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Two-dimensional 'discrete hydrodynamics' and Monge–Ampère equations

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Abstract. An integrable discrete-time Lagrangian system on the group of area-preserving plane diffeomorphisms *SDiff* $(\mathbb{R}^2)$ is considered. It is shown that non-trivial dynamics exists only for special initial data and the corresponding mapping can be interpreted as a Bäcklund transformation for the (simple) Monge–Ampère equation. In the continuous limit, this gives the isobaric (constant pressure) solutions of the Euler equations for an ideal fluid in two dimensions. In the Appendix written by E. V. Ferapontov and A. P. Veselov, it is shown how the discrete system can be linearized using the transformation of the simple Monge–Ampère equation going back to Goursat.

1. *Introduction*

It is well known that the Euler equations for the motion of an ideal (i.e. incompressible, inviscid, homogeneous) fluid in the domain D can be derived from a natural variational principle on the group *SDiff* (D) of the volume-preserving diffeomorphisms of D (see [**1**]). In [**7**], a natural discrete Lagrangian system on the group *SDiff* (D) was introduced in the case when D is a domain in the Euclidean plane $\mathbb{R}^2$. The choice of the corresponding Lagrangian was motivated by the results of [**5**], where the integrable discretizations of some classical integrable systems have been discussed. The main question was how far the analogy with the finite-dimensional case can go for the group *SDiff* (D) and what can we say about the dynamics. The fact that the Euler equations in two-dimensional hydrodynamics are known to be non-integrable made the situation even more intriguing.

In this paper we investigate this problem in more detail. The main results are the following. The corresponding Lagrangian map has two branches. The first branch determines trivial dynamics without a continuous limit. The second branch is defined only for special initial data corresponding to the diffeomorphisms whose differentials have constant spectrum. We show that it can be interpreted as a Bäcklund transformation of a simple Monge–Ampère (MA) equation

$$G_{xx}G_{yy} - G_{xy}^2 = c,$$

where c is a constant. In the continuous limit, this gives the special solutions of two-dimensional Euler equations for the ideal fluid corresponding to the case when the pressure is constant in space (isobaric flows).

The first version of this paper was circulated as a preprint of FIM (ETH, Zurich) in April 1993. Since then we have received several very useful comments from B. Khesin, O. Mokhov and E. Ferapontov, which substantially clarified the situation. We had planned to come back to this problem together again, but unfortunately it did not happen ...

The second author has taken the responsibility (not without hesitation) to prepare a slightly revised version of the original preprint [**6**] for publication in this special issue. In the Appendix, which was written in collaboration with E. Ferapontov, it is explained how one can linearize the discrete dynamics by a suitable contact transformation going back to Goursat and the continuous limit is discussed in more detail.

2. *Motivations: discrete system on SL_2*

As was shown in [**5, 8**] the discrete-time Lagrangian system

$$\delta S = 0,$$

$S = \sum_{k\in\mathbb{Z}} \mathcal{L}(X_k, X_{k+1})$, $\mathcal{L}(X, Y) = \mathrm{tr}(XJY^{\mathrm{T}})$, $X, Y \in \mathcal{O}(3)$, $J = J^{\mathrm{T}}$ can be considered as an integrable discrete version of the rigid body's dynamics.

In this case, the function $\mathcal{L}$ has the following properties which determine it uniquely (see [**7**]):

(1) symmetry, $\mathcal{L}(X, Y) = \mathcal{L}(Y, X)$;

(2) left-invariance, $\mathcal{L}(gX, gY) = \mathcal{L}(X, Y)$, $g \in \mathcal{O}(3)$;

(3) $\mathcal{L}(X, Y)$ is bilinear as a function of the matrices X and Y.

