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# THREE UTILITIES FOR THE EQUAL SIGN

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*We compare the activity of young children using a microworld and a JavaScript relational calculator with the literature on children using traditional calculators. We describe how the children constructed different meanings for the equal sign in each setting. It appears that the nature of the meaning constructed is highly dependent on specificities of the task design and the tools available. In particular, the microworld offers the potential for children to adopt a meaning of equivalence for the equal sign.*

## CHILDREN'S PERCEPTIONS OF THE EQUAL SIGN

There exist "two intuitive meanings of equality among preschoolers" that correspond to equivalence and operation (Kieran, 1981). When conceived of as an operator the equal sign would be expected to generate an answer from a sum. In contrast, when conceived of as equivalence the equal sign would be regarded as a static relation between two expressions that are the same in value.

However school children attend to the operator meaning more readily than equivalence (Behr et al, 1980). The reasons for this have been identified by Ginsburg (1977) as based in the way the equal sign is encountered in the classroom. Invariably it is the operator meaning of the equal sign that is appealed to through the way sums are framed in textbooks, as well as the use of the = button on calculators.

Hughes (1986) points out that in the case of using a calculator it is actually valid to consider the equal sign as an operator. The = button can be viewed as a function that takes a 'number-operator sequence' and outputs a 'number'. In order to discuss this functionality in contrast to others desribed later we have found it helpful to consider the notion of 'utility' (Ainley et al, in press), specifically the construction of meaning for the ways in which the equal sign is useful. The utility of the equal sign in calculators is to obtain the answer to programmed sequences. Calculators arguably contribute significantly to children's adoption of the operator meaning. Behr et al. (1980) in interviews of six to twelve year olds report

> children consider the symbol = as a 'do something signal' that 'gives the answer' on the right hand side. There is a strong tendency among all the children to view the = symbol as being acceptable when one (or more) operation signs precede it.

The correlation between the function of calculator = buttons and children's conceptions of the equal sign is clear. In addition Behr, Erlwanger and Nichols note that children consider the equal sign as unacceptable without an operator and respond by "completing the sum". For example, when a 12 year old child was asked about the meaning of 3 = 3 she said that it could mean "0 minus 3 equals 3". This is consistent with the = button on a calculator lacking utility if the input is a single number.

Calculators may also contribute to children's "conception of an equation as a temporal event, corresponding to a verbal left-right reading, rather than a static state" (Pirie & Martin, 1997). The programming of a sequence of operations followed by pressing = and the appearance of a 'final' answer can be expected to reinforce dynamic, directional conceptions of equations.

In this paper we explore the implications of exposing children to technologies which ascribe alternative functionalities to the equal sign. One of those technologies is a microworld in which an = object takes two number inputs and outputs a Boolean state; the other is a 'relational calculator' in which the = button takes two 'number-operator sequence' inputs and outputs a graphic, which may be blank or an =. In functional notation the role of = can be considered as follows:

**Traditional Calculator** $e^C(num - op\ sequence) \rightarrow number$

**Microworld** $e^M(num, num) \rightarrow Boolean$

**Relational Calculator** $e^{RC} num - op\ sequence, num - op\ sequence) \rightarrow [graphic]$

Our discussion of children's conceptions in relation to different functionalities of the equal sign will adopt a perspective in which various meanings can be held simultaneously though with varying priorities. Accordingly, we are using a theoretical framework based on diSessa (1993), in which knowledge is seen as made up of a multitude of small pieces, which gradually become connected through learning. We believe this approach will enable fruitful analysis of the subtleties and apparent inconsistencies to be expected when children interact with technologically supported representations of mathematical equality.

## THE MICROWORLD

The *Visual Fractions* microworld allows interaction with a variety of arithmetical objects including fractions, operators, relations, regions and Boolean flags (Figure 1). The user may create, manipulate, destroy and connect objects on screen. All objects have functionality: they can take one or more inputs and produce a numerical or Boolean output.



