Generation of internal undular bores by transcritical flow over topography

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Abstract

In both the ocean and the atmosphere, the interaction of a density stratified flow with topography can generate large-amplitude, horizontally propagating internal solitary waves. Often these waves appear as a wave-train, or undular bore. In this article we focus on the situation when the flow is critical, that is, the flow speed is close to that of a linear long wave mode. In the weakly nonlinear regime, this is modeled by the forced Korteweg de Vries equation. We will demonstrate how Whitham's modulation theory may be applied to obtain an analytical description of undular bores, for flow over isolated obstacles and for flow over a step.

1 Introduction

Solitary waves are nonlinear waves of quasi-permanent form, first observed by Russell (1844) in a now famous report on his observations of a free surface solitary wave in a canal, and his subsequent experiments. Theoretical work by Boussinesq (1871) and Rayleigh (1876) later established a theoretical model, and then Korteweg and de Vries (1895) derived the well-known equation which now bears their names. But it was not until the second half of the twentieth century that it was realised that the Korteweg-de Vries equation was a valid model for solitary waves in a wide variety of physical contexts. Of principal concern here are the large-amplitude solitary waves which propagate in density-stratified fluids such as the ocean and atmosphere (see, e.g., Apel (1995), Grimshaw (2001), Holloway et al (2001) and Rottmann and Grimshaw (2001)). They owe their existence to a balance between nonlinear wave-steepening effects and linear wave dispersion, and hence can be effectively modeled by nonlinear evolution equations of the Korteweg-de Vries (KdV) type.

Often, these waves are generated by critical flow over topography, and in this situation the waves appear as upstream and downstream wavetrains, each having the character of an undular bore. In this situation the appropriate model equation is the forced Korteweg-de Vries equation (fKdV), given by

$$-\frac{1}{c}(A_t + \Delta A_x) + \mu A A_x + \lambda A_{xxx} + \frac{1}{2}F_x = 0\,, \tag{1}$$

Here $A(x,t)$ is the amplitude of the wave, and x, t are space and time variables respectively. The coefficients c and Δ are the relevant linear long wave speed (equal to the flow speed U at criticality) and the departure from criticality (i.e. $\Delta = U - c$); the coefficients μ and λ of the nonlinear and dispersive terms are determined by the waveguide properties of the specific physical system being considered, while the forcing term $F(x)$ is the projection of the topography onto the relevant waveguide mode. The fKdV equation was derived by Akylas (1984) for water waves and by Grimshaw and Smyth (1986) for internal waves.

In this paper, we shall review the theory of the undular bore based on the Korteweg-de Vries equation in Section 2, and then in Section 3 review how that theory has been used to describe the generation of undular bores by flow over an isolated obstacle. Then in Section 4 we extend that theory to describe how upstream undular bores are generated by flow over a forward-facing topographic step, and downstream by a backward-facing step.

2 Undular bore

The term "undular bore" is widely used in the literature in a variety of contexts. Here, we are concerned with non-dissipative flows, in which case an undular bore is intrinsically unsteady. In general, an undular bore is an oscillatory transition between two different basic states. A simple representation of an undular bore can be obtained from the solution of the unforced KdV equation, that is (1) with $F(x) \equiv 0$, written here in the canonical form,

$$A_t + 6AA_x + A_{xxx} = 0\,. \tag{2}$$

with the initial condition that

$$A = A_0 H(-x)\,, \tag{3}$$

where we assume at first that $A_0 > 0$. Here $H(x)$ is the Heaviside function (i.e. $H(x) = 1$ if $x > 0$ and $H(x) = 0$ if $x < 0$). The solution can in principle be obtained through the inverse scattering transform. However, it is more instructive to use the asymptotic method developed by Gurevich and Pitaevskii (1974), and Whitham (1974). In this approach, the solution of (2) with this initial condition is represented as the modulated periodic wave train

$$A = a\{b(m) + \mathrm{cn}^2(\gamma(x - Vt); m)\} + d\,, \tag{4}$$

$$\text{where} \qquad b = \frac{1-m}{m} - \frac{E(m)}{mK(m)}\,, \quad a = 2m\gamma^2\,, \tag{5}$$

$$\text{and} \quad V = 6d + 2a\left\{\frac{2-m}{m} - \frac{3E(m)}{mK(m)}\right\}\,. \tag{6}$$

Here $cn(x;m)$ is the Jacobian elliptic function of modulus $m, 0 < m < 1$, $K(m), E(m)$ are the elliptic integrals of the first and second, γ is a wavenumber such that the spatial period is $2K(m)/\gamma$ and a, d are the amplitude and mean level respectively, As the modulus $m \to 1$, this becomes a solitary wave, since then $b \to 0$ and $\mathrm{cn}^2(x) \to \mathrm{sech}^2(x)$, but as $m \to 0$ it reduces to sinusoidal waves of small amplitude $a \sim m$ and wavenumber 2γ.

