Soliton dynamics in a strong periodic field: the Korteweg-de Vries framework

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Abstract

Nonlinear long wave propagation in a medium with periodic parameters is considered in the framework of a variable-coefficient Korteweg-de Vries equation. The characteristic period of the variable medium is varied from slow to rapid, and its amplitude is also varied. For the case of a piece-wise constant coefficient with a large scale for each constant piece, explicit results for the damping of a soliton damping are obtained. These theoretical results are confirmed by numerical simulations of the variable-coefficient Korteweg-de Vries equation for the same piece-wise constant coefficient, as well as for a sinusoidally-varying coefficient. The resonance curve for soliton damping is predicted, and the maximum damping is for a soliton whose characteristic timescale is of the same order as the coefficient inhomegeneity scale. If the variation of the nonlinear coefficient is very large, and includes the a critical point where the nonlinear coefficient equals to zero, the soliton breaks and is quickly damped.

1. Introduction

The Korteweg-de Vries equation (KdV) is a canonical equation in nonlinear wave physics demonstrating the existence of solitons and their elastic interactions. Many papers deal with the weakly perturbed KdV equation due to smoothly varying coefficients, weak dissipation and forcing (see for instance Karpman, 1979; Kaup and Newell, 1978; Gorshkov and Ostrovsky, 1981; Kivshar and Malomed, 1989; Grimshaw and Mitsudera, 1993; Abdulaev, 1994; Grimshaw et al, 1994; 1998; Garnier, 2001; El and Grimshaw, 2002). In this case various perturbation methods have been effectively used to describe the soliton dynamics and the tail formed behind the soliton. This present paper analyzes strong periodic variability of the coefficients of the KdV equation. An explicit analytical solution describing the leading-order soliton dynamics is obtained for piece-wise constant coefficients of the KdV equation. Numerical simulations performed for this same situation confirm the theoretical results. Further the predictions of the analytical model compare well with the numerical results for sinusoidal coefficients if the period of the oscillations is large enough. The resonant character of soliton damping is investigated.

2. Soliton transformation for piecewise-constant periodic variation of the coefficients

The variable-coefficient KdV equation in general form, after suitable scaling, can include only one variable coefficient only (if the dispersion parameter is never zero),

$$\frac{\partial u}{\partial t} + p(t)u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0 . \qquad (1)$$

This equation may be solved exactly for a constant coefficient, p, and its soliton solution is

$$u(x,t) = A\mathrm{sech}^2\left[\frac{x - Vt}{\lambda}\right], \qquad \lambda = \sqrt{\frac{12}{pA}}, \qquad V = \frac{pA}{3}. \qquad (2)$$

Unsteady wave dynamics is determined by the Usell parameter

$$U = pA\lambda^2, \qquad (3)$$

for the soliton $U = 12$.

If the nonlinear coefficient is variable, there is no exact theory of the soliton transformation. If the nonlinear coefficient varies smoothly, then an asymptotic method can be applied, and it leads to a variable soliton amplitude (Grimshaw et al, 1998),

$$A(t) = A_0\left[\frac{p(t)}{p_0}\right]^{1/3}, \qquad (4)$$

while the trailing tail can also be analytically described (El and Grimshaw, 2002).

In order to obtain an explicit analytical expression, we consider piece-wise constant periodic variation of the coefficient *p(t),* with a period *T,* as shown in Figure 1. We will assume that duration of each step is the same and its mean value is p_0, so that

