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Teaching mathematics to address fundamental human rights

Barbara Jaworski
F. C. Donders Chair

INTRODUCTION

Mathematics is one of the most important subjects in the curriculum, central to so many areas of life and academic disciplines. Yet students – and people widely – struggle with mathematics possibly more than with any other subject. It is the right of every human being to know and understand mathematics relative to the context and purpose for which it is needed. These statements have profound implications for the teaching of mathematics.

In this short monograph (a written version of my inaugural lecture for the Donders Chair at the University of Utrecht) I address the following questions about the role of teaching:

- What does it mean to teach mathematics?
- What are the characteristics of "good" teaching of mathematics?
- How does/can "good teaching" develop?

Because it is hard to address "what is *good* teaching?" in a simple way at the outset, I will start with another question:

> How can we teach mathematics for the *effective* learning of our students?

I am assuming here that the purpose of teaching is to cause, stimulate or create learning. The words used here betray something of the theoretical perspectives that we bring to talking about learning. I shall say more about this later. However, for the moment let me say what I mean by the *effective learning of mathematics* that I desire for our students. I see there being three qualities or dimensions: enjoyment, understanding and proficiency.

Enjoyment is about experiencing stimulating activity in interesting contexts; gaining inspiration and motivation from seeing the beauty of the subject; and it includes affective factors such as ease of access and comfort in engagement. Students should be able to enter readily into mathematical experiences and thinking and not feel threatened or excluded.

Understanding involves insight into mathematical concepts and conceptual relationships, and an appreciation of mathematical activity and process that goes beyond the instrumental. By 'instrumental' I refer to 'rules without reasons' – limited, short term understanding which depends on simple recall and lacks depth of structure or relationships to other mathematical ideas (Skemp, 1976).

Proficiency includes skill in being able to use mathematical rules and procedures, knowing when and how to apply these to problems and being able to use mathematics in everyday lives, other disciplinary areas and the world of work.

Understanding and proficiency are deeply related, and one without the other leaves the student at a disadvantage. Without enjoyment, the processes of learning can be painful with students seeking avoidance and coming to believe that mathematics is beyond their capabilities.

Teaching for enjoyment, understanding and proficiency is demanding and challenging for a teacher. How can a teacher achieve these goals? In order to set the scene for a further discussion later, I will start with an example from a mathematics classroom.

INVESTIGATING IN DOING MATHEMATICS AND IN TEACHING MATHEMATICS

A mathematics lesson focusing on 'perimeter'

This lesson was recorded as part of a research project into the use of investigational activities in mathematics lessons to promote students' mathematical engagement (Jaworski, 1994). The teacher had designed, or chosen a task on which he invited his students (aged 12-13) to work. The class was seated around tables in which students worked together in friendship groups. Design of teaching involves a didactic process in which the abstract ideas of mathematics are (re)conceptualised by the teacher into mathematical tasks and activity for students. I suggest that the didactic goals demonstrated in this lesson included the following

- To provide opportunity for students to engage with the topic;
- To stimulate language patterns and imagery to contribute to understanding;
- To provide a need to practice and apply procedures – not just practice for its own sake;
- To promote students' own exploration and inquiry for motivation and purposeful engagement.

The teacher had chosen a task named "Four square perimeter". It was stated simply as follows:

> What perimeters can we get with *four* squares placed edge to edge or corner to corner, but not overlapping? (5? 6? 10? 99?)

Two examples of legitimate arrangement of the four squares can be seen in Figure 1.

Figure 1: Ways of arranging four squares

The class had worked on the task for four squares, trying out different arrangements; they had moved on to consider larger numbers of squares as suggested in the question. In all cases, according to the rules of arrangement, the perimeter they found was an even number. This resulted in a conjecture, "the perimeter will always be even", and led to a question, "is it possible to find an odd perimeter?"

I focus now on an episode, involving three girls' approach to tackling this question, which was recorded on video. In discussion with the teacher they had suggested that, rather than lining the squares up with full sides touching, they might consider the situation with half squares touching. The teacher encouraged them to explore this. They talked about and drew various diagrams, and then one girl offered the diagram in Figure 2 and started to count its sides.

Figure 2: Four squares with sides half touching

The other girls joined in the counting. They counted sides and counted again. It seemed to add up to 13. They then counted systematically together – all the whole sides first (there were 10) and then the half sides (6), so altogether 13 sides – a perimeter of 13. One of the girls said, "So you can. If you take half squares you can get an odd number", and the other two nodded in agreement. We then see the teacher return to this group and the girls eager to tell him what they had found. The girls spoke all at once "you can … ", "if you add the half squares …", "you can get an odd number". The teacher looked at their diagram and started to count: one, two, two-and-a-half, …". "No" said the girls, "No, No, count like this", and they demonstrated their systematic form of counting. The teacher followed their instructions; he counted 10 whole sides, wrote down 10; he counted the half sides, wrote down 3, then he wrote 13, and said "Hey!". The "Hey" seemed to acknowledge their success. They were all smiling and seemed pleased with themselves.

I have described this episode in detail to acknowledge certain aspects or qualities of this lesson. The girls were fully engaged in their investigation. Of course this may have had something to do with their being video-recorded, but nevertheless, there was an unforced spontaneity in their words and actions. They wanted to be sure of what they were finding: I draw this conclusion from the ways in which they drew and re-drew their figures and checked and rechecked their counting. They bounced ideas off each other through half-formed sentences. When the teacher returned to them, they were insistent that he should do the counting in their way – telling him clearly what to do. This demonstrated a confidence in their finding that an odd number was indeed possible with this kind of arrangement.