In fact, the first two conditions are already very restrictive. Indeed from item (2) one has $\mathcal{L}(X, Y) = \mathcal{L}(Y^{-1}X, I) = F(\omega)$, where $\omega = Y^{-1}X$, $F(\omega) = \mathcal{L}(\omega, I)$. The symmetry $\mathcal{L}(X, Y) = \mathcal{L}(Y, X)$ implies the following property for F:

$$F(\omega) = F(\omega^{-1}).$$

For the groups $G = \mathcal{O}(N)$, $U(N)$, $Sp(N)$ there exists a *linear* involution $*$ on the space of the matrices such that

$$\omega^{-1} = \omega^* \Leftrightarrow \omega \in G.$$

In the orthogonal case $\omega^* = \omega^{\mathrm{T}}$, in the symplectic case

$$\omega^* = \Omega^{-1}\omega^{\mathrm{T}}\Omega, \quad \Omega = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}. \tag{1}$$

In all these cases the function $\mathcal{L}$ can be taken in the form

$$\mathcal{L} = \mathrm{tr}(XJY^*), \quad J^* = J \tag{2}$$

which corresponds to

$$F = \mathrm{tr}(J\omega), \quad J^* = J. \tag{3}$$

For the general matrix Lie group there is no homogeneous polynomial function F with the property $F(\omega) = F(\omega^{-1})$. For example, one can show that for the group $G = SL(N)$ the polynomial function F with this property exists only if $N \leq 2$. The existence of F for $N = 2$ is explained by the isomorphism

$$SL(2) \simeq Sp(2).$$

The function $\mathcal{L}$ in this case has the form

$$\mathcal{L}(X, Y) = \mathrm{tr}(XY^*), \quad X, Y \in SL(2) \tag{4}$$

where

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}. \tag{5}$$

The variation of the functional

$$S(X) = \sum_{k\in\mathbb{Z}} \mathrm{tr}(X_k X_{k+1}^*) \tag{6}$$

leads to the equation

$$X_{k+1} + X_{k-1} = \lambda_k X_k, \tag{7}$$

where λ_k is the Lagrange multiplier determined by the constraints

$$\det X_k = 1 \quad \text{for all } k \in \mathbb{Z}.$$

Multiplying (7) by X_k^{-1} one has

$$X_{k+1}X_k^{-1} + X_{k-1}X_k^{-1} = \lambda_k I \tag{8}$$

or

$$\omega_{k+1}^{-1} + \omega_k = \lambda_k I, \tag{9}$$

where $\omega_k = X_{k-1}X_k^{-1}$ is the 'discrete angular velocity'. The condition $\det \omega_{k+1}^{-1} = \det(\lambda_k I - \omega_k) = 1$ leads to a quadratic equation for λ_k with the solutions

$$\lambda_k = 0 \quad \text{and} \quad \lambda_k = \mathrm{tr}\,\omega_k.$$

Thus the dynamics on $SL(2)$ with the Lagrangian (4) is described by the following two-valued mapping $\omega_k \mapsto \omega_{k+1}$:

(1) $\omega_{k+1} = -\omega_k^{-1}$ $(\lambda_k = 0)$;

(2) $\omega_{k+1} = (\lambda_k I - \omega_k)^{-1} = \omega_k$ $(\lambda_k = \mathrm{tr}\,\omega_k)$.

The first case corresponds to the periodic dynamics $X_{k+1} = -X_{k-1}$ and has no continuous limit.

The trajectories of the second branch are the shifted one-generator subgroups

$$X_k = \omega^k X_0, \quad \omega = \omega_0^{-1}.$$

In the continuous limit $\omega \longrightarrow I$, one has the standard geodesic flow on $SL(2)$ with the bi-invariant (but indefinite) metric

$$\langle \dot{X}, \dot{X} \rangle = \tfrac{1}{2}\,\mathrm{tr}(\dot{X}\dot{X}^*) = \det \dot{X}.$$