The asymptotic method of Gurevich and Pitaevskii (1974) and Whitham (1974) is to let the expression (4) describe a modulated periodic wavetrain in which the amplitude a, the mean level d, the speed V and the wavenumber γ are all slowly varying functions of x and t. The relevant asymptotic solution corresponding to the initial condition (3) can now be constructed in terms of the similarity variable x/t, and is given by

$$\frac{x}{t} = 2A_0\left\{1 + m - \frac{2m(1-m)(K(m)}{E(m)-(1-m)K(m)}\right\}\,,$$

$$\text{for} \quad -6A_0 < \frac{x}{t} < 4A_0\,, \tag{7}$$

$$a = 2A_0 m\,, \quad d = A_0\left\{m - 1 + \frac{2E(m)}{K(m)}\right\}\,. \tag{8}$$

Ahead of the wavetrain where $x/t > 4A_0, A = 0$ and at this end, $m \to 1$, $a \to 2A_0$ and $d \to 0$; the leading wave is a solitary wave of amplitude $2A_0$ relative to a mean level of 0. Behind the wavetrain where $x/t < -6A_0$, $A = A_0$ and at this end $m \to 0$, $a \to 0$, and $d \to A_0$; the wavetrain is now sinusoidal with a wavenumber $2\gamma = 2\sqrt{A_0}$. Further, it can be shown that on any individual crest in the wavetrain, $m \to 1$ as $t \to \infty$. In this sense, the undular bore evolves into a train of solitary waves.

If $A_0 < 0$ in the initial condition (3), then an "undular bore" solution analogous to that described by (4, 7) does not exist. Instead, the asymptotic solution is a rarefaction wave,

$$\begin{aligned} &A = 0 \quad \text{for} \quad x > 0\,, \\ &A = \frac{x}{6t} \quad \text{for} \quad A_0 < \frac{x}{6t} < 0\,, \\ &A = A_0, \quad \text{for} \quad \frac{x}{6t} < A_0(< 0)\,. \end{aligned} \tag{9}$$

Small oscillatory wavetrains are needed to smooth out the discontinuities in A_x at $x = 0$ and $x = -6A_0$ (for further details, see Gurevich and Pitaevskii 1974).

3 The generation of undular bores by flow over localized topography

The fKdV equation in canonical form is obtained by putting

$$t = \frac{t^*}{\lambda c}\,,\; A = \frac{6\lambda}{\mu}A^*\,,\; F = \frac{12\lambda^2}{\mu}F^*\,,\; \Delta = \lambda c \Delta^*\,, \tag{10}$$

in (1). Omitting the superscript we get

$$-A_t - \Delta A_x + 6AA_x + A_{xxx} + F_x(x) = 0\,. \tag{11}$$

This is to be solved with the initial condition that $A(x,0) = 0$, which corresponds to a slow introduction of the topographic obstacle. An important issue here is the polarity of the forcing in (11), that is, whether it has positive (negative) polarity $F(x) \geq 0 (\leq 0)$. The following summary is based on Grimshaw and Smyth (1986).

First we recall the typical solution of (11) when the forcing $F(x)$ is positive and localized. That is, $F(x)$ is positive, and non-zero only in a vicinity of $x = 0$, with a maximum value of $F_M > 0$. A typical solution at exact criticality ($\Delta = 0$) is shown in Figure 1. The solution is characterised by upstream and downstream wavetrains connected by a locally steady solution over the obstacle. For supercritical flow ($\Delta < 0$) the upstream wavetrain weakens, and for sufficiently large $|\Delta|$ detaches from the obstacle, while the downstream wavetrain intensifies and for sufficiently large $|\Delta|$ forms a stationary lee wave field. On the other hand, for supercritical flow ($\Delta > 0$) the upstream wavetrain develops into well-separated solitary waves while the downstream wavetrain weakens and moves further downstream (for more details see Grimshaw and Smyth 1986 and Smyth 1987).