In terms of what was achieved in this lesson, we might say that these students *knew* perimeter – that perimeter had been 'reified' as I shall explain below. They could count it, talk about it, work with it and manipulate it. They showed evidence of mathematical thinking: of trying out special cases, making conjectures and moving towards generalisation (e.g., Mason, Burton and Stacey, 1982). They worked well together within a group, built on each other's suggestions, and looked critically at what they had found. We could argue that all they had found was one special case. We did not see them check other numbers of squares. However, their systematic mode of counting could be seen as generic. We might believe they could have applied this to any number of squares. The teacher did not push them to check further. In fact they had answered the question, "is an odd number of squares possible?" The answer was "yes".

With hindsight, it could have been valuable to push them further to address whether this arrangement would reveal an odd perimeter for *any* number of squares and then towards a proof. In fact, an odd perimeter only arises when the number of squares is even which might have been revealed with further exploration. However, this is just speculation.

Teacher collaboration

The teacher here was one of a team of mathematics teachers in the mathematics department of their school. I was working as a researcher with several of them and it was common for us to sit together to view a video episode from a lesson and discuss aspects and issues in teaching. Usually the teacher concerned started discussion with

reasons for choosing the particular episode. The teacher, George, chose and introduced the episode described above. As part of his introduction, he uttered the following words:

> "I was ad-libbing – I didn't know what would happen for half squares"

> "These girls were teaching me something"

So, it appeared that, in encouraging the girls to explore further with the half squares, the teacher was on unknown ground, but willing to take a risk; perhaps, later, in using this task with other students he would be more aware of possibilities and able to judge whether to push towards them. He certainly seemed to have learned from the girls' activity and reasoning. Thus we might say that he was in the process, overtly, of developing his teaching. We might even say that he was acting in an inquiry mode in trying out possibilities in his classroom and learning from outcomes. We might be less positive and say that he was taking too many risks, and that without the requisite knowledge he might not advise or support his students in the best possible way.

Such issues arose in the discussion of the teachers and researcher. While respectful of George's activity and decisions as a teacher, the other teachers probed teaching decisions and outcomes. One issue, raised by one of George's colleagues, was a challenge to friendship groups, suggesting they might be too "comfortable" and possibly not challenging enough. This was debated, with this teacher and George choosing to disagree.

There was some agreement as to what activity with this task had afforded, and I relate this to the dimensions of students' learning above.

> The task and context encouraged students' meaningful engagement with mathematics:
>
> - They *engaged* actively with the topic and with mathematical process and seemed to be enjoying themselves;
>
> - They *asked questions and explored* possibilities and seemed confident with their understanding of the concepts involved;
>
> - They *practised* finding perimeter and seemed to have a good grasp of both how to find it and what it meant.

Through being open with his students, the teacher also learned. Through working with his colleagues and a researcher he had the chance to develop didactical/pedagogical knowledge, such as an awareness that friendship groups might not be the only way to organise his students, or that it might be worth pushing students more overtly towards generalisation and proof.

We might relate these observations to the words of Hans Freudenthal, who wrote

> It is a not so new, but still rarely fulfilled requirement that mathematics is taught, not as a created subject but as a subject to be created (1978, p. 72)

I suggest that, in this classroom, mathematics can be seen as "a subject to be created" and that the teacher and his students were all engaged in creating it. I shall address below the need for an associated critical dimension in examining what has been created and its validity and rigour.

PROBLEMS WITH MATHEMATICS

We might ask, can or should *all* teaching look like what we observe in the episode above? It would be far too bold to suggest that it should, and to ask if it *can* is to raise

many questions about what this means. However, we know undoubtedly that it does not. And we have known this for some time.

In the UK in 1982, a government report, the Cockcroft Report, from an inquiry into *the teaching of mathematics in schools*, stated

> Mathematics is a difficult subject to teach and to learn

The authors of the Cockcroft report drew on research and experience widely, and one of the studies they consulted was a 1978 research study into adult innumeracy, conducted by researchers in association with Yorkshire Television. As part of this study, researchers went out into the streets in Yorkshire and asked passers by

> How many 7p stamps can you buy for £1 (100p)?

A resulting television programme showed members of the public responding to this question. Many were unable to answer the question, their faces showing varying degrees of puzzlement, embarrassment, or apology. Despite being given time to think and work out an answer, many could not give an answer. The people's responses, especially their facial expressions and jokey responses were amusing to watch, but revealed a serious issue. Here, a sample of the public, those walking down this street on this day at this time, showed a high percentage of people unable to answer a seemingly simple mathematical question in an everyday context. This speaks to some level of inadequacy of their past experience of learning mathematics and could be seen as an indictment of the teaching they had experienced.

I relate this to words reportedly from an eminent mathematician-philosopher, Poincaré.

> How is it that there are so many minds that are incapable of understanding mathematics? Is there not something paradoxical in this?

> Here is a science which appeals only to the fundamental principles of logic, to the principle of contradiction for instance, to what forms, so to speak, the skeleton of our understanding, to what we could not be deprived of without ceasing to think, and yet there are people who find it obscure, and actually they are in the majority.

