The Unambiguity of Segmented Morphisms

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Abstract. A segmented morphism $\sigma_n : \Delta^* \longrightarrow \{\mathtt{a},\mathtt{b}\}^*$, $n \in \mathbb{N}$, maps each symbol in Δ onto a word which consists of n distinct subwords in $\mathtt{ab}^+\mathtt{a}$. In the present paper, we examine the impact of n on the unambiguity of σ_n with respect to any $\alpha \in \Delta^+$, i.e. the question of whether there does not exist a morphism τ satisfying $\tau(\alpha) = \sigma_n(\alpha)$ and, for some symbol x in α, $\tau(x) \neq \sigma_n(x)$. To this end, we consider the set $U(\sigma_n)$ of those $\alpha \in \Delta^+$ with respect to which σ_n is unambiguous, and we comprehensively describe its relation to any $U(\sigma_m)$, $m \neq n$. Our paper thus contributes fundamental (and, in parts, fairly counter-intuitive) results to the recently initiated research on the ambiguity of morphisms.

1 Introduction

This paper deals with morphisms that map a *pattern*, i.e. a finite string over an infinite alphabet Δ of *variables*, onto a finite *word* over $\{\mathtt{a},\mathtt{b}\}$; for the sake of convenience, we choose $\Delta := \mathbb{N}$. With regard to such a morphism σ, we ask whether it is *unambiguous* with respect to any pattern α, i.e. there is no morphism $\tau : \mathbb{N}^* \longrightarrow \{\mathtt{a},\mathtt{b}\}^*$ satisfying $\tau(\alpha) = \sigma(\alpha)$ and, for some symbol x in α, $\tau(x) \neq \sigma(x)$. As recently demonstrated in the initial paper on the ambiguity of morphisms by Freydenberger, Reidenbach and Schneider [5], for every pattern α, there is a particular morphism $\sigma^{\mathrm{su}}_\alpha$ such that $\sigma^{\mathrm{su}}_\alpha$ is unambiguous with respect to α if and only if α is *succinct*, i.e. a shortest generator of its E-pattern language, which, in turn, is equivalent to the fact that α is not a fixed point of a nontrivial morphism $\phi : \mathbb{N}^* \longrightarrow \mathbb{N}^*$. Since there is no single morphism which is unambiguous with respect to all succinct patterns, the morphism $\sigma^{\mathrm{su}}_\alpha$ has to be tailor-made for α. More precisely, for various patterns $\alpha \in \mathbb{N}^+$, $\sigma^{\mathrm{su}}_\alpha$ must be *heterogenous* with respect to α, which means that there exist certain variables x, y in α such that the first (or, if appropriate, the last) letter of $\sigma^{\mathrm{su}}_\alpha(x)$ differs from the first (or last, respectively) letter of $\sigma^{\mathrm{su}}_\alpha(y)$. In addition to this, $\sigma^{\mathrm{su}}_\alpha$ has a second important feature: it maps each variable in α onto a word that consists

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of exactly *three* distinct *segments*, i.e. subwords taken from $\mathtt{ab}^+\mathtt{a}$ (or, in order to guarantee heterogeneity, $\mathtt{ba}^+\mathtt{b}$).

A closer look at the approach by Freydenberger et al. [5] – which is mainly meant to prove the *existence* of an unambiguous morphism with respect to any succinct pattern – reveals that it is not optimal, as there exist numerous patterns with respect to which there is a significantly less complex unambiguous morphism. For instance, as demonstrated by Reidenbach [11], the standard morphism σ_0 given by $\sigma_0(x) := \mathtt{ab}^x$, $x \in \mathbb{N}$, is unambiguous with respect to *every* pattern α satisfying, for some $m \in \mathbb{N}$ and $e_1, e_2, \ldots, e_m \geq 2$, $\alpha = 1^{e_1} \cdot 2^{e_2} \cdot \ldots \cdot m^{e_m}$ (where the superscripts ^{e_j} refer to the concatenation). With regard to this result, it is noteworthy that, first, σ_0 maps each variable onto a much shorter word than $\sigma_\alpha^{\mathrm{su}}$ and, second, σ_0 is *homogeneous*, i. e. for all variables $x, y \in \mathbb{N}$, $\sigma_0(x)$ and $\sigma_0(y)$ have the same first and the same last letter. Consequently, σ_0 is unambiguous with respect to each pattern in a reasonably rich set, although it does not show any of the two decisive properties of $\sigma_\alpha^{\mathrm{su}}$.

In the present paper, we wish to further develop the theory of unambiguous morphisms. In accordance with the structure of $\sigma_\alpha^{\mathrm{su}}$, we focus on *segmented* morphisms σ_n, which map every variable onto n distinct segments. More precisely, for every $x \in \mathbb{N}$, we define the homogeneous morphism σ_n by $\sigma_n(x) := \mathtt{ab}^{nx-(n-1)}\mathtt{a}\,\mathtt{ab}^{nx-(n-2)}\mathtt{a} \ldots \mathtt{ab}^{nx-1}\mathtt{a}\,\mathtt{ab}^{nx}\mathtt{a}$. With regard to such morphisms, we introduce the set $U(\sigma_n) \subseteq \mathbb{N}^+$ of all patterns with respect to which σ_n is unambiguous, and we give a characterisation of $U(\sigma_m)$ for $m \geq 3$. Furthermore, for every $n \in \mathbb{N}$, we compare $U(\sigma_n)$ with every $U(\sigma_m)$, $m \neq n$, and, since every σ_n is a biprefix code, we complement our approach by additionally considering the set $U(\sigma_0)$ of the suffix code σ_0 as introduced above. Our corresponding results yield comprehensive insights into the relation between any two sets $U(\sigma_m), U(\sigma_n)$, $m, n \in \mathbb{N} \cup \{0\}$.