$$p_1 = p_0(1+m), \qquad p_2 = p_0(1-m). \tag{5}$$

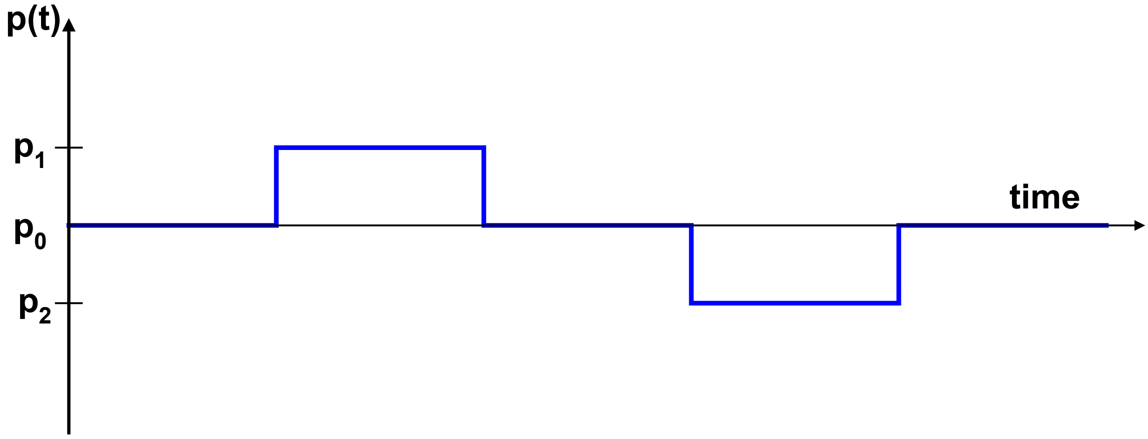

Figure 1. Periodic piece-wise constant variation of the nonlinear coefficient

An explicit analytical solution for such a geometry can be obtained by matching the solutions of the KdV equation on the each step. The boundary condition on the each step is easily found from equation (1) after integration across the step, and leads to

$$u(x,t_+) = u(x,t_-). \tag{6}$$

Therefore the wave shape is conserved at the step transmission. But this wave is not a steady-state soliton after the step, because the Ursell parameter is changed, and in particular after the first step it is

$$U_1 = U_0 \frac{p_1}{p_0}, \tag{7}$$

where index 0 is for the incident soliton and 1 for the wave transmitted after the first step. As a result, this wave will transform into solitons and a dispersive wave train; this process has been described many times in the literature. The number and amplitudes of the secondary solitons can be found with the use of the inverse scattering method. In particular, the amplitude of the leading soliton (of maximum amplitude) is found algebraically (in fact, we may find the amplitudes of the next solitons as well, but their amplitudes will be smaller)

$$A_1 = \frac{A_0}{4}\frac{p_0}{p_1}\left[\sqrt{1+8p_1/p_0}-1\right]^2 = \frac{A_0}{4(1+m)}\left[\sqrt{1+8(1+m)}-1\right]^2. \qquad (8)$$

It is important to mention that such a soliton can form only if the duration between two steps is large (when coefficient in the KdV equation is constant) and exceeds the characteristic timescale of soliton formation, T_s (Gurevich and Pitaevsky, 1974; Pelinovsky and Stepanyants, 1981),

$$T_s \sim \frac{10-100}{(pA)^{3/2}}. \qquad (9)$$

It means that the wave amplitude should be relatively large, or that the steps are located very far from each other.

At the second step, the nonlinear parameter returns to the initial state, and the amplitude of the leading soliton formed after the second step is

$$A_2 = \frac{A_1(1+m)}{4}\left[\sqrt{1+\frac{8}{1+m}}-1\right]^2 = \frac{A_0}{16}\left[\sqrt{1+8(1+m)}-1\right]^2\left[\sqrt{1+\frac{8}{1+m}}-1\right]^2 \qquad (10)$$

The third and fourth steps are analyzed similarly, replacing m with $-m$. The final formula for amplitude of the leading soliton after the full cycle of variation of the nonlinear coefficient is

$$A_I = \frac{A_0}{256}\left[\sqrt{1+8(1+m)}-1\right]^2\left[\sqrt{1+8(1-m)}-1\right]^2\left[\sqrt{1+\frac{8}{1+m}}-1\right]^2\left[\sqrt{1+\frac{8}{1-m}}-1\right]^2. \qquad (11)$$

In the limiting case of weak variation ($m << 1$) the formula (11) simplifies to

$$A_I \approx A_0\left[1-\frac{8m^2}{27}\right]. \qquad (12)$$

Formula (11) is displayed in Figure 2 by solid line and the approximate expression (12) by the dashed line. The approximant expression is valid up to about $m = 0.4$.

