Classification of integrable (2+1)-dimensional quasilinear hierarchies

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Abstract. We investigate (2+1)-dimensional hierarchies associated with integrable PDE's of the form

$$\Omega_{tt} = F(\Omega_{xx}, \Omega_{xt}, \Omega_{xy}),$$

which generalize the dispersionless KP hierarchy. The integrability is understood as the existence of infinitely many hydrodynamic reductions.

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1. Introduction

Let us consider a function Ω of infinitely many independent variables t^0, t^1, t^2, ... satisfying a system of second order PDE's

$$\Omega_{nk} = \Phi_{nk}(\Omega_{00},\, \Omega_{01},\, \Omega_{02},\, ...,\, \Omega_{0,n+k});$$

here $\Omega_{nk} \equiv \partial_{t^n}\partial_{t^k}\Omega$, $n \geqslant 1$, $k \geqslant 1$. Explicitly, one has

$$\begin{aligned}
\Omega_{11} &= \Phi_{11}(\Omega_{00}, \Omega_{01}, \Omega_{02}),\\
\Omega_{12} &= \Phi_{12}(\Omega_{00}, \Omega_{01}, \Omega_{02}, \Omega_{03}),\\
\Omega_{13} &= \Phi_{13}(\Omega_{00}, \Omega_{01}, \Omega_{02}, \Omega_{03}, \Omega_{04}),\\
\Omega_{22} &= \Phi_{22}(\Omega_{00}, \Omega_{01}, \Omega_{02}, \Omega_{03}, \Omega_{04}),
\end{aligned} \tag{1}$$

etc. Equations of this type generalize the dispersionless KP hierarchy

$$\begin{aligned}
\Omega_{11} &= \Omega_{02} - \frac{1}{2}\Omega_{00}^2,\\
\Omega_{12} &= \Omega_{03} - \Omega_{00}\Omega_{01},\\
\Omega_{13} &= \Omega_{04} - \Omega_{00}\Omega_{02} - \frac{1}{2}\Omega_{01}^2,\\
\Omega_{22} &= \Omega_{04} + \frac{1}{3}\Omega_{00}^3 - \Omega_{00}\Omega_{02} - \Omega_{01}^2,
\end{aligned}$$

etc. Further examples arise in the theory of Dirichlet's problem in multi-connected domains [8]. The compatibility conditions of the equations (1) impose strong restrictions on the functions Φ_{nk} implying, in particular, that Φ_{11} uniquely determines the rest of the functions Φ_{nk} [10], [2]. The function Φ_{11} itself satisfies a complicated over-determined system of third order PDE's (see Sect. 2 where we re-derive this system based on the method of hydrodynamic reductions [7], [4], [6], [9]). Its general solution can be reduced to either of the four essentially different canonical forms

$$\begin{aligned}
\Phi_{11} &= \Omega_{02} + \frac{1}{4A}(A\Omega_{01} + 2B\Omega_{00})^2 + Ce^{-A\Omega_{00}},\\
\Phi_{11} &= \frac{\Omega_{02}}{\Omega_{00}} + \left(\frac{1}{\Omega_{00}} + \frac{A}{4\Omega_{00}^2}\right)\Omega_{01}^2 + \frac{B}{\Omega_{00}^2}\Omega_{01} + \frac{B^2}{A\Omega_{00}^2} + Ce^{A/\Omega_{00}},\\
\Phi_{11} &= \frac{\Omega_{02}}{\Omega_{01}} + \frac{1}{6}\eta(\Omega_{00})\Omega_{01}^2,\\
\Phi_{11} &= \ln\Omega_{02} - \ln\theta_1\left(\Omega_{01}, \Omega_{00}\right) - \frac{1}{4}\int^{\Omega_{00}}\eta(\tau)d\tau,
\end{aligned}$$

