus \{0\}$ and we determine how many solutions of (4.2) are in $\mathbb{C}_\infty$. Since $w \neq \infty$ and from (4.1), we could rewrite (4.2) as

$$a_0 + a_1 z + \cdots + a_s z^s = w(b_0 + b_1 z + \cdots + b_q z^q). \tag{4.4}$$

Here we have three cases to consider. The first is when $s \neq q$, the second when $s = q$ and $a_s \neq wb_q$ and the third case is when $s = q$ and $a_s = wb_q$. In the first two cases, we show that ∞ is not a preimage of w. This implies, since in (4.4) the polynomial is of degree $d = \max(s, q)$, that equation (4.2) has d solutions in $\mathbb{C} \subset \mathbb{C}_\infty$.

For the first case, let us consider first $q > s$, which yields $\lim_{z\to\infty} R(z) = 0 \neq w$. If $s > q$, then $\lim_{z\to\infty} R(z) = \infty \neq w$. The second case is when $s = q$ and $a_s \neq wb_q$, which leads to $\lim_{z\to\infty} R(z) = \dfrac{a_s}{b_q} \neq w$. The conclusion here is that ∞ is not one of the preimages of w and according to (4.3) and (4.4), $\mathrm{P}(R, w) = \max(s, q) = d$ in $\mathbb{C} \subset \mathbb{C}_\infty$.

The third case is when $s = q$ and $a_s = wb_s$, which implies (after taking the limit and using polynomial long division)

$$\lim_{z\to\infty} R(z) = \lim_{z\to\infty} \frac{a_0 + \cdots + a_s z^s}{b_0 + \cdots + b_s z^s} = \lim_{z\to\infty} \left(\frac{a_s}{b_s} + \frac{c_0 + \cdots + c_r z^r}{b_0 + \cdots + b_s z^s} \right) = w, \tag{4.5}$$

where $r < s$ and $c_r \neq 0$. This shows that ∞ is one of the preimages of w. In this case, to show the multiplicity of w at $z = \infty$ under $R(z)$, let us consider

$$R(z) = w + \frac{c_0 + \cdots + c_r z^r}{b_0 + \cdots + b_s z^s}, \tag{4.6}$$

where $w, c_r, b_s \neq 0$ and $r < s$. In the finite plane $\mathbb{C}$, the equation $R(z) = w$ is valid if and only if $c_0 + \cdots + c_r z^r = 0$ which means that $\mathrm{P}(R, w) = r$ in $\mathbb{C}$. Hence, it implies that equation (4.2) has r solutions in $\mathbb{C}$. Now in the extended complex plane $\mathbb{C}_\infty$ as $z \longrightarrow \infty$, we have

$$\lim_{z\to\infty} R(z) = \lim_{z\to\infty} w + \frac{c_0 + \cdots + c_r z^r}{b_0 + \cdots + b_s z^s} = \lim_{z\to\infty} \left(w + \frac{c_r}{b_s}\left(\frac{1}{z}\right)^{s-r} + \ldots \right).$$

This allows us to say that the equation $R(z) - w = 0$ has a zero of multiplicity $(s - r)$ at $z = \infty$. Using (4.3) to calculate the number of preimages of w, we find that it equals $r + (s - r) = s = \max(s, q) = d$, since $(s = q)$. In conclusion, we were able to show that the equation $R(z) = w$ has d preimages in $\mathbb{C}_\infty$. It means that $R(z) - w = 0$ has d roots in $\mathbb{C}_\infty$ which equals $\deg_z(R)$ in $\mathbb{C}_\infty$.

Now we outline the settings used in the next section to measure $\deg_z(x_n)$ in a certain class of difference equations. This class of difference equations is

$$(x_{n-1} + x_n)(x_n + x_{n+1}) = \frac{P_n(x_n)}{(x_n - a_n)(x_n - b_n)} = \frac{x_n^4 + \alpha_n x_n^3 + \beta_n x_n^2 + \gamma_n x_n + \eta_n}{(x_n - a_n)(x_n - b_n)}, \quad (4.7)$$

where a_n, b_n are not identically zero and $\alpha_n, \beta_n, \gamma_n$, η_n, a_n and b_n are rational functions of $n \in \mathbb{Z}$. Also, the right hand side of (4.7) is irreducible at every $n \in \mathbb{Z}$ and $a_n \neq b_n$ for all n. Note that we consider the iterate $x_n(z)$ as a non-constant rational function of the variable $z \in \mathbb{C}_\infty$. We define a degree growth function $D_r(x_n)$ below for some integer r_0 such that $r \geq r_0$.

Definition 4.1.1. *The degree growth function $D_r(x_n)$ of a sequence of non-constant rational functions $(x_n(z)) \subset \mathbb{C}_\infty$ is*

$$D_r(x_n) = \sum_{n=r_0}^{r} deg_z(x_n), \quad (4.8)$$

for some integer r_0.

Since $\deg_z(x_n) = \mathrm{P}(x_n, \nu)$, where $\mathrm{P}(x_n, \nu)$ is the number of preimages of ν under x_n in $\mathbb{C}_\infty$ counting multiplicities, we get the following relation:

$$D_r(x_n) = \sum_{n=r_0}^{r} \mathrm{P}(x_n, \nu). \quad (4.9)$$

Now the following inequality holds for any value $\nu_i \in \mathbb{C}_\infty$ $(i \in \{1,2,3\})$ and $\tau = 2$:

$$\sum_{n=r_0}^{r} \mathrm{P}(x_n, \nu_1) + \sum_{n=r_0}^{r} \mathrm{P}(x_n, \nu_2) \leq \tau \sum_{n=r_0-1}^{r+1} \mathrm{P}(x_n, \nu_3), \tag{4.10}$$

Suppose that (4.10) is true for some constant $\tau < 2$. Generally, ν_1, ν_2, ν_3 are any values in $\mathbb{C}_\infty$, we choose them to be special values for the difference equation (4.7). Specifically, we choose $\nu_1 = a_n$, $\nu_2 = b_n$ and $\nu_3 = \infty$. Using the notation of (4.9) in (4.10) leads to

$$D_{r+1}(x_n) + \deg_z(x_{r_0-1}) \geq \frac{2}{\tau} D_r(x_n). \tag{4.11}$$

Since for every n $(\forall n \geq r_0 - 1)$ x_n is a non-constant rational function of z, $\deg_z(x_n) \geq 1$, $\forall n \geq r_0 - 1$ where in particular, $\deg_z(x_{r_0-1}) \geq 1$. This implies $D_r(x_n) \geq r - r_0 + 1$, $\forall r \geq r_0$.
For a given $0 < \mu \ll 1$, $\exists R > r_0$ such that

$$\deg_z(x_{r_0-1}) \leq \mu D_{r+1}(x_n) \quad \forall r > R. \tag{4.12}$$

Using (4.12) in (4.11) yields

$$\begin{aligned} D_{r+1}(x_n) &\geq \frac{2}{\tau} D_r(x_n) - \deg_z(x_{r_0-1}) \geq \frac{2}{\tau} D_r(x_n) - \mu D_{r+1}(x_n), \\ D_{r+1}(x_n) &\geq \left(\frac{2}{\tau(1+\mu)}\right) D_r(x_n). \end{aligned}$$

By introducing the shift $r \longrightarrow r - 1$ and using the recurrence relation repeatedly implies that

$$D_r(x_n) \geq \left(\frac{2}{\tau(1+\mu)}\right) D_{r-1}(x_n) \geq \left(\frac{2}{\tau(1+\mu)}\right)^2 D_{r-2}(x_n) \geq \cdots \geq \left(\frac{2}{\tau(1+\mu)}\right)^m D_{r-m}(x_n),$$

where $r - m \geq R$. Hence,

$$D_r(x_n) \geq K \left(\frac{2}{\tau(1+\mu)}\right)^r, \quad \forall r > R \text{ and an arbitrary } 0 < \mu \ll 1. \tag{4.13}$$

Since x_ns are non-constant rational functions, the $\deg_z(x_n)$s are positive integers, which implies that $K > 0$. The inequality in (4.13) implies exponential growth for the function $D_r(x_n)$ with r, provided that $\tau(1 + \mu) < 2$. The inequality in (4.10) is global (i.e. if it is true, then it holds for all z), since the multiplicity of a zero (or a pole) that $x_n(z_0) - u = 0$ (or $x_n(z_0) = u$ if u is ∞) has at some $z_0 \in \mathbb{C}_\infty$ for any $u \in \mathbb{C}_\infty$ is less than or equals $\deg_z(x_n) = \mathrm{P}(x_n, u)$. Therefore, to show that the global inequality (4.10) holds, we start locally by showing that when $x_n(z_0) = u$ for $u = a_n$ or b_n with multiplicity k, then there is a pole $x_{n-1}(z_0)$ or $x_{n+1}(z_0)$ with multiplicity k. If this pole is not associated with any value a_m or b_m where $m \in \{n-2, n+2\}$, then we can associate its multiplicity to the value $x_n(z_0)$ and this leads to $\tau = 1$ in (4.10) provided it happens for all $z = z_0$. If this pole is shared with other values a_m or b_m, then we can associate half of its multiplicity to the value $x_n(z_0)$ which leads to $\tau = 2$ given that it happens to all $z = z_0$.

The strategy that we adopt in the next section is aimed at showing that (4.10) holds with $\tau < 2$. Note that ∞ in this inequality is linked to any source of singularity that equation (4.7) has. We consider two types of singularities. The first happens if $x_n(z_0) = a_n$ or b_n at some z_0, then $x_{n+\theta}$ is ∞ where $\theta = -1$ or 1. The second happens if $x_n(z_0) = g$ at some z_0 where g is a root of the numerator of (4.7). Then $x_{n+1}(z_0) + x_n(z_0) = 0$ or $x_n(z_0) + x_{n-1}(z_0) = 0$ which implies $x_{n+2}(z_0) = \infty$ or $x_{n-2}(z_0) = \infty$, provided $P_{n+1}(-g) \neq 0$ or $P_{n-1}(-g) \neq 0$, respectively. Using this strategy we prove the main two results in the next section stated in Theorem 4.2.1 and Theorem 4.2.2 respectively.