Our studies are largely motivated by the intrinsic interest involved in the examination of the unambiguity of *fixed* instead of tailor-made morphisms. Thereby, we face a task which gives less definitional leeway than the original setting studied by Freydenberger et al. [5], and therefore our paper reveals new elementary phenomena related to the ambiguity of morphisms that have not been discovered by the previous approach. The choice of segmented morphisms as main objects of our considerations, in turn, is primarily derived from the observation that σ_3 is simply the homogeneous version of $\sigma_\alpha^{\mathrm{su}}$. Hence, the insights gained into $U(\sigma_3)$ immediately yield a deeper understanding of the necessity of the heterogeneity of $\sigma_\alpha^{\mathrm{su}}$ and, thus, of a crucial concept introduced in [5]. In addition to this, our partly surprising results on the relation between the number of segments of a morphism σ_n and the set of patterns for which σ_n is unambiguous suggest that – in a similar manner as the work by, e. g., Halava et al. [6] with respect to the Post Correspondence Problem, which is loosely related to our subject – we deal with a vital type of morphisms that addresses some of the very foundations of the problem field of ambiguity of morphisms. Finally, it is surely worth mentioning that the properties of segmented morphisms have also been studied in the context of *pattern languages* (cf., e. g., Jiang et al. [8]); in particular, recent papers prove the substantial impact of the (un-)ambiguity of such morphisms on pattern *inference* (cf. Reidenbach [10, 11]). Thus, our results provide a worthwhile starting point for further considerations in a prominent algorithmic research field related to pattern languages. In the present paper, however, we do not explicitly discuss this aspect of our work.

2 Definitions and Basic Notes

We begin the formal part of this paper with a number of basic definitions. A major part of our terminology is adopted from the research on pattern languages (cf. Mateescu and Salomaa [9]). Additionally, for notations not explained explicitly, we refer the reader to Choffrut and Karhumäki [3].

Let $\mathbb{N} := \{1, 2, 3, ...\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Let Σ be an *alphabet*, i.e. an enumerable set of symbols. We regard two different alphabets: $\mathbb{N}$ and $\{\mathtt{a}, \mathtt{b}\}$ with $\mathtt{a} \neq \mathtt{b}$. Henceforth, we call any symbol in $\mathbb{N}$ a *variable* and any symbol in $\{\mathtt{a}, \mathtt{b}\}$ a *letter*. A *string (over Σ)* is a finite sequence of symbols from Σ. For the *concatenation* of two strings w_1, w_2 we write $w_1 \cdot w_2$ or simply $w_1 w_2$. The notation $|x|$ stands for the size of a set x or the length of a string x, respectively. We denote the *empty string* by λ, i.e. $|\lambda| = 0$. In order to distinguish between a string over $\mathbb{N}$ and a string over $\{\mathtt{a}, \mathtt{b}\}$, we call the former a *pattern* and the latter a *word*. We name patterns with lower case letters from the beginning of the Greek alphabet such as α, β, γ. With regard to an arbitrary pattern α, $V(\alpha)$ denotes the set of all variables occurring in α. For every alphabet Σ, Σ^* is the set of all (empty and non-empty) strings over Σ, and $\Sigma^+ := \Sigma^* \setminus \{\lambda\}$. Furthermore, we use the regular operations $+$, $*$ and $\cdot$ on sets and letters in the usual way. For any $w \in \Sigma^*$ and any $n \in \mathbb{N}$, w^n describes the n-fold concatenation of w, and $w^0 := \lambda$. We say that a string $v \in \Sigma^*$ is a *substring* of a string $w \in \Sigma^*$ if and only if, for some $u_1, u_2 \in \Sigma^*$, $w = u_1 v u_2$. Subject to the concrete alphabet considered, we call a substring a *subword* or *subpattern*.

Since we deal with word semigroups, a *morphism* σ is a mapping that is compatible with the concatenation, i.e. for patterns $\alpha, \beta \in \mathbb{N}^+$, a morphism $\sigma : \mathbb{N}^* \longrightarrow \{\mathtt{a}, \mathtt{b}\}^*$ satisfies $\sigma(\alpha \cdot \beta) = \sigma(\alpha) \cdot \sigma(\beta)$. Hence, a morphism is fully explained as soon as it is declared for all variables in $\mathbb{N}$. Note that we restrict ourselves to total morphisms, even though we normally declare a morphism only for those variables explicitly that, in the respective context, are relevant.

For any pattern $\alpha \in \mathbb{N}^+$ with $\sigma(\alpha) \neq \lambda$, we call $\sigma(\alpha)$ *unambiguous (with respect to α* or *on α)* if there is no morphism $\tau : \mathbb{N}^* \longrightarrow \{\mathtt{a}, \mathtt{b}\}^*$ such that $\tau(\alpha) = \sigma(\alpha)$ and, for some $x \in V(\alpha)$, $\tau(x) \neq \sigma(x)$; otherwise, we call σ *ambiguous (with respect to α* or *on α)*. For a given morphism σ, let $U(\sigma)$ denote the set of all $\alpha \in \mathbb{N}^+$ such that σ is unambiguous on α.

We continue the definitions in this section with a partition of the set of all patterns subject to the following criterion that is due to Freydenberger et al. [5]:

Definition 1. *We call any $\alpha \in \mathbb{N}^+$* prolix *if and only if there exists a decomposition $\alpha = \beta_0 \gamma_1 \beta_1 \gamma_2 \beta_2 \ldots \beta_{n-1} \gamma_n \beta_n$ with $n \geq 1$, $\beta_k \in \mathbb{N}^*$ and $\gamma_k \in \mathbb{N}^+$, $k \leq n$, such that*

1. *for every k, $1 \leq k \leq n$, $|\gamma_k| \geq 2$,*
2. *for every k, $1 \leq k \leq n$, and for every k', $0 \leq k' \leq n$, $V(\gamma_k) \cap V(\beta_{k'}) = \emptyset$,*
3. *for every k, $1 \leq k \leq n$, there exists an $x_k \in V(\gamma_k)$ such that x_k occurs exactly once in γ_k and, for every k', $1 \leq k' \leq n$, if $x_k \in V(\gamma_{k'})$ then $\gamma_k = \gamma_{k'}$.*

We call $\alpha \in \mathbb{N}^+$ succinct *if and only if it is not prolix.*

Succinct and prolix patterns possess several interesting characteristic properties. First, Freydenberger et al. [5] demonstrate that a pattern α is succinct if and only if there is an injective morphism $\sigma^{\mathrm{su}}_{\alpha}$ such that $\sigma^{\mathrm{su}}_{\alpha}$ is unambiguous on α. Furthermore, there is no injective morphism that is unambiguous on all succinct patterns, and all nonerasing morphism are ambiguous on all prolix patterns. These results serve as the main fundament of the present work. In addition to this aspect, the set of prolix patterns exactly corresponds to the set of finite fixed points of nontrivial morphisms, i.e. for every prolix pattern α there exists a morphism $\phi : \mathbb{N}^* \longrightarrow \mathbb{N}^*$ such that, for an $x \in V(\alpha), \phi(x) \neq x$ and yet $\phi(\alpha) = \alpha$ (cf., e. g., Hamm and Shallit [7]). Finally, according to Reidenbach [10], the succinct patterns are the shortest generators for their respective E-pattern language – this explains the terms "succinct" and "prolix".