Figure 1 Arithmetical objects: Fraction; Operator; Relation; Region; Flag

The user creates an input by dragging the small diamond above an object to another object. For example, in Figure 2 the + object takes the inputs $1\frac{3}{6}$ and $\frac{8}{12}$ and produces the output $\frac{26}{12}$ which is displayed just above the right diamond. (Note the curved arrows have been added to screen excerpts for the purposes of clarity.)
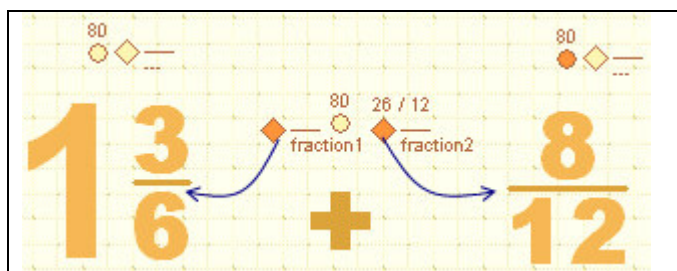
Figure 2 An operator takes two numbers as inputs and produces a number as output

A fraction object connected to the + object takes as input the output from + and produces $\frac{26}{12}$ which is displayed as the fraction itself (Figure 3). This fraction is dependent on the output of the + object and changes with it.

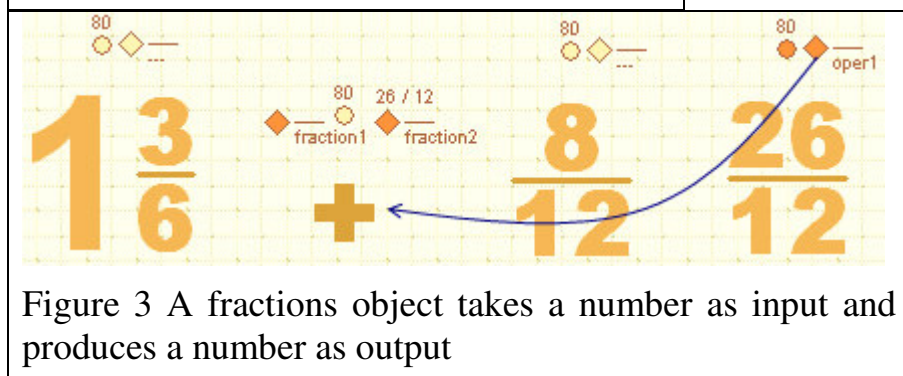An = object connected to the + object and a fraction object takes the inputs $\frac{26}{12}$ and $\frac{26}{12}$ and produces the Boolean output 'true' which is displayed just above the right diamond (Figure 4). Regions (Figure 1) have a similar functionality to relations, taking two or more numbers as inputs



Figure 3 A fractions object takes a number as input and produces a number as output

and producing a Boolean output. Flags (Figure 1) take a Boolean input and simply display that input graphically as a 'thumbs-up' for true and 'thumbs-down' for false.

The function of the equal sign is different to that of a calculator's = button. Thus, changing its position allows construction of an expression that lacks left-right spatial sequencing.
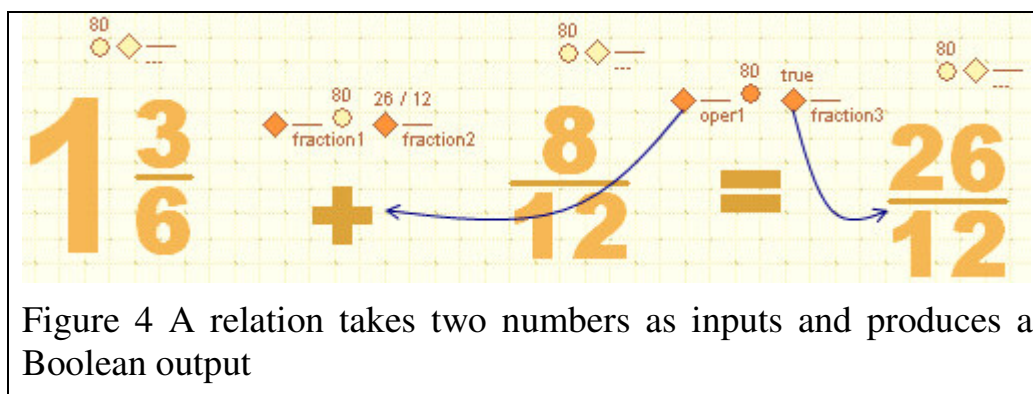


Figure 4 A relation takes two numbers as inputs and produces a Boolean output

## The Microworld - Trial

A trial of the microworld took place with two girls, C & L, aged 13 years. The activity lasted 80 minutes and was recorded using a microphone and screen capture software. A researcher introduced the addition operator and asked C & L to investigate its connector ports. After practising a little they were challenged to design a task for peers of the same age. They went on to construct an arithmetic task in which users are required to correct answers to expressions. The activity was transcribed and analysed by identifying key incidents associated with the equal sign. A trace of the evolution of conceptions for the equal sign was subsequently constructed.