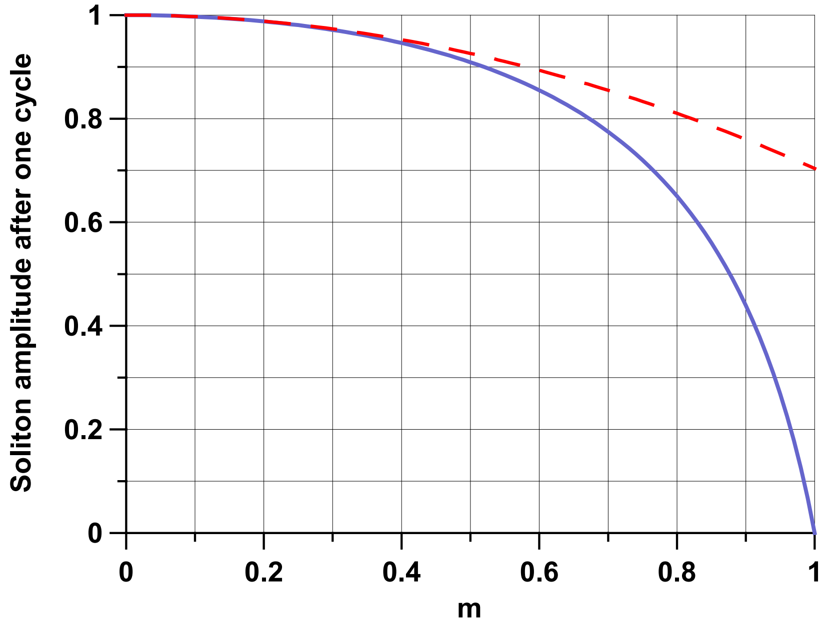

Figure 2. The soliton amplitude after transmission of a full cycle of the variation of the nonlinear coefficient; exact formula (11) – solid line, and approximate formula (12) – dashed line

If now we consider the periodic variation of the nonlinear coefficient, the soliton amplitude after N cycles is

$$A_N = \frac{A_0}{16^{2N}}\left[\sqrt{1+8(1+m)}-1\right]^{2N}\left[\sqrt{1+8(1-m)}-1\right]^{2N}\left[\sqrt{1+\frac{8}{1+m}}-1\right]^{2N}\left[\sqrt{1+\frac{8}{1-m}}-1\right]^{2N}. \quad (13)$$

In particular, for a weak variation of the nonlinear coefficient, the simplified formula follows from (13)

$$A_N = A_0\left[1-\frac{8m^2}{27}\right]^N, \quad (14)$$

and for large number of cycles we can replace N on t/T and write the final formula,

$$A(t) = A_0 \exp\left(-\frac{8m^2 t}{27T}\right). \quad (15)$$

Our approach has enabled us to calculate analytically the amplitude of the leading soliton with no assumptions on the magnitude of the coefficient variation. The full reconstruction of the wave field behind the leading soliton would require us to analyze the continuous spectrum in the inverse-scattering method on each shelf. The scattered field can contain a dispersive wave train and also some small solitons if these had time to form between the two steps. The energy (momentum) and mass of the tail behind the soliton can be found from the two conservation laws for equation (1)

$$M = \int u(x,t)dx = \text{const}, \qquad E = \int u^2(x,t)dx = \text{const}. \tag{16}$$

The mass and energy of the leading soliton dampen with time

$$M_s \sim \exp\left(-\frac{4m^2 t}{27T}\right), \quad E_s \sim \exp\left(-\frac{4m^2 t}{9T}\right), \tag{17}$$

transferring mass and energy to the wave tail formed behind the soliton.

As we have pointed out above, this approach requires us to have long shelves between the steps. If the steps are short, the wave will not have time to evolve into the soliton and a dispersive tail, and its parameters will not be significantly changed. In this case the soliton will not "feel" the variable coefficient and propagates as in a homogeneous medium. Such a situation has been investigated in detail for a surface wave above rapidly varying topography; in particular for the Korteweg-de Vries and Boussinesq equations specially derived for the this geometry (Hamilton, 1977; Rosales and Papanicolaou, 1983; Nachbin and Papanicolaou, 1992; Nachbin, 2003). It is important to say that the same situation should be realized in the final stage of wave transformation on very long shelves, because due to the iinitial damping, the width of the soliton will become comparable with shelf width, and its damping will then be stopped. It means that the soliton in the general case of a periodic medium damps for a finite time only, and then propagates as a smaller-amplitude steady-state soliton.