Whithin the scope of the present paper, we call a morphism $\sigma : \mathbb{N}^* \longrightarrow \{\mathtt{a}, \mathtt{b}\}^*$ *homogeneous* if there exist $p, s \in \{\mathtt{a}, \mathtt{b}\}^+$ such that for all $x \in \mathbb{N}$, p is a prefix of $\sigma(x)$ and s is a suffix of $\sigma(x)$. Otherwise, σ is *heterogeneous*.

For every $n \in \mathbb{N}$, we define σ_n (the *segmented morphism with n segments*) by $\sigma_n(x) := \mathtt{ab}^{nx-(n-1)}\mathtt{a}\,\mathtt{ab}^{nx-(n-2)}\mathtt{a}\,\ldots\,\mathtt{ab}^{nx-1}\mathtt{a}\,\mathtt{ab}^{nx}\mathtt{a}$ for every $x \in \mathbb{N}$ and refer to the subwords $\mathtt{ab}^+\mathtt{a}$ as *segments*. In this work, we mostly concentrate on the morphisms σ_1, σ_2, σ_3 given by $\sigma_1(x) := \mathtt{ab}^x\mathtt{a}$, $\sigma_2(x) := \mathtt{ab}^{2x-1}\mathtt{a}\,\mathtt{ab}^{2x}\mathtt{a}$ and $\sigma_3(x) := \mathtt{ab}^{3x-2}\mathtt{a}\,\mathtt{ab}^{3x-1}\mathtt{a}\,\mathtt{ab}^{3x}\mathtt{a}$. Although it is not a segmented morphism, we also study the morphism σ_0 given by $\sigma_0(x) := \mathtt{ab}^x$, as it is quite similar to σ_1 and often used to encode words over infinite alphabets using only two letters.

There is an interesting property of all σ_n with $n \geq 3$ that can be derived from the proof of Lemma 28 by Freydenberger et al. [5]:

Lemma 1. *Let $\alpha \in \mathbb{N}^+$ succinct, $n \geq 3$ and $\tau(\alpha) = \sigma_n(\alpha)$ for some morphism $\tau \neq \sigma_n$. Then, for every $x \in V(\alpha)$, $\tau(x)$ contains* $\mathtt{a}\,\mathtt{ab}^{nx-(n-2)}\mathtt{a}\,\ldots\,\mathtt{ab}^{nx-1}\mathtt{a}\,\mathtt{a}$.

This lemma is of great use in the next section, and the fact that there is no similar property for $n \leq 2$ is the very reason for the existence of Section 4.

3 Homogeneous Morphisms with Three or More Segments

Due to Freydenberger et al. [5], we know that the characteristic regularities in prolix patterns render every injective morphism ambiguous on these patterns. Although succinctness prohibits those regularities, some other structures supporting ambiguity of segmented morphisms can occur. For example, it is easy to see that σ_1 is ambiguous on the succinct pattern $\alpha := 1 \cdot 2 \cdot 1 \cdot 3 \cdot 3 \cdot 2$, e. g. by considering morphisms τ_1 or τ_2 which are given by $\tau_1(1) := \mathtt{ab}$, $\tau_1(2) := \mathtt{a}\,\mathtt{ab}^2\mathtt{a}$

and $\tau_1(3) := \mathtt{a\,ab}^3$ and $\tau_2(1) := \mathtt{aba\,a}$, $\tau_2(2) := \mathtt{b}^2\mathtt{a}$ and $\tau_2(3) := \mathtt{b}^3\mathtt{a\,a}$. In both cases, the arising ambiguity can be understood (albeit rather metaphorically) as some kind of communication where occurrences of 1 decide which modification is applied to their image under σ_1 and communicate this change to occurrences of 2, where applicable using 3 as a carrier. The patterns that show such a structure can be generalised as follows:

Definition 2. *Let $\alpha \in \mathbb{N}^+$. An* SCRN-partition for α *is a partition of $V(\alpha)$ into pairwise disjoint sets S, C, R and N such that $\alpha \in (N^*SC^*R)^+ N^*$. We call α* SCRN-partitionable *if and only if it has an SCRN-partition.*

As demonstrated by the above example, the existence of an SCRN-partition of a pattern α is a sufficient condition for the ambiguity of any segmented morphism (and σ_0 as well). In fact, it holds for every homogeneous morphism:

Proposition 1. *Let $\alpha \in \mathbb{N}^+$. If α is SCRN-partitionable, then every homogeneous morphism σ is ambiguous on α.*

Proof. As σ is homogeneous, there exist a $p \in \{\mathtt{a},\mathtt{b}\}^+$ and, for every $x \in \mathbb{N}$, an $s_x \in \{\mathtt{a},\mathtt{b}\}^*$ such that $\sigma(x) = p\, s_x$. Let S, C, R, N be an SCRN-partition for α. We define τ by, for all $x \in S$, $\tau(x) := \sigma(x)\, p$, for $x \in R$, $\tau(x) := s_x$, for $x \in C$, $\tau(x) := s_x\, p$. For $x \in N$, we simply define $\tau(x) := \sigma(x)$. As we are using an SCRN-partition, $\alpha \notin N^*$; therefore, $\tau \neq \sigma$ holds. It is easy to see that $\tau(\alpha) = \sigma(\alpha)$. Thus, σ is ambiguous on α. □

We now wish to demonstrate that, for σ_n with $n \geq 3$, this condition is even characteristic. If σ_n is ambiguous on some succinct $\alpha \in \mathbb{N}^+$ (i.e., there is some $\tau \neq \sigma_n$ with $\tau(\alpha) = \sigma_n(\alpha)$), every variable possessing different images under τ and σ_n still keeps all its characteristic inner segments under τ (cf. Lemma 1). Any change is therefore limited to some gain or loss of its (or its neighbours') outer segments and has to be communicated along subpatterns resembling the SC^*R-sequences of a SCRN-partition. This allows to construct an SCRN-partition from τ and leads to the following theorem:

Theorem 1. *Let $\alpha \in \mathbb{N}^+$. Then, for every $n \geq 3$, σ_n is ambiguous on α if and only if α is prolix or SCRN-partitionable.*

Proof. As mentioned above, [5] demonstrates that we can safely restrict ourselves to succint α, since every injective morphism is ambiguous on every prolix $\alpha \in \mathbb{N}^+$. We begin with the only-if-direction. Assume σ_n is ambiguous on some succinct $\alpha \in \mathbb{N}^+$; then there exists some morphism $\tau \neq \sigma_n$ with $\tau(\alpha) = \sigma_n(\alpha)$. Lemma 1 guarantees that every $\tau(x)$ contains the inner segments of $\sigma_n(x)$. This allows us to distinguish the following cases: For every $x \in V(\alpha)$, let $x \in N$ if and only if $\tau(x) = \sigma_n(x)$. If x has neither lost nor gained to its left, but has lost or gained to the right, let $x \in S$, if its the other way around, let $x \in R$. Finally, if $\tau(x)$ is different from $\sigma_n(x)$ on both sides, let $x \in C$. To show that $\alpha \in (N^*SC^*R)^+ N^*$, we read α from the left to the right. As the first variable has no left neighbour, it cannot have gained or lost some word on its left side; thus, it must belong to

N or S. If it belongs to N, the same is true for the next variable, but as $\alpha \in N^+$ would contradict $\tau \neq \sigma_n$, sooner or later some variable from S must occur. As this variable has a changed right segment, its right neighbour experienced the corresponding change on its left segment. Consequently, that variable must belong to C or R. If it is from C instead, again variables from C must follow until a variable from R is encountered; so α has a prefix from N^*SC^*R. But as variables from R do not change their right segments under τ, we now have the same situation as when we started. We conclude $\alpha \in (N^*SC^*R)^+ N^*$; therefore, α is SCRN-partitionable. The if-direction follows from Proposition 1. □

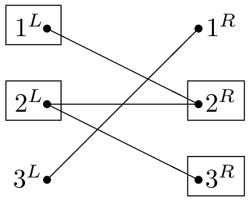
Consequently, ambiguity of morphisms with at least three segments on succinct patterns is always only a transfer of parts of segments in blocks consisting of a sender, a receiver and possibly some carriers between them.[3] As a sidenote, consider *generalised segmented morphisms with n segments* as morphisms $\sigma_G : \mathbb{N}^* \longrightarrow \Sigma^*$ where $\sigma_G(x) \in (\mathtt{ab}^+\mathtt{a})^n$ for all $x \in \mathbb{N}$, and for every $w \in \mathtt{ab}^+\mathtt{a}$, there is at most one $x \in \mathbb{N}$ such that w is a subword of $\sigma_G(x)$. It can be shown that if $n \geq 3$, Lemma 1 holds for σ_G as well. Thus, for every generalised segmented morphism σ_G with at least three segments, $U(\sigma_G) = U(\sigma_3)$. Furthermore, as σ_3 is the homogeneous version of the heterogeneous unambiguous morphism $\sigma_\alpha^{\mathrm{su}}$ constructed by Freydenberger et al. [5], Theorem 1 precisely distinguishes the patterns for which there is an unambiguous *homogeneous* morphism from those patterns where an unambiguous morphism has to be *heterogeneous*. Thus, our result significantly contributes to a deeper understanding of the impact of the heterogeneity of a segmented morphism on its unambiguity.

Theorem 1 demonstrates, that for σ_n with $n \geq 3$, ambiguity on succinct patterns is inherently related to the occurrence of global regularities that depend on local interactions between *neighbouring variables* only. In fact, these regularities can be described by the equivalence classes $L_i^\sim$ and $R_i^\sim$ on $V(\alpha)$ introduced by Freydenberger et al. [5] as fundamental tools to construct tailor-made unambiguous morphisms $\sigma_\alpha^{\mathrm{su}}$. In the present paper, we describe these equivalence classes using an equivalent but simpler definition that is based on the adjacency graph of a pattern, a construction that has first been employed by Baker et al. [1] to simplify the Bean-Ehrenfeucht-McNulty-Zimin characterisation of avoidable words, cf. Cassaigne [2]. Like Baker et al., we associate a pattern $\alpha \in \mathbb{N}^+$ with a bipartite graph $\mathrm{AG}(\alpha)$, the *adjacency graph of* α: The vertex set consists of two marked copies of $V(\alpha)$, V^L and V^R (for left and right, respectively); for each $x \in V(\alpha)$, there is an element $x^L \in V^L$ and an element $x^R \in V^R$. There is an edge $x^L - y^R$ for $x, y \in V(\alpha)$ if and only if xy is a subpattern of α.

Unlike Baker et al., we consider a partition of $V^L \cup V^R$ into sets $H_1, \ldots, H_n$ such that each H_i is the set of vertices of a maximal and connected subgraph of $\mathrm{AG}(\alpha)$. We call such a set H_i a *neighbourhood in* α and refer to the set of all neighbourhoods as $H(\alpha)$. For every neighbourhood H_i, the *left neighbourhood class* $L_i^\sim$ denotes the set of all x such that x^L is in H_i and likewise the *right neighbourhood class* $R_i^\sim$ the set of all x such that x^R is in H_i.

[3] Hence the letters S, C, R, N stand for sender, carrier, receiver and neutral, respectively.

Example 1. Let $\alpha := 1 \cdot 2 \cdot 3 \cdot 1 \cdot 2 \cdot 2 \cdot 3$. We obtain $H_1 = \{1^L, 2^L, 2^R, 3^R\}$ and $H_2 = \{3^L, 1^R\}$ and therefore, $L_1^\sim = \{1, 2\}$, $L_2^\sim = \{3\}$, $R_1^\sim = \{2, 3\}$ and $R_2^\sim = \{1\}$. In the following figure, we display the adjacency graph of α. Boxes mark the elements of H_1:



As no injective morphism is unambiguous on a prolix pattern, we mainly deal with succinct patterns. It is useful to note that, apart from patterns of length 1 (like 1), no succinct pattern contains variables that occur only once. Therefore, in succinct patterns every neighbourhood contains elements from V^L and V^R, and every variable belongs to exactly one left and one right neighbourhood class.

