The stability of Killing–Cauchy horizons in colliding plane wave space-times

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Abstract

It is confirmed rigorously that the Killing–Cauchy horizons, which sometimes occur in space-times representing the collision and subsequent interaction of plane gravitational waves in a Minkowski background, are *unstable* with respect to *bounded* perturbations of the initial waves, at least for the case in which the initial waves have constant aligned polarizations.

1 Introduction

Many classes of explicit exact solutions are known which model the collision and subsequent interaction between shock-fronted plane gravitational waves which propagate and collide in a Minkowski background (for reviews see [1] or [2]). In all these solutions, some kind of singularity always appears in the interaction region. This is generally a spacelike curvature singularity (like the time-reverse of an initial cosmological singularity). However large classes of solutions with an infinite number of parameters exist in which the scalar polynomial curvature singularity is replaced by a horizon. In these cases, the space-time can be extended through the horizon, but the extension is not unique. It has become widely believed that such Killing–Cauchy horizons are unstable with respect to small changes in the initial data. For example, Yurtsever [3] has shown that they are unstable with respect to the addition of some perturbative linear field which preserves the G_2 symmetry. However, this result does not answer the question of whether they are unstable with respect to variations in the approaching purely gravitational waves when the vacuum field equations are satisfied exactly. It is the purpose of the present paper to investigate this question in detail.

It must first be pointed out that, in the vast amount of work that was undertaken on this topic in the 1970s and 1980s, and in all the exact solutions that were then produced, the approach was adopted of first solving the field equations in the interaction region. Once a family of such solutions had been obtained, the free parameters were constrained to satisfy the junction conditions that have to be imposed in order to extend the solution to the prior regions. Thus, the approaching waves which physically give rise to the solutions were only determined once the solution had been obtained. Within this context, it was argued that those solutions which contained horizons were unstable with respect to perturbations of the initial data. However, it was not then appreciated that the perturbations which transform a Killing–Cauchy horizon to a scalar polynomial curvature singularity also normally introduce singularities in the initial waves prior to their collision. The question therefore still needs to be addressed as to whether or not

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the horizons are stable with respect to regular (bounded) perturbations of physically acceptable initial waves.

It is only very recently that techniques have been developed by which colliding plane wave solutions can be explicitly constructed from their characteristic initial data representing the approaching waves [4]–[8]. It is therefore only now that tools are available to reconsider this question. In this paper, only the linear vacuum case in which two approaching gravitational waves have constant aligned polarizations will be considered. This situation is much easier to analyse, but it already demonstrates the essential features of the physical situation. The basic result, not surprisingly, is that the Killing–Cauchy horizons that sometimes appear in colliding plane wave space-times are unstable with respect to bounded perturbations of the initial waves that generate these solutions.
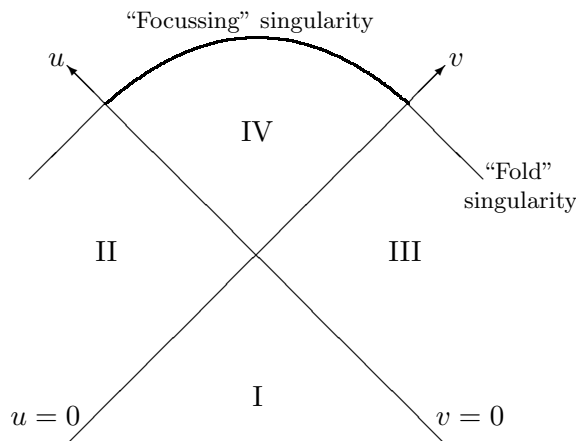

Figure 1: The structure of colliding plane wave space-times. Region I is the background Minkowski space, regions II and III contain the approaching plane waves, and region IV represents the interaction region following the collision. The singularity structure following the collision is known from the study of families of exact solutions and is described in [1].

2   Initial data

Colliding plane wave space-times are naturally divided into four regions as indicated in figure 1. It is found to be convenient to use two future-pointing null coordinates u and v throughout the space-time. The four distinct regions can then be identified as those that are separated by two null hypersurfaces taken as $u = 0$ and $v = 0$ which represent the wavefronts of the approaching waves.