4.2 Growth of rational iterates in a certain class of difference equations

In this section we study equation (4.7) where our main results are proven in Theorem 4.2.1 and Theorem 4.2.2. Recall that in the previous section we considered the x_ns as rational functions of $z \in \mathbb{C}_\infty$, and that all the coefficients of (4.7) are rational functions of n and constant with respect to z. Here we discuss the $D_r(x_n)$ growth with r. The strategy we adopt here is aimed at showing that the inequality (4.10) holds with $\tau = 1$. First we define for any rational function $f(z)$ and any $z_0 \in \mathbb{C}_\infty$, the function $\mathrm{ord}_{z_0} f$ such that $\mathrm{ord}_{z_0} f = 0$ if f is analytic at $z = z_0$, otherwise, $\mathrm{ord}_{z_0} f$ equals the multiplicity of the pole of f at $z = z_0$. Since $\sum_{z_0 \in \mathbb{C}_\infty} \mathrm{ord}_{z_0} f = \deg_z f$, the inequality (4.10) is written as follows:

$$\sum_{n=r_0}^{r} \sum_{z_0 \in \mathbb{C}_\infty} \mathrm{ord}_{z_0} \frac{1}{x_n - a_n} + \sum_{n=r_0}^{r} \sum_{z_0 \in \mathbb{C}_\infty} \mathrm{ord}_{z_0} \frac{1}{x_n - b_n} \leq \tau \sum_{n=r_0-1}^{r+1} \sum_{z_0 \in \mathbb{C}_\infty} \mathrm{ord}_{z_0} x_n. \tag{4.14}$$

To show that (4.14) holds with $\tau = 1$ we start locally at some z_0 with the following inequality:

$$\sum_{n=r_0}^{r} \mathrm{ord}_{z_0} \frac{1}{x_n - a_n} + \sum_{n=r_0}^{r} \mathrm{ord}_{z_0} \frac{1}{x_n - b_n} \leq \tau \sum_{n=r_0-1}^{r+1} \mathrm{ord}_{z_0} x_n, \tag{4.15}$$

where $\tau = 2$ for some integer r_0 such that $r \geq r_0$. If we show that (4.15) holds with $\tau = 1$ for all $z_0 \in \mathbb{C}_\infty$ and for all sufficiently large r_0, then this implies (4.14) holds where $\tau = 1$.

In the proof of Theorem 4.2.1 it is enough to consider the first type of singularity only to ensure that (4.14) holds with $\tau = 1$, while in the proof of Theorem 4.2.2 we consider both types of singularity to show that (4.14) holds with $\tau = 1$. Now we are ready to state and prove Theorem 4.2.1.

Theorem 4.2.1. *Let $(x_n) \subset \mathbb{C}_\infty$ be a sequence of non-constant rational functions of z solving*

$$(x_{n-1} + x_n)(x_n + x_{n+1}) = \frac{x_n^4 + \alpha_n x_n^3 + \beta_n x_n^2 + \gamma_n x_n + \eta_n}{(x_n - a_n)(x_n - b_n)},$$

where a_n, b_n are not identically zero, $\alpha_n, \beta_n, \gamma_n$, η_n, a_n and b_n are rational functions of n and constant with respect to z. Furthermore, assume that for all sufficiently large n the right hand side of the equation is irreducible and $a_n \neq b_n$ for every n . Let $X(z_0) = \{n \in \mathbb{Z} : x_n(z_0) = a_n$ or $x_n(z_0) = b_n\}$. Then

$$\sum_{n=r_0}^{r} ord_{z_0} \frac{1}{x_n(z) - a_n} + \sum_{n=r_0}^{r} ord_{z_0} \frac{1}{x_n(z) - b_n} \leq \tau \sum_{\substack{n = r_0 - 1 \\ \{n-1, n+1\} \cap X(z_0) \neq \emptyset}}^{r+1} ord_{z_0} x_n(z), \quad (4.16)$$

where $\tau = 2$. If there are infinitely many points $(n, z_0) \in \mathbb{Z} \times \mathbb{C}_\infty$ such that

1. *$x_{n-1}(z_0) = a_{n-1}$, $x_n(z_0) = \infty$ and $x_{n+1}(z_0) = a_{n+1}$, then* $\alpha_n \equiv a_{n+1} - a_n - b_n + a_{n-1}$;

2. *$x_{n-1}(z_0) = a_{n-1}$, $x_n(z_0) = \infty$ and $x_{n+1}(z_0) = b_{n+1}$, then* $\alpha_n \equiv b_{n+1} - a_n - b_n + a_{n-1}$;

3. *$x_{n-1}(z_0) = b_{n-1}$, $x_n(z_0) = \infty$ and $x_{n+1}(z_0) = a_{n+1}$, then* $\alpha_n \equiv a_{n+1} - a_n - b_n + b_{n-1}$;

4. *$x_{n-1}(z_0) = b_{n-1}$, $x_n(z_0) = \infty$ and $x_{n+1}(z_0) = b_{n+1}$, then* $\alpha_n \equiv b_{n+1} - a_n - b_n + b_{n-1}$.

If for all sufficiently large n, there are no sequences of the type described in $1 - 4$ above, then the inequality (4.16) holds with $\tau = 1$ and $D_r(x_n(z)) \geq \left(\frac{2}{1+\mu}\right)^r \cdot K$, where $K > 0$ and $0 < \mu \ll 1$.

Proof We choose n sufficiently large such that it is greater than all the real poles and zeros of α_n, β_n, γ_n, η_n, a_n and b_n if they have any, otherwise, n is any integer. First, we show that if we have any of the sequences stated above in $1 - 4$, then we get the corresponding expression for α_n. Suppose that $x_n(z_0) = \infty$, then as $z \to z_0$ equation (4.7) can be written as

$$x_{n+1}(x_{n-1} + x_n) = (\alpha_n + a_n + b_n - x_{n-1})x_n + O(1).$$

Note that if $x_{n-1}(z_0) = a_{n-1}$, then $x_{n+1}(z_0) = \alpha_n + a_n + b_n - a_{n-1}$, while if $x_{n-1}(z_0) = b_{n-1}$, then $x_{n+1}(z_0) = \alpha_n + a_n + b_n - b_{n-1}$. Equating the appropriate expression for $x_{n+1}(z_0)$ to either a_{n+1} or b_{n+1} as required in the sequences $1-4$ gives the forms for α_n given in the theorem above. Since a_n, b_n and α_n are rational functions of n and the forms of α_n in $1-4$ hold for an infinite number of choices of n, these forms are true for all n.

Let $X(z_0) = \{n \in \mathbb{Z} : x_n(z_0) = a_n \text{ or } x_n(z_0) = b_n\}$. Suppose without loss of generality that $x_n(z) - a_n$ has a zero of multiplicity k at $z = z_0$. Note that in the following argument any value a_i could be replaced by b_i. Since the right hand side of (4.7) is irreducible, $(x_{n+1} + x_n)(x_{n-1} + x_n)$ has a pole of multiplicity k at $z = z_0$. It follows that

$$\text{ord}_{z_0} x_{n+1} + \text{ord}_{z_0} x_{n-1} \geq k.$$

In order to obtain an inequality of the form (4.16), we wish to associate at least some of the multiplicity of the pole(s) at x_{n-1} or x_{n+1} with the fact that x_n takes the value a_n with multiplicity k at $z = z_0$. If $x_{n-2}(z_0) = a_{n-2}$ or $x_{n+2}(z_0) = a_{n+2}$, then we could "share" the poles between the points x_{n-2}, x_n and x_{n+2} by only associating half the sum of the multiplicities of the pole(s) at x_{n-1} or x_{n+1} with a_n at x_n. This can be done consistently at all points where x_i is either a_i or b_i where $i \in \mathbb{Z}$. This proves (4.16) with $\tau = 2$.

If none of the choices for (x_{n-1}, x_n, x_{n+1}) listed in $1-4$ above occurs for all sufficiently large n, then we are free to associate the pole(s) at x_{n-1} or x_{n+1} with a_n or b_n at x_n without any need to "share". This gives an inequality of the form (4.16) with $\tau = 1$ by considering only the first type of singularity:

$$\sum_{n=r_0}^{r} \text{ord}_{z_0} \frac{1}{x_n(z) - a_n} + \sum_{n=r_0}^{r} \text{ord}_{z_0} \frac{1}{x_n(z) - b_n} \leq \tau \sum_{\substack{n = r_0 - 1 \\ \{n-1, n+1\} \cap X(z_0) \neq \emptyset}}^{r+1} \text{ord}_{z_0} x_n(z).$$

Since this holds for all z_0, provided r_0 is sufficiently large,this proves (4.14) with $\tau = 1$. From the previous section (4.10)-(4.13) we have

$$D_r(x_n(z)) \geq \left(\frac{2}{1+\mu} \right)^r \cdot K,$$

where $0 < \mu \ll 1$ and $K > 0$ since all the x_ns are non-constant rational functions of z, which proves the theorem. □

The above theorem says that if α_n is not one of the forms stated in $1-4$, then $D_r(x_n(z))$ grows exponentially with r. This implies that $\deg_z(x_n)$ is not growing polynomially which suggests that (4.7) is non-integrable provided that α_n is not one of the forms given in $1-4$.

In the next theorem, we study (4.7) further where we write it as

$$(x_{n+1}+x_n)(x_{n-1}+x_n)=\frac{P_n(x_n)}{(x_n-a_n)(x_n-b_n)}=\frac{\prod_{i=1}^{4}(x_n-r_{i,n})}{(x_n-a_n)(x_n-b_n)}, \tag{4.17}$$

where we assume that $\{r_{1,n}, r_{2,n}, r_{3,n}, r_{4,n}\}$ is the set of distinct roots of P_n. We wrote (4.7) in the above way so that it will be clear which root we mean when we are discussing the second type of singularity. In the next theorem we consider a smaller class of equations in (4.7) subject to the assumptions in the theorem. Now we state and prove Theorem 4.2.2.