The origin of the upstream and downstream wavetrains can be found in the structure of the locally steady solution over the obstacle. In the transcritical regime this is characterised by a transition from a constant state A_- upstream of the obstacle to a constant state A_+ downstream of the obstacle, where $A_- < 0$ and $A_+ > 0$. It is readily shown that $\Delta = 3(A_+ + A_-)$ independently of the details of the forcing term $F(x)$. Explicit determination of A_+ and A_- requires some knowledge of the forcing term $F(x)$. However, in the "hydraulic" limit when the linear dispersive term in (11) can be neglected, it is readily shown that

$$6A_\pm = \Delta \mp (12F_M)^{1/2}\,. \tag{12}$$

This expression also serves to define the transcritical regime, which is

$$|\Delta| < (12F_M)^{1/2}\,. \tag{13}$$

Thus upstream of the obstacle there is a transition from the zero state to A_-, while downstream the transition is from A_+ to 0; each transition is effectively generated at $X = 0$.

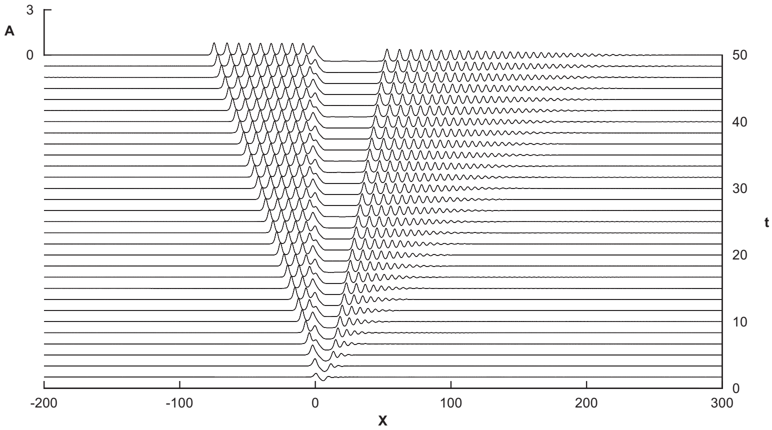

Figure 1: Numerical solution of the fKdV equation (11) for flow over a localized obstacle, with $\Delta = 0$. The obstacle is not shown, but is located at $x = 0$

Both transitions are resolved by "undular bore" solutions as described in section 2. That in $x < 0$ is exactly described by (4) to (8) with x replaced by $\Delta t - x$, and A_0 by A_-. It occupies the zone

$$\Delta - 4A_- < \frac{x}{t} < \min\{0,\ \Delta + 6A_-\}\,. \tag{14}$$

Note that this upstream undular bore is constrained to lie in $x < 0$, and hence is only fully realised if $\Delta < -6A_-$. Combining this criterion with (12) and (13)) defines the regime

$$-(12F_M)^{1/2} < \Delta < -\frac{1}{2}(12F_M)^{1/2}\,, \tag{15}$$

where a fully developed undular bore solution can develop upstream. On the other hand, the regime $\Delta < -6A_-$ or

$$-\frac{1}{2}(12F_M)^{1/2} < \Delta < (12F_M)^{1/2}\,, \tag{16}$$

is where the upstream undular bore is only partially formed, and is attached to the obstacle. In this case the modulus m of the Jacobian elliptic function varies from 1 at the leading edge (thus describing solitary waves) to a value m_- (< 1) at the obstacle, where m_- can be found from (7) by replacing x/t with Δ and A_0 with A_-. For instance, when $\Delta = 0$, $m_- = 0.8$ independently of F_M.

The transition in $x > 0$ can also be described by (4) to (8) where we now replace x with $(\Delta - 6A_+)t - x$, A_0 with $-A_+$, and d with $d - A_+$. This "undular bore" solution occupies the zone

$$\max\{0,\ \Delta - 2A_+\} < \frac{x}{t} < \Delta - 12A_+\,. \tag{17}$$

Here, this downstream wavetrain is constrained to lie in $x > 0$, and hence is only fully realised if $\Delta > 2A_+$. Combining this criterion with (12) and (13) defines the regime (16), and so a fully detached downstream undular bore coincides with the case when the upstream undular bore is attached to the obstacle. On the other hand, in the regime (15), when the upstream undular bore is detached from the obstacle, the downstream undular bore is attached to the obstacle, and the modulus m varies from $m_+(< 1)$ at the obstacle, where m_+ can be founding from (7) by replacing x/t with $\Delta - 6A_+$ and A_0 with $-A_+$ to $m = 0$ at the trailing edge. Further, a stationary lee wavetrain may develop just behind the obstacle (for further details, see Smyth, 1987). At the transition to non-resonant subcritical flow, where $\Delta = -\sqrt{12F_M}, m_+ = 0.96$ independently of F_M.