The background region I ($u < 0$, $v < 0$) is a flat vacuum represented by the line element

$$ds^2 = 2\,\mathrm{d}u\,\mathrm{d}v - \mathrm{d}x^2 - \mathrm{d}y^2. \tag{1}$$

Region II ($u \geq 0$, $v < 0$) contains one of the approaching plane waves. If this wave has constant (linear) polarization, it can be described by a metric in the Brinkmann form

$$ds^2 = 2\,\mathrm{d}u\,\mathrm{d}r - \mathrm{d}X^2 - \mathrm{d}Y^2 + h_+(u)(X^2 - Y^2)\,\mathrm{d}u^2, \tag{2}$$

where $h_+(u) = \Psi_{4+}(u)$, which is the only non-zero component of the Weyl tensor (relative to an appropriate tetrad). This arbitrary function explicitly represents the profile of the plane gravitational wave as a function of the retarded time u. (The subscripts $+$ and $-$ are used here and below to denote functions that are defined in regions II and III respectively.)

The form of the metric (2) may be taken to include region I with $h_+ = 0$ for $u < 0$. However, to analyse the collision of plane waves, it is appropriate to initially transform the line element

(2) to the form

$$ds^2 = 2\,\mathrm{d}u\,\mathrm{d}v - e^{2\psi_+}\,\mathrm{d}x^2 - \alpha_+^2\,e^{-2\psi_+}\,\mathrm{d}y^2. \tag{3}$$

As explained in [8], the coordinate u is an affine parameter on the null geodesics $v = \text{const.}$ $x = \text{const.}$ $y = \text{const.}$ which cross the wavefront from the background region. Also, the function $\psi_+(u)$ must satisfy the linear scattering equation

$$(e^{\psi_+})_{,uu} + h_+(u)\,e^{\psi_+} = 0. \tag{4}$$

In order to join (3) smoothly with (1), it is necessary that $\psi_+(0) = 0$ and that $\psi_{+,u}(0)$ must vanish except for any non-zero component which arises from a possible impulsive component on the wavefront. With these conditions, $\psi_+(u)$ is determined uniquely from the initial wave profile $h_+(u)$. The function $\psi_+(u)$ can therefore be taken to represent the initial data to be prescribed on the characteristic $v = 0$.

With $\psi_+(u)$ determined, the initial gravitational wave is given by the Weyl tensor component

$$\Psi_{4+}(u) = -\psi_{+,u}\delta(u) - \left[\psi_{+,uu} + (\psi_{+,u})^2\right]\Theta(u). \tag{5}$$

The other metric function $\alpha_+(u)$ in (3) is then determined for region II from the linear equation

$$\alpha_{+,uu} - 2\,\psi_{+,u}\,\alpha_{+,u} + 2(\psi_{+,u})^2\,\alpha_+ = 0, \tag{6}$$

whose solution is uniquely specified by the initial data $\alpha_+(0) = 1$, and $\alpha_{+,u}(0) = 0$. It can be seen that $\alpha_+(u)$ must be a monotonically decreasing function in this region.

Region III ($u < 0,\ v \geq 0$) contains the other plane wave which approaches from the opposite direction. If this wave also has constant polarization which is aligned with that in region II, the metric can simply be taken as having the same form as that in that region, but with the roles of u and v reversed. Thus, the initial data that is specified in this region will be taken as the function $\psi_-(v)$. The non-zero component of the Weyl tensor $\Psi_{0-}(v)$ representing the second initial gravitational wave is given by an equation equivalent to (5), and the metric function $\alpha_-(v)$ is given by the equivalent of (6) and the corresponding initial data.