3. Numerical simulations of soliton transformation

Numerical simulations of the variable-coefficient KdV equation (1) are performed for an initial soliton amplitude, $A_0 = 1$, mean value of the nonlinear coefficient, $p_0 = 1$, and various T and m. The first series of runs is done for the geometry presented in Figure 1 for $m = 0.2$ and various T. If $T > 200$, the soliton dynamics follows the scenario described above; see Figure 3. The wave tail is increased with reduction of T, due to increased damping of the leading soliton (Figure 4). For $T < 200$ the soliton has no time to form, and the wave amplitude varies only weakly (Figure 5), while the theory overestimates the wave damping. The wave tail becomes more sinusoidal. When T = 1, wave amplitude is almost constant with variations less 0.5% (this plot is not shown). Increasing of the step height does not change this scenario.

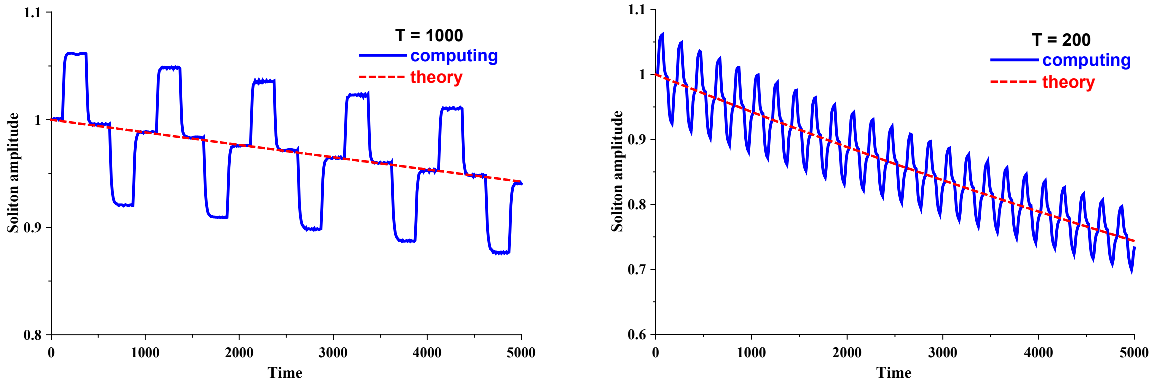

Figure 3. Time evolution of the soliton amplitude for a piece-wise constant coefficient