Utilising our definition of neighbourhood classes, we now give a second characterisation of $U(\sigma_n)$, $n \geq 3$:

Theorem 2. *For every $\alpha \in \mathbb{N}^+$ with first variable f and last variable l and any $n \geq 3$, σ_n is ambiguous on α if and only if α is prolix or there is a neighbourhood $H_i \in H(\alpha)$ such that $f \notin R_i^\sim$ and $l \notin L_i^\sim$.*

Proof. First, assume that, for some succinct $\alpha := f\alpha' l$ with $\alpha' \in \mathbb{N}^*$, there is some $\tau \neq \sigma_n$ such that $\tau(\alpha) = \sigma_n(\alpha)$. Now we construct an SCRN-partition S, C, R, N of $V(\alpha)$ like in the proof to Theorem 1. Let $x \in S$ and choose i such that $x \in L_i^\sim$. Then $\tau(x)$ can be seen as the result of $\sigma_n(x)$ either loosing a word $\mathtt{b}^*\mathtt{a}$ to or gaining some word $\mathtt{ab}^*$ from every right neighbour of an occurrence of x in α. Therefore, all those neighbours must reflect this change on the left side of their image under τ, as anything else would contradict Lemma 1 or $\tau(\alpha) = \sigma_n(\alpha)$. Likewise, all those neighbours' left neighbours must change their right segment in the same way as x. This has to propagate through all of H_i; so all elements of $L_i^\sim$ show the same change to their right segment, and all elements of $R_i^\sim$ show the corresponding change to their left segment. Now assume $f \in R_i^\sim$. As f is the first variable of α and due to Lemma 1, $\tau(f)$ can differ from $\sigma_n(x)$ only to the right of the middle segment, and only by some part of a segment. But this contradicts our previous observation that all elements of $R_i^\sim$ are afflicted by a change to their left segment. This leads to $f \notin R_i^\sim$. Likewise, $l \notin L_i^\sim$, which concludes this direction of the proof. For the other direction, let $\alpha := f\alpha' l$ be succinct with some neighbourhood H_i such that $f \notin R_i^\sim$ and $l \notin L_i^\sim$. Now define S, C, R, N by $S = L_i^\sim \setminus R_i^\sim$, $C = L_i^\sim \cap R_i^\sim$, $R = R_i^\sim \setminus L_i^\sim$ and $N = V(\alpha) \setminus (L_i^\sim \cup R_i^\sim)$. The four sets form a partition of $V(\alpha)$, so it merely remains to be shown that their elements occur in α in the right order. First observe that, by definition, $f \in S \cup N$ and $l \in R \cup N$. Furthermore, for any subpattern xy of α, if $x \in S$ or $x \in C$, then $x \in L_i^\sim$. Therefore, $y \in R_i^\sim$ and thus $y \in C \cup R$. Likewise, $x \in N$ or $x \in R$ implies $x \notin L_i^\sim$ and $y \notin R_i^\sim$, which leads to $y \in N \cup S$ and $\alpha \in (N^* S C^* R)^+ N^*$. By Theorem 1, we conclude that σ_n is ambiguous on α. $\square$

Consequently, in order to decide ambiguity of σ_n, $n \geq 3$, on a succinct pattern α, it suffices to construct $H(\alpha)$ and check the classes of the first and last variable of α. The construction can be done efficiently, e. g. by using a Union-Find-algorithm.

This theorem provides a useful corollary for a class of patterns first described by Baker et al. [1]. We call a pattern $\alpha \in \mathbb{N}^+$ *locked* if and only if $|H(\alpha)| = 1$ and thus $L_1^\sim = R_1^\sim = V(\alpha)$. We observe the following consequence:

Corollary 1. *Let $\alpha \in \mathbb{N}^+$. If α is succinct and locked, then $\alpha \in U(\sigma_3)$.*

This corollary is of use in the next section, where we shall see that having less than three segments entails other types of ambiguity than the one described in the previous section.

4 Homogeneous Morphisms with Less than Three Segments

In this section, we examine the effects caused by reducing the number of segments. One might expect no change in the corresponding sets of unambiguous patterns, or a small hierarchy that reflects the number of segments, but as we shall see, neither is the case. To this end, we construct the following five patterns:

Definition 3. *We define α_0, α_1, α_2 and $\alpha_{0\backslash 2}$ as follows:*

$$\begin{aligned}
\alpha_0 &:= 1 \cdot 2 \cdot 3 \cdot 1 \cdot 3 \cdot 2,\\
\alpha_1 &:= 1 \cdot 2 \cdot 2 \cdot 3 \cdot 1 \cdot 1 \cdot 3 \cdot 1,\\
\alpha_2 &:= (1 \cdot 2 \cdot 3 \cdot 3 \cdot 4)^2 \cdot 5 \cdot 2 \cdot 6 \cdot 5 \cdot 7 \cdot (8 \cdot 6)^2 \cdot (9 \cdot 7)^2 \cdot 10 \cdot 4 \cdot 11 \cdot 4 \cdot 10 \cdot 12 \cdot\\
&\quad\ 11 \cdot 12 \cdot (3 \cdot 13)^2 \cdot (14 \cdot 3 \cdot 2 \cdot 15)^2,\\
\alpha_{0\backslash 2} &:= (1 \cdot 2 \cdot 3)^2 \cdot (4 \cdot 5 \cdot 4)^2 \cdot (6 \cdot 7 \cdot 6 \cdot 8)^2 \cdot 1 \cdot 7 \cdot 3 \cdot (9 \cdot 6 \cdot 6 \cdot 10)^2 \cdot (11 \cdot 12)^2 \cdot\\
&\quad\ (13 \cdot 7 \cdot 7 \cdot 4 \cdot 14 \cdot 12)^2 \cdot (15 \cdot 14)^2 \cdot 9 \cdot 6.
\end{aligned}$$