Region IV ($u \geq 0,\ v \geq 0$) represents the region in which the waves interact. In the vacuum case for the collision of plane gravitational waves with constant aligned polarization, this region can always be described by the line element

$$ds^2 = 2\,f\,\mathrm{d}u\,\mathrm{d}v - e^{2\psi}\,\mathrm{d}x^2 - \alpha^2\,e^{-2\psi}\,\mathrm{d}y^2, \tag{7}$$

where $\psi(u,v)$, $\alpha(u,v)$ and $f(u,v)$ are now functions of both null coordinates. If characteristic initial data is taken as described above, these functions must satisfy the initial data:

$$
\begin{array}{lll}
\psi(u,0) = \psi_+(u) & \alpha(u,0) = \alpha_+(u) & f(u,0) = 1 \\
\psi(0,v) = \psi_-(v) & \alpha(0,v) = \alpha_-(v) & f(0,v) = 1
\end{array}
\tag{8}
$$

with $\psi(0,0) = 0$, $\alpha(0,0) = 1$ and $f(0,0) = 1$. The necessary (vacuum) field equations will be considered in the following section.

3 The field equations

One of the vacuum field equations for the metric (7) is

$$\alpha_{,uv} = 0,$$

whose solution, satisfying the necessary initial conditions, is given by

$$\alpha(u,v) = \alpha_+(u) + \alpha_+(u) - 1. \tag{9}$$

It is convenient to introduce the functions $\xi(u)$ and $\eta(v)$ such that $2\alpha = \xi(u) - \eta(v)$, where

$$\xi(u) = \begin{cases} 1 & \text{in regions I and III} \\ 2\alpha_+(u) - 1 & \text{in regions II and IV} \end{cases}, \qquad \eta(v) = \begin{cases} -1 & \text{in regions I and II} \\ 1 - 2\alpha_-(v) & \text{in regions III and IV} \end{cases}$$

Then, since $\alpha_+(u)$ and $\alpha_-(v)$ are monotonically decreasing, $\xi(u)$ must be a monotonically decreasing function and $\eta(v)$ monotonically increasing in the interaction region.

The other main vacuum field equation is

$$2\,\alpha\,\psi_{,uv} + \alpha_{,u}\,\psi_{,v} + \alpha_{,v}\,\psi_{,u} = 0, \tag{10}$$

where α is given by (9) and the initial data by (8). For any solution of (10) satisfying the initial data, the additional metric function f can be obtained by integrating the remaining field equations which are

$$\frac{f_{,v}}{f} = \frac{\alpha_{,vv}}{\alpha_{,v}} + \frac{2\,\alpha\,\psi_{,v}{}^2}{\alpha_{,v}} - 2\,\psi_{,v}, \qquad \frac{f_{,u}}{f} = \frac{\alpha_{,uu}}{\alpha_{,u}} + \frac{2\,\alpha\,\psi_{,u}{}^2}{\alpha_{,u}} - 2\,\psi_{,u}. \tag{11}$$

The non-zero components of the Weyl tensor in the interaction region are then

$$\Psi_0 = -\psi_{,v}\delta(v) - \frac{\psi_{,vv} + \psi_{,v}{}^2}{f} + \frac{2\,\psi_{,v}\,f_{,v}}{f^2}$$

$$\Psi_2 = \frac{1}{2\alpha f}\left(2\alpha\,\psi_{,v}\,\psi_{,u} - \alpha_{,u}\,\psi_{,v} - \alpha_{,v}\,\psi_{,u}\right) \tag{12}$$

$$\Psi_4 = -\psi_{,u}\delta(u) - \frac{\psi_{,uu} + \psi_{,u}{}^2}{f} + \frac{2\,\psi_{,u}\,f_{,u}}{f^2}$$

It may be observed that a solution can be determined in region IV up to the spacelike hypersurface on which $\alpha = \frac{1}{2}(\xi - \eta) = 0$, on which some kind of singularity will occur. This is referred to as a "focussing" singularity in figure 1. Its character will be discussed below.

In view of the properties described above, it is always possible to adopt ξ and η as coordinates throughout the interaction region, in which $\xi < 1$, $\eta > -1$ and $\xi - \eta > 0$. In terms of these coordinates, the main field equation (10) becomes

$$(\xi - \eta)\,\psi_{,\xi\eta} - \tfrac{1}{2}\,\psi_{,\xi} + \tfrac{1}{2}\,\psi_{,\eta} = 0, \tag{13}$$

which is an Euler–Poisson–Darboux equation with non-integer coefficients, whose solution $\psi(\xi, \eta)$ must satisfy the initial data