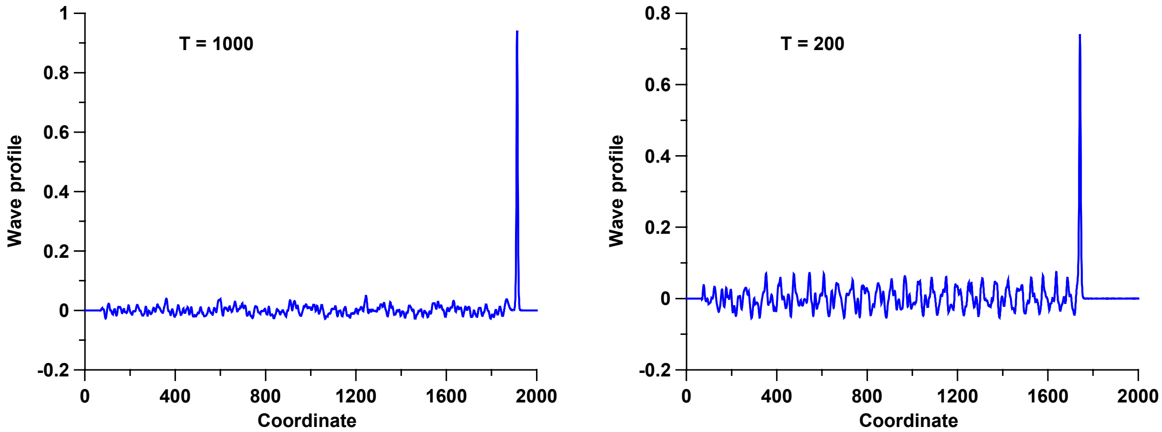

Figure 4. Wave field at $t = 5000$ and for a piece-wise constant coefficient

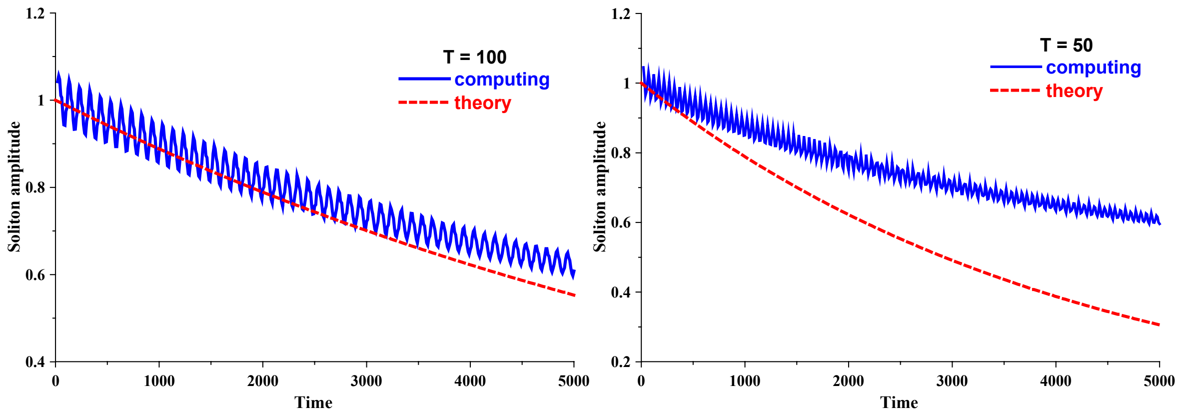

Figure 5. Time evolution of the soliton amplitude for a rapidly varying piece-wise constant coefficient

The second series is performed for a sinusoidally varying coefficient

$$p(t) = 1 + m \sin(2\pi t / T) . \tag{18}$$

Figure 6 displays the soliton amplitude versus time for m = 0.5 and T = 1000 and 300. As expected for a smoothly-varying coefficient (T = 1000) the soliton amplitude is well described by the adiabatic formula (4). The "piece-wise" formula (15) overestimates the soliton damping but the agreement is quite good, and therefore, the developed analytical theory can be applied for more complicated bahaviour of the coefficient variations.

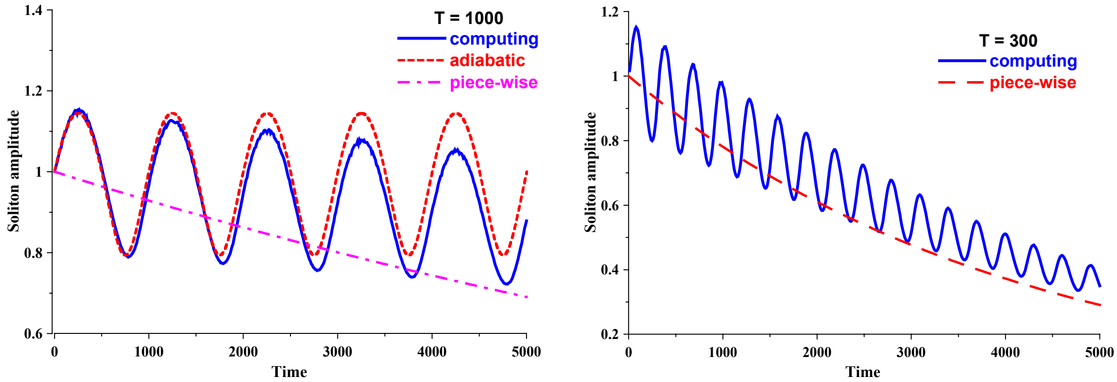

Figure 6. Time evolution of the soliton amplitude for a sinusoidal coefficient

For a rapidly-varying coefficient the soliton damping is again almost zero, and the tail is not generated (Figure 7). Figure 8 demonstrates the resonant character of the soliton damping for a sinusoidal coefficient. The maximum damping of the soliton is for a period T = 100 (characteristic scale for the formation of the soliton), just as for a piece-wise constant coefficient. The existence of this maximum effect for coincident time-scales was demonstrated for periodic cnoidal waves, when the periodic topographic variations with a scale similar to the nonlinear characteristic scale, induce the breaking of the quasi-stationary structure of a cnoidal wave (Agnon et al, 1998).

