On the real zeroes of the Hurwitz zeta-function and Bernoulli polynomials

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Abstract. The behaviour of real zeroes of the Hurwitz zeta function

$$\zeta(s,a) = \sum_{r=0}^{\infty}(a+r)^{-s} \qquad a > 0$$

is investigated. It is shown that $\zeta(s,a)$ has no real zeroes $(s = \sigma, a)$ in the region $a > \frac{-\sigma}{2\pi e} + \frac{1}{4\pi e}\log(-\sigma) + 1$ for large negative σ. In the region $0 < a < \frac{-\sigma}{2\pi e}$ the zeroes are asymptotically located at the lines $\sigma + 4a + 2m = 0$ with integer m. If $N(p)$ is the number of real zeroes of $\zeta(-p,a)$ with given p then

$$\lim_{p\to\infty}\frac{N(p)}{p} = \frac{1}{\pi e}.$$

As a corollary we have a simple proof of Inkeri's result that the number of real roots of the classical Bernoulli polynomials $B_n(x)$ for large n is asymptotically equal to $\frac{2n}{\pi e}$.

1 Introduction.

The classical *Hurwitz zeta-function* is defined for any positive real a as an analytic continuation of the series

$$\zeta(s,a) = \sum_{r=0}^{\infty}(a+r)^{-s}.$$

When $a = 1$ it reduces to the Riemann zeta-function.

We should note that sometimes the definition of the Hurwitz zeta-function is restricted to $0 < a \leq 1$ (see e.g. [1, 2]). From our point of view this is not natural and we follow the definition of the Hurwitz zeta-function from [3] where all positive a are allowed (cf. also the original Hurwitz paper [4]).

It is known (see e.g. [2], volume 1, page 27) that in the special cases when s is a negative integer this function (as a function of the parameter a) reduces, up to a factor, to a Bernoulli polynomial: explicitly when $s = -m, m = 0, 1, 2, 3, ...$

$$\zeta(-m, a) = -\frac{B_{m+1}(a)}{m+1}.$$

The Bernoulli polynomials $B_k(a)$ can be defined through the generating function:

$$\frac{ze^{za}}{e^z - 1} = \sum_{k=0}^{\infty} \frac{B_k(a)}{k!} z^k$$

giving, for example:

$$B_0(a) = 1 \qquad B_1(a) = a - \frac{1}{2} \qquad B_2(a) = a^2 - a + \frac{1}{6} \qquad B_3(a) = a^3 - \frac{3a^2}{2} + \frac{a}{2}, ...$$

Bernoulli polynomials possess many interesting properties and arise in many areas of mathematics (see [1, 2]).

Inkeri [5] proved a remarkable fact that the number $N(n)$ of real roots of the Bernoulli polynomials B_n for large n asymptotically equals to $\frac{2n}{\pi e}$. More precise estimates for $N(n)$ have been found by Delange [6, 7].

In this paper we investigate the behaviour of the real zeroes of the Hurwitz zeta function $\zeta(\sigma, a)$ in the upper half-plane $a > 0$. As a corollary we have a simple proof of Inkeri's result. Our approach is different from [5, 6, 7] and we believe is more elementary. It is based on the remarkable Hurwitz representation of the $\zeta(s, a)$ on the interval $0 < a \leq 1$ and $\text{Re}(s) = \sigma < 0$:

$$\zeta(s, a) = \frac{2\Gamma(1-s)}{(2\pi)^{1-s}} \sum_{r=1}^{\infty} \frac{\sin(2\pi ra + \frac{1}{2}\pi s)}{r^{1-s}},$$

where Γ is the Euler gamma-function (see e.g [2], volume 1, page 26).

Our main observation is that for a large negative σ this formula gives a good approximation for $\zeta(\sigma, a)$ on a much wider interval: $0 < a < -\frac{\sigma}{2\pi e}$. As a result we prove that

$$\frac{\zeta(\sigma, a)}{Q(\sigma)} = \sin(2\pi a + \frac{1}{2}\pi\sigma) + o(1) \qquad \text{as} \quad \sigma \to -\infty \qquad \text{provided} \quad 0 < a < -\frac{\sigma}{2\pi e}$$

where $Q(s) = \frac{2\Gamma(1-s)}{(2\pi)^{1-s}}$. We show also that in the region $a > -\frac{\sigma}{2\pi e} + \frac{1}{4\pi e}\log(-\sigma) + 1$ the Hurwitz zeta-function has no real zeroes for large negative σ.