For the case when the obstacle has negative polarity (that is $F(x)$ is negative, and non-zero only in the vicinity of $x = 0$), the upstream and downstream solutions are qualitatively similar. However, the solution in the vicinity of the obstacle remains transient, and this causes a modulation of the "undular bore" solutions.

4 Generation of solitary waves by flow over a step

Here we consider the situation when the forcing term in (1) has a step-like structure, that is,

$$\begin{aligned} F(x) &= 0\,, \quad \text{for} \quad 0 < x < L\,, \\ F(x) &= F_M\,, \quad \text{for} \quad x > L\,, \end{aligned} \tag{18}$$

and $F(x)$ varies monotonically in $0 < x < L$. A positive (negative) step has $F_M > (<)0$. Strictly $F(x)$ should return to zero for some $L_1 >> L$. Here we ignore this, and in effect assume that $L_1 \to \infty$. In practice it means that the solutions constructed below are only valid for some limited time, determined by how long it takes for a disturbance to travel a distance L_1.

We shall sketch how the solution for the localized forcing described above becomes modified for a step, and adapt the approach used by Grimshaw and Smyth (1986) where we first construct the local steady-state solution in the forcing region, $0 < x < L$, using the "hydraulic" limit. In this limit $A = A(x), 0 < x < L$ while

$$A = A_- \quad \text{for} \quad x < 0\,, \tag{19}$$

$$\text{and} \quad A = A_+ \quad \text{for} \quad x > L\,. \tag{20}$$

It is readily found that

$$-\Delta A + 3A^2 + F = C\,. \tag{21}$$

Here the constant C is determined by considering the long-time limit of the unsteady hydraulic solution, as in Grimshaw and Smyth (1986). But note that

$$C = -\Delta A_- + 3A_-^2 = -\Delta A_+ + 3A_+^2 + F_M \,,$$

giving a connection between A_- and A_+.

Suppose first that the step is positive, $F_M > 0$. The the local hydraulic solution is,

$$\Delta \leq 0: \quad 6A_- = \Delta + (\Delta^2 + 12F_M)^{1/2}\,, \quad 6A_+ = 0\,, \tag{22}$$

$$0 < \Delta < (12F_M)^{1/2}: \quad 6A_- = \Delta + (12F_M)^{1/2}\,, \quad 6A_+ = \Delta\,, \tag{23}$$

$$\Delta > (12F_M)^{1/2}: \quad 6A_- = 0 \quad 6A_+ = \Delta - (\Delta^2 - 12F_M)^{1/2}\,, \tag{24}$$

Here the constant in (21) is $C = F_M, F_M - \Delta^2/12, 0$ respectively. In all cases, the upstream solution $A_- > 0$ is a "shock" in the hydraulic limit (although in (24) the shock has zero strength and so can be ignored), which needs to be replaced with an "undular bore" as in section 3. But importantly note that the upstream elevation A_- is different from that found for flow over a localized obstacle. The undular bore is again given by (4) to (8) with x replaced by $\Delta t - x$ and A_0 by A_- and occupies the zone (compare (14))

$$\Delta - 4A_- < \frac{x}{t} < \min\{0,\ \Delta + 6A_-\}\,. \tag{25}$$

But now A_- is given by (22, 23) in place of (12). For a fully detached undular bore, $\Delta + 6A_- < 0$, and combining this criterion with (22, 23), we get the regime

$$\Delta < -2(F_M)^{1/2} < 0\,. \tag{26}$$

On the other hand the regime where $\Delta + 6A_- > 0$ but $\Delta - 4A_- < 0$, or

$$-2(F_M)^{1/2} < \Delta < (12F_M)^{1/2}\,, \tag{27}$$

is where the upstream undular bore is only partially formed and is attached to the obstacle. As for the localized forcing case described in section 3, the modulus m of the Jacobian elliptic function varies from 1 at the leading edge (thus describing solitary waves) to a value m_- (< 1) at the obstacle, where m_- can be found from (7) by replacing x/t with Δ and A_0 with A_-. As before, when $\Delta = 0$, $m_- = 0.8$ independently of F_M.

Downstream, for $\Delta > 0, A_+ > 0$, and the hydraulic solution is terminated by a rarefraction wave; hence, no undular bore solution is needed. Instead a weak oscillatory wave train is needed to smooth the corners. When $\Delta < 0, A_- = 0$, but stationary lee waves may form. Thus the overall scenario that is predicted from these considerations is the formation of a detached upstream undular bore when (26) holds, an attached undular bore when (27) holds, and the possibility of some weak lee waves downstream when $\Delta < 0$.