> That they should be incapable of discovery we can understand, but that they should fail to understand the demonstrations expounded to them, that they should remain blind when they are shown a light that seems to us to shine with a pure brilliance, it is this that is altogether miraculous.
> (Poincaré, 1952, cited in Sierpinska, 1994, p. 112)

These words seem to foreshadow what we experience in many countries today of people who struggle to understand mathematics. I would like to highlight two phrases from the words above: a) "the demonstrations expounded to them" and b) "when they are shown a light that seems to us to shine with a pure brilliance". I wonder, what do these demonstrations consist of, and why are they not successful? And, what does "showing the light" look like? Why is it not successful. Thus, I change the emphasis from people who, so surprisingly, are not able to appreciate mathematics to those who *do* appreciate mathematics, but are unable to communicate it in ways that others can understand. Because surely the responsibility lies with those who can, rather than with those who cannot. This again points to problems with teaching mathematics.

The Cockcroft report led to a number of television programmes which aimed to communicate its messages to a wider audience than the academic community. One of these programme showed a boy, Charlie, who seemed to be having problems with

mathematics. An interviewer asked Charlie to take away seventy from one hundred and nine. Charlie wrote this down as in Figure 3a.

$$(a) \quad \begin{array}{r} 109 \\ \underline{70} - \\ \end{array} \qquad (b) \quad \begin{array}{r} 109 \\ \underline{70} - \\ \underline{100} \end{array}$$

Figure 3: Charlie's subtraction calculation

He then proceeded to work right to left with the following words:

> Nought from nine, you can't do that, so you put nought down.

> Then it's seven take nought, you can't do that either so put nought down again.

> There's nothing to take from one, so just put one down. (Figure 3b)

It is tempting to say that Charlie got this wrong and that he does not understand subtraction. However, what happened next is very revealing. The interviewer then said "OK Charlie, if you had a hundred and nine pounds, and you took seventy pounds away, would you have that that amount left (she pointed to the number 100 – Figure 3b)?" Charlie shook his head and said "No, I realised …". She asked him, "Do you know in fact how much you would have left?" With only a slight pause, Charlie said, "thirty nine". The interviewer asked him how he worked that out and he replied, counting with his fingers, "It's seventy, eighty, ninety, a hundred. Then there's another nine, so it's thirty nine". In coordination with his fingers, he had used an informal 'counting-on' strategy with which he seemed quite comfortable.

So, the mathematics was not a problem for Charlie; his problem seemed to lie in a misremembered algorithm for subtraction. I wonder why his first approach to the problem was via the formal algorithm, rather than through the use of his own informal, correct procedure. The way such algorithms are taught could be one of the problems here. It could also be that the introduction of the money context triggered the use of the informal procedure. Without this context, the problem was more of a classroom problem and for classroom problems you need the formal algorithm – because this is the way mathematics is often taught. This again is speculation.

PROBLEMS WITH MATHEMATICS TEACHING

Mathematics as a set of rules

I have raised above a number of issues which suggest problems with mathematics teaching. When one knows some mathematics, it is relatively easy to present the rules and procedures – and the conditions under which the rules and procedures apply. The responsibility then rests with the learner to retrieve the rules and procedures and use them correctly. However, the learner is the one who is more vulnerable, and so the one less able to take this responsibility. Is this fair or reasonable? Problems that arise include the following

- 'remembering' the rules incorrectly
- applying rules in the wrong circumstances
- not *linking* one set of rules with another
- not *having a feel* for what the rules are about
- psychological barriers – anxiety, fear
- not really *engaging* with mathematics

    ▪ not understanding …

We saw above with Charlie an example of mis-remembering. Figure 4 shows an example of following a rule in the wrong circumstances:

$$\frac{a\ +\ b}{b\ +\ c} \qquad\qquad \frac{a\ +\ \cancel{b}}{\cancel{b}\ +\ c} = \frac{a}{c}$$

Figure 4: Incorrect use of the operation of cancelling

This use of cancelling across the addition sign is attributed to a mathematics undergraduate (Joint Mathematical Council and the Royal Society, 1997) which suggests that instrumental application of rules without associated conceptual thinking is not limited to people who struggle with mathematics. In order to avoid such errors arising from a teaching approach of trying to convey rules, we might ask what more is needed from the person offering the rules?

So this brings me to a didactic/pedagogic question

> In what ways can **we** be more successful in communicating mathematics?

My "we" here refers to teachers, mathematicians and educators, because surely the responsibility is ours. What are the problems with the teaching that seems to be currently experienced by students?

An exercise culture

Scandinavian professors of mathematics education, Ole Skovsmose and Roger Säljö, write about classrooms in which an *exercise culture* prevails.

> This implies that the activities engaged in the [mathematics] classroom to a large extent involve struggling with pre-formulated exercises that get their meaning through what the teacher has just lectured about. ...

> An exercise traditionally has one, and only one, correct answer, and finding this answer will steer the whole cycle of classroom activities and the obligations of the partners involved. (Skovsmose and Säljö, 2008, p.40)

These words suggest perhaps that examples of an exercise culture might be found in Scandinavian classrooms. What are the "obligations" of which Skovsmose and Säljö speak and are these special to Scandinavia? Elena Nardi and Susan Steward (2003) conducted research into students' attitudes to mathematics in a number of secondary classrooms in the UK. They found what they called "quiet disaffection", and asked "Is mathematics T.I.R.E.D.? The following is a selection of quotations from the students they interviewed (p.355-360):

Tedium
> *I want to enjoy maths but I can't because it's so boring*

Isolation
> *When he sets it as a class, it's individual – the whole class do it, but individual*

Rule and cue following – rote learning
> *It's like parrot work – it is parrot work*

Elitism
> *I hate maths because I'm not very good at it*

Depersonalisation
> *We don't get any attention at all.*

These quotations refer overtly or implicitly to the teaching experienced by these students. They suggest that some students in the UK experience mathematics teaching negatively – lacking enjoyment, not encouraging conceptual thinking, not promoting a sense of proficiency.