$$\psi(\xi, -1) = \psi_+(\xi), \qquad \psi(1, \eta) = \psi_-(\eta), \tag{14}$$

with $\psi(1, -1) = 0$. This is sufficient to uniquely determine the solution of the colliding plane wave problem in the interaction region. However, as explained in [8], to satisfy the initial data, $\psi(\xi, \eta)$ must be nonanalytic on each wavefront where it behaves like $\sqrt{1 - \xi}$ or $\sqrt{1 + \eta}$ respectively. Also, in terms of these coordinates, the non-zero Weyl tensor components are

$$\Psi_0 = -\eta_{,v}\,\psi_{,\eta}\delta(v) - \frac{\eta_{,v}^2}{f}\left(\psi_{,\eta\eta} + 2(\xi - \eta)\,\psi_{,\eta}{}^3 + 3\psi_{,\eta}{}^2\right)$$

$$\Psi_2 = -\frac{\xi_{,u}\eta_{,v}}{2f(\xi - \eta)}\left(2(\xi - \eta)\psi_{,\xi}\,\psi_{,\eta} + \psi_{,\xi} - \psi_{,\eta}\right) \tag{15}$$

$$\Psi_4 = -\xi_{,u}\,\psi_{,\xi}\delta(u) - \frac{\xi_{,u}^2}{f}\left(\psi_{,\xi\xi} - 2(\xi - \eta)\,\psi_{,\xi}{}^3 + 3\psi_{,\xi}{}^2\right).$$

4   The singularity in the interaction region

Now consider the singularity which occurs in region IV on the spacelike hypersurface on which $\alpha = \frac{1}{2}(\xi - \eta) = 0$. It can be shown (see references in [1]) that this is generically a scalar polynomial curvature singularity on which the invariant $\Psi_0\Psi_4 - 4\Psi_1\Psi_3 + 3\Psi_2^2$ diverges. However, families of solutions for colliding plane waves exist, even having infinitely many parameters, in which this invariant does not diverge. In these cases, the singularity at $\alpha = 0$ corresponds to a Killing–Cauchy horizon through which the space-time can be extended.

The general solution of the main equation (13) can be expressed (exactly as for vacuum Gowdy cosmologies) in terms of the coordinates $\alpha = \frac{1}{2}(\xi - \eta)$ and $\beta = \frac{1}{2}(\xi + \eta)$ in the form

$$\psi = \int_{-\infty}^{\infty} \Big[a(\omega)J_0(\omega\alpha) + b(\omega)Y_0(\omega\alpha) \Big] e^{i\omega\beta} \mathrm{d}\omega, \tag{16}$$

where $J_0(\omega\alpha)$ and $Y_0(\omega\alpha)$ are Bessel functions of the first and second kinds of zero order, and $a(\omega)$ and $b(\omega)$ are arbitrary functions of the parameter ω. The terms which involve $J_0(\omega\alpha)$ remain well behaved on the singularity at $\alpha = 0$, while the terms which involve $Y_0(\omega\alpha)$ diverge logarithmically (the presence of some terms of this type is necessary to satisfy the initial data).

In his analysis of the asymptotic behaviour of colliding plane wave space-times, Yurtsever [9] has shown that, for any timelike geodesic which approaches the singularity $\alpha = 0$, the metric asymptotically approaches that of a particular Kasner solution. However, the particular Kasner exponents vary for geodesics which approach different points on the singularity. It follows that the focussing singularity is generally a curvature singularity. The only exception is that in which the metric approaches that of the degenerate (flat) Kasner solution at all points over the hypersurface $\alpha = 0$. In this case, the focussing hypersurface is a Killing–Cauchy horizon across which space-time can be extended. Such solutions are characterised by the asymptotic behaviour

$$\psi(\xi, \eta) \simeq \log\alpha - B(\beta) \qquad \text{or} \qquad \psi(\xi, \eta) \simeq B(\beta) \tag{17}$$

as $\alpha \to 0$, where $B(\beta)$ is an arbitrary function which is independent of α. However, the substitution $\psi \to \log\alpha - \psi$ simply corresponds to an interchange of the x, y coordinates. Thus, the two possible conditions in (17) are physically equivalent, and the second can be adopted as the criteria for a colliding plane wave solution with a horizon.