see [10]. Here $\eta(\tau)$ is a solution of the Chazy equation

$$\eta''' + 2\eta\eta'' = 3\eta'^2, \tag{2}$$

which can be represented in parametric form

$$\eta(\tau) = \frac{4}{\pi^2}\mathbf{K}(s)[(2-s^2)\mathbf{K}(s) - 3\mathbf{E}(s)], \qquad \tau = -\pi^2\frac{\mathbf{K}(\sqrt{1-s^2})}{\mathbf{K}(s)},$$

where $\mathbf{K}(s)$ and $\mathbf{E}(s)$ are complete elliptic integrals of the first and second kind, respectively [1]. The theta-function

$$\theta_1(z,\tau) = 2\sum_{n=0}^{\infty}(-1)^n e^{-(n+1/2)^2\tau}\sin[(2n+1)z]$$

is defined as a solution of the involutive system

$$\partial_z\theta_1 = -k\theta_1, \quad \partial_\tau\theta_1 = \tfrac{1}{4}(k^2-l)\theta_1,$$

$$\partial_z k = l, \quad \partial_\tau k = \tfrac{1}{4}\sqrt{4l^3-4\eta l^2-8\eta' l-\tfrac{8}{3}\eta''}-\tfrac{1}{2}kl, \tag{3}$$

$$\partial_z l = \sqrt{4l^3-4\eta l^2-8\eta' l-\tfrac{8}{3}\eta''}, \quad \partial_\tau l = l^2-\eta l-\eta'-\tfrac{1}{2}k\sqrt{4l^3-4\eta l^2-8\eta' l-\tfrac{8}{3}\eta''},$$

where, again, η solves the Chazy equation (2). We emphasize, however, that one does not need the explicit formulae for θ_1 and η to work with the above expressions for Φ_{11}: what one actually needs are the equations (2), (3).

The dKP hierarhy corresponds to a simple degeneration of the first canonical form: $A=-2B^2$, $B\to 0$, $C=0$. Similarly, the hierarchy of the modified dKP equation,

$$\Omega_{11} = \Omega_{02} + B\Omega_{00}\Omega_{01} + \frac{B^2}{3}\Omega_{00}^3,$$

can be obtained by the degeneration $C=-2B^2A^{-3}$, $A\to 0$ (along with an appropriate linear change of the variable t^2).

In Sect. 2 we concentrate on the first equation

$$\Omega_{11} = \Phi_{11}(\Omega_{00},\Omega_{01},\Omega_{02}), \tag{4}$$

dropping any assumptions on the structure of higher flows of the hierarchy. Introducing the notation $t^0\equiv x$, $t^1\equiv t$, $t^2\equiv y$, $\Phi_{11}\equiv G$, $\Omega_{00}\equiv a$, $\Omega_{01}\equiv b$, $\Omega_{02}\equiv c$ we rewrite (4) in a quasilinear form

$$a_t = b_x, \ a_y = c_x, \ b_y = c_t, \ b_t = G(a,b,c)_x. \tag{5}$$

Applying to (5) the method of hydrodynamic reductions (as outlined in [6]), we arrive at the same system of PDE's for Φ_{11} as the one obtained in [10]. This confirms that the symmetry approach of [10] based on equations (1) yields a *complete list* of integrable PDE's of the form (4).

In Sect. 3 we discuss scalar pseudopotentials

$$\psi_t = Q(\psi_x, \ \Omega_{00}, \ \Omega_{01}), \ \psi_y = L(\psi_x, \ \Omega_{00}, \ \Omega_{01}, \ \Omega_{02})$$

which play a role of dispersionless Lax pairs [12] for equations (4). To calculate pseudopotentials we introduce *negative* times t^{-1}, t^{-2}, ... and consider the corresponding *negative* flows of the hierarchy.