Moral Education

The *obligations* referred to by Skovsmose and Säljö, as with the *responsibilities* that I refer to above, depend on the *values* a teacher brings to the classroom. Richard Pring, philosopher of education in Oxford, writes about education as a "moral practice" (Pring, 2004).

> I shall argue that education itself is a moral practice … Ideally the 'practice' should be in the hands of moral educators (who themselves should manifest the signs of moral development). (p. 12)

He makes reference to a particular teacher, and says

> [T]he teacher was helping the young people to make sense, to develop a serious and authentic response to the real, sometimes threatening and practical situations in which they found themselves. (p. 16)

We might interpret this in terms of George, designing tasks that can help his students to make sense of the concept of perimeter, and interacting with them to support their own exploration. Indeed, we know that many students find mathematics threatening, so it behoves a teacher help students overcome such feelings. Pring goes on:

> [Teaching is a] social practice with its own principles of conduct and values … a commitment to helping young people to learn those things which are judged to be worthwhile. (p. 16)

> The teacher, in helping the learner to make sense, both respects what is inherited and at the same time helps the learner to engage critically with such a tradition. (p.17)

Presumably learning about 'perimeter' is judged to be worthwhile, as it is a topic in most mathematics curricula. Students must make sense of the concept relative to what has gone before in its historical development, but at the same time develop ability to question results and relationships and make their own judgements. Pring says that without such a moral stance, teaching is "impoverished" (p. 18). In these terms, we might see classrooms in which experience does not go beyond the exercise culture as impoverished.

Skovsmose and Säljö contrast the exercise culture with a culture based on mathematical inquiry. They write

> The ambition of promoting *mathematical inquiry* can be seen as a general expression of the idea that there are many educational possibilities to be explored beyond the exercise paradigm. (Skovsmose and Säljö, 2008, p. 40)

I compare this with the words of Freudenthal quoted above:

> It is a not so new, but still rarely fulfilled requirement that mathematics is taught, not as a created subject but as a subject to be created (Freudenthal, 1978, p. 72)

It seems to me that Freudenthal's "subject to be created" can be related clearly to the idea of a culture of mathematical inquiry in the classroom and indeed to the ideas of moral education.

> Mathematical Inquiry <--> A subject to be created <--> Moral Education

We might see George's classroom activity described above to fit into such a culture.

The context of teaching

However, we must not under-rate the complexity of classroom situations and the many factors that impinge on classroom activity. Sally Brown and Donald McIntyre (1993) in a study of secondary classrooms observed that activity in lessons settles down to what they call "normal desirable states". The normal desirable state is what is most amenable or comfortable for all participants and is negotiated between teacher and students, often implicitly. It often results in a reduction of cognitive load as it means students have less demand placed on them and are therefore more amenable to being cooperative with the teacher.

Walter Doyle and colleagues (e.g., Doyle, 1988, pp. 173/4) speak of the kinds of tasks presented in mathematics classrooms and their demand on students. They characterise *Familiar tasks*, based primarily in memory, formulas, search and match strategies, as having routinised recurring exercises and outcomes that are predictable. These are compared with *Novel Tasks* which require higher cognitive processes, understanding and transfer, and decisions about how to use knowledge; and whose predictability is low and emotional demands high. When *familiar* work is being done, they say that the flow of classroom activity is typically smooth and well ordered. Tasks are initiated easily and quickly, work involvement and productivity are typically high, and most students are able to complete tasks successfully. When *novel* work is being done, activity flow is slow and bumpy. Rates for student errors and non-completion of work are high. Students negotiate directly with teachers to increase explicitness of product specifications or reduce strictness of grading standards. Tasks which appear to elicit comprehension or analytical skills are often subverted to become routine or algorithmic.

From this discussion, it seems clear that teachers are faced with a confusing range of issues and choices. A question to address is how they can navigate this ocean of complexity. I offer ideas of collaborative inquiry as a way to start to address these issues.

INQUIRY COMMUNITY AND COMMUNITY OF PRACTICE

Inquiry is a relatively simple idea. It involves

- Asking questions and seeking answers;
- Recognising problems and seeking solutions;
- Wondering, imagining, inventing, exploring investigating …;
- Looking critically at outcomes and results.

When a community takes inquiry seriously in an explicit way, we might refer to *an inquiry community*. Figure 5 suggests examples of inquiry communities.

An inquiry community starts to form when participants use inquiry as a tool – asking questions, trying out and evaluating new approaches, looking critically at results. Over time such actions lead to participants developing an inquiry way of being, an inquiry stance (Cochran Smith and Lytle, 1999, Jaworski 2004).

The idea of inquiry community can be seen to develop from the theory of *community of practice* introduced by Jean Lave and Etienne Wenger (Lave and Wenger, 1991, Wenger 1998). The theory is based on ideas of learning through participation and reification. Participation is about mutual engagement in practice in which each individual contributes alongside others in the community, negotiating their shared
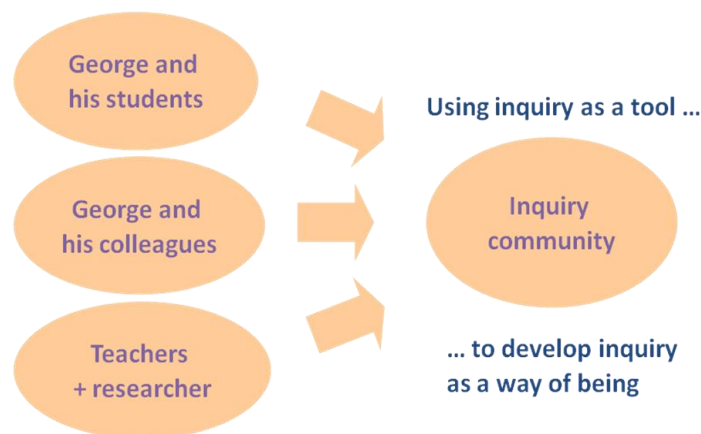
Figure 5: Examples of inquiry communities

activity, its purposes and characteristics. Reification involves a process of turning participatory experiences into objects of knowledge which can be manipulated and re-formed. I suggested above that we might see George's students as having *reified* the concept of perimeter, being able to treat perimeter as an object which they could manipulate.