In considering perturbations of such solutions, Yurtsever [9] has rigorously shown that arbitrarily small changes in the amplitude functions (specifically in the coefficients $b(\omega)$ in (16)) will cause variations in the Kasner parameters. He has reasonably concluded that, although there are colliding plane wave space-times which contain a Killing–Cauchy horizon rather than a space-like curvature singularity, these space-times are unstable against small perturbations of the initial data and that 'generic' initial data leads to curvature singularities. However, it may be observed that the perturbations which induce a curvature singularity in the interaction region also generally have the effect that the relevant Weyl tensor component becomes unbounded at the fold singularities in the initial regions II and III. In other words, the components which introduce a curvature singularity into a space-time which otherwise has a horizon are associated with unphysical initial data. Thus, the question that still needs to be addressed is that of the stability of the horizon with respect to perturbations which correspond to bounded variations of the initial gravitational waves prior to their collision.

5   Explicit solutions with horizons

Now consider the colliding plane wave solutions [10] for which

$$\psi = \frac{(1+a)}{2}\log\alpha - \frac{k_+}{2}\cosh^{-1}\left(\frac{1-\beta}{\alpha}\right) - \frac{k_-}{2}\cosh^{-1}\left(\frac{1+\beta}{\alpha}\right), \tag{18}$$

where a, k_+ and k_- are constants. When $a = 0$ and $k_+ = k_- = 1$, this is the Khan–Penrose solution [11] which represents the collision of purely impulsive waves. It also includes the degenerate Ferrari–Ibáñez solutions [12], for which the interaction region is locally isomorphic to part of the Schwarzschild solution inside the horizon.

By showing that, as $\alpha \to 0$, the scalar invariant behaves as

$$\Psi_0\Psi_4 + 3\Psi_2^2 \to \frac{\xi_{,u}{}^2\eta_{,v}{}^2}{2^8 f^2}\Big((a + k_+ + k_-)^2 - 1\Big)^2\Big((a + k_+ + k_-)^2 + 3\Big)\alpha^{-4},$$

Feinstein and Ibáñez [13] have shown that colliding plane wave solutions given by (18) have a horizon at $\alpha = 0$ in region IV rather than a curvature singularity if $a + k_+ + k_- = \pm 1$. (This also follows from the condition (17).)

It can be seen that the last two terms in the expansion (18) are required to describe the necessary behaviour of the solution near the wavefronts $\xi = 1$ and $\eta = -1$ respectively, on which they also become zero. Also, all three components of (18) diverge logarithmically as $\alpha \to 0$. Thus, these terms must include components which arise from the $Y_0(\omega\alpha)$ terms in (16) together with possible components which arise from the regular terms. In addition, it is clear that the solutions (18) arise from the initial data functions

$$\psi_+(\xi) = \frac{(1+a)}{2}\log\left(\frac{1+\xi}{2}\right) - \frac{k_+}{2}\cosh^{-1}\left(\frac{3-\xi}{1+\xi}\right),$$

$$\psi_-(\eta) = \frac{(1+a)}{2}\log\left(\frac{1-\eta}{2}\right) - \frac{k_-}{2}\cosh^{-1}\left(\frac{3+\eta}{1-\eta}\right). \tag{19}$$

Thus the constants are determined by the initial data. Moreover, to satisfy the initial conditions for colliding plane waves, it can be shown that $k_\pm$ must satisfy $1 \le |k_\pm| < \sqrt{2}$. It can then be seen that the initial wave in region II is

$$\Psi_{4+} = -\frac{k_+\xi_{,u}}{2\sqrt{2(1-\xi)}}\delta(u) - \frac{\xi_{,u}{}^2}{4(1+\xi)^2}\left[a(1-a^2) - \frac{3\sqrt{2}a^2k_+}{\sqrt{1-\xi}} - \frac{6\,a\,k_+^2}{(1-\xi)} + \frac{2\sqrt{2}k_+(1-k_+^2)}{(1-\xi)^{3/2}}\right],$$

which, as $\xi \to -1$, behaves as

$$\Psi_{4+} \to \frac{(a + k_+)}{4}\left[(a + k_+)^2 - 1\right]\frac{\xi_{,u}{}^2}{(1+\xi)^2}.$$