Finally, we define $\alpha_{1\backslash 2}$ by $\alpha_{1\backslash 2} := 1^2 \cdot \delta \cdot 1 \cdot p(\delta) \cdot 1$, where

$$\begin{aligned}
\delta := &\beta_1 \cdot 1 \cdot \beta_2 \cdot 1 \cdot \beta_3 \cdot 1 \cdot \beta_4 \cdot 1 \cdot \beta_5 \cdot 1 \cdot \gamma_1 \cdot \beta_6 \cdot 1 \cdot \beta_7 \cdot 1 \cdot \gamma_2 \cdot 1 \cdot \beta_8 \cdot\\
&1 \cdot \gamma_3 \cdot 1 \cdot \beta_9 \cdot 1 \cdot \beta_{10} \cdot 1 \cdot \beta_{11} \cdot 1 \cdot \beta_{12} \cdot 1 \cdot \beta_{13} \cdot 1 \cdot \beta_{14},
\end{aligned}$$

and $p(1) := \lambda$, $p(x) := x$ for all $x \in \mathbb{N} \setminus \{1\}$, and furthermore

$\beta_1 := 2 \cdot 3 \cdot 3 \cdot 4$, $\beta_2 := 3 \cdot 2 \cdot 2 \cdot 5$,
$\beta_3 := 6 \cdot 7$, $\beta_4 := 8 \cdot 9$,
$\beta_5 := 10 \cdot 11$, $\gamma_1 := (12 \cdot 1)(13 \cdot 1) \cdot \ldots \cdot (17 \cdot 1)$,
$\beta_6 := 18 \cdot 19$, $\beta_7 := 6 \cdot 20 \cdot 9$,
$\gamma_2 := 21 \cdot 1 \cdot 22$, $\beta_8 := 6 \cdot 23 \cdot 11$,
$\gamma_3 := 24 \cdot 1 \cdot 25 \cdot 1 \cdot 26$, $\beta_9 := 27 \cdot 2 \cdot 20 \cdot 2 \cdot 20 \cdot 28 \cdot 2 \cdot 29$,

$\beta_{10} := 30 \cdot 2 \cdot (20 \cdot 2 \cdot 23 \cdot 2)^3 \cdot 20 \cdot (31)^4 \cdot 32$, $\beta_{11} := 33 \cdot 3 \cdot 34 \cdot 23 \cdot 3 \cdot 23 \cdot 3 \cdot 35$,
$\beta_{12} := 36 \cdot 20 \cdot 20 \cdot 28 \cdot 2 \cdot 29$, $\beta_{13} := 33 \cdot 3 \cdot 34 \cdot 23 \cdot 23 \cdot 37$,
$\beta_{14} := 18 \cdot 31 \cdot 32$.

We begin by establishing the relation between $U(\sigma_3)$ and the other sets:

Theorem 3. *The sets $U(\sigma_0)$, $U(\sigma_1)$ and $U(\sigma_2)$ are strictly included in $U(\sigma_3)$.*

Proof. For all three languages, the inclusion directly follows from Theorem 1: If $\alpha \notin U(\sigma_3)$ then α is prolix or SCRN-partitionable. In the former case, every injective morphism is ambiguous on α, and, due to Proposition 1, the existence of an SCRN-partition is sufficient for ambiguity of segmented morphisms and σ_0. To prove strictness, we show that σ_0, σ_1, σ_2 are ambiguous on the patterns α_0, α_1, α_2, respectively, from Definition 3. All three patterns are succinct and – as demonstrated by their adjacency graphs – have only one neighbourhood class each. Hence, due to Corollary 1, σ_3 is unambiguous on each of the patterns.

We start with α_0 and define τ by $\tau(1) := \sigma_0(1{\cdot}2)$, $\tau(2) := \sigma_0(2)$ and $\tau(3) := \mathtt{b}$. Then $\tau \neq \sigma_0$, but $\tau(\alpha_0) = \sigma_0(1 \cdot 2) \cdot \sigma_0(2) \cdot \mathtt{b} \cdot \sigma_0(1 \cdot 2) \cdot \mathtt{b} \cdot \sigma_0(2) = \sigma_0(\alpha_0)$. Therefore, σ_0 is ambiguous on α_0. For α_1, we set $\tau(1) := \mathtt{a}$, $\tau(2) := \mathtt{baab}$ and $\tau(3) := \mathtt{ba}\,\sigma_1(3)\,\mathtt{ab}$. It is easy to see that $\tau \neq \sigma_1$ and $\tau(\alpha_1) = \sigma_1(\alpha_1)$. With regard to σ_2, we consider the morphism τ given by

$$\tau(1) := \sigma_2(1 \cdot 2 \cdot 3)\,\mathtt{ab}^5\mathtt{a}\,\mathtt{ab}^3, \qquad \tau(2) := \mathtt{b}^3\mathtt{a}\,\mathtt{ab}^3,$$
$$\tau(3) := \lambda, \qquad \tau(4) := \mathtt{b}^4\mathtt{a}\,\mathtt{ab}^8\mathtt{a},$$
$$\tau(5) := \sigma_2(5)\,\mathtt{a}, \qquad \tau(6) := \mathtt{ba}\,\sigma_2(6),$$
$$\tau(7) := \mathtt{b}^{13}\mathtt{a}\,\mathtt{ab}^{14}\mathtt{a}, \qquad \tau(8) := \mathtt{ab}^{15}\mathtt{a}\,\mathtt{ab}^{15},$$
$$\tau(9) := \sigma_2(9)\,\mathtt{a}, \qquad \tau(10) := \sigma_2(10)\,\mathtt{ab}^3,$$
$$\tau(11) := \sigma_2(11)\,\mathtt{ab}^3, \qquad \tau(12) := \mathtt{b}^{20}\mathtt{a}\,\mathtt{ab}^{24}\mathtt{a},$$
$$\tau(13) := \sigma_2(3 \cdot 13), \qquad \tau(14) := \mathtt{ab}^{27}\mathtt{a}\,\mathtt{ab}^{25},$$
$$\tau(15) := \mathtt{b}^2\mathtt{a}\,\mathtt{ab}^6\mathtt{a}\,\sigma_2(2 \cdot 15).$$

Then $\tau \neq \sigma_2$. Proving $\tau(\alpha_2) = \sigma_2(\alpha_2)$ is less obvious, but straightforward. □