Wenger (1998) suggests that a community of practice has three main elements, Mutual Engagement, Joint Enterprise and Shared Repertoire. I have interpreted these in terms of mathematics education:

Mutual engagement

- Doing mathematics together -- with recognisable norms and expectations and collaborative relationships

Joint enterprise

- Working with a shared understanding of mathematical objectives and outcomes

Shared repertoire

- Using common resources--material or symbolic--which are recognized as central to engagement with mathematics

It seems important to point out the these elements of community of practice can apply to situations described by Brown and McIntyre and by Nardi and Steward (quoted above) as well as those in George's school.

For the individual who belongs to a community of practice, Wenger suggests that *belonging* implies *engagement*, *imagination* and *alignment*. Relating again to mathematics education, we *engage* with mathematics (or with the teaching of mathematics), use *imagination* to interpret our own roles in community activity, and *align* with the norms and expectations of the community. In a mathematics classroom with an exercise culture, aligning would imply being a part of the joint enterprise, and participating in all that is involved in working with exercises. Exercises would be a key part of the shared repertoire, and members of the community would develop a reified sense of the exercise, its meaning and purpose. For a teacher in a school where the focus is on achieving high scores in tests and exams, possibly at the expense of moral values in learning and teaching, alignment would involve participating fully in the expected practices and sharing objectives in the joint enterprise of achieving high scores. The shared repertoire would involve a discourse around scores and their

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importance for the joint enterprise. I have chosen examples here deliberately to show that the theory of community of practice *describes* the practice that is in place whether or not the practice is seen to be effective according to given objectives, or indeed whether the practice has moral values.

A community of inquiry can be seen to bring an element of inquiry into a community of practice. When participants develop inquiry ways of being, this can be said to *transform* the practice. Inquiry permeates the community and changes the very nature of the community. It changes "alignment" to "critical alignment" – a process of asking questions about what we do and if there are possibilities to do it differently or better. Critical alignment implies a process not only of aligning with normal practice, but also of looking critically at what we do as we do it

- taking a questioning attitude;
- engaging in reflection-in-action (Schön,1987);
- trying out new possibilities and looking critically at outcomes in relation to objectives.

This might involve teachers in taking on a research role within their community of (teaching) practice in collaboration with their colleagues. Reflection in action is a process of recognising issues in what we do as we do it and offering possibilities, in the moment, to do it differently (Mason, 2008; Schön,1987)

The suggestion of teachers taking on a research role has been criticised by a number of eminent researchers, the argument being that teachers are not trained to be researchers, and teaching is a demanding enough job, without asking teachers to take on yet other demanding roles. Donald McIntyre writes:

> it seems unreasonable to demand of teachers that they be researchers as well as teachers, when the expertise required for the two activities is so different (McIntyre, 1993, p. 43).

Michael Eraut suggests that time constraints limit the opportunity for reflection-in-action (Eraut, 1995), and Fred Korthagen writes

> Teachers need quick and concrete answers to situations in which they have little time to think (Korthagen, 1999, p. 5)

It is clear to me that these are all reasonable points of view. Teacher are indeed under pressure in a demanding and challenging job. Teachers also have fundamental human rights and can expect not to have unreasonable demands made of them. The established communities of practice in which they work and the norms and expectation with which they are expected to align, often do not afford opportunity for critical alignment. My own experience shows that teachers who have a sincere desire to interpret a moral stance in their classrooms are often pressured towards behaving in ways contrary to their best intentions (Potari & Jaworski, 2002). It is hard for any teacher to act against the system, differently from the ways in which others act within the system. A community of practice can be a juggernaut in forcing the alignment of its participants. It is also the case that teachers sometimes do not have all the knowledge that is important to effecting certain ways of working or making changes to practice (Rowland, 2008).

Collaborative inquiry between teachers and didacticians

One response to these issues seems to be to recognize another group of practitioners who also have a stake in the teaching of mathematics for the effective learning of students. These people are university mathematics educators, academics and

researchers, sometimes called didacticians; they do research in mathematics education and many work also in programmes for educating both prospective and practising teachers. Their work brings them of necessity into contact with teachers in schools, and there are often mutual interests in supervising prospective teachers or in engaging jointly in developmental projects. It is possible for didacticians and teachers to support each other in demanding roles, working together with mutual respect, sharing knowledge and learning together. For example:

- a small group of teachers working with a university researcher (e.g. George and his colleagues);

- a group of teachers agreeing to conduct research into their own teaching in collaboration with educators from a university (Jaworski, 1998)

- a specially designed project to promote teacher learning and culture change (e.g., LCM and TBM in Norway – Jaworski, Fuglestad, Bjuland, Breiteig, Goodchild & Grevholm, 2007)

To demonstrate some of the possibilities of such collaboration, I will describe briefly the activity in the Norwegian projects LCM and TBM.