2. Classification of integrable PDEs of the form $\Omega_{tt} = G(\Omega_{xx},\Omega_{xt},\Omega_{xy})$

In this section we demonstrate how the classification results of [10] (see also [2]) follow from the method of hydrodynamic reductions as proposed in [6]. Introducing the notation $\Omega_{xx}=a$, $\Omega_{xt}=b$, $\Omega_{xy}=c$, $\Omega_{tt}=G(a,b,c)$ we first rewrite our PDE in the quasilinear form (5). Looking for hydrodynamic reductions in the form $a = a(R^1, ..., R^n)$, $b = b(R^1, ..., R^n)$, $c = c(R^1, ..., R^n)$ where the Riemann invariants satisfy the equations

$$R^i_t = \lambda^i(R)\ R^i_x, \quad R^i_y = \mu^i(R)\ R^i_x \tag{6}$$

and substituting into (5), one arrives at

$$\partial_i b = \lambda^i \partial_i a, \ \partial_i c = \mu^i \partial_i a$$

along with the dispersion relation

$$(\lambda^i)^2 = G_a + G_b \lambda^i + G_c \mu^i.$$

The commutativity conditions of the flows (6) are of the form

$$\frac{\partial_j \lambda^i}{\lambda^j - \lambda^i} = \frac{\partial_j \mu^i}{\mu^j - \mu^i}, \ i \neq j, \ \partial_j = \partial/\partial_{R^j}, \tag{7}$$

see [11]. Differentiating the dispersion relation and taking into account (7) one obtains the expressions for $\partial_j \lambda^i$ in the form

$$\begin{gathered}\partial_j \lambda^i = \tfrac{\partial_j a}{G_c(\lambda^i - \lambda^j)} \big(G_{aa} G_c + G_{ab} G_c (\lambda^i + \lambda^j) + G_{bb} G_c \lambda^i \lambda^j + \\ (G_{ac} + \lambda^i G_{bc})((\lambda^j)^2 - G_b \lambda^j - G_a) \\ + ((\lambda^i)^2 - G_b \lambda^i - G_a)[G_{ac} + G_{bc} \lambda^j + \tfrac{G_{cc}}{G_c}((\lambda^j)^2 - G_b \lambda^j - G_a)] \big).\end{gathered} \tag{8}$$

The compatibility conditions of the equations $\partial_i b = \lambda^i \partial_i a$ and $\partial_i c = \mu^i \partial_i a$ imply

$$\partial_i \partial_j a = \frac{\partial_j \lambda^i}{\lambda^j - \lambda^i} \partial_i a + \frac{\partial_i \lambda^j}{\lambda^i - \lambda^j} \partial_j a. \tag{9}$$

One can see that the consistency conditions of the equations (8), that is, $\partial_k \partial_j \lambda^i - \partial_j \partial_k \lambda^i = 0$, are of the form $P\ \partial_j a \partial_k a = 0$ where P is a complicated rational expression in $\lambda^i, \lambda^j, \lambda^k$ whose coefficients depend on partial derivatives of $G(a, b, c)$ up to third order (to obtain the integrability conditions it suffices to consider 3-component reductions setting $i = 1, \ j = 2, \ k = 3$). Requiring that P vanishes identically we obtain the expressions for all third order partial derivatives of G. Similarly, the compatibility conditions of the equations (9), that is, $\partial_k(\partial_i \partial_j a) - \partial_j(\partial_i \partial_k a) = 0$, take the form $S\ \partial_i a \partial_j a \partial_k a = 0$ where, again, S is rational in $\lambda^i, \lambda^j, \lambda^k$. Equating S to zero one obtains exactly the same conditions as the ones obtained on the previous step. The resulting set of integrability conditions looks as follows:

$$
\begin{aligned}
G_{ccc} &= \frac{2G_{cc}^2}{G_c}, \quad G_{acc} = \frac{2G_{ac}G_{cc}}{G_c}, \quad G_{bcc} = \frac{2G_{bc}G_{cc}}{G_c}, \\
G_{aac} &= \frac{2G_{ac}^2}{G_c}, \quad G_{abc} = \frac{2G_{ac}G_{bc}}{G_c}, \quad G_{bbc} = \frac{2G_{bc}^2}{G_c}, \\
G_{bbb} &= \frac{2}{G_c^2}\left(G_bG_{bc}^2 + G_{bc}(G_cG_{bb} + 2G_{ac}) - G_{cc}(G_bG_{bb} + 2G_{ab})\right), \\
G_{abb} &= \frac{2}{G_c^2}\left(G_aG_{bc}^2 + G_{ac}(G_cG_{bb} + G_{ac}) - G_{cc}(G_aG_{bb} + G_{aa})\right), \\
G_{aab} &= \frac{2}{G_c^2}\left(G_{cc}(G_bG_{aa} - 2G_aG_{ab}) - G_{ac}(G_bG_{ac} - 2G_cG_{ab}) - G_{bc}(G_cG_{aa} - 2G_aG_{ac})\right), \\
G_{aaa} &= \frac{2}{G_c^2}\big((G_a + G_b^2)G_{ac}^2 + G_a^2G_{bc}^2 + G_c^2(G_{ab}^2 - G_{aa}G_{bb}) \\
&\quad + G_{ac}G_c(G_{aa} + 2(G_aG_{bb} - G_bG_{ab})) + 2G_{bc}(G_b(G_cG_{aa} - G_aG_{ac}) - G_aG_cG_{ab}) \\
&\quad - G_{cc}((G_a + G_b^2)G_{aa} - 2G_aG_bG_{ab})\big).
\end{aligned} \tag{10}
$$

This system is in involution and its general solution depends on 10 integration constants (indeed, the values of G and its partial derivatives up to second order can be prescribed arbitrarily at any point a_0, b_0, c_0).

The integration of the first six equations in (10) yields

$$G(a,b,c) = \frac{1}{\varepsilon}\ln(\alpha a + \beta b + \gamma + \varepsilon c) + F(a,b).$$

The substitution of this ansatz into the remaining equations imposes further constraints on the function $F(a,b)$,

$$
\begin{gathered}
F_{bbb} - 4\varepsilon F_{ab} - 2\varepsilon F_bF_{bb} = 0, \\
F_{abb} - 2\varepsilon F_{aa} - 2\varepsilon F_aF_{bb} = 0, \\
F_{aab} + 2\varepsilon F_bF_{aa} - 4\varepsilon F_aF_{ab} = 0, \\
F_{aaa} - 2\varepsilon F_aF_{aa} + 2F_{aa}F_{bb} - 2\varepsilon F_b^2F_{aa} - 2F_{ab}^2 + 4\varepsilon F_aF_bF_{ab} - 2\varepsilon F_a^2F_{bb} = 0,
\end{gathered} \tag{11}
$$

which identically coincide with the ones derived in [10]. The first equation in (11) has the general solution

$$F = -\frac{1}{4\varepsilon}\int \eta(a)da - \frac{1}{\varepsilon}\ln\theta(a,\, b), \qquad 4\varepsilon\theta_a = \theta_{bb}, \tag{12}$$

and the substitution of (12) into the remaining equations (11) and further integration lead to the four essentially different cases as shown in [10].

This confirms that the method of higher symmetries adopted in [10] gives *all* integrable hierarchies of the form (1).

3. Pseudopotentials

Let us introduce the *negative* times t^{-1}, t^{-2}, t^{-3}, ... and extend the hierarchy (1) by the equations

$$
\begin{aligned}
\Omega_{n,-k} &= \Phi_{n,-k}(\Omega_{0,-k},\, \Omega_{0,-k+1},\, ...,\, \Omega_{00},\, \Omega_{01},\, ...,\, \Omega_{0n}), \\
\Omega_{-n,-k} &= \Phi_{-n,-k}(\Omega_{0,-n-k},\, \Omega_{0,-n-k+1},\, ...,\, \Omega_{00});
\end{aligned}
$$

in particular,

$$\begin{aligned}\Omega_{1,-1} &= \Phi_{1,-1}(\Omega_{0,-1},\, \Omega_{00},\, \Omega_{01}),\\ \Omega_{2,-1} &= \Phi_{2,-1}(\Omega_{0,-1},\, \Omega_{00},\, \Omega_{01},\, \Omega_{02}).\end{aligned}$$