3. *Discrete Lagrangian system on SDiff* $(\mathbb{R}^2)$

Let us now consider the case when $G = SDiff(D)$ is a group of the area-preserving diffeomorphisms of a domain $D \subseteq \mathbb{R}^2$ (in most of the paper we will assume that D is $\mathbb{R}^2$) and define the Lagrangian by the formula [**7**]:

$$\mathcal{L}(f, g) = \iint_D \operatorname{tr}(J(f)J(g)^*)\, d\sigma, \tag{10}$$

where $J(f)$ and $J(g)$ are the Jacobi matrices of $f, g \in SDiff(D)$ and $d\sigma$ is the standard measure on $\mathbb{R}^2$. This Lagrangian has the following properties:

(1) $\mathcal{L}(f, g) = \mathcal{L}(g, f)$;

(2) $\mathcal{L}(f \circ \varphi, g \circ \varphi) = \mathcal{L}(f, g)$, $\varphi \in SDiff(D)$,

and can be considered as an infinite-dimensional analogue of the previous case.

A formal variation of the function $\mathcal{L}(f, g) + \mathcal{L}(g, h)$ with respect to $g \in SDiff(D)$ leads to the equation

$$J(f) + J(h) = \lambda J(g)$$

with a Lagrange multiplier $\lambda = \lambda(x_1, x_2)$. Introducing $\varphi = g \circ f^{-1}$, $\psi = h \circ g^{-1}$, $\chi = \varphi^{-1} = f \circ g^{-1}$ one has

$$J(\chi) + J(\psi) = \lambda I$$

and by the same arguments as above, the following two possibilities arise:

$$\begin{aligned} &\lambda = 0 \Longrightarrow J(\psi) = -J(\chi), \\ &\lambda = \operatorname{tr} J(\chi) = \operatorname{tr} J(\varphi) \Longrightarrow J(\psi) = J(\chi)^{-1} = J(\chi)^*. \end{aligned} \tag{11}$$

In the first case

$$\psi = \sigma_a \circ \varphi^{-1}$$

where σ_a is a central symmetry: $\sigma_a(x) = a - x$, $a, x \in \mathbb{R}^2$, or equivalently $h = \sigma_a \circ f$. Thus the dynamics in this case is trivial:

$$(f, g) \mapsto (g, \sigma_a \circ f) \mapsto (\sigma_a \circ f, \sigma_b \circ g) \mapsto \cdots$$

and has no continuous limit.

The second case is much more interesting. We have

$$\begin{aligned} \partial_1 \psi_1 &= \partial_2 \chi_2, \qquad & \partial_1 \psi_2 &= -\partial_1 \chi_2, \\ \partial_2 \psi_1 &= -\partial_2 \chi_1, \qquad & \partial_2 \psi_2 &= \partial_1 \chi_1, \end{aligned} \tag{12}$$

where $\psi(x) = (\psi_1(x), \psi_2(x))$, $\chi(x) = (\chi_1(x), \chi_2(x))$, $x = (x_1, x_2)$.

The compatibility conditions for (12) have the form

$$\partial_1(\partial_1 \chi_1 + \partial_2 \chi_2) = 0, \quad \partial_2(\partial_1 \chi_1 + \partial_2 \chi_2) = 0, \tag{13}$$

which implies for the connected domain D

$$\operatorname{tr} J(\chi) = \partial_1 \chi_1 + \partial_2 \chi_2 = \operatorname{tr} J(\chi) = \operatorname{tr} J(\varphi) = \tau, \quad \tau = \text{constant}. \tag{14}$$

Thus the second map exists only for special initial data, namely when the Jacobi matrices $J(\varphi)$ of the corresponding map $\varphi = g \circ f^{-1}$ have constant spectrum:

$$\operatorname{tr} J(\varphi) = \tau, \quad \det J(\varphi) = 1.$$

In this case, the mapping ψ defined by (12) has the form

$$\psi_1 = \tau x_1 + a_1 - \chi_1, \quad \psi_2 = \tau x_2 + a_2 - \chi_2, \tag{15}$$

where $\chi = \varphi^{-1}$.