LEARNING COMMUNITIES OF MATHEMATICS (LCM)

LCM was a four-year developmental research project, funded by the Research Council of Norway[1] and involving a team of didacticians (~12) from a university in Norway with teachers (~30) from 8 schools, lower primary to upper secondary. The project aimed to promote *inquiry* in three layers (explained below) in order to develop new ways of working in mathematics teaching in school classrooms. The activity in the project included:

- Workshops in the university – all participants doing mathematics together, demonstrating inquiry processes and leading to discussion about didactics and pedagogy – together with input on various topics from both didacticians and teachers.

- Teacher teams in schools designing innovations for their classrooms and students, with didactician support as requested by the teachers. Feedback to the project community from school activity through video recording of lessons and presentations at workshops.

- Shared knowledge and expertise as summarised below.

After three of the four years of LCM, a subsequent proposal to the research council resulted in a second four-year project, entitled *Teaching Better Mathematics*, which involved extending the activity from the LCM project to other schools and teachers, and taking the LCM model into four further areas of Norway. See Jaworski et al. (2007), for a detailed account of the LCM project and the developmental model involved.

Inquiry in three layers

The projects are based on ideas of inquiry in three layers:

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[1] The LCM Project, along with associated projects Teaching Better Mathematics (TBM) and Information and Communications Technology in Mathematics Learning (ICTML/IKTML) were funded by the Research Council of Norway http://www.forskningsradet.no/en/Home_page/1177315753906 Details of these projects and links to related papers can be found at the following websites:
http://prosjekt.uia.no/lcm/   http://prosjekt.uia.no/iktml/   http://prosjekt.hia.no/tbm/

Inquiry in learning mathematics:

- Teachers and didacticians exploring mathematics together in tasks and problems in workshops;
- Students in schools learning mathematics through exploration in tasks and problems in classrooms.

Inquiry in teaching mathematics:

- Teachers using inquiry in the design and implementation of tasks, problems and mathematical activity in classrooms in association with didacticians.

Inquiry in developing the teaching of mathematics:

- Teachers and didacticians researching the processes of using inquiry in mathematics and in the teaching and learning of mathematics.

The three layers are deeply related in a nested form as can be seen in the diagram in Figure 6.
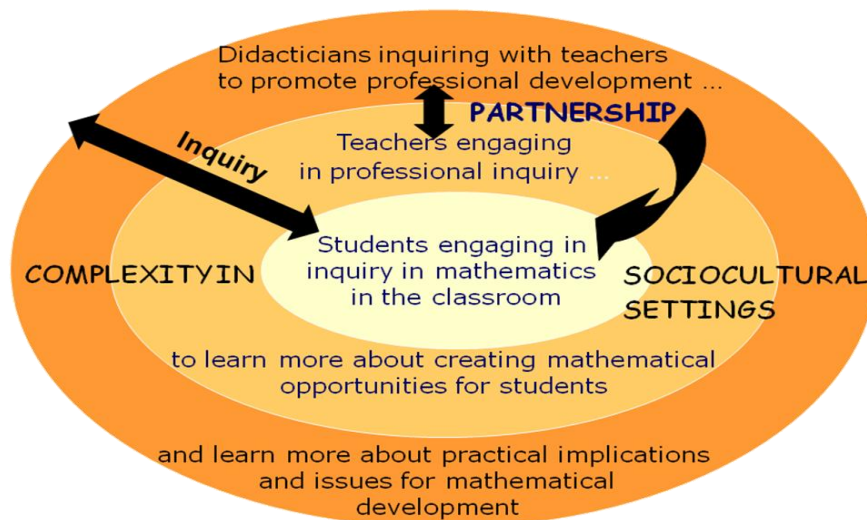

Figure 6: The LCM Model of inquiry-based practice in three layers

I asked at the beginning of this text, "How can we teach mathematics for the effective learning of our students"? At the centre of all activity in LCM were the students, and the entire enterprise was based on seeking the best possible opportunities for students' learning of mathematics. It was important that we did not start from prescribed approaches, although theories of inquiry did inform our activity. We started from a genuine desire to explore possibilities together and to look critically at what we do and what is possible. Approaches to engaging students in mathematics to achieve understanding and proficiency were rooted in inquiry-based tasks and activity. We sought to engage our students in inquiry in ways that they could find stimulating and suitably challenging and which would offer an enjoyable learning experience.

Student classroom activity was the focus of the teachers who sought to design suitable learning experiences for their students. This involved design of tasks and organization of classrooms to facilitate engagement in prepared tasks. Workshops for which tasks were prepared and offered by didacticians provided a basis for thinking about tasks for the classroom (Jaworski, Goodchild, Daland and Eriksen, in press). Teachers in

workshops joined with their colleagues from other schools to prepare tasks for students at a similar school level.

Teachers' activity in preparing for lessons with their students was the focus of didacticians in preparing activity for workshops. The workshops had to provide examples of inquiry-based tasks and promote an inquiry culture in the project. They had to be a forum for exchanging ideas and experiences, and for input relating to areas of knowledge and expertise: for example, mathematical input relating to the teaching of algebra or probability. Didacticians also collected data from all activity and organized a rigorous analysis of data within the university environment. Teachers who were interested in studying their own development joined with didacticians in analysis of data from their classrooms (e.g., Jørgensen & Goodchild, 2007)