The third series of experiments is performed for a large variation (m = 1), when the nonlinear coefficient equals zero on part of the cycle (Figure 9). At the critical point (p = 1) the soliton breaks and a new soliton of opposite polarity is generated; this process is studied in detail in the paper by Grimshaw et al (1998) for various durations of the transition zone between $p = \pm 1$. In our case the critical point is passed immediately for a piece-wise constant coefficient, and the soliton damping is bigger than for a sinusoidal coefficient, if the period is large enough; see Figure 9, left. If the coefficient varies rapidly, there is no difference in the results (Figure 9, right). Meanwhile, the leading soliton damps quickly, and it becomes comparable with the wave tail (Figure 10); the last figure contains soliton-like and oscillatory waves.

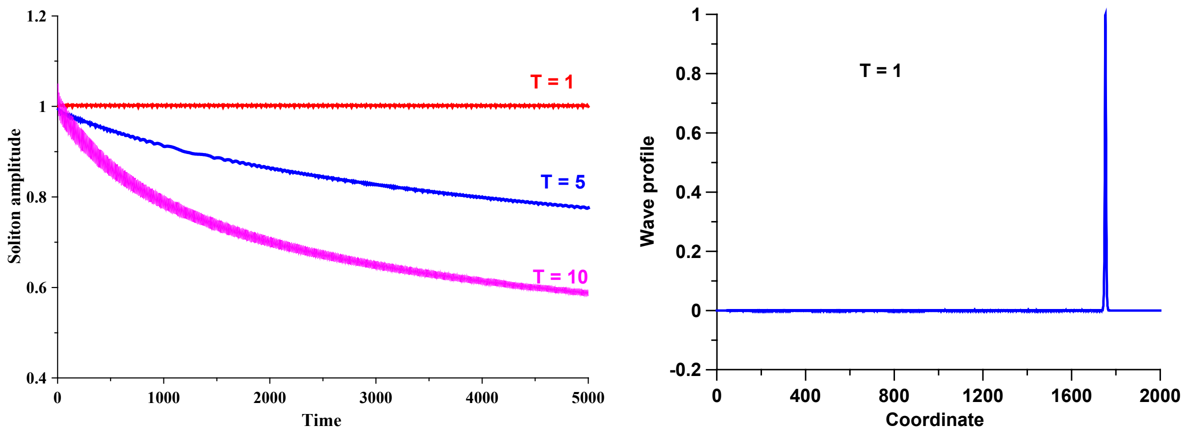

Figure 7. Time evolution of the soliton amplitude for a rapidly-varying sinusoidal coefficient

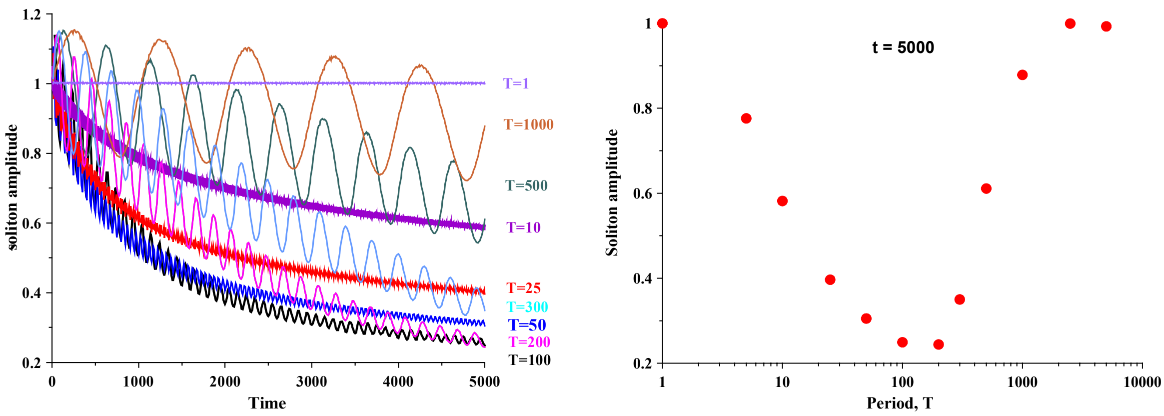

Figure 8. Summary of the wave damping and the resonance curve for the soliton amplitude at $t = 5000$ (sinusoidal coefficient, m = 0.5)