In the notation $t^0 \equiv x$, $t^1 \equiv t$, $t^2 \equiv y$, $t^{-1} \equiv z$, $\Phi_{1,-1} \equiv Q$, $\Phi_{2,-1} \equiv L$ we can rewrite these equations in the form

$$\begin{aligned}\Omega_{tz} &= Q(\Omega_{xz},\, \Omega_{xx},\, \Omega_{xt}),\\ \Omega_{yz} &= L(\Omega_{xz},\, \Omega_{xx},\, \Omega_{xt},\, \Omega_{xy}).\end{aligned}$$

In terms of $\psi = \Omega_z$ this provides a pseudopotential

$$\begin{aligned}\psi_t &= Q(\psi_x,\, \Omega_{xx},\, \Omega_{xt}),\\ \psi_y &= L(\psi_x,\, \Omega_{xx},\, \Omega_{xt},\, \Omega_{xy})\end{aligned}$$

for the equation (4). Below we demonstrate how one can obtain explicit expressions for both functions Q and L.

Remark: The hierarchy of the Boyer-Finley equation [3]

$$\Omega_{1,\,-1} = \exp \Omega_{00}$$

can be obtained by a simple degeneration $B = C = 0,\ A = -2$ in the first canonical form; the first commuting flow of this hierarchy is

$$\Omega_{11} = \Omega_{02} - \frac{1}{2}\Omega_{01}^2.$$

Let us first derive the explicit form for the function Q applying the method of hydrodynamic reductions to the PDE

$$\Omega_{zt} = Q(\Omega_{xz}, \Omega_{xx}, \Omega_{xt}).$$

Introducing the notation $\Omega_{xx} = a,\ \Omega_{xt} = b,\ \Omega_{xz} = e,\ \Omega_{zt} = Q(e, a, b)$, one can rewrite this PDE in the quasilinear form

$$a_t = b_x,\ a_z = e_x,\ b_z = e_t = Q(e, a, b)_x. \tag{13}$$

Looking for reductions in the form $a = a(R^1, ..., R^n),\ b = b(R^1, ..., R^n),\ e = e(R^1, ..., R^n)$ where the Riemann invariants satisfy the equations

$$R^i_t = \lambda^i(R)\ R^i_x, \quad R^i_z = \zeta^i(R)\ R^i_x$$

and substituting into (13), one arrives at

$$\partial_i b = \lambda^i \partial_i a,\ \partial_i e = \zeta^i \partial_i a$$

along with the dispersion relation

$$\zeta^i \lambda^i = Q_a + Q_b \lambda^i + Q_e \zeta^i.$$

As before, the commutativity conditions (7) lead to the expressions for $\partial_j \lambda^i,\ (i \neq j)$, and the compatibility conditions of the equations $\partial_i b = \lambda^i \partial_i a,\ \partial_i e = \zeta^i \partial_i a$ yield (9). The consistency conditions $\partial_k \partial_j \lambda^i - \partial_j \partial_k \lambda^i = 0$ are of the form $R\ \partial_j a \partial_k a = 0$ where R is a rational expression in $\lambda^i, \lambda^j, \lambda^k$ whose coefficients depend on partial derivatives of $Q(e, a, b)$ up to the third order. Requiring that R vanishes identically we obtain the expressions for all third order partial derivatives of Q. Similarly, the compatibility conditions of the equations (9), that is, $\partial_k(\partial_i \partial_j a) - \partial_j(\partial_i \partial_k a) = 0$, take the form $M\ \partial_i a \partial_j a \partial_k a = 0$ where, again, M is a rational expression in $\lambda^i, \lambda^j, \lambda^k$. Equating M to zero one obtains exactly the same conditions as on the previous step. The final set of integrability conditions looks as follows:

$$
\begin{gathered}
Q_{bbb} = \frac{Q_{bb}(Q_bQ_{be} + Q_eQ_{bb} + Q_{ab})}{Q_bQ_e + Q_a}, \quad Q_{eee} = \frac{Q_{ee}(Q_{ee}Q_b + Q_eQ_{be} + Q_{ae})}{Q_bQ_e + Q_a},\\
Q_{bbe} = \frac{Q_{bb}(Q_{ee}Q_b + Q_eQ_{be} + Q_{ae})}{Q_bQ_e + Q_a}, \quad Q_{bee} = \frac{Q_{ee}(Q_bQ_{be} + Q_eQ_{bb} + Q_{ab})}{Q_bQ_e + Q_a},\\
Q_{aee} = \frac{Q_{ee}(Q_bQ_{ae} + Q_eQ_{ab} + Q_{aa})}{Q_bQ_e + Q_a}, \quad Q_{abb} = \frac{Q_{bb}(Q_bQ_{ae} + Q_eQ_{ab} + Q_{aa})}{Q_bQ_e + Q_a},\\
Q_{aab} = \frac{Q_{ab}(2Q_eQ_{ab} + Q_{aa}) + Q_{bb}(2Q_aQ_{ae} - Q_eQ_{aa}) - Q_{be}(2Q_aQ_{ab} - Q_bQ_{aa})}{Q_bQ_e + Q_a},\\
Q_{abe} = \frac{Q_{ae}Q_{ab} + Q_a(Q_{bb}Q_{ee} - Q_{be}^2) + Q_{be}(Q_bQ_{ae} + Q_eQ_{ab})}{Q_bQ_e + Q_a}, \qquad (14)\\
Q_{aae} = \frac{2Q_aQ_{ab}Q_{ee} + Q_{aa}Q_{ae} + Q_{be}(Q_eQ_{aa} - 2Q_aQ_{ae}) + Q_b(2Q_{ae}^2 - Q_{aa}Q_{ee})}{Q_bQ_e + Q_a},\\
Q_{aaa} = \big(Q_{aa}[2(Q_b^2Q_{ee} + Q_e^2Q_{bb}) + Q_bQ_{ae} + Q_eQ_{ab} + Q_{aa} - 2Q_{be}(Q_a + 2Q_bQ_e)]\\
+2Q_{ab}[Q_a(Q_{ae} + 2Q_eQ_{be}) + 2Q_b(Q_eQ_{ae} - Q_aQ_{ee}) - Q_e^2Q_{ab}]\\
+2Q_{ae}[Q_a(2Q_bQ_{be} - 2Q_eQ_{bb}) - Q_b^2Q_{ae}] + 2Q_a^2[Q_{bb}Q_{ee} - Q_{be}^2]\big)/(Q_bQ_e + Q_a).
\end{gathered}
$$

This system is in involution and its general solution depends on 10 integration constants.

It is easy to see that the general solution of the first six equations is

$$Q = \frac{1}{4}\ln\frac{U(a,p)}{V(a,q)}, \qquad 4U_a = U_{pp}, \qquad 4V_a = V_{qq},$$

where

$$p = b - e, \qquad q = b + e.$$

The general solution of the system (14) can be obtained by the substitution of this ansatz into the remaining equations. Let us introduce the notation

$$Q_p = -\frac{1}{4}k, \qquad Q_{pp} = -\frac{1}{4}l, \qquad Q_q = \frac{1}{4}m, \qquad Q_{qq} = \frac{1}{4}n,$$

here $\partial_b = \partial_q + \partial_p$, $\partial_e = \partial_q - \partial_p$. Then the equation $(14)_8$ takes the form

$$4nn_a - n_q(n_q - 2mn) + 2n[2l_a + kl_p - 2l^2] = 4ll_a - l_p(l_p - 2kl) + 2l[2n_a + mn_q - 2n^2] \quad (15)$$

where

$$
\begin{aligned}
Q_a &= \frac{1}{16}[-l + n + k^2 - m^2], \qquad Q_{be} = \frac{1}{4}(n + l), \qquad Q_b = \frac{1}{4}(m - k),\\
Q_e &= \frac{1}{4}(m + k), \qquad Q_{bb} = \frac{1}{4}(n - l), \qquad Q_{ee} = \frac{1}{4}(n - l), \qquad (16)\\
Q_{ae} &= \frac{1}{16}[n_q - 2mn + l_p - 2kl], \qquad Q_{ab} = \frac{1}{16}[n_q - 2mn - l_p + 2kl].
\end{aligned}
$$