Theorem 4.2.2. *Let $(x_n) \subset \mathbb{C}_\infty$ be a sequence of non-constant rational functions of z satisfying (4.17) where $r_{i,n}$ $\forall i = 1, \dots, 4$, a_n and b_n are rational functions of n independent of z. Assume that for sufficiently large n, the right hand side of (4.17) is irreducible for every n and $a_n \neq b_n$, $\forall n$. Furthermore, assume that $P_n(y)$ has distinct roots for all n, $\{r_{1,n}, r_{2,n}, r_{3,n}, r_{4,n}\}$. Let $a_{n\pm1} \neq -r_{i,n}$ and $b_{n\pm1} \neq -r_{i,n}$ for all $i \in \{1, \dots, 4\}$ and for all n. Also, let $r_{i,n} \notin \{a_{n+3}-\alpha_{n+2}-a_{n+2}-b_{n+2}, b_{n+3}-\alpha_{n+2}-a_{n+2}-b_{n+2}, a_{n-3}-\alpha_{n-2}-a_{n-2}-b_{n-2}, b_{n-3}-\alpha_{n-2}-a_{n-2}-b_{n-2}\}$ for all n and $\forall i \in \{1, \dots, 4\}$. Then either $D_r(x_n(z)) \geq \left(\frac{2}{1+\mu}\right)^{\lfloor r/2 \rfloor} K$ for some $K > 0$ and an arbitrary $0 < \mu \ll 1$ or $P_n(x_n)$ has one of the following forms:*

1. *$P_n(x_n) = x_n^4 + \beta x_n^2 + \eta$,*
2. *$P_n(x_n) = (x_n - f_n)(x_n - g_n)(x_n + f_{n-1})(x_n + g_{n-1})$,*
3. *$P_n(x_n) = (x_n - f_n)(x_n + f_{n-1})(x_n + f_{n+1})(x_n - f_{n+2})$,*
4. *$P_n(x_n) = x_n(x_n - f_n)(x_n + f_{n+1})(x_n + f_{n-1})$,*

for some constants β, η or some arbitrary rational functions of n, f_n and g_n.

Proof Since $r_{1,n}$, $r_{2,n}$, $r_{3,n}$ and $r_{4,n}$ are rational functions of n it follows that α_n, β_n, γ_n and η_n are rational functions of n. We choose n greater than all the real poles and zeros of α_n, β_n, γ_n, η_n, a_n and b_n if there are any, otherwise, n could be any integer. Since using a Möbius transformation any value $z \in \mathbb{C}_\infty$ could be mapped to 0, without loss of generality in the following argument we choose $z_0 = 0$. Let

$$x_n(z) = r_{i,n} + \xi z^k + \dots, \tag{4.18}$$

for any one of the roots of P_n, $r_{i,n}$ where $i \in \{1, 2, 3, 4\}$ and $\xi \neq 0$. Now we show that we get at least one pole at x_{n+2} or x_{n-2} or at both such that their multiplicity sum is at least k subject to some conditions. Substituting (4.18) into (4.17) yields

$$(x_{n+1}(z) + x_n(z))(x_{n-1}(z) + x_n(z)) = \phi z^k + \dots, \tag{4.19}$$

for non-zero ϕ. At least one of the factors $x_{n+1}(z) + x_n(z)$ or $x_{n-1}(z) + x_n(z)$ vanishes at $z = 0$. Without loss of generality, let $x_{n+1}(z) + x_n(z)$ vanishes with multiplicity m_1, where $0 < m_1 \leq k$ if $x_{n-1}(z) + x_n(z)$ is a finite value and $m_1 > k$ if $x_{n-1}(z) + x_n(z)$ is infinite. This implies that

$$x_{n+2}(z) = \frac{P_{n+1}(-r_{i,n})}{(r_{i,n} + a_{n+1})(r_{i,n} + b_{n+1})z^{m_1}} + \dots,$$

where $r_{i,n} \neq -a_{n+1}$ and $r_{i,n} \neq -b_{n+1}$ for all i and n by assumption. Suppose that for some $i = 1, \dots, 4$,

$$P_{n+1}(-r_{i,n}) \neq 0,$$

for some n. Then for all sufficiently large n, $P_{n+1}(-r_{i,n}) \neq 0$. A similar argument follows if $x_{n-1}(z)+x_n(z)$ vanishes with multiplicity m_2 using the previous reasoning, where $0 < m_2 \leq k$ if $x_{n+1}(z) + x_n(z)$ is a finite value and $m_2 > k$ if $x_{n+1}(z) + x_n(z)$ is infinite. Hence,

$$x_{n-2}(z) = \frac{P_{n-1}(-r_{i,n})}{(r_{i,n} + a_{n-1})(r_{i,n} + b_{n-1})z^{m_2}} + \dots,$$

where $r_{i,n} \neq -a_{n-1}$ and $r_{i,n} \neq -b_{n-1}$ for all i and n by assumption. Note that if both factors in the left hand side of the equation in (4.19) vanish, then $m_1+m_2 = k$. Suppose that for some $i = 1, \dots, 4$, $P_{n-1}(-r_{i,n}) \neq 0$ for some n. Then for sufficiently large n, $P_{n-1}(-r_{i,n}) \neq 0$. This implies

$$\mathrm{ord}_0(x_{n+2}) + \mathrm{ord}_0(x_{n-2}) \geq k.$$

It is worth noting that in the above argument if $x_{n-1}(z)$ or $x_{n+1}(z)$ has a pole at $z = 0$, then if this pole is from the first type singularity it contributes in the sum (4.14), otherwise it will not contribute to the sum (4.14). Now we need to show that the pole(s) at x_{n-2} or x_{n+2} are not shared with a_j or b_j where $j \in \{n-3, n+3\}$ from the first type of singularity, so that we are not double counting the poles when we sum to get the inequality (4.14). Recall that we are considering both types of singularities to show that (4.14) holds with $\tau = 1$ here. Without loss of generality if the pole is at x_{n+2}, then the next iterate x_{n+3} is

$$x_{n+3} = (\alpha_{n+2} + a_{n+2} + b_{n+2} + r_{i,n}) + O(z).$$

Since by assumption $r_{i,n} \neq a_{n+3} - \alpha_{n+2} - a_{n+2} - b_{n+2}$ and $r_{i,n} \neq b_{n+3} - \alpha_{n+2} - a_{n+2} - b_{n+2}$, the iterate x_{n+3} is not equal to a_{n+3} or b_{n+3} at $z = z_0 = 0$. Therefore, the pole at x_{n+2} is not counted twice in (4.14). Similarly, we get the same result about the pole at x_{n-2}.

Considering the second type of singularity we have the following relation

$$\begin{aligned}
&\sum_{z_0\in\mathbb{C}_\infty}\sum_{n=r_0}^{r}\left[\mathrm{ord}_{z_0}\left(\frac{1}{x_n-a_n}\right)+\mathrm{ord}_{z_0}\left(\frac{1}{x_n-b_n}\right)\right] = 2\sum_{n=r_0}^{r}\deg_z(x_n)\\
&= 2\sum_{z_0\in\mathbb{C}_\infty}\sum_{n=r_0}^{r}\mathrm{ord}_{z_0}\left(\frac{1}{x_n-r_{i,n}}\right)\\
&\leq 2\sum_{z_0\in\mathbb{C}_\infty}\sum_{\substack{n=r_0-2\\ \{n-1,n+1\}\cap X(z_0)=\emptyset}}^{r+2}\mathrm{ord}_{z_0}x_n, \qquad (4.20)
\end{aligned}$$

where the set $X(z_0)$ is defined in Theorem 4.2.1. Recall in Theorem 4.2.1 we showed considering the first type of singularity only that we get (4.16) with $\tau = 2$, consequently

$$\sum_{z_0\in\mathbb{C}_\infty}\sum_{n=r_0}^{r}\left[\mathrm{ord}_{z_0}\left(\frac{1}{x_n-a_n}\right)+\mathrm{ord}_{z_0}\left(\frac{1}{x_n-b_n}\right)\right] \leq 2\sum_{z_0\in\mathbb{C}_\infty}\sum_{\substack{n=r_0-1\\ \{n-1,n+1\}\cap X(z_0)\neq\emptyset}}^{r+1}\mathrm{ord}_{z_0}x_n. \qquad (4.21)$$

Adding (4.21) and (4.20) yields

$$4D_r(x_n) \leq 2\sum_{z_0\in\mathbb{C}_\infty}\sum_{n=r_0-2}^{r+2}\mathrm{ord}_{z_0}x_n = 2D_{r+2}(x_n) + 2\mathrm{ord}_{z_0}(x_{r_0-2}) + 2\mathrm{ord}_{z_0}(x_{r_0-1}).$$

Using the shift $r \to r-2$ then iterating this recurrence relation repeatedly and following the same reasoning as in the previous section (4.10)-(4.13) yields

$$D_r(x_n(z)) \geq \left(\frac{2}{1+\mu}\right)^{\lfloor r/2\rfloor} K,$$

where $0 < \mu \ll 1$ is an arbitrary constant and $K > 0$, since all $x_n(z)$ are non-constant rational functions.

Now the rest of the proof is dealing with cases and sub-cases such that for each $i = 1, \dots, 4$ either $P_{n+1}(-r_{i,n}) = 0$ or $P_{n-1}(-r_{i,n}) = 0$ for all n. This prevents having poles at either x_{n-2} or x_{n+2}. Note that here the assumption $r_{i,n} \not\equiv -a_{n\pm1}$ and $r_{i,n} \not\equiv -b_{n\pm1}$ for all $i \in \{1, \dots, 4\}$ and all n is necessary, otherwise we have a contradiction to our irreducibility condition.

Case 1: Either for all $i = 1, \dots, 4$, $P_{n+1}(-r_{i,n}) = 0$ for all n or for all $i = 1, \dots, 4$, $P_{n-1}(-r_{i,n}) = 0$, $\forall n$.

Without loss of generality, let $P_{n+1}(-r_{i,n}) = 0$ $\forall n$ and for all $i = 1, \dots, 4$. Let $Q_{n+1}(y) \equiv P_{n+1}(-y)$, $\forall n$. Since $P_n(r_{i,n}) \equiv 0$, $\forall i \in \{1, \dots, 4\}$ and $Q_{n+1}(r_{i,n}) \equiv P_{n+1}(-r_{i,n}) \equiv 0$, we have two quartic polynomials Q_{n+1} and P_n that have the same set of distinct roots and their leading coefficients equal 1. Therefore, these two polynomials are equal. Equating the corresponding coefficient for each power of the independent variable we get the following relations: $\alpha_n = -\alpha_{n+1}$, $\beta_n = \beta_{n+1}$, $\gamma_n = -\gamma_{n+1}$ and $\eta_n = \eta_{n+1}$. Since the coefficients are rational functions of n, we get $\alpha_n = 0$, $\beta_n = \beta$ (constant), $\gamma_n = 0$, and $\eta_n = \eta$ (constant) for all n. Hence,

$$P_n(x_n) = x_n^4 + \beta x_n^2 + \eta,$$

which is the polynomial given in case 1 in the statement of the theorem. The same conclusion is obtained if we start with $P_{n-1}(-r_{i,n}) = 0$, $\forall n$ and for all $i = 1, \dots, 4$.