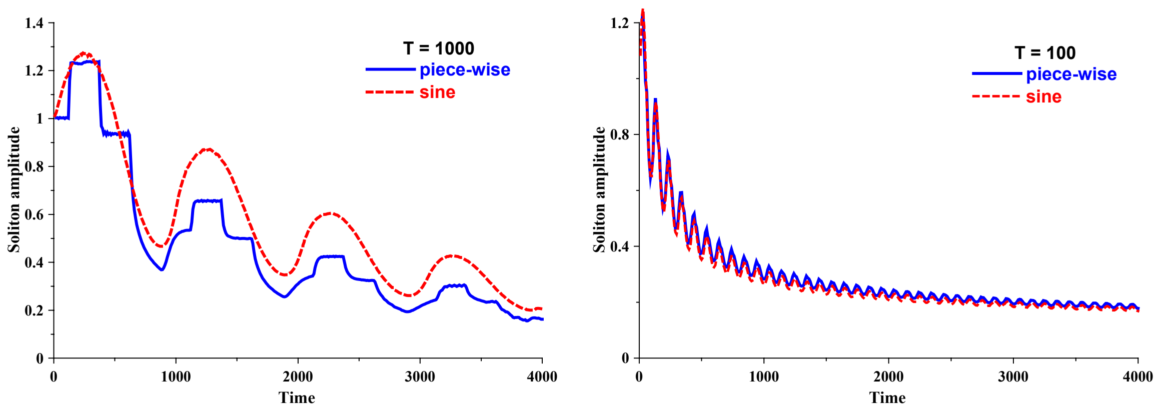

Figure 9. Soliton damping for presence of the turning point ($m = 1$)

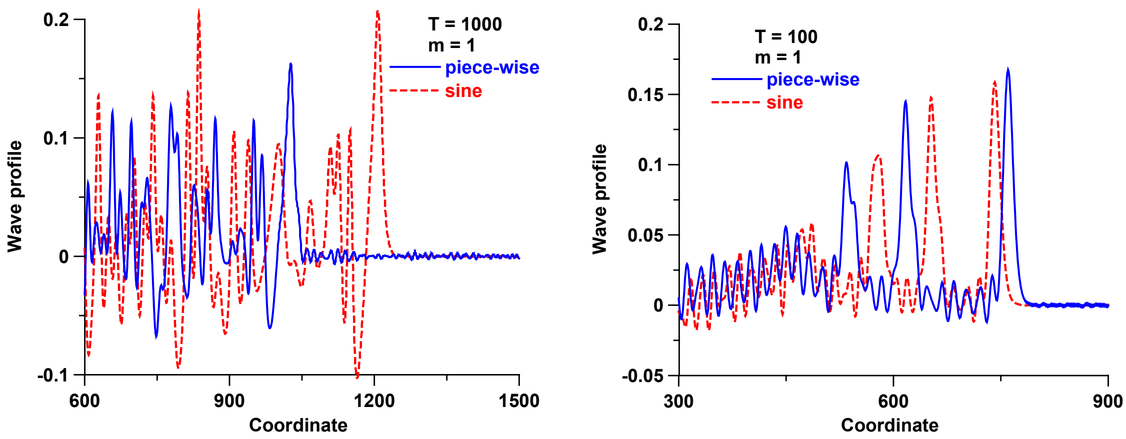

Figure 10. Wave profiles after multiple passages through the critical point (t = 4000)

4. Conclusion

Usually, the variable-coefficient Korteweg-de Vries equation is solved for smoothly and slowly varying coefficients, when asymptotic methods for the solitons are effective. Here we considered a periodic coefficient, varying slowly and rapidly. For the case of a piece-wise constant periodic coefficient, with large constant shelves, explicit analytical results for soliton damping are obtained. These theoretical results are confirmed by numerical simulations of the variable-coefficient Korteweg-de Vries equation, piece-wise constant and for sinusoidal periodic coefficient. The resonance curve of soliton damping is predicted, and the maximum damping is for the soliton whose characteristic timescale of the formation of the soliton is of the same order of the inhomogeneity scale. If the variation of the nonlinear coefficient is large enough to include a critical point (where the nonlinear coefficient equals zero), the soliton breaks and is quickly damped.

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