4. *Constant J-spectrum mappings and the Monge-Ampère equation*
Let φ be such a mapping, i.e.

$$\partial_1\varphi_1 + \partial_2\varphi_2 = \tau,$$
$$\partial_1\varphi_1 \cdot \partial_2\varphi_2 - \partial_1\varphi_2 \cdot \partial_2\varphi_1 = 1.$$

Using the first relation one can introduce the 'stream' function G such that

$$\varphi_1 = \frac{\tau}{2}x_1 + \partial_2 G, \quad \varphi_2 = \frac{\tau}{2}x_2 - \partial_1 G. \tag{16}$$

The relation $\det J(\varphi) = 1$ then takes the form

$$\left(\frac{\tau}{2} + \partial_1\partial_2 G\right)\left(\frac{\tau}{2} - \partial_1\partial_2 G\right) + \partial_1^2 G \cdot \partial_2^2 G = 1$$

or in the notation $G_i = \partial_i G$, $G_{ij} = \partial_i\partial_j G$,

$$G_{11}G_{22} - G_{12}^2 = c, \quad c = 1 - \frac{\tau^2}{4}. \tag{17}$$

This equation is known as the Monge–Ampère (MA) equation (more precisely it is a special case of the MA equation which is sometimes called the simple MA equation).

In spite of the nonlinearity of the MA equation, it turns out that the notions of the ellipticity, hyperbolicity and parabolicity do not depend on the solution considered (see [**2**]):

- elliptic case: $c > 0$ $(-2 < \tau < 2)$;
- hyperbolic case: $c < 0$ $(|\tau| > 2)$;
- parabolic case: $c = 0$ $(\tau = \pm 2)$.

This classification is in good agreement with the spectral type of the Jacobi matrices $J(\varphi)$.

In the elliptic case, there exists a result of Jörgens [**4**], which says that all global solutions of (17) are quadratic polynomials

$$G = \alpha_{11}x_1^2 + 2\alpha_{12}x_1x_2 + \alpha_{22}x_2^2 + \beta_1 x_1 + \beta_2 x_2 + \gamma. \tag{18}$$

In the parabolic case, all global solutions were described by Hartman and Nirenberg [**3**], who proved that all such solutions have the form

$$G = \varphi(\alpha_1 x_1 + \alpha_2 x_2) + \beta_1 x_1 + \beta_2 x_2 + \gamma \tag{19}$$

with some function φ, which can be arbitrary.

The corresponding mappings φ for the solutions (18) are simply affine symplectic transformations. For the solutions (19), after a suitable rotation they have a triangular form

$$\varphi : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} \alpha x_1 + f(x_2) \\ \alpha^{-1}x_2 + \beta \end{pmatrix}. \tag{20}$$

In all these cases, the dynamics can be easily described. For the solutions (18), it coincides with the dynamics on SL_2 described in §2.

In the general case, we can observe that the mapping $\varphi \mapsto \psi$ is the composition of two involutions on the space of the (local) solutions of MA equation. The first one, σ, maps the function G into $G^* = \sigma(G)$ such that

$$y = \varphi(x) = \frac{\tau x}{2} + \Omega \nabla G(x)$$
$$x = \varphi^{-1}(y) = \frac{\tau y}{2} + \Omega \nabla G^*(y), \quad \Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{21}$$

The second involution is just

$$\varepsilon : G \mapsto -G. \tag{22}$$

It is easy to see that the functions G and $\widetilde{G}$ corresponding to φ and ψ are related by

$$\widetilde{G} = (\varepsilon \cdot \sigma)(G) = -G^*.$$

For $\tau = 0$, the function G^* is the 'symplectic' Legendre transformation of the function G

$$G^*(y) = \max_x(\langle y, x\rangle + G(x)) = \langle y, \varphi^{-1}(y)\rangle + G \circ \varphi^{-1}(y),$$

where $\langle x, y\rangle$ denotes the symplectic product $\langle x, y\rangle = (\Omega x, y)$. It would be interesting to find an analogous formula for G^* for an arbitrary τ.