Case 2: Either $P_{n+1}(-r_{1,n}) \equiv P_{n+1}(-r_{2,n}) \equiv P_{n+1}(-r_{3,n}) \equiv 0$ and $P_{n+1}(-r_{4,n}) \not\equiv 0$ for all n or $P_{n-1}(-r_{1,n}) \equiv P_{n-1}(-r_{2,n}) \equiv P_{n-1}(-r_{3,n}) \equiv 0$ and $P_{n-1}(-r_{4,n}) \not\equiv 0$ for all n.

Without loss of generality, let $P_{n+1}(-r_{1,n}) \equiv P_{n+1}(-r_{2,n}) \equiv P_{n+1}(-r_{3,n}) \equiv 0$ and $P_{n+1}(-r_{4,n}) \not\equiv 0$ for all n. This implies that $P_{n-1}(-r_{4,n}) \equiv 0$

Since P_{n+1} has 4 distinct roots $\{r_{1,n+1}, r_{2,n+1}, r_{3,n+1}, r_{4,n+1}\}$ by assumption, our case implies $\{-r_{1,n}, -r_{2,n}, -r_{3,n}\} \subset \{r_{1,n+1}, r_{2,n+1}, r_{3,n+1}, r_{4,n+1}\}$. Suppose that $r_{i,n+1} = -r_{j,n}$, $\forall n$ for some $i, j \in \{1, 2, 3\}$, then

$$0 = P_{n-1}(r_{j,n-1}) = P_{n-1}(-r_{i,n}).$$

So if $\{-r_{1,n}, -r_{2,n}, -r_{3,n}\} = \{r_{1,n+1}, r_{2,n+1}, r_{3,n+1}\}$, then $P_{n-1}(-r_{l,n}) \equiv 0$ for all $l = 1, \dots, 4$ which is case 1. Hence, $r_{4,n+1} = -r_{i,n}$ for some $i = 1, 2, 3$. By re-indexing if necessary $\{-r_{1,n}, -r_{2,n}, -r_{3,n}\}$, we choose $r_{4,n+1} = -r_{1,n}$, so we have $\{-r_{2,n}, -r_{3,n}\} \subset \{r_{1,n+1}, r_{2,n+1}, r_{3,n+1}\}$.

Sub-case 2.1: Let $\{-r_{2,n}, -r_{3,n}\} = \{r_{2,n+1}, r_{3,n+1}\}$. If we let $r_{2,n+1} = -r_{2,n}$ and $r_{3,n+1} = -r_{3,n}$, then it implies that $r_{2,n} \equiv r_{3,n} \equiv 0$ since $r_{i,n}$ for $i = 1, \dots, 4$ are rational functions of n. We have a contradiction since the roots are distinct. So $r_{2,n+1} = -r_{3,n}$ and $r_{3,n+1} = -r_{2,n}$. This yields $r_{2,n+1} = -(-r_{2,n-1}) = r_{2,n-1}$

which means that $r_{2,n}$ is a constant, r_2. Similarly, $r_{3,n} = -r_2$. Recall that we let $r_{4,n+1} = -r_{1,n}$, which means that $P_n(x_n) = (x_n - r_{1,n})(x_n - r_2)(x_n + r_2)(x_n + r_{1,n-1})$. Let $r_{1,n} = f_n$ for an arbitrary rational function of n and $r_2 = \alpha$, it implies

$$P_n(x_n) = (x_n - f_n)(x_n - \alpha)(x_n + \alpha)(x_n + f_{n-1}).$$

This polynomial is a special case of the form given in case 2 in the statement of the theorem with $g_n = \alpha$.

Sub-case 2.2: Let $\{-r_{2,n}, -r_{3,n}\} = \{r_{1,n+1}, r_{j,n+1}\}$ where $j = 2$ or 3. We could re-index 2 by 3 or vice versa if necessary. Without loss of generality, let $\{-r_{2,n}, -r_{3,n}\} = \{r_{1,n+1}, r_{2,n+1}\}$.

- If $r_{1,n+1} = -r_{2,n}$ and $r_{2,n+1} = -r_{3,n}$, recall that we have $r_{4,n+1} = -r_{1,n}$, then setting $r_{1,n} \equiv f_n$ for an arbitrary rational function in n we get $r_{2,n} = -f_{n+1}$, $r_{3,n} = f_{n+2}$ and $r_{4,n} = -f_{n-1}$. Since the roots of P_n are distinct then f_n is non-constant rational function of n. This implies

$$P_n(x_n) = (x_n - f_n)(x_n + f_{n+1})(x_n - f_{n+2})(x_n + f_{n-1}),$$

  which is the form given in case 3 in the statement of the theorem.

- If $r_{1,n+1} = -r_{3,n}$ and $r_{2,n+1} = -r_{2,n}$, this yields $r_{2,n} \equiv 0$. Let $r_{1,n} = f_n$, then we have $r_{3,n} = -f_{n+1}$ and $r_{4,n} = -f_{n-1}$. Note that f_n is a non-constant rational function of n. Hence,

$$P_n(x_n) = (x_n - f_n)x_n(x_n + f_{n+1})(x_n + f_{n-1}),$$

  which is the form given in case 4 in the statement of the theorem.

Case 3: Either $P_{n+1}(-r_{1,n}) \equiv P_{n+1}(-r_{2,n}) \equiv 0$ and $P_{n+1}(-r_{3,n}) \not\equiv 0$, $P_{n+1}(-r_{4,n}) \not\equiv 0$ or $P_{n-1}(-r_{1,n}) \equiv P_{n-1}(-r_{2,n}) \equiv 0$ and $P_{n-1}(-r_{3,n}) \not\equiv 0$, $P_{n-1}(-r_{4,n}) \not\equiv 0$.

Without loss of generality, let $P_{n+1}(-r_{1,n}) \equiv P_{n+1}(-r_{2,n}) \equiv 0$ and $P_{n+1}(-r_{3,n}) \not\equiv 0$, $P_{n+1}(-r_{4,n}) \not\equiv 0$. This implies $P_{n-1}(-r_{3,n}) \equiv P_{n-1}(-r_{4,n}) \equiv 0$. We have $\{-r_{1,n}, -r_{2,n}\} \subset \{r_{1,n+1}, r_{2,n+1}, r_{3,n+1}, r_{4,n+1}\}$. Suppose that $r_{i,n+1} = -r_{j,n}$ for some $i, j \in \{1, 2\}$, this means that $P_n(r_{j,n}) = P_n(-r_{i,n+1}) \neq 0$ which is a contradiction since $r_{j,n}$ is a root of P_n. So

$$\{-r_{1,n}, -r_{2,n}\} = \{r_{3,n+1}, r_{4,n+1}\}. \tag{4.22}$$

Similarly,

$$\{-r_{3,n}, -r_{4,n}\} = \{r_{1,n-1}, r_{2,n-1}\}, \tag{4.23}$$

if we start with $\{-r_{3,n}, -r_{4,n}\} \subset \{r_{1,n-1}, r_{2,n-1}, r_{3,n-1}, r_{4,n-1}\}$. Re-indexing if necessary in (4.22) we have

$$r_{3,n+1} = -r_{1,n} \text{ and } r_{4,n+1} = -r_{2,n}. \tag{4.24}$$

Sub-case 3.1 From (4.23) we set $r_{1,n-1} = -r_{3,n}$ and $r_{2,n-1} = -r_{4,n}$ and from (4.24) we have $r_{3,n+1} = -r_{1,n}$ and $r_{4,n+1} = -r_{2,n}$. Let $r_{1,n} = f_n$ and $r_{2,n} = g_n$ it implies that $r_{3,n} = -f_{n-1}$ and $r_{4,n} = -g_{n-1}$. Hence,

$$P_n(x_n) = (x_n - f_n)(x_n - g_n)(x_n + f_{n-1})(x_n + g_{n-1}),$$

which is the form given in case 2 in the statement of the theorem.

Sub-case 3.2 From (4.23) if we set $r_{1,n-1} = -r_{4,n}$, $r_{2,n-1} = -r_{3,n}$ and let $r_{1,n} = f_n$, then $r_{2,n} = -r_{3,n+1} = r_{1,n} = f_n$ where we used (4.24), $r_{3,n} = -f_{n-1}$ and $r_{4,n} = -f_{n-1}$. This is a contradiction since P_n has no repeated roots. The proof of the theorem is completed. □

From Theorem 4.2.1 if we have infinitely many of the sequences (a_n, ∞, b_{n+2}), then combining the expression of α_n from this theorem with the first form of $P_n(x_n)$ given in Theorem 4.2.2, we have $0 = \alpha_n = (b_{n+1} - a_n) - (b_n - a_{n-1})$. It yields

$$b_{n+1} = a_n + c_1, \tag{4.25}$$

where c_1 is a constant. Similarly, if we have infinitely many of the sequences (b_n, ∞, a_{n+2}), then we have $0 = \alpha_n = a_{n+1} - a_n - b_n + b_{n-1}$. It yields

$$a_{n+1} = b_n + c_2, \tag{4.26}$$

where c_2 is a constant. From (4.25) and (4.26) we have $a_n = (c_1 + c_2)n + c_3$ and $b_n = (c_1 + c_2)n + c_4$ where c_3, c_4 are constants. Let $c_1 + c_2 = \phi$, $c_3 = \omega$ and $c_4 = \nu$, then (4.7) is of the form $(x_{n+1} + x_n)(x_{n-1} + x_n) = \frac{x_n^4+\beta x_n^2+\eta}{(x_n-(\phi n+\omega))(x_n-(\phi n+\nu))} = \frac{(x_n^2-p^2)(x_n^2-q^2)}{(x_n-(\psi n+\xi))^2-u^2}$. Hence, $(x_{n+1} + x_n)(x_{n-1} + x_n) = \frac{x_n^4+\beta x_n^2+\eta}{(x_n-a_n)(x_n-b_n)}$ reduces to a discrete analogue of P_{IV} if this equation has infinitely many sequences of the forms (a_n, ∞, b_{n+2}) and (b_n, ∞, a_{n+2}) and all the assumptions in Theorem 4.2.1 and Theorem 4.2.2 are satisfied.