Although in the beginnings of the project, the didacticians were largely the leaders, gradually over time, teachers became more aware and confident of their own knowledge and of having important contributions to the project. The diagram in Figure 7 tries to capture this distribution of knowledge, albeit in rather a simplistic way (Jaworski, 2008a).
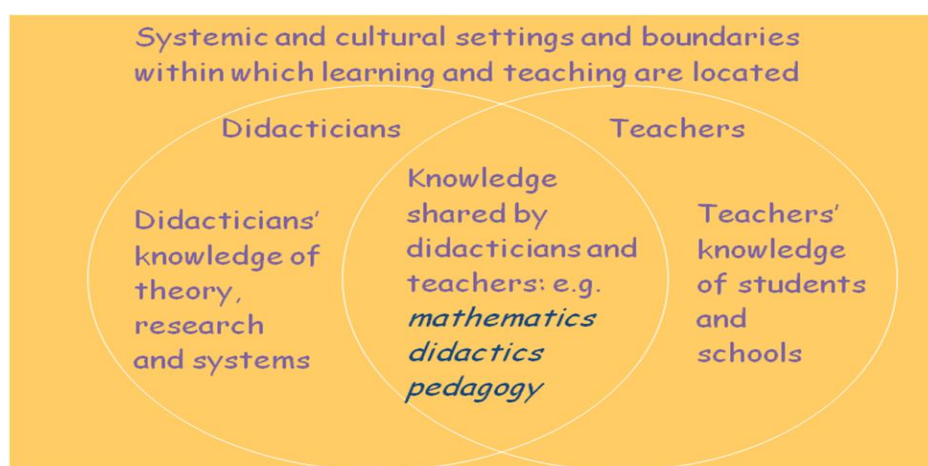

Figure 7: Distribution of knowledge between teachers and didacticians in LCM

Didacticians and teachers share knowledge about mathematics, and about aspects of the teaching and learning of mathematics that contribute to didactics and pedagogy. Of course this knowledge looks different for the two groups since it is closely related to the different kinds of activity in which they each engage. However, the apparently shared aspects of this knowledge serve as an important base for dialogue between the two groups. In the beginnings of the project, didacticians, as the people with most power and therefore most responsibility, had to temper their language to fit more closely with that of teachers. As the community developed over time, it became more possible to talk from the separate perspectives and expect that the other group would not be alienated or intimidated, but would seek clarification or offer a challenge. This happened forcefully on a number of occasions when teachers challenged didacticians about their actions or intentions. Some of these are recorded and analysed in Jaworski and Goodchild, 2006.

In addition to the so-called 'shared' knowledge, both groups brought specialist knowledge to the partnership. In both cases it was knowledge related to the specialist activity in which each engaged. Didacticians knowledge of theory, research and the associated literatures, and teachers' knowledge of the school system and characteristics and cultural aspects of their students, were essentially important areas

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of knowledge which contributed to the partnership community. Over time it became clearer how these kinds of knowledge were important, and that both groups contributed at equally important levels to project as a whole. Details can be found in Jaworski, 2008b.

A particular mention should be made of *mathematical* knowledge. There was a considerable variety of mathematical knowledge and experience in the project with some teachers and didacticians having studied mathematics to degree level (bachelor or masters, and one didactician to PhD level). For others, particularly for some teachers in primary schools, there were feelings of insecurity with mathematics and a corresponding unwillingness to expose uncertainty or insecurity in group activity. Teachers made it clear almost from the beginning that they preferred to be in small groups with other teachers from their own level (lower or upper primary, lower or upper secondary) so that they could address with confidence the mathematics they would work on with students. Didacticians were encouraged to offer mathematical input during workshops and to provide tasks and problems related to particular areas of mathematics. Teachers planned mathematical activities for the classroom together with colleagues working at the same level. In doing so, mathematical and didactical knowledge together were an ongoing focus of project activity.

The distribution of knowledge just discussed can be seen to relate closely to a distribution of activity in the project. The diagram in Figure 8, similar to that in Figure 7, tries to capture, again in a somewhat simplistic way, a distribution of activity.
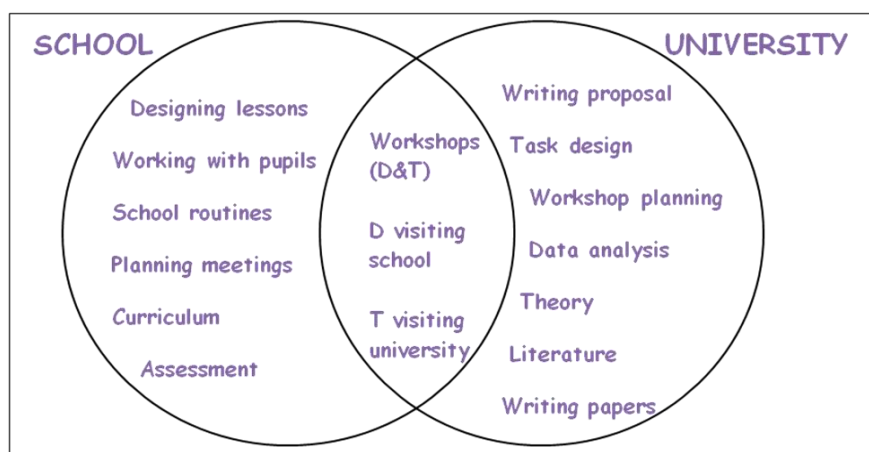

Figure 8: Distribution of activity in the LCM Project

On the left we see the school community with a long history of school activity in which teachers are immersed in school norms and expectations and their own perspectives on didacticians (who didacticians are and what they do). On the right we see the university community with a long history of university activity in which didacticians are immersed in university norms and expectations with their own perspectives on teachers (who teachers are and what they do). The school and university communities are established communities of practice, and the whole is overlaid with the project community which seeks to develop as an inquiry community.