This clearly diverges unless $a + k_+$ has the values 0, $+1$ or -1. The opposing wave similarly diverges as $\eta \to 1$ unless $a + k_-$ has the values 0, $+1$ or -1. Thus, within this family of solutions, initial waves which are bounded as $\xi \to -1$ or $\eta \to 1$ and which give rise to solutions with a Killing–Cauchy horizon can only occur in a very limited number of possible cases. In fact, the only possibility up to a sign occurs when $a = -1$ and $k_+ = k_- = 1$. This is precisely the degenerate Ferrari–Ibáñez solution with a horizon [12].

Feinstein and Ibáñez [13] have also shown that the above result can be extended to solutions in which ψ is given by (18) plus an arbitrary regular function that can be expressed as $\int_{-\infty}^{\infty} a(\omega)J_0(\omega\alpha)e^{i\omega\beta}d\omega$ as in (16). (This is currently the widest explicitly known class of colliding plane wave solutions with a horizon.) It follows that perturbations which correspond to variations in the arbitrary function $a(\omega)$ do not alter the character of the horizon at $\alpha = 0$. On the other hand, perturbations of any of the parameters a or $k_\pm$ lead to the occurrence of a curvature singularity in region IV and also give rise to to divergencies in the initial waves. Perturbations corresponding to additional components of $b(\omega)$ in (16) similarly introduce both a curvature singularity in the interaction region and divergencies in the initial waves.

6   The stability of solutions with horizons

From the above discussion, it may be concluded that some perturbations which indicate that horizons in colliding plane wave space-times are unstable correspond to unbounded perturbations

of the initial data. On the other hand, such horizons appear to be stable with respect to perturbations which correspond to arbitrary variations in the function $a(\omega)$. In this case, however, although the corresponding perturbations of the initial waves are bounded, this corresponds to a situation in which the perturbation of one wave is exactly correlated with the perturbation of the other. Thus, it is not possible to perturb the initial data on one characteristic while keeping constant those on the other and at the same time preserving the character of the horizon in the interaction region. It follows that, for the above explicit solutions at least, the horizon must be *unstable* with respect to small *independent* perturbations of the initial data.

To demonstrate the generic instability of any colliding plane wave solution with a horizon, consider a perturbation for which the initial data is given by

$$\psi_+(\xi) = \psi_{b+}(\xi) + R_+(\xi), \qquad \psi_-(\eta) = \psi_{b-}(\eta), \tag{20}$$

where $\psi_{b+}(\xi)$ and $\psi_{b-}(\eta)$ are the initial data which lead to a background solution with a horizon, and $R_+(\xi)$ is a perturbation function which is bounded for $\xi \in [-1, 1]$ and such that $R_{+,\xi} \to 0$ as $\xi \to -1$. i.e. a perturbation has been included in one initial wave only and, when $R_+(\xi) = 0$, a horizon occurs in the interaction region.

Use can now be made of the solution of the characteristic initial value problem for colliding plane waves given by Hauser and Ernst [4] and implemented in [7]. However, it is appropriate to present this in a slightly modified form which is more clearly related to that of the general non-linear case described in [8]. In this method, it is first necessary to calculate the initial spectral amplitude functions $A_+(\zeta_+)$ and $A_-(\zeta_-)$ from the initial data functions $\psi_+(\xi)$ and $\psi_-(\eta)$ using the integrals

$$A_+(\zeta_+) = -\frac{1}{\pi\sqrt{1-\zeta_+}} \int_{\zeta_+}^1 \frac{\psi_{+,\xi}}{\sqrt{\xi-\zeta_+}} \, d\xi, \qquad A_-(\zeta_-) = \frac{1}{\pi\sqrt{\zeta_-+1}} \int_{-1}^{\zeta_-} \frac{\psi_{-,\eta}}{\sqrt{\zeta_- - \eta}} \, d\eta. \tag{21}$$