If (4.7) has infinitely many of the sequence (a_n, ∞, b_{n+2}) and none of the other sequence (b_n, ∞, a_{n+2}), then we expect that the solution x_n of (4.7) solves a difference Riccati equation of the form:

$$x_{n+1} = \frac{b_{n+1}x_n + f_n}{x_n - a_n},$$

where f_n is an arbitrary rational function of n. A similar argument follows if we exchange all a_n by b_n in the above argument. To prove this we need some extra tools and analysis concerning the nature of the singularities. We expect that it is somehow analogous to our treatment of the Riccati case in the previous chapter. We leave this part for future publications since it needs further investigations and study.

Chapter 5

Ultra-discrete equations

In this chapter, we consider ultra-discrete equations. We present some results that have some indications and suggestions towards a new integrability detector for ultra-discrete equations. These results are based on numerical simulations using Mathematica software. First, we need to understand the algebraic settings that these equations are linked to. Hence, in section 5.1, we lay the base of the max-plus semi-field. In section 5.2, we illustrate our numerical findings.

5.1 Max-plus semi-field

The real number system with the ordinary operations of addition and multiplication $(\mathbb{R}, +, \times)$ has a well known classical algebra. Any system of numbers with operations that has sufficiently many of the axiomatic properties of $(\mathbb{R}, +, \times)$ could be used as a starting point for analogue theories of the classical algebra over the new system. Some of these systems appeared in the middle of the last century, e.g. the max-plus algebra (appeared first in Kleene's paper on nerve sets and automata [49]), the min-plus algebra and the minimax algebra [16]. There are lots of fields (e.g. computer science, computer languages, finite automata, optimisation problems on graph, stochastic systems....etc.) that used successfully the settings of these algebras to recast their problems in a simpler manner. In [17], there are illustrations for a number of applications in different fields where the experts in them recast their problems using these new algebras and solved them.

In this section, we give a formal definition of the max-plus semi-field over $\mathbb{R}_{max} = \mathbb{R} \cup \{-\infty\}$ with the addition and multiplication operations. The addition and multiplication operations are defined as follows:

$$\begin{aligned} a \oplus b &= \max(a, b), \\ a \otimes b &= a + b, \end{aligned} \tag{5.1}$$

where a and $b \in \mathbb{R}$. By definition

$$\begin{aligned} a \oplus -\infty &= a \qquad \forall a \in \mathbb{R}_{max}, \\ a \otimes -\infty &= -\infty \qquad \forall a \in \mathbb{R}_{max}. \end{aligned} \tag{5.2}$$

From (5.2), it is clear that the additive identity is $-\infty$. The multiplicative identity is 0 and the multiplicative inverse is $-a$ for any $a \in \mathbb{R}_{max}$ except for $-\infty$. There is no additive inverse for any element $a \in \mathbb{R}$. There are some suggested solutions for this problem of lack of inverses discussed in [62] and in the references of [56]. In this semi-field, all the usual commutative, associative and distributive axioms hold [56]:

$$\begin{aligned} a \oplus b &= b \oplus a, \\ a \oplus (b \oplus c) &= (a \oplus b) \oplus c, \\ a \otimes b &= b \otimes a, \\ a \otimes (b \otimes c) &= (a \otimes b) \otimes c, \\ a \otimes (b \oplus c) &= (a \otimes b) \oplus (a \otimes c), \end{aligned} \tag{5.3}$$

where a, b and $c \in \mathbb{R}_{max}$. There is also one property that holds for the operation $\oplus$ called the *idempotent* law:

$$a \oplus a = a, \tag{5.4}$$

where $a \in \mathbb{R}_{max}$. This property opens the door for a very rich algebra and analysis called the idempotent algebra and idempotent analysis. Idempotent analysis was established by Maslov and his collaborators in the 1980s, more details of which can be found in [52] and [53]. Sometimes, the max-plus semi-field is called idempotent semi-field because of this property (5.4). Another term used for max-plus semi-field is *tropical semi-field.* According to [52], this term was first introduced in computer science to represent the discrete version of max-plus algebra $\mathbb{R}_{max}$ or min-plus algebra $\mathbb{R}_{min}$ (i.e. $a \oplus b = \min(a, b)$, $a \otimes b = a + b$, $\forall a, b \in \mathbb{R} \cup \{\infty\}$) and their sub-algebras.

We denote the $r-$fold (product of an element a with itself r times) by a power:

$$a^{(r)} = a \otimes a \otimes \cdots \otimes a \quad (r \text{ times}). \tag{5.5}$$

To distinguish the regular exponent produced from the ordinary multiplication operation and the exponent that is a result of the operation $\otimes$ we used brackets around the exponent in (5.5). From (5.1), it is clear that $a^{(r)} = ra$, $\forall a \in \mathbb{R}_{max}$ when r is a positive integer. Let us define the zero and negative exponents by

$$\begin{aligned} a^{(0)} &= 0, \\ a^{(-r)} &= -ra \quad (r > 0). \end{aligned} \tag{5.6}$$

Another property could be added here [18]:

$$(a \oplus b)^{(r)} = a^{(r)} \oplus b^{(r)} \quad (r \geq 0), \tag{5.7}$$

for which the proof is straightforward, from (5.1) and (5.5) we have

$$(a \oplus b)^{(r)} = r\max(a, b) = \max(ra, rb) = a^{(r)} \oplus b^{(r)} \quad (r \geq 0).$$

Now we are ready to introduce a new notation and concept for a quotient in our max-plus algebra. We denote the quotient by $\oslash$ to distinguish it from the usual quotient in the classic algebra. Now if we have two expressions in the max-plus algebra U and V, then we define the quotient by [17]

$$U \oslash V = U \otimes V^{(-1)} = U - V. \tag{5.8}$$

To make this definition clearer, let us take an example

$$(4 \oplus (7 \otimes x^{(-1)})) \oslash ((3 \otimes x) \oplus x) = \max(4, 7 - x) - \max(3 + x, x), \tag{5.9}$$

where we have used the definitions in (5.8) and (5.6). In addition, we could define the operator min in the max-plus semi-field as follows:

$$\min(a, b) = a + b - \max(a, b) = (a \otimes b) \oslash (a \oplus b). \tag{5.10}$$

The max-plus semi-field can be introduced from a different perspective similar to quantum theory. The parameter h below plays a role similar to Planck's constant in quantum theory. Consider the semi-ring $(\mathbb{R}_+, +, \times)$, where $\mathbb{R}_+$ is the set of all non-negative real numbers and $+$ and $\times$ are the classic addition and multiplication operations. Define a map $\Phi_h : \mathbb{R}_+ \to \mathbb{R} \cup \{-\infty\}$ by

$$\Phi_h(x) = h\log(x) \quad (h > 0). \tag{5.11}$$

Now let the classic addition and multiplication be mapped from $\mathbb{R}_+$ to $\mathbb{R} \cup \{-\infty\}$ by Φ_h. Hence, for $a = \Phi_h(x) = h\log(x)$ and $b = \Phi_h(y) = h\log(y)$, where $x = e^{a/h}$, $y = e^{b/h}$, we have

$$\Phi_h(x+y) = h\log(x+y) = h\log(\exp(a/h) + \exp(b/h)) = a \oplus_h b,$$
$$\Phi_h(xy) = h\log(xy) = h\log(x) + h\log(y) = \Phi_h(x) + \Phi_h(y) = a + b = a \otimes b.$$

Also, note that from (5.11) the additive identity 0 mapped to $-\infty$ and the multiplicative identity 1 mapped to 0 . Note that

$$a \oplus b = \lim_{h\to 0^+} a \oplus_h b = \lim_{h\to 0^+} (h\log(\exp(a/h) + \exp(b/h))) = \max(a,b). \tag{5.12}$$

Therefore, $\mathbb{R} \cup \{-\infty\}$ forms a max-plus semi-field with respect to the operations $a \oplus b = \max(a,b)$ and $a \otimes b = a + b$ with identity $-\infty$ for the operation $\oplus$ and 0 as the identity for the operation $\otimes$. All the axioms and the properties discussed above are valid here. By analogy with quantum theory, $\mathbb{R}_+$ could be viewed as a quantum object and $\mathbb{R}_{max}$ as a result of its dequantisation [53]. The process in identity (5.12) is known as dequantisation or ultra-discretisation.

Recently, strong links between integrable cellular automata and tropical geometry were found. In the next section, we discuss the ultra-discrete equations and highlight some of the known integrability detectors. We also present some numerical results that might give some suggestion or indication towards a new integrability detector for ultra-discrete equations.

5.2 Ultra-discrete equations

The last section ended by introducing the ultra-discretisation identity

$$\lim_{\epsilon\to 0^+} \epsilon\log(e^{A/\epsilon} + e^{B/\epsilon}) = \max(A,B), \tag{5.13}$$

where $A, B \in \mathbb{R}$ and $\epsilon > 0$. Applying this identity to discrete equations introduces a new type called ultra-discrete equations. Naturally, these equations are expressed in terms of max-plus semi-field expressions. An integrable ultra-discrete equation was given by Takahashi and Satsuma in 1990 [78]. Their equation is related to soliton cellular automata which is known as box and ball system. Integrable ultra-discrete equations usually arise from the ultra-discretisation of known integrable discrete equations, according to the authors of [29].

Recently, mathematicians are interested in exploring the integrability of ultra-discrete equations and introducing detectors for it. Joshi and Lafortune proposed a detector of integrability based on singularity confinement in [46]. This method was investigated in depth in [29] and the authors found a non-integrable system that passed this test. Basically, our work here is an attempt to extend the Ablowitz *et al.* [1] idea to the ultra-discrete equations' settings. The work in this chapter was motivated by preliminary results in [37]. The preliminary results in [37] suggest that the existence of finite order max-plus meromorphic solutions could be used as a good detector of the integrability of ultra-discrete equations. The numerical results that we have support this suggestion. The tools we used here are ultra-discrete versions of the Nevanlinna theory functions. These functions are derived in [37].