## The Microworld - Findings

Initially the students placed two fractions and an operator on screen. The operator's constraint of possessing two connector ports guided the students to connecting the three objects correctly. An equal sign was placed on screen and the students discussed possible connections. Figure 5 captures L attempting to drag the second equal sign connector to empty space [see mouse cursor]:

1    L:      I don't know where you connect that.

2    C:      No you leave that one…put a box or something.

3    L:      We really need to put a region



Figure 5 C & L tried to connect the equal sign
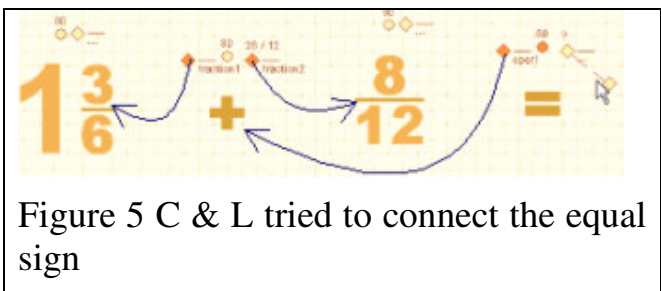
Lines 1-3 show that the existence of two connector ports above = affords making two connections. It seems the girls wanted an object to which = could be connected and decided upon using a region. A fraction was positioned inside the region to act as the answer. The equal sign was then connected to it in a way that parallels the operator's connections. Next a flag was placed on screen and connected to the region (Figure 6).
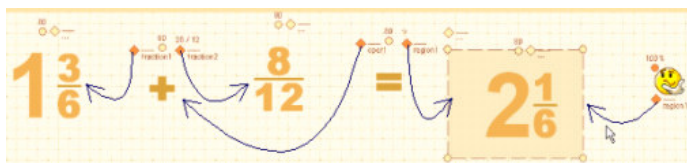


Figure 6 C & L connect the equal object and a flag to a region containing a fraction

At this point the students had made progress largely due to experimenting with objects that afford connection. However the region was technically superfluous to realising their design and consequently the flag did not display 'true' as the students expected. As it emerged that experimenting with possible connections was a flawed strategy, a switch to thinking mathematically was evoked. C considered how else the flag might be connected, at which point L attempted to connect the answer fraction to =.
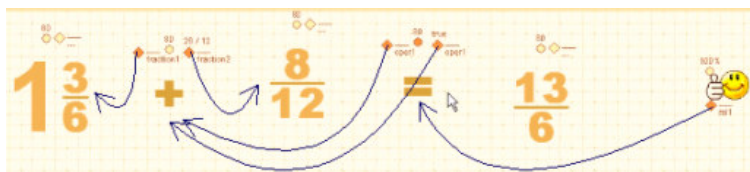


Figure 7 The students connect the flag to the equal sign

The connection failed and an error sound played. The students had now hit an impasse, and went on to try connecting the answer to = another three times throughout the trial - despite the clear error sound that played on each attempt. A researcher suggested that they try without using the region and the students deleted it. Further attempts to attach the answer to the equal sign were disallowed and the students reconnected the objects from scratch, this time with both the equal sign's connector ports linked to the operator and the flag connected to the equal sign (Figure 7). The flag became true and the students believed that they had

succeeded, but when they tested it by changing the answer the flag remained true. The students tried to fix the expression, starting with a disallowed attempt to connect the answer to the flag. The experimental nature of their trials is evidenced by dialogue:

4    L:    Maybe that needs to be on the other side.

5    C:    This [answer] isn't connected to anything.

6    L:    Perhaps if you do that to the smiley face. Or try it to the equals

After another failed attempt to connect the answer to = a researcher asked what was written above the equal sign.