The solution in the interaction region can then be expressed as

$$\psi(\xi,\eta) = \int_\xi^1 \frac{\sqrt{1-\zeta_+}\sqrt{\zeta_++1}}{\sqrt{\zeta_+-\xi}\sqrt{\zeta_+-\eta}} A_+(\zeta_+) \, d\zeta_+ + \int_{-1}^\eta \frac{\sqrt{1-\zeta_-}\sqrt{\zeta_-+1}}{\sqrt{\xi-\zeta_-}\sqrt{\eta-\zeta_-}} A_-(\zeta_-) \, d\zeta_-. \tag{22}$$

This method, which is based on the Abel transform, is linear. Thus, when the perturbation vanishes (i.e. when $R_+(\xi) = 0$), the resulting solution would be just the background solution $\psi_b(\xi,\eta)$ which contains a horizon. However, the perturbation term in (20) would lead to an additional perturbation component in the initial spectral data given by

$$A_{p+}(\zeta_+) = -\frac{1}{\pi\sqrt{1-\zeta_+}} \int_{\zeta_+}^1 \frac{R_{+,\xi}}{\sqrt{\xi-\zeta_+}} \, d\xi, \qquad A_{p-}(\zeta_-) = 0,$$

and this would lead to the solution in the interaction region being given by

$$\psi = \psi_b + \psi_p, \qquad \text{where} \qquad \psi_p = \int_\xi^1 \frac{\sqrt{1-\zeta_+}\sqrt{\zeta_++1}}{\sqrt{\zeta_+-\xi}\sqrt{\zeta_+-\eta}} A_{p+}(\zeta_+) \, d\zeta_+. \tag{23}$$

It can then be shown that the scalar invariant $\Psi_0\Psi_4 + 3\Psi_2^2$ diverges as $\alpha \to 0$ (actually, it is the component Ψ_2 which diverges), unless $\psi_{p,\xi} - \psi_{p,\eta} \to 0$ in this limit. (This is equivalent to the condition (17).)

However, the perturbation $R_+(\xi)$ is arbitrary and it is therefore possible to consider the component $A_{p+}(\zeta_+)$ to be an arbitrary bounded function satisfying $\lim_{\zeta_+\to 1} (\sqrt{1-\zeta_+}\, A_{p+}(\zeta_+)) = 0$. Moreover, it can be seen that

$$\psi_{p,\xi} - \psi_{p,\eta} = \int_\xi^1 \frac{\sqrt{\zeta_+-\eta}}{\sqrt{\zeta_+-\xi}} \frac{d}{d\zeta_+} \left[\frac{\sqrt{1-\zeta_+}\sqrt{\zeta_++1}}{(\zeta_+-\eta)} A_{p+}(\zeta_+) \right] d\zeta_+, \tag{24}$$

which does not necessarily approach zero as $\alpha \to 0$ for an arbitrary choice of $A_{p+}(\zeta_+)$. (Indeed, it may well be unbounded, but this stronger result is not needed.) It can therefore be concluded that a perturbation of one initial wave will generically lead to a scalar polynomial curvature singularity in the interaction region in place of the horizon. i.e. the Killing–Cauchy horizon is *unstable* with respect to the perturbation (20).

7 Conclusion

Colliding plane wave solutions which contain a Killing–Cauchy horizon, only occur in very particular circumstances. Moreover, colliding plane wave space-times which are generated by bounded initial waves (apart from impulsive components on the shock fronts) and possess a Killing–Cauchy horizon form a highly restricted subclass of these solutions. It has been shown above that the only permitted perturbations of this subclass which preserve such a horizon are those in which perturbations of the two approaching waves are strongly correlated. However, when the initial waves are perturbed independently, then the horizon in the interaction region is necessarily replaced by a scalar polynomial curvature singularity. It is therefore concluded that the Killing–Cauchy horizon which sometimes occurs in colliding plane wave space-times is unstable with respect to general bounded perturbations of the initial waves.

The above discussion has concentrated entirely on the "linear case" in which the approaching gravitational waves have constant and aligned polarization. However, it is expected that the behaviour of the nonlinear case in which the initial waves have variable or non-aligned polarizations, or include electromagnetic waves, would be qualitatively similar. This expectation is supported by the arguments of Yurtsever [14] for the vacuum case, but further analysis is required.

Acknowledgments

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