In the finite-dimensional case, the integrable discrete versions of the classical systems discussed in [**5**] can also be represented as the compositions of two involutions. This is related to certain factorization problems for matrix polynomials: one of the involutions is the permutation of the factors, another corresponds to the dual factorization (see [**5**]). Finally, this leads to the linearization of the dynamics on the Jacobi variety of the corresponding spectral curve.

To understand the corresponding analogue of this procedure in our infinite-dimensional case is a very interesting problem†.

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A. *Appendix*

The aim of this Appendix is to answer some of the questions raised in the paper (in particular, the last one about linearization of the dynamics). We also discuss the continuous limit of the system.

Let us first rewrite the relations (21) defining the transformation $(x, G) \mapsto (y, G^*)$ in the form

$$y_1 = \frac{\tau}{2}x_1 + G_2(x), \quad y_2 = \frac{\tau}{2}x_2 - G_1(x),$$
$$G_1^*(y) = -cx_2 - \frac{\tau}{2}G_1(x), \quad G_2^*(y) = cx_1 - \frac{\tau}{2}G_2(x).$$

† See Appendix below for further discussion of this problem.

One can check that

$$dG^* = d\left(\left(1-\frac{\tau^2}{2}\right)G - c(x_1G_1 + x_2G_2)\right) - \frac{\tau}{2}(G_1\,dG_2 - G_2\,dG_1 - c(x_1\,dx_2 - x_2\,dx_1)).$$

Let us note that the differential $(G_1\,dG_2 - G_2\,dG_1 - c(x_1\,dx_2 - x_2\,dx_1))$ is closed on the solutions of the MA equation (17), which can be rewritten as $dG_1 \wedge dG_2 = c\,dx_1 \wedge dx_2$. When $\tau = 0$, we have the relation

$$G^* = G - (x_1G_1 + x_2G_2) = y_2x_1 - y_1x_2 + G$$

mentioned above, but for the general τ such a formula for G^* probably does not exist.

It turns out that the dynamics $G \mapsto \widetilde{G} = -G^*$ can be linearized to the following contact transformation of the simple MA equation going back to Goursat [**9**].

Let us restrict ourselves to the hyperbolic case $c < 0$. Introduce $\mu = (-c)^{1/2}$ or, equivalently, the relation

$$\mu^2 = \frac{\tau^2}{4} - 1$$

and define (cf. [**9, 10**]) the following 'half-Legendre' transformation $(x, G(x)) \mapsto (\xi, H(\xi))$:

$$\xi_1 = \mu^{-1}G_1, \quad \xi_2 = x_2,$$
$$H = \mu^{-1}(x_1G_1 - G), \quad H_1 = x_1, \quad H_2 = -\mu^{-1}G_2.$$

One can check that it transforms the MA equation

$$G_{11}G_{22} - G_{12}^2 = c$$

into the linear wave equation

$$H_{11} - H_{22} = 0.$$

The map $(\xi, H) \mapsto (\widetilde{\xi}, \widetilde{H})$ corresponding to the transformation $G \mapsto \widetilde{G} = -G^*$ is described by the following formulas:

$$\widetilde{\xi}_1 = -\frac{\tau}{2}\xi_1 + \mu\xi_2, \quad \widetilde{\xi}_2 = -\mu\xi_1 + \frac{\tau}{2}\xi_2,$$
$$\widetilde{H}_1 = \frac{\tau}{2}H_1 - \mu H_2, \quad \widetilde{H}_2 = \mu H_1 - \frac{\tau}{2}H_2.$$

Notice that it is linear on the space of solutions of the wave equation. If we parametrize this space by two functions, f, g, of one variable according to the standard formula

$$H(\xi_1, \xi_2) = f(\xi_1 + \xi_2) + g(\xi_1 - \xi_2)$$

then the dynamics becomes very simple:

$$\widetilde{f}(z) = \alpha f(\beta z), \quad \widetilde{g}(z) = \alpha^{-1}g(\beta^{-1}z),$$

where

$$\alpha = \frac{\tau^2}{2} - 1 - \mu\tau, \quad \beta = \frac{\tau}{2} + \mu.$$

This answers the question about the linearization of the dynamics and explains the nature of integrability of the system.