In the early days it was didacticians who promoted the ideas of inquiry community, based on their own theoretical perspectives and views of what is or could be possible for school classrooms. Early activity in workshops was designed around ideas of inquiry and creating an inquiry community within the project. In the beginning

teachers needed to understand what was meant by "inquiry", and it seemed that there was no *one* word in the Norwegian language that would translate the English word 'inquiry' exactly. This led to much debate, and many ways of expressing 'inquiry' in Norwegian. What emerged was that the word "inquiry" seemed to enter the Norwegian language, and we would hear Norwegian sentences in which the word "inquiry" appeared. This seemed to indicate that Norwegian teachers were assimilating the ideas of inquiry into their own language and culture – perhaps having reified the notion of inquiry into an object that they can use and manipulate. Of course, most of the didacticians spoke Norwegian as well, so this became another area of shared knowledge and expertise – together they had decided how to speak about inquiry in Norwegian.

It is probably clear from the discussion above that an important finding from this project was what it means for teachers and didacticians to work together, the kinds of activity that took place and what each group learned, and the resulting knowledge generated through the project. Published papers, some of which are referenced above, address findings and associated knowledge and learning in much more detail than is possible here (see the websites listed in footnote 1 for a list of papers). Here are just a few of the areas of development demonstrated in the project:

- Teachers learning to design tasks and engage students more conceptually in mathematics.
- Didacticians learning what is possible for mathematics teaching within a school; what teachers can/will do and what is not possible within the norms and expectations.
- Perceiving what goals we are working to – and questioning the goals if they seem not to focus on creating opportunity for students' mathematical development.
- Respecting each other's knowledge and possibility to contribute.
- Developing "voice" within the project (particularly for the teachers) to allow more confident and fruitful participation.

In conclusion

I began by asking questions about what makes good teaching of mathematics. I am ready to offer a tentative definition. For a mathematics teacher, good teaching involves

- A desire to offer the best possible opportunities to students to achieve enjoyment, understanding and proficiency with mathematics at the level at which teaching is offered;
- Confidence with mathematics at an appropriate level, with the design of tasks and with the use of resources.
- A willingness to engage with inquiry and critical alignment with respect to the school environment, its norms and practices and ways of approaching mathematical activity in classrooms;
- Collaboration with colleagues to provide (critical) support and generate ideas and, if possible, with educators who bring other areas of knowledge which can be useful in inquiring into teaching.

Collaboration between teachers and educators emphasises educators also as learners in the developmental process in relation to ways in which they can best support teachers in achieving the best for their students.

In this paper I have tried to address mathematics and its teaching and learning as a human practice in which the *rights* of participants are addressed centrally. These include

- the rights of all students at all levels to be provided with opportunity to engage with and be successful with mathematics.
- the rights (and responsibilities) of teachers and educators to become more knowledgeable about what makes good teaching of mathematics and to have opportunities and possibilities to work with and develop this knowledge. (Goodchild, 2007; Jaworski, 2008b)

Fulfilling these rights and responsibilities can be seen as taking a *moral* stance towards mathematics education. I have suggested that practice based in inquiry in collaborative communities offers a framework for such activity and its development.

I am happy to discuss these ideas further with teachers and educators/didacticians who are interested.

Barbara Jaworski
2nd March 2011

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Barbara Jaworski

Barbara Jaworski worked as a school teacher for 15 years. Four of these were in developing countries, Kenya and the Philippines. Having a first degree in Physics, she taught both physics and mathematics (and chemistry, art, music, English language and computer science as needed). When it became clear that teaching mathematics was what satisfied her most, she studied for a mathematics degree with the *Open University* (OU) in the UK. OU Summer Schools in various courses introduced her to investigational activities in the teaching and learning of mathematics. Her last five years of teaching were in a large comprehensive upper school in which she was head of faculty of mathematics and computing. This provided opportunity to explore with colleagues the kinds of teaching that contributed to pupils' mathematical understanding. From here she moved to the *Centre for Mathematics Education* at the OU, where she became involved in preparing materials for development of teaching by practising teachers. During this time she worked on her own PhD focusing on investigative approaches to teaching mathematics and working very closely with classroom teachers to study their thinking and practice (Jaworski, 1994).

These experiences led to a long period as a teacher educator at the universities of Birmingham and Oxford working in initial teacher education programmes, supervising PhD students and doing research into mathematics teaching and its development. At Oxford, working with mentors from the initial teacher education programme, she studied the processes and practices of teachers engaging in research into their own teaching – the Mathematics Teacher Enquiry Project (Jaworski, 1998). From here she took up a post as Professor of Mathematics Education at the University of Agder in Norway, where she contributed to building a new PhD programme and to teaching courses for doctoral students. Substantial research funding from the Norwegian Research Council led to several major research projects in which mathematics educators from the university (didacticians) worked in partnership with school teachers to develop mathematics teaching in school classrooms (Jaworski, Fuglestad, Bjuland, Breiteig, Goodchild and Grevholm, 2007). Here, ideas of *co-learning* and *community of inquiry* between teachers and didacticians developed (Jaworski, 2006, 2008b). Returning to the UK from Norway, she took up a post as Professor of Mathematics Education at Loughborough University in the Mathematics Education Centre, a part of the School of Mathematics. Here she works with mathematics educators and mathematicians to teach mathematics to university students and do research into the teaching of mathematics at university level.

In 2011, Barbara Jaworski was invited, for three months, to take up the F. C. Donders Chair at the University of Utrecht, Freudenthal Institute for Science and Mathematics Education, in the Netherlands.