The differentiation of (15) twice with respect to p and q implies

$$n_q[2l_a + kl_p - 2l^2]_p = l_p[2n_a + mn_q - 2n^2]_q.$$

Assuming that $n_q \neq 0$ and $l_p \neq 0$ one obtains

$$l_a = l^2 - \frac{1}{2}kl_p - \eta(a)l - C(a), \qquad n_a = n^2 - \frac{1}{2}mn_q - \eta(a)n - B(a),$$

where $\eta(a)$, $B(a)$ and $C(a)$ are functions to be determined. Substituting n_a and l_a into (15) one has

$$l_p^2 = 4l^3 - 4\eta(a)l^2 - 4[B(a)+C(a)]l - E(a), \qquad n_q^2 = 4n^3 - 4\eta(a)n^2 - 4[B(a)+C(a)]n - E(a),$$

where $E(a)$ is yet another undetermined function. Checking the compatibility conditions $(l_p)_a = (l_a)_p$, $(n_q)_a = (n_a)_q$ one obtains

$$B = C = \eta', \quad E = \frac{8}{3}\eta'',$$

where η is a solution of the Chazy equation (2).With these formulas the remaining expressions for Q_{aaa}, Q_{aab} and Q_{aac} hold identically.

Thus, the general solution of the involutive system (14) yields the main classification result of this section:

$$\Omega_{zt} = \frac{1}{4}\ln\frac{\theta_1(\Omega_{xx}, \Omega_{xt} - \Omega_{xz})}{\theta_1(\Omega_{xx}, \Omega_{xt} + \Omega_{xz})},$$

where θ_1 is the Jacobi theta-function defined by (3).

Remark: This formula provides a pseudopotential for the general case

$$\Omega_{tt} = \ln \Omega_{xy} - \ln \theta_1(\Omega_{xt}, \Omega_{xx}) - \frac{1}{4}\int^{\Omega_{xx}} \eta(\tau)d\tau,$$

see the Introduction. All other particular cases can be obtained by appropriate degenerations, see [2].

Remark: Under linear transformation of independent variables (z, t) this equation can be written in more symmetric form

$$e^{\Omega_{zz}}\theta_1(\Omega_{xx}, \Omega_{xz}) = e^{\Omega_{tt}}\theta_1(\Omega_{xx}, \Omega_{xt}).$$

To calculate L we consider the compatibility condition

$$\partial_z G(a, b, c) = \partial_t Q(e, a, b)$$

which implies

$$G_a e_x + G_b Q(a, b, e)_x + G_c L(e, a, b, c)_x = Q_e Q(a, b, e)_x + Q_a b_x + Q_b G(a, b, c)_x;$$

recall that $c_z = L(e, a, b, c)_x$. Therefore,

$$L = Q_b c + f(e, a, b).$$

along with

$$\begin{aligned} Q_{bb} &= \varepsilon(Q_a + Q_b Q_e), \qquad Q_{ab} = \varepsilon(Q_a Q_e + Q_b F_a - F_b Q_a), \\ Q_{be} &= \varepsilon(Q_e^2 - F_b Q_e - F_a), \qquad f_b = (\alpha a + \beta b + \gamma)(Q_a + Q_b Q_e), \\ f_a &= (\alpha a + \beta b + \gamma)(Q_a Q_e + Q_b F_a - F_b Q_a) + \frac{1}{\varepsilon}(\alpha Q_b - \beta Q_a), \\ f_e &= (\alpha a + \beta b + \gamma)(Q_e^2 - F_b Q_e - F_a) - \frac{1}{\varepsilon}(\beta Q_e + \alpha). \end{aligned}$$

These relations are sufficient for the reconstruction of L.

4. Conclusion

We have proved that the method of hydrodynamic reductions applied to quasilinear equations (5) and (13) yields the same classification results as the symmetry approach used in [10]. Thus, both methods allow to classify integrable (2+1)-dimensional equations and find their commuting flows, see [10] for the details.

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