Let us now discuss what happens with this discrete dynamics in the continuum limit. Take $\tau = 2$ and consider $y = \varphi(x) = x + \epsilon v(x)$ with a small ϵ. The condition that $\operatorname{tr} J(\varphi) = 2$ implies that the vector field v is divergence free: $\operatorname{div} v = 0$. The condition $\det J(\varphi) = 1$ implies that the stream function G defined by the relations

$$v_1 = -\partial_2 G, \quad v_2 = \partial_1 G$$

satisfies the homogeneous MA equation

$$G_{11}G_{22} - G_{12}^2 = 0. \tag{A.1}$$

We have $\chi(y) = x = y - \epsilon v(y - \epsilon v + \cdots) = y - \epsilon v(y) + \epsilon^2(v, \partial)v(y) + \cdots$. The map $\psi = 2\,\mathrm{Id} - \chi$ up to order two in ϵ has the form $\psi = \varphi - \epsilon^2(v, \partial)v$. Now as usual assume that $\varphi = \varphi(t)$ and $\psi = \varphi(t + \epsilon)$. Then we have $v(t + \epsilon) = v(t) - \epsilon(v, \partial)v$.

In the limit $\epsilon \to 0$, we have the equations

$$v_t + (v, \partial)v = 0, \quad \operatorname{div} v = \partial_1 v_1 + \partial_2 v_2 = 0, \tag{A.2}$$

which are the Euler equations for an ideal fluid in the case when the pressure is constant in space (isobaric flows). One can check that the compatibility condition of these two equations is

$$\partial_1 v_1 \cdot \partial_2 v_2 - \partial_1 v_2 \cdot \partial_2 v_1 = 0, \tag{A.3}$$

which is equivalent to the homogeneous MA equation (A.1) for the stream function G.

In gas and fluid dynamics, the isobaric flows are known to be very special and admit some exact descriptions (see, for example, [**11–13**]). In particular, for an arbitrary function f of one variable and constants $\alpha_1, \alpha_2, \beta_1, \beta_2$, the formulas

$$\begin{aligned} v_1 &= -\alpha_2 f(\alpha_1 x_1 + \alpha_2 x_2 + \gamma t) - \beta_2, \\ v_2 &= \alpha_1 f(\alpha_1 x_1 + \alpha_2 x_2 + \gamma t) + \beta_1, \end{aligned} \tag{A.4}$$

with $\gamma = \alpha_2\beta_1 - \alpha_1\beta_2$, give the global solutions of the system (A.2), (A.3).

The geometrical origin of this system has been clarified by Bao and Ratiu in [**14**], who investigated the extrinsic geometry of the volume-preserving group *SDiff*(M) considered as a submanifold of the full group of diffeomorphisms *Diff*(M) of a Riemannian manifold M. The system (A.2), (A.3) describes the geodesics on the group *Diff*$(\mathbb{R}^2)$, which are also geodesics on the subgroup *SDiff*$(\mathbb{R}^2)$.

We would also like to mention that the simple MA equation (17) has a natural geometrical meaning in classical affine differential geometry: it describes the so-called *improper affine spheres*. Namely, if $x_3 = G(x_1, x_2)$ is the equation of such a sphere with the affine normals parallel to the x_3-axis, then G must satisfy the MA equation $G_{11}G_{22} - G_{12}^2 =$ constant (see, for example, [**15**, p. 219]). It would be interesting to understand the geometric nature of the mapping $G \mapsto G^*$ from this point of view.

Another interesting question is to find the examples of discrete Lagrangian systems with integrable dynamics on other infinite-dimensional groups. An interesting particular case is the Virasoro group *Vir*, which is a central extension of the group *Diff*$_+(S^1)$ of the diffeomorphisms of a circle preserving the orientation. First steps in this direction have been carried out in [**16, 17**], where some Lagrangian discrete systems on *Vir* are discussed.

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