7    C:    It says true. But we don't know what...

8    L:    Click on the true thing [text above =]

C clicked on the word true and the equal sign's right-hand connector was disconnected and the word true changed to ?. C connected = to the answer and the word true reappeared. This left the flag unconnected and an attempt to connect it to the answer was made. The students did not relate the required Boolean state of the flag to the word 'true' written above =. It would seem the girls expected the flag to simply 'know' if the expression was correct by virtue of being tagged on to the end. When the attempted connection failed, C momentarily dragged the flag's connector round the screen and placed it on the equals. This circuitous route suggested her choice of where to place the connection was somewhat experimental and playful. The flag became true.

9    L:    He still always has his thumbs up

10   C:    No he won't. It's worked now because look it says now. There, that's done it now.

The students tested the activity by changing the value of the answer. The flag (and =) displayed false.

## THE RELATIONAL CALCULATOR

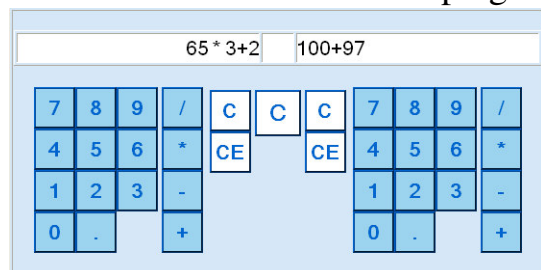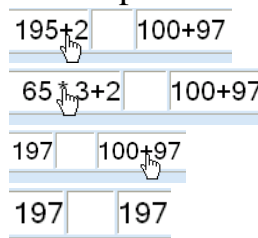The first author is developing a JavaScript relational calculator, which allows number-operator sequences to be entered into left and right screens using two keypads (Figure 8). The user may click on the operators that appear in the screens in order to collapse 'number-operator-number' trios into equivalent single numbers (Figure 9).The



Figure 8: Number-operator sequences are entered into two screens



Figure 9: Effect of clicking on operators

user may click on the smaller, central screen to establish the relation between the left and right screens. Doing so causes a graphic to appear (Figure 10). Further adjustments to the left or right screens clear the central screen.
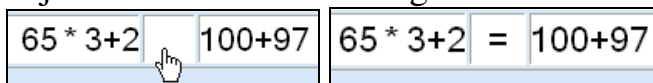


Figure 10: Clicking on the central screen displays =

## The Relational Calculator – Trial

Three trials of the microworld took place with pairs of students, aged 8 and 9 years. Each trial lasted about 60 minutes and was recorded using a microphone and screen capture software. The students were allowed free exploration of the calculator and the researcher pointed out features as appropriate. They were then challenged to create a worksheet of up to 20 'really hard' number sentences for other children to check using the relational calculator. Each number sentence created on the relational calculator had to be hand-written on a worksheet.

## The Relational Calculator - Findings

The relational calculator data offer no window on student interactions (Noss & Hoyles, 1996) but do provide insights into students' interpretations and responses to = as an outputted graphic.

*Temporal Sequencing Conceptions.* One pair of students, Ch & H, began by creating a single operator expression and writing it down exactly as it appeared on screen:

*Worksheet:*  *Screenshot:* 

When the researcher later suggested they write down the answer to the left of = on their worksheet they were reluctant to do so despite having seen the relational calculator allow this on occasion:



Another pair, U & Lu, displayed conceptions of right-left directionality when writing down their first expression: 

    11    U:        But we write it that way [9/3=3], not three equals nine.
    12    Lu:       I'd do it that way [3=9/3] so it would be harder for them.

However, left-right conceptions became prioritised in U & Lu's later expressions, perhaps because they attended to numeracy as they tried to make the worksheet progressively more difficult by using longer numbers. Left-right temporal sequencing is also evident when the researcher requested they create an expression with operators on each side of =, resulting in '6781+2976=9757+39=9796'.