In [37], Halburd and Southall developed the tropical Nevanlinna theory where the role of meromorphic functions is played by piecewise linear functions. These functions are of real variable with one-sided derivatives which are integers at every point. A tropical version of the Nevanlinna functions (m, n, N and T functions defined in Appendix B) had been derived. In addition, the authors of [37] gave an interpretation of these functions under the new setting, i.e. the max-plus semi-field. We present here tools from tropical Nevanlinna theory given in [37]. Then we present our numerical results.

First, let us start by giving a definition of a max-plus rational function

$$R(x) = \{a_0 \oplus (a_1 \otimes x) \oplus \cdots \oplus (a_p \otimes x^{(p)})\} \oslash \{b_0 \oplus (b_1 \otimes x) \oplus \cdots \oplus (b_q \otimes x^{(q)})\}, \quad (5.14)$$

where $x \oslash y = x - y$, $x^{(n)} = nx$ and p and q are non-negative integers. Geometrically, any max-plus rational function is a continuous piecewise linear function with a finite number of distinct linear segments, each with integer slope, as shown in Figure 5.1.

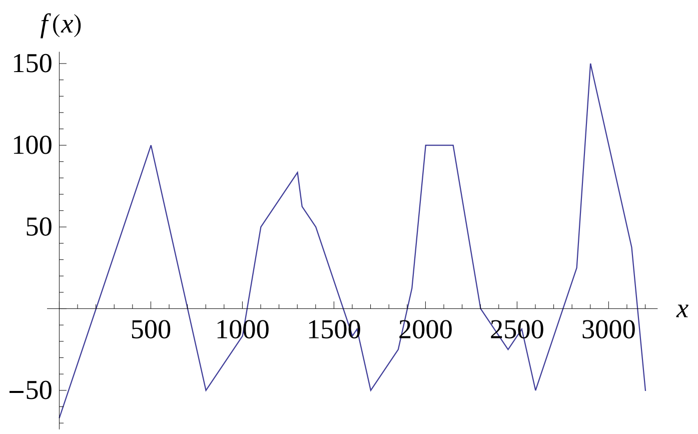


Figure 5.1: Continuous piecewise linear function $f(x)$

Definition 5.2.1. *A continuous piecewise linear function $f : \mathbb{R} \to \mathbb{R}$ is said to be max-plus meromorphic on $\mathbb{R}$ if both one-sided derivatives are integers at each point $x \in \mathbb{R}$.*

Definition 5.2.2. *For any $x \in \mathbb{R}$ the term $\omega_f(x) = \lim_{\epsilon \to 0^+}\{f'(x+\epsilon) - f'(x-\epsilon)\}$ represents the change of slope at the point x in the graph of the f function.*

Now if $\omega_f(x) > 0$, then x is called a root of f with multiplicity $\omega_f(x)$. If $\omega_f(x) < 0$, then x is called a pole of f with multiplicity $-\omega_f(x)$. In general, if f is a max-plus meromorphic function on $\mathbb{R}$ and $f'(x) = m \; \forall x < x_0$, for a constant $m \in \mathbb{Z}$ and $x_0 \in \mathbb{R}$, then f is called a max-plus meromorphic function on $\mathbb{R} \cup \{-\infty\}$. The point $-\infty$ is called a root of multiplicity m if $m > 0$, a pole of multiplicity $-m$ if $m < 0$ and an ordinary point if $m \geq 0$.

Let $x^+ = \max(x, 0)$ for any $x \in \mathbb{R} \cup \{-\infty\}$. For any function $f : \mathbb{R} \to \mathbb{R}$, define f^+ function by $f^+(x) = \max(f(x), 0)$. Now we define the max-plus proximity function

$$m(f, r) = \frac{f^+(r) + f^+(-r)}{2}. \tag{5.15}$$

The max-plus proximity function $m(f, r)$ is an average of f^+ function at the end points of the interval $(-r, r)$ which is similar to the definition of the classic Nevanlinna mean proximity function (discussed in Appendix B). The max-plus counting function, $n(f, r)$ counts the number of poles of f in the interval $(-r, r)$ counting multiplicities. The integrated max-plus counting function is defined by

$$N(f, r) = \frac{1}{2}\int_0^r n(f, t)dt = \frac{1}{2}\sum_{\nu=1}^{K}(r - |b_\nu|), \tag{5.16}$$

where b_νs represent the poles of the function f in the interval $(-r, r)$ and K is the total number of the poles counting multiplicities. The max-plus characteristic function is defined by

$$T(f, r) = m(f, r) + N(f, r). \tag{5.17}$$

Definition 5.2.3. *A max-plus meromorphic function is of finite order if there exist positive numbers σ and r_0 such that $T(f, r) \leq r^\sigma$, for all $r > r_0$.*

We could extend the definition of the max-plus Nevanlinna characteristic to any arbitrary continuous piecewise linear function (not just with integer slopes) by allowing the counting function $n(f, r)$ to count poles of non-integer multiplicities (i.e. the difference of slopes). The class of equations

$$y(x+1) \otimes y(x-1) = R(x, y(x)), \tag{5.18}$$

where R is max-plus rational in x, y and $x \in \mathbb{R}$, was discussed in [37] and was shown to admit infinitely many max-plus meromorphic solutions. It was also shown that a large class of equations (5.18) admits infinite order solutions. Joshi and Lafortune considered in their paper [46] an equation of this class,

$$y_{n+1} + 3y_n + y_{n-1} = \max(y_n + K, 0), \tag{5.19}$$

where K is a constant and n, y are integers. This equation was used as an example for not possessing their ultra-discrete analogue of singularity confinement. In [37], Halburd and Southall explored the extended version of equation (5.19) where the independent variable is a real number. They proved the following lemma:

Lemma 5.2.1. *Let K be a positive constant and let y be a max-plus meromorphic solution of*

$$y(x + 1) + 3y(x) + y(x - 1) = \max(y(x) + K, 0), \tag{5.20}$$

such that $y(0) > 0$ and $y(1) < -K$. Then y has infinite order.

The proof of this lemma is straightforward using induction and is given in [37], hence we omit it. Another equation has been explored by Joshi and Lafortune in their paper [46]:

$$y_{n-1} + y_n + y_{n+1} = \max(y_n + \phi_n, 0), \tag{5.21}$$

where n, $y \in \mathbb{Z}$. They showed that for this equation to pass the singularity confinement, then ϕ_n should satisfy the following condition:

$$\phi_{n+5} - \phi_{n+3} - \phi_{n+2} + \phi_n = 0.$$

Hence,

$$\phi_n = \alpha + \beta n + \gamma(-1)^n + \delta \cos\left(\frac{2\pi n}{3}\right) + \omega \sin\left(\frac{2\pi n}{3}\right),$$

where $\alpha, \beta, \gamma, \delta$ and ω are all arbitrary constants. In [37], the authors considered the extended version of equation (5.21)

$$y(x + 1) + y(x) + y(x - 1) = \max(y(x) + \phi(x), 0). \tag{5.22}$$

They found that the confinement condition is

$$\phi(x) = \pi_2(x) + \pi_3(x) + Nx + C,$$

where π_2, π_3 are arbitrary periodic max-plus meromorphic functions of periods 2 and 3, respectively, $N \in \mathbb{Z}$ and $C \in \mathbb{R}$. The authors commented on equation (5.22) and made some observations about the order of its solution depending on the type of the ϕ function. Here, we explore equation (5.22) further numerically using Mathematica software and we give the result of our findings. We found that if ϕ is only a periodic function of periods 2 or 3 (or their sum), then the order of the solution $y(x)$ tends to be finite. If we choose ϕ to be a periodic function of period higher than 3, like $4, 5$ or 7, then the counting function $n(y, r)$ shows a sign of a fast growth but eventually it grows slower, which indicates the finite order of the solution. If we choose ϕ to have the form $x + \psi(x)$, where ψ is a periodic function of any period, then $y(x)$ tends to be of finite order. Nevertheless, if x is sufficiently large, then we notice that the $\max(y(x)+\phi(x), 0)$ term is switched off and the solution of equation (5.22) becomes identical to the solution of a simpler ultra-discrete equation

$$y(x+1) + y(x-1) = \phi(x),$$

given that $\phi(x) = x + \psi(x)$. In all the numerical simulations done here, we divided every unit of real number in the real axis into $30,000$ distinct points and we have done that for intervals of different lengths r to get a total of over $1,000,000$ distinct points. We gave initial conditions of various piecewise linear functions to the ultra-discrete equations in hand to get their solutions which are continuous piecewise linear functions. If we calculate the max-plus Nevanlinna characteristic function for the solutions of these functions, then we get the following results.

For equation (5.22) with ϕ is purely a periodic function of period 3 (π_3), we get a slow growth of the T function which therefore suggests that the solution $y(x)$ is of finite order.

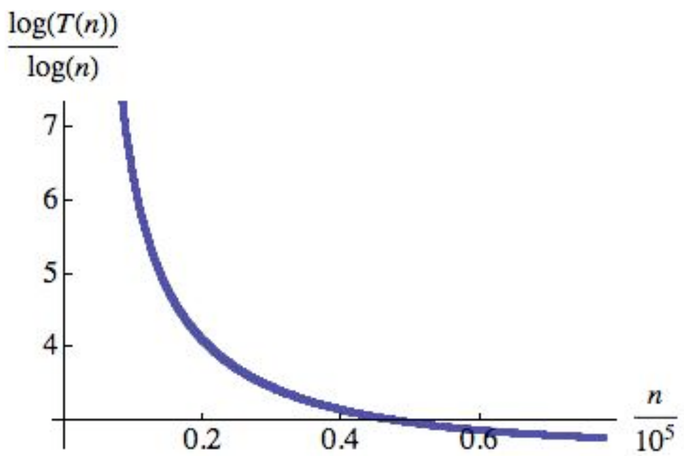


Figure 5.2: Plot of $\frac{\log(T(y(x),r))}{\log r}$ of equation (5.22) with $\phi = \pi_3$

We get a similar result if ϕ is purely a periodic function of period 5 (π_5).