Next consider a negative step, $F_M < 0$. Now the local hydraulic solution is,

$$\Delta \geq 0: \; 6A_+ = \Delta - (\Delta^2 - 12F_M)^{1/2}\,, \; 6A_- = 0\,, \quad (28)$$

$$-(-12F_M)^{1/2} < \Delta < 0: \; 6A_+ = \Delta - (-12F_M)^{1/2}\,, \; 6A_- = \Delta\,, \quad (29)$$

$$\Delta < -(-12F_M)^{1/2}: \; 6A_+ = 0\,, \; 6A_- = \Delta + (\Delta^2 + 12F_M)^{1/2}\,. \quad (30)$$

Here the constant in (21) is $C = 0, -\Delta^2/12, F_M$ resepctively. In all cases the downstream solution $A_+ < 0$ is a shock (in (30) the shock has zero strength), and needs to be replaced by an undular bore solution as in Section 3, but again note that the downstream depression A_+ differs from that found for flow over a localized obstacle. The undular bore is given by (4) to (8) with x replaced by $(\Delta - 6A_+)t - (x - L)$, A_0 by $-A_+$ and d with $d - A_+$. It occupies the zone (compare (17))

$$\max\{0,\; \Delta - 2A_+\} < \frac{x - L}{t} < \Delta - 12A_+\,. \quad (31)$$

But now A_+ is given by (28, 29) in place of (12). For a fully detached undular bore, $\Delta - 2A_+ > 0$, and combining with the criteria (29, 30) we get the regime

$$\Delta > -(-3F_M)^{1/2}\,. \quad (32)$$

On the other hand, the regime where $\Delta - 2A_+ < 0$ but $\Delta - 12A_+ > 0$, or

$$-(-12F_M)^{1/2} < \Delta < -(-3F_M)^{1/2} < 0\,. \quad (33)$$

is where the undular bore is only partially formed and is attached to the step in the same manner described above in Section 3. That is, the modulus m varies from $m_+(< 1)$ at the step, where m_+ can be founding from (7) by replacing x with $\Delta - 6A_+$ and A_0 with $-A_+$ to $m = 0$ at the trailing edge. Further, a stationary lee wavetrain may develop just behind the obstacle. At the transition to non-resonant subcritical flow, where $\Delta = -\sqrt{-12F_M}$, again $m_+ = 0.96$ independently of F_M.

For $\Delta < 0, A_- < 0$ and the hydraulic solution is terminated by a rarefraction wave; hence no undular bore is needed. But an oscillatory wave train is needed to smooth out the corners. For $\Delta > 0$ the upstream solution is zero. Thus the overall scenario that is now predicted from these considerations is the formation of a detached downstream undular bore when (32) holds, an attached undular bore when (33) holds, and the possibility of some weak oscillatory waves u[stream when $\Delta < 0$.

These predictions are confirmed by direct numerical simulations of the fKdV equation (11) for the case when the forcing term is given by (18) with $F_M > 0$, terminated by an analogous step down at the pont $x = L_1 >> L$. A typical scenario is shown in Figure 2 for exact criticality ($\Delta = 0$). We see a well-formed attached undular bore upstream of the positive step, zero over the step itself, and a well-formed detached undular bore downstream of the negative step, in agreement with the theoretical predictions made above. This scenario has also been found in direct numerical simulations of the Euler equations for free surface flow over a step by Zhang and Chwang (2001).

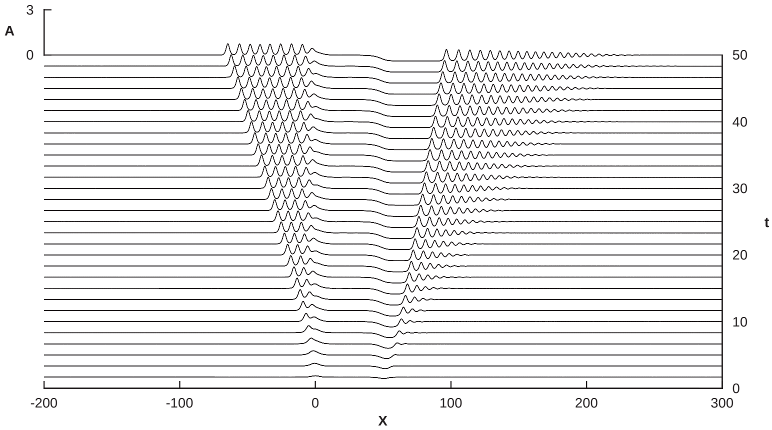

Figure 2: Numerical solution of the fKdV equation (11) for flow over a step, with $\Delta = 0$. The step is not shown, but occupies the zone $0 < x < 50$.

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