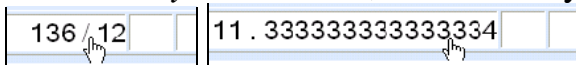The final pair, K & M, began entering '19*12' into the left screen and clicked the operator. They decided to test it and without comment entered '19*12' into the right screen and then clicked the central, relation screen.
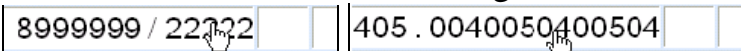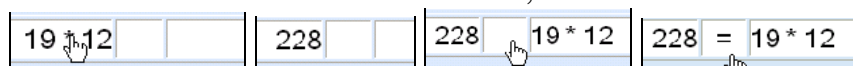
The appearance of '=' satisfied K & M that the expression was correct; but they wrote it on the worksheet as '19*12=228' (i.e. with left-right temporal sequencing). It would seem that K & M's conception of left-right directionality prioritises itself over and above the on-screen right-left directionality. Left-right directionality is also evident when K & M were requested to use an operator on each side of an expression, resulting in '1200-34=1162+4=1166'. A principle factor of prioritisation of left-right conceptions would appear to be the students' requirement for the equal sign to be followed by an answer. When the researcher requested K & M write down '1200-34=1162+4' as it appeared on screen they considered it to be impossible for other children to understand:

13 M:  Because otherwise the answer's going to be in the middle of the... and then the sum's going to be on the side and they won't know which...

14 K:  People aren't going to know which answer's which.

*Operational Conceptions.* The relational calculator equal sign affords no interactivity. Nonetheless, the data contain some evidence of students attending to operator notions. For example at one point Lu reads an expression while inputting it, "*One three six... divided by... twelve...*", and then says "*equals*" while clicking '/' to get the answer:
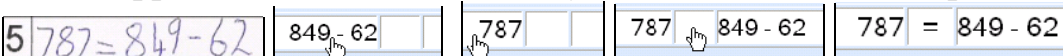


The data contain examples of all pairs of students saying "*equals*" while clicking on an operator. However, perhaps more interestingly, the data abound with evidence of students not attending to = at all. Ch & H created and wrote down an expression of the form 'a/b=c' without using the central button or even mentioning equals:
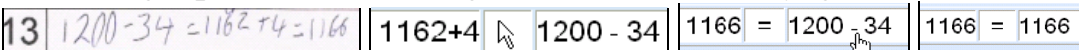


Ch & H went on to create a further four expressions without any attention to =. The data contain examples of all students doing this to a greater or lesser extent. It would seem that when = lacks affordance there are times children do not attend to = at all.

*Relational Conceptions.* K & M were the only pair to make substantial use of the = button of their own volition throughout most of the trial. Typically they got an answer by clicking on an operator then re-entered the expression to check that = would appear. When it did so they were satisfied the expression was correct:



In addition they were the only pair who managed to create a valid expression containing operators to the right of = without any guidance from the researcher:



Although it cannot be inferred whether or not K & M actually prioritised or even attended to relational meanings during the trial it is notable that they are the only students who did not repeatedly violate equivalence.

## DISCUSSION

In the case of traditional calculators the equal sign has the function $e^C(num - op\ sequence) \rightarrow number$. The outputted number is inevitably an answer to a

sum and it is this aspect that has been shown to generate the operational utility (Hughes, 1986; Behr et al, 1980). Traditional calculators also reinforce left-right temporal sequencing as experienced in writing.

In the case of the microworld the equal sign has the function $e^M(num, num) \rightarrow Boolean$. The Boolean output has mathematical potential to be re-employed in developmental sequences. This potential seems to allow, through experimentation, a utility for the equal sign of equivalence and can challenge students' prioritisation of operational utility of the equal sign. The mathematical potential of interactions with the equal sign can invoke thinking about relational meanings.

In the case of the relational calculator the equal sign has the function $e^{RC} num - op\ sequence, num - op\ sequence) \rightarrow [graphic]$. The output has no potential for further interactivity and sometimes leads to children ignoring = altogether. There is a potential utility of allowing expressions to be checked for equivalence, though in practice only one pair seemed to appreciate this possibility. The relational calculator can however challenge left-right temporal sequencing conceptions. We now wonder whether the next design of the relational calculator should automatically display <, = or > according to the relative value of the two expressions. Perhaps students would then be less inclined to ignore the equal sign though there would still be a lack of potential for the graphic to be used in contrast either to the outputted number on a traditional calculator or the outputted Boolean in the microworld.

Our experience of using the relational calculator and the Visual Fractions microworld has highlighted the close relationship between the particular utility that is constructed and the nature of the task and the structuring resources in the setting (Lave, 1988). We see the pedagogic challenge not as one of eliminating the operational utility but as providing new experiences, carefully designed, to optimise the possibility that the child may construct new utilities for the equal sign such as that of equivalence.

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