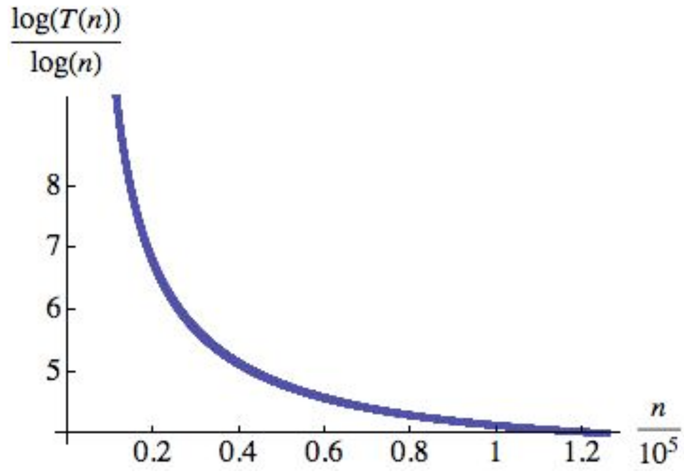


Figure 5.3: Plot of $\frac{\log(T(y(x),r))}{\log r}$ of equation (5.22) with $\phi = \pi_5$

If we choose $\phi = \pi_5 + x$ or any other periodic term of different period, then we get the T function of the solution of equation (5.22) is of a slow growth, which suggests that $y(x)$ is of finite order.

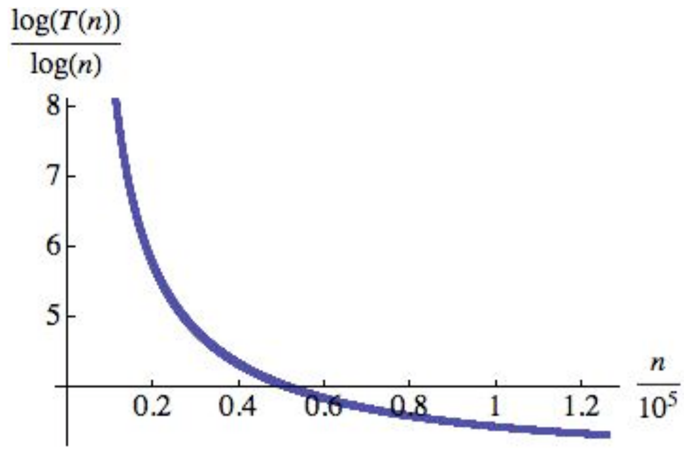


Figure 5.4: Plot of $\frac{\log(T(y(x),r))}{\log r}$ of equation (5.22) with $\phi = \pi_5 + x$

We applied this method for other ultra-discrete equations, in particular an ultra-discrete analogue of the first Painlevé equation discussed in [29].

$$y_{n+1} + y_{n-1} = A - 2y_n + \max(y_n, 0).$$

This equation is integrable and when we applied our method we found that the T function of the solution is growing slowly, which implies that the solution is of finite order.

We applied this method to an ultra-discrete equation that is derived from non-integrable discrete equation as shown in [80]. This equation is

$$y_{n+1} = y_{n-1} + |y_n|. \tag{5.23}$$

The numerical results are shown in Figure 5.5.

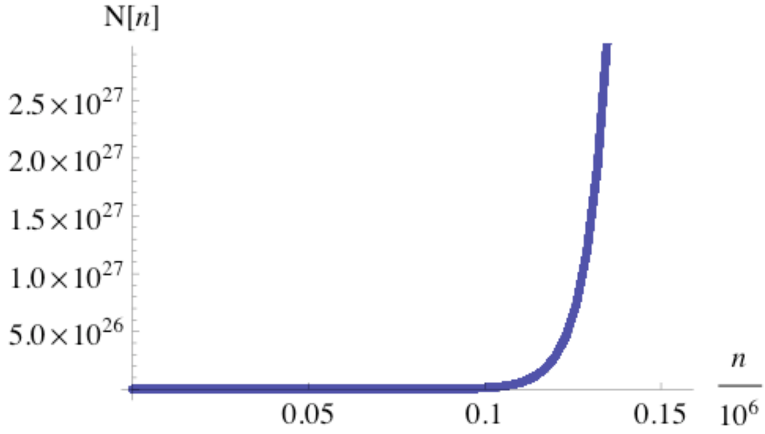


Figure 5.5: Plot of $N(y, r)$ of equation (5.23).

Since the function $N(y, r)$ is growing exponentially, $T(y, r)$ is growing exponentially, hence the solution of (5.23) is of infinite order. We applied our method for (5.20). Recall that in Lemma 5.2.1, the solution of this equation is shown to have infinite order. We get a similar result to Figure 5.5.

In this chapter, we presented numerical results for a proposed integrability detector of ultra-discrete equations. The work here is motivated by encouraging preliminary results given in [37] and our numerical simulations support these results and their indications. Our method simply is to investigate the growth of the max-plus Nevanlinna characteristic function T of a solution of different ultra-discrete equations. Consequently, our numerical results suggest that if the equation is believed to be integrable, then its solution is of a finite order in the max-plus Nevanlinna theory sense.

Chapter 6

Summary and future work

The framework that we adopt in this thesis is based on investigating the growth of certain characteristics of certain discrete equations which enables us to draw some conclusions about their integrability. In Chapter 3, the characteristic that we analyse is the logarithmic height of the equation solution $h(y_n)$, while in Chapter 4 we analyse the degree $\deg_z(x_n)$ of the equation solution in terms of an external variable z. The first discrete equation we studied in Chapter 3 is

$$y_{n+1} + y_{n-1} = \frac{a_n + b_n y_n + c_n y_n^2}{1 - y_n^2}.$$

We investigated the growth of the logarithmic height of an admissible solution y_n. The main goal of this chapter is to prove Theorem 1.4.1. We showed, using a rigorous analysis in Theorem 3.1.1. (provided that all the assumptions in the theorem are satisfied), that $h_r(y_n)$ is growing exponentially with r. If $c_n = 0\ \forall n$ and all the assumptions in Theorem 1.4.2 and Corollary 3.2.1 are satisfied, then either $h_r(y_n)$ grows very fast with r. Or the coefficients of the equation satisfy the following relation: $a_n - \theta b_n - \theta(\theta a_{n-2} + b_{n-2} - 2b_{n-1}) = 0$, $\forall n$ where $\theta = -1$ or 1, which reduces the above equation to a discrete analogue of P_{II}: $y_{n+1} + y_{n-1} = \frac{(\alpha n+\beta)y_n+\lambda}{1-y_n^2}$. Or the admissible solution y_n solves a difference Riccati equation of the form $y_{n+1} = \frac{\frac{1}{2\theta}(\theta a_n + b_n - 2) + y_n}{1-\theta y_n}$ as well. If $c_n = 2\theta\ \forall n$ (where $\theta = -1$ or 1) and the assumptions of Theorem 3.3.1 and Corollary 3.3.1 are satisfied, then $h_r(y_n)$ grows exponentially with r. The result of this chapter provides further evidence that the height growth of the solution of a discrete equation is a good indicator for the integrability of the equation.

In Chapter 4, we studied the following difference equation,

$$(x_{n-1} + x_n)(x_n + x_{n+1}) = \frac{P_n(x_n)}{(x_n - a_n)(x_n - b_n)} = \frac{x_n^4 + \alpha_n x_n^3 + \beta_n x_n^2 + \gamma_n x_n + \eta_n}{(x_n - a_n)(x_n - b_n)}.$$

We used the same framework with the characteristic $\deg_z(x_n)$, where we explored the growth of $D_r(x_n)$ of a non-constant rational function solution $x_n(z)$ in the above equation. We showed in Theorem 4.2.1, given that all the assumptions are satisfied that if $\alpha_n \not\equiv \mu_{-1} + \mu_1 - a_n - b_n$ (where $\mu_i \in \{a_{n+i}, b_{n+i}\}$) for all $n \in \mathbb{Z}$, then $D_r(x_n) \geq K2^r$ for some $K > 0$. This implies that $\deg_z(x_n)$ is growing fast, which suggests that this equation is not integrable with these assumptions on α_n. If all the assumptions in Theorem 4.2.2 are satisfied, then either $D_r(x_n)$ grows exponentially with r or $P_n(x_n)$ is one of the four special forms stated in the theorem . If the equation $(x_{n-1} + x_n)(x_n + x_{n+1}) = \frac{x_n^4 + \beta x_n^2 + \eta}{(x_n - a_n)(x_n - b_n)}$ has infinitely many of the sequences (a_n, ∞, b_{n+2}) and (b_n, ∞, a_{n+2}), then the equation reduces to a discrete analogue of P_{IV}.

In Chapter 5, we continued using our framework but this time we investigated the growth of the max-plus Nevanlinna characteristic function T of solutions of different ultra-discrete equations numerically. The numerical results suggest that if the equation is integrable, then its solution is of a finite order in the max-plus Nevanlinna theory sense. These results are encouraging for a necessary condition for the integrability of ultra-discrete equations. A future work in this area will be aiming at further study of this criterion of ultra-discrete equations numerically and analytically. We hope that we could construct an integrability detector for ultra-discrete equations, where this area of research is very active recently. Another direction for our future work is to extend the idea of Diophantine integrability to include discrete equations with coefficients in number fields, not just the field of rational numbers.