The proof for Theorem 3 is of additional interest as Freydenberger et al. [5] propose to study a morphism $\sigma_\alpha^{\text{2-seg}}$ that maps each variable x in a succinct pattern α onto a word that merely consists of the left and the right segment of $\sigma_\alpha^{\text{su}}(x)$ (recall that $\sigma_\alpha^{\text{su}}$ is a heterogeneous morphism which maps every variable x onto *three* segments). In [5] it is asked whether, for every succinct pattern α, $\sigma_\alpha^{\text{2-seg}}$ is unambiguous on α, thus suggesting the chance for a major improvement of $\sigma_\alpha^{\text{su}}$. With regard to this question, we now consider the pattern α_2. In the above proof, it is stated that α is a locked pattern, which implies that $\sigma_{\alpha_2}^{\text{2-seg}}$ only maps the variable 1 onto a word $\mathtt{b}\ldots\mathtt{b}$ and all other variables in α_2 onto words $\mathtt{a}\ldots\mathtt{a}$. Consequently, for each $x \in V(\alpha_2) \setminus \{1\}$, $\sigma_{\alpha_2}^{\text{2-seg}}(x) = \sigma_2(x)$. Therefore – and since, for the corresponding τ introduced in the proof of Theorem 3, the word $\tau(1)$ completely contains $\sigma_2(1)$ – we can define a morphism τ' by $\tau'(1) :=$

$\sigma_{\alpha_2}^{\text{2-seg}}(1 \cdot 2 \cdot 3)\,\mathtt{ab}^5\mathtt{a}\,\mathtt{ab}^3$ and $\tau'(x) := \tau(x)$, $x \in V(\alpha_2) \setminus \{1\}$, and this definition yields $\tau'(\alpha_2) = \sigma_{\alpha_2}^{\text{2-seg}}(\alpha_2)$. So, there exists a succinct pattern α (namely α_2) such that $\sigma_{\alpha}^{\text{2-seg}}$ is ambiguous on α. Thus, α_2 does not only prove $U(\sigma_2) \subset U(\sigma_3)$, but it also provides a negative answer to an intricate question posed in [5].

Returning to the focus of the present paper, the examples in the proof for Theorem 3 demonstrate ambiguity phenomena that are intrinsic for their respective kind of morphisms and cause ambiguity on patterns that are neither prolix nor SCRN-partitionable: With regard to σ_0, the fact that for each x, y with $x < y$, $\sigma_0(x)$ is a prefix of $\sigma_0(y)$ can be used to achieve ambiguity, as demonstrated by α_0. Concerning σ_1, a variable x can achieve $\tau(x) = \mathtt{a}$ both by giving $\mathtt{ab}^x$ to the left or $\mathtt{b}^x\mathtt{a}$ to the right, which can be prefix or suffix of some $\sigma_1(y)$. In α_1, we use this for $\tau(1)$. The situation is less obvious and somewhat more complicated for σ_2, as suggested by the fact that we do not know a shorter pattern serving the same purpose as α_2. Here, a variable x can obtain an image $\tau(x) \in \mathtt{b}^*\mathtt{aab}^*$, which can be used both as a middle part of some $\sigma_2(y)$, and as the borderline between some $\sigma_2(y)$ and some $\sigma_2(z)$. In the proofs for Theorem 5 and Theorem 6 we utilise further examples for complicated cases of σ_2-ambiguity.

It is natural to ask whether these phenomena can be used to find patterns where one of the three morphisms σ_0, σ_1, σ_2 is ambiguous, and another is not. We begin with a comparison of $U(\sigma_0)$ and $U(\sigma_1)$:

Theorem 4. *The sets $U(\sigma_0)$ and $U(\sigma_1)$ are incomparable.*

Proof. We have already established the ambiguity of σ_0 on α_0 and of σ_1 on α_1 in the proof of Theorem 3. The proofs for the unambiguity of σ_1 on α_0 and of σ_0 on α_1 are left out due to space reasons. □

This result is perhaps somewhat counter-intuitive, but the fact that $U(\sigma_0)$ and $U(\sigma_1)$ can be separated by two very short examples might be considered evidence that the two languages are by far not as similar as the two morphisms. We proceed with a comparison of $U(\sigma_0)$ and $U(\sigma_2)$. Surprisingly, the same result holds (although one of the examples is considerably more involved):

Theorem 5. *The sets $U(\sigma_0)$ and $U(\sigma_2)$ are incomparable.*

Proof. Here, we use α_0 and $\alpha_{0\backslash 2}$. In spite of the NP-completeness of the problem (cf. Ehrenfeucht, Rozenberg [4]), $\alpha_{0\backslash 2} \in U(\sigma_0)$ and $\alpha_0 \in U(\sigma_2)$ can be verified by a computer; therefore (and due to space constraints), we omit the corresponding proof. Contrary to this, the length of $\sigma_2(\alpha_{0\backslash 2})$ does not allow for the use of a computer. With regard to the ambiguity of σ_2, we thus refer to the morphism τ given by

$$\begin{aligned}
\tau(1) &= \sigma_2(1)\,\mathtt{ab}^2, & \tau(2) &= \mathtt{ba}\,\mathtt{ab}^2,\\
\tau(3) &= \mathtt{b}^2\mathtt{a}\,\sigma_2(1), & \tau(4) &= \lambda,\\
\tau(5) &= \sigma_2(4 \cdot 5 \cdot 4), & \tau(6) &= \mathtt{a},\\
\tau(7) &= \mathtt{b}^{11}\mathtt{a}\,\mathtt{ab}^{12}, & \tau(8) &= \sigma_2(7 \cdot 6 \cdot 8),\\
\tau(9) &= \sigma_2(9 \cdot 6), & \tau(10) &= \sigma_2(6 \cdot 10),
\end{aligned}$$

$$\tau(11) = \sigma_2(11)\,\mathtt{ab}^{23}\mathtt{a}\,\mathtt{ab}^{12}, \qquad \tau(12) = \mathtt{b}^{12}\mathtt{a},$$
$$\tau(13) = \sigma_2(13 \cdot 7 \cdot 7 \cdot 4)\,\mathtt{ab}^{27}\mathtt{a}\,\mathtt{ab}^{17}, \qquad \tau(14) = \lambda,$$
$$\tau(15) = \sigma_2(15 \cdot 14),$$

which yields $\tau(\alpha_{0\backslash 2}) = \sigma_2(\alpha_{0\backslash 2})$ and, hence, the ambiguity of σ_2 on α. □

We conclude this section by the examination of the last open case, namely the relation between $U(\sigma_1)$ and $U(\sigma_2)$. Again, one might conjecture that the more complex morphism σ_2 is "stronger" than σ_1, but our most sophisticated example pattern $\alpha_{1\backslash 2}$ shows that this expectation is not correct:

Theorem 6. *The sets $U(\sigma_1)$ and $U(\sigma_2)$ are incomparable.*

Proof. For this proof we use the patterns α_1 and $\alpha_{1\backslash 2}$. Recall that σ_1 is ambiguous on α_1 (cf. proof of Theorem 3). With little effort, it can be seen that σ_2 is unambiguous on α_1. Thus, we know that $\alpha_1 \in U(\sigma_2) \setminus U(\sigma_1)$. The fact that σ_1 is unambiguous on $\alpha_{1\backslash 2}$ requires extensive reasoning, which is left out due to space reasons. Showing that σ_2 is ambiguous on $\alpha_{1\backslash 2}$ is more straightforward. Let $\tau(x) := \lambda$ for $x \in \{2, 3, 28, 31, 34\}$ and $\tau(x) := \sigma_2(x)$ for $x \in V(\gamma_1) \cup V(\gamma_2) \cup V(\gamma_3)$. For all other $x \in V(\alpha_{1\backslash 2})$, define $\tau(x)$ as follows:

$$\tau(4) := \sigma_2(2 \cdot 3 \cdot 3 \cdot 4), \qquad \tau(5) := \sigma_2(3 \cdot 2 \cdot 2 \cdot 5),$$
$$\tau(6) := \sigma_2(6)\,\mathtt{ab}^{11}, \qquad \tau(7) := \mathtt{b}^2\mathtt{a}\,\mathtt{ab}^{14}\mathtt{a},$$
$$\tau(8) := \mathtt{ab}^{15}\mathtt{a}\,\mathtt{ab}^{3}, \qquad \tau(9) := \mathtt{b}^{13}\mathtt{a}\,\sigma_2(9),$$
$$\tau(10) := \mathtt{ab}^{19}\mathtt{a}\,\mathtt{ab}^{8}, \qquad \tau(11) := \mathtt{b}^{12}\mathtt{a}\,\sigma_2(11),$$
$$\tau(18) := \sigma_2(18)\,\mathtt{ab}^{27}, \qquad \tau(19) := \mathtt{b}^{10}\mathtt{a}\,\mathtt{ab}^{38}\mathtt{a},$$
$$\tau(20) := \mathtt{b}^{28}\mathtt{a}\,\mathtt{ab}^{27}, \qquad \tau(23) := \mathtt{b}^{34}\mathtt{a}\,\mathtt{ab}^{34},$$
$$\tau(27) := \sigma_2(27 \cdot 2 \cdot 20 \cdot 2)\,\mathtt{ab}^{39}\mathtt{a}\,\mathtt{ab}^{12}, \qquad \tau(29) := \mathtt{b}^{29}\mathtt{a}\,\sigma_2(2 \cdot 29),$$
$$\tau(30) := \sigma_2\left(30 \cdot 2 \cdot (20 \cdot 2 \cdot 23 \cdot 2)^3\right)\,\mathtt{ab}^{39}\mathtt{a}\,\mathtt{ab}^{12}, \qquad \tau(32) := \mathtt{b}^{34}\mathtt{a}\,\mathtt{ab}^{62}\mathtt{a} \cdot \sigma_2(32),$$
$$\tau(33) := \sigma_2(33 \cdot 3)\,\mathtt{ab}^{33}, \qquad \tau(35) := w\,\sigma_2(3 \cdot 23 \cdot 3 \cdot 35),$$
$$\tau(36) := \sigma_2(36 \cdot 20)\,\mathtt{ab}^{39}\mathtt{a}\,\mathtt{ab}^{12}, \qquad \tau(37) := w\,\sigma_2(23 \cdot 37),$$

where $w := \mathtt{b}^{11}\mathtt{a}\,\mathtt{ab}^{46}\mathtt{a}$. Obviously $\tau \neq \sigma_2$. As $\tau(x) = \sigma_2(x)$ for $x \in V(\gamma_1) \cup V(\gamma_2) \cup V(\gamma_3)$, especially for $x = 1$, it suffices to show $\tau(\beta_i) = \sigma_2(\beta_i)$ for all $i \in \{1, 2, \ldots, 14\}$. For β_1 and β_2, the claim holds trivially. For the other β_i, the process is straightforward but somewhat lenghty. Thus, $\alpha_{1\backslash 2} \in U(\sigma_1) \setminus U(\sigma_2)$, and therefore $U(\sigma_1)$ and $U(\sigma_2)$ are incomparable. □

Note that we do not know any nontrivial characterisation of $U(\sigma_0)$, $U(\sigma_1)$ and $U(\sigma_2)$. Moreover, we cannot refer to a computationally feasible method to successfully seek for any patterns in $U(\sigma_1) \setminus U(\sigma_2)$, $U(\sigma_0) \setminus U(\sigma_2)$ and $U(\sigma_3) \setminus U(\sigma_2)$. Therefore, we cannot answer the question of whether there exist shorter examples than α_2, $\alpha_{0\backslash 2}$ and $\alpha_{1\backslash 2}$ suitable for proving Theorems 3, 5 and 6, respectively. The intricacy of the ambiguity phenomena relevant for the construction of such patterns, however, suggests that our examples cannot be shortened significantly.

5 Conclusion and Open Problems

In the present paper, we have studied the unambiguity of an important type of injective morphisms. More precisely, we have examined the impact of the number n of segments of a segmented morphism σ_n on the set $U(\sigma_n)$ of patterns for which σ_n is unambiguous. Our main results show that a change of n, surprisingly, does not give rise to a "real" hierarchy of sets of patterns, as the three pairwise incomparable languages $U(\sigma_0)$, $U(\sigma_1)$ and $U(\sigma_2)$ are all contained in one common superset $U(\sigma_3)$, that is also the maximum any homogeneous morphism can achieve. We have established the result on $U(\sigma_3)$ by two characteristic criteria on $U(\sigma_3)$, which additionally entail a substantial improvement of the main technique introduced in the initial paper [5] on the unambiguity of morphisms.

Contrary to this, a major part of our results on σ_0, σ_1 and σ_2 are not based on criteria, but on example patterns. We regard it as a very interesting problem to find characterisations of $U(\sigma_0)$, $U(\sigma_1)$ and $U(\sigma_2)$. In consideration of the remarkable complexity of the patterns $\alpha_{0\backslash 2}$, $\alpha_{1\backslash 2}$ and α_2, however, we expect this to be an extraordinarily cumbersome task.

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