Appendix A

Overview of Nevanlinna theory

Nevanlinna theory is a branch of complex analysis, the basics of which arose in the late 1920s from Nevanlinna [58]. A lot of development has been added since then to make it what we know now as Nevanlinna theory. The backbone of the theory is its two main theorems. Generally, Nevanlinna theory in one variable describes the value distribution theory of a non-constant meromorphic function $f : \mathbb{C} \to \mathbb{P}^1$. The main aim here is to give an introductory overview of the principles of this theory.[1]

To present Nevanlinna theory in a simpler way, we have chosen to discuss its link with the Fundamental theorem of algebra. Understanding the basis of this theory relies on understanding the connection between it and the Fundamental theorem of algebra. The latter states that a non-constant polynomial of degree d in one complex variable takes on every complex value exactly d times, provided that the values are counted with their proper multiplicities. Picard generalised this theorem by proving that a transcendental entire function (i.e. a sort of polynomial of infinite degree [13]) must take on all but at most one complex value infinitely many times. The main question is what type of infinite degree he meant. There are many infinities and after Picard's work mathematicians tried to distinguish between different infinite degrees. The development of this topic relies on viewing the degree of complex polynomial $P(z)$ as the rate at which the maximum modulus of $P(z) \to \infty$ as $|z| \to \infty$. Then Hadamard proved that there was a connection between the growth order of an entire function and the distribution of the function's zeros. Later, Borel proved the connection between the growth rate of the maximum modulus of an entire function and the asymptotic frequency with which it must attain all but at most one complex value. Finally, R. Nevanlinna discovered the right way to express and measure the growth of meromorphic functions and developed the base of Nevanlinna

[1] Almost all the historic events and facts mentioned in the development of Nevanlinna theory are closely following the treatment in [13], where the interested reader could find all the original references cited there.

theory (First and Second main theorems).

A.1 Nevanlinna Functions

Nevanlinna theory could simply be viewed as a generalisation for the Fundamental theorem of algebra to meromorphic and holomorphic functions in $\mathbb{C}$. To achieve this, we need to develop the tools used by this theory, i.e. Nevanlinna functions. The four fundamental functions are the counting function $n(f, a, r)$, the integrated counting function $N(f, a, r)$, the mean proximity function $m(f, a, r)$ and the Nevanlinna characteristic function (or Nevanlinna height) $T(f, a, r)$, where f is a meromorphic function and $a \in \mathbb{C}$ is in a disc of radius r.

The function $n(f, a, r)$ counts the number of times f takes on the value a (counting multiplicities) in the closed disc of radius r, $D(r)$. Also, $n(f, \infty, r)$ counts the number of poles of f in the disc $D(r)$ counted with their multiplicities. Note that

$$n(f, a, r) = n\left(\frac{1}{f-a}, \infty, r\right).$$

We define the integrated counting function by

$$N(f, a, r) = n(f, a, 0) \log r + \int_0^r [n(f, a, t) - n(f, a, 0)] \frac{dt}{t}. \tag{A.1}$$

Hence, it counts as a logarithmic average the number of times f takes on the value a in the closed disc of radius r, $D(r)$. The mean proximity function $m(f, a, r)$ measures how often f is close on average to a but not equal to it on the circle of radius r. We define $m(f, a, r)$ by

$$\begin{aligned} m(f, a, r) &= \int_0^{2\pi} \log^+ \left| \frac{1}{f(re^{i\theta}) - a} \right| \frac{d\theta}{2\pi}, \qquad a \neq \infty \\ m(f, \infty, r) &= \int_0^{2\pi} \log^+ \mid f(re^{i\theta}) \mid \frac{d\theta}{2\pi}, \end{aligned} \tag{A.2}$$

where for any positive number x, $\log^+ x = \max(0, \log x)$. The Nevanlinna characteristic function is defined as the sum of the integrated counting function and the mean proximity function

$$T(f, a, r) = m(f, a, r) + N(f, a, r). \tag{A.3}$$

The function $T(f, a, r)$ plays in Nevanlinna theory the same role that the degree of polynomials plays in the Fundamental theorem of algebra. Another way to think of this function is as a measure of the area on the Riemann sphere covered by the image of the disc of radius r under the mapping f. Now we state the First and the Second Main theorems, the proofs of which are given in most Nevanlinna theory books such as [13], [51].

First Main Theorem. *If f is a non-constant meromorphic function on $\mathbb{C}$ and a is a point in $\mathbb{C} \cup \{\infty\}$, then*

$$T(f,r) - m(f,a,r) - N(f,a,r) = O(1),$$

as $r \to \infty$.

This theorem shows that the function T is independent of the choice of the value a, except for a bounded term independent of r. This theorem gives an upper bound on the number of times that the function f takes on the value a by $T(f,r)$. Note that this is similar to the fact that a non-constant polynomial of degree d takes on every value a at most d times. Moreover, d does not depend on which value a the function f takes. Similarly, T is independent of the choice of the value a.

Second Main Theorem. *If f is a non-constant meromorphic function on $\mathbb{C}$ and $a_1, \ldots, a_q$ are distinct points in $\mathbb{C} \cup \{\infty\}$, then*

$$(q-2)T(f,r) - \sum_{j=1}^{q} N(f,a_j,r) + N_{ram}(f,r) \leq o(T(f,r)),$$

for a sequence of $r \to \infty$.

The term $N_{\mathrm{ram}}(f,r)$ is positive (at least for $r \geq 1$) and measures how often the function f is ramified. This theorem provides a lower bound on the sum of any finite collection of integrated counting functions $N(f,a_j,r)$ for certain arbitrary large radii r. If we combine the results from both theorems then this will serve as a generalisation of the Fundamental theorem of algebra. Also, T gives for most values as upper and lower bounds on the number of times the function f takes on the value a, where the Second Main Theorem provides a precise limit on how much the lower bound can fail for all as taken together. This is an analogue to Picard's generalisation of the Fundamental theorem of algebra.

For any meromorphic function f in the whole plane (i.e. $0 < r \leq \infty$), we define its order σ by [57]

$$\sigma = \limsup_{r\to\infty} \frac{\log T(f,r)}{\log r}, \tag{A.4}$$

and if $0 < \sigma < \infty$, then it is of type ν, where

$$\nu = \limsup_{r\to\infty} \frac{T(f,r)}{r^{\sigma}}. \tag{A.5}$$

Theorem A.1.1. *A meromorphic function f is rational if and only if $T(f,r) = O(\log r)$.*

The proof is given in most Nevanlinna theory books, in particular in [51]. Theorem A.1.1 implies that the order of a rational function is 0 using (A.4).

Consider $f(z) = e^z$, since the exponential function is an entire function, then it has no poles, hence e^z never takes the value ∞. Therefore, $n(e^z, \infty, r) = 0$ and consequently, $N(e^z, \infty, r) = 0$, while

$$m(e^z, \infty, r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log^+ |\exp(re^{i\theta})| d\theta = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} r \cos\theta d\theta = \frac{r}{\pi}. \quad \text{(A.6)}$$

Thus, $T(e^z, r) = \frac{r}{\pi}$. As a consequence, the order of e^z is 1. We call the rational function and the exponential function functions of finite order. If a function f has an order $\sigma = \infty$, then we say f has an infinite order.

Appendix B

Differential and difference Riccati equations

An ordinary differential equation of the following form:

$$w'(t) = a(t)w^2(t) + b(t)w(t) + c(t), \tag{B.1}$$

where $a(t) \neq 0$, is known as the *Riccati equation.* It took its name from Jacopo Francesco, Count Riccati (1676-1754), who considered a class of equations of the form

$$w'(t) + t^{-n}w^2(t) - nt^{m+n-1} = 0,$$

where m and n are constants. The history of this equation can be found in [73] and its references. In [73], historically, James Bernoulli (1654-1705) expressed the solution of the equation

$$w'(t) = t^2 + w^2(t), \tag{B.2}$$

as a quotient of 2 infinite series. It is a procedure for relating the solution of equation (B.2) into another equation with a new variable $u \not\equiv 0$ such that the relation between the two variables is $w = -\frac{u'}{u}$ [11]. In general, in the eighteenth century, the Riccati equation had a lot of attention and much of mathematicians' works were concerned with finding a solution of the Riccati equation in a finite form or expressing it in terms of specified types of functional transforms. Many mathematicians contributed at that time to the study of the Riccati equation like James, John and Daniel Bernoulli, Leonhard Euler, Jean-le-Rond d'Alembert and Adrian Marie Legendre, according to [73].

The Riccati equation got a lot of attention through its history for two main reasons. The first is that it appeared in many disciplines of mathematics such as calculus of variations, optimisation theory, dynamic programming and mathematical physics. The other reason is its special properties distinguishing it from all other differential equations in the following class of equations:

$$w' = F(z, w), \tag{B.3}$$

where F is non-linear in $w(z)$ and $z \in \mathbb{C}_\infty$. The main two properties that concern us here are [81]

1. It is linearisable in terms of a new variable $u \not\equiv 0$ by introducing the change of variable $w = -\frac{u'(t)}{a(t).u(t)}$ in (B.1), (i.e., $u''(t) - \left(b(t) + \frac{a'(t)}{a(t)}\right) u'(t) + a(t)c(t)u(t) = 0$).
2. It has the Painlevé property, i.e., the only movable singularities of its solution are poles.

The Riccati equation is the only linearisable equation in the (B.3) class of equations. In difference equations, there is a difference Riccati equation with properties analogous to the differential Riccati equation. A class of difference equations

$$x_{n+1} = R(n, x_n), \tag{B.4}$$

where $R(n, x_n)$ is a rational function in x_n and all its coefficients are expressed freely in n, includes the difference Riccati equation. The difference Riccati equation is

$$x_{n+1} = \frac{A_n x_n + B_n}{C_n x_n + D_n}, \tag{B.5}$$

where A_n, B_n, C_n and D_n are arbitrary functions in n and $C_n \neq 0$ $\forall n \in \mathbb{Z}$. The difference Riccati equation properties are

1. It is linearisable in terms of a new variable $u_n \not\equiv 0$ by introducing the change of variable $x_n = g_n \frac{u_n - u_{n-1}}{u_n}$, where $g_n = \frac{A_{n-1}}{C_{n-1}}$, (i.e., $(g_{n+1}D_n - B_n)u_{n+1} - g_{n+1}(g_n C_n + D_n)u_n + (g_{n+1}g_n C_n)u_{n-1} = 0$).
2. It passes the known discrete integrability detector tests, proposed as a discrete analogue to the Painlevé property in differential equations, such as singularity confinement [28], algebraic entropy [71], Diophantine integrability [34] and Nevanlinna theory approach [84].

Here we include the linear function $x_{n+1} = \frac{A_n}{D_n} x_n + \frac{B_n}{D_n}$ (i.e. $C_n = 0$ and $D_n \neq 0$ in (B.5)) in our classification of the difference Riccati type equation. Note that the linear equation has the same properties stated above, except that now we do not need to associate it with any linear system of higher order since it is already linear. The difference Riccati equation is believed to be the only integrable equation in the class of difference equations (B.4) as shown in [84] .

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