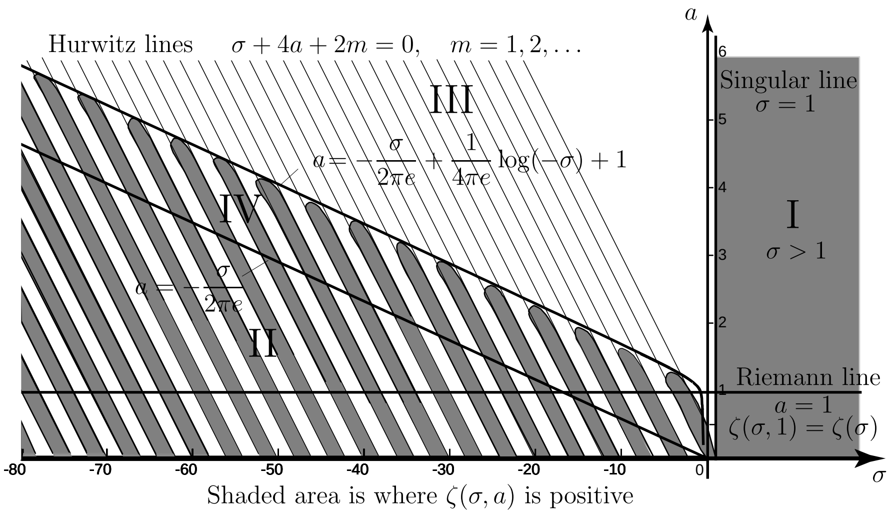

Figure 1

Our results are illustrated in Figure 1 which shows the behaviour of the real zeroes of $\zeta(\sigma, a)$. Since $\frac{1}{2\pi e}$ is a small number we have chosen different scales on the axes to make the picture more illuminating. Notice that $\zeta(s, 1)$ coincides with the Riemann zeta-function $\zeta(s)$ which only has real zeroes (for $s < 0$) at negative even numbers $s = -2, -4, -6, \ldots$. Also, using the well-known identity $\zeta(s, 1/2) = (2^s - 1)\zeta(s)$, we see that the only real zeroes of $\zeta(s, a)$ on $a = 1$ are $s = -2, -4, -6, \ldots$ and on $a = \frac{1}{2}$ they are $s = 0, -2, -4, \ldots$ As we have mentioned above when s is a non-positive integer $\zeta(s, a)$ reduces to a Bernoulli polynomial. We have used this fact to compute numerically the corresponding values of $\zeta(s, a)$ and to draw the picture in the regions II and IV. When $s = 1$ the Hurwitz zeta-function has a simple pole with the residue 1 and for $\sigma = \mathrm{Re}(s) > 1$ it is given by the convergent series with positive elements and therefore has no zeroes.

2 Asymptotic behaviour of the Hurwitz zeta-function $\zeta(\sigma, a)$ for large negative σ.

The Hurwitz zeta-function (or generalised Riemann zeta-function) is defined as a series

$$\zeta(s, a) = \sum_{r=0}^{\infty} (a + r)^{-s} \qquad a > 0$$

in the complex domain $\mathrm{Re}(s) > 1$ and can be analytically continued to a meromorphic function in the whole complex plane with the only pole at $s = 1$ (see [1],[2],[3]). When $a = 1$ it reduces to the Riemann zeta-function

$$\zeta(s) = \sum_{k=1}^{\infty} k^{-s}.$$

The Hurwitz zeta-function can be extended to the whole of the complex s-plane through the formula

$$\zeta(s,a) = -\frac{\Gamma(1-s)}{2\pi \mathrm{i}} \int_{\infty}^{(0+)} \frac{(-z)^{s-1} e^{-az}}{1-e^{-z}} \mathrm{d}z \qquad a > 0$$

in which the integral is taken over a curve starting at 'infinity' on the real axis, encircles the origin in a positive direction and returns to the starting point (see [2]). By using an alternative integral formulation for $\zeta(s,a)$ it can be shown that $\zeta(s,a)$ is analytic everywhere except for the simple pole at $s = 1$.

The Hurwitz zeta-function obviously satisfies the functional relation:

$$\zeta(s,a) = \zeta(s, n+a) + \sum_{r=0}^{n-1} (r+a)^{-s} \qquad n = 1, 2, \ldots \tag{1}$$

Since each term in this relation is analytic we can assume this relation is true for the whole of the complex s plane, except for $s = 1$.

In this paper we restrict ourselves to the case when s is real: $s = \sigma \in \mathbf{R}$. When σ is negative Hurwitz has found the following Fourier representation for $\zeta(\sigma, a)$ on the interval $0 < a \leq 1$:

$$\zeta(\sigma, a) = \frac{2\Gamma(1-\sigma)}{(2\pi)^{1-\sigma}} \sum_{r=1}^{\infty} \frac{\sin(2\pi r a + \frac{1}{2}\pi\sigma)}{r^{1-\sigma}} \tag{2}$$

From this formula we see that

$$\frac{\zeta(\sigma, a)}{Q(\sigma)} = \sin(2\pi a + \frac{1}{2}\pi\sigma) + o(1) \qquad \text{when} \quad \sigma \to -\infty,$$

where $Q(\sigma) = \frac{2\Gamma(1-\sigma)}{(2\pi)^{1-\sigma}}$. Our first theorem proves that this is actually true on a much larger interval.

As part of the theorem proofs we will use the following inequality for the function $S(p,n) = 1^p + 2^p + \ldots + n^p$:

$$S(p,n) < n^p \left(\frac{1-e^{-p}}{1-e^{-p/n}} \right)$$

Indeed,

$$S(p,n) = n^p \left(1 + (1 - \frac{1}{n})^p + (1 - \frac{2}{n})^p + \ldots + (1 - \frac{n-1}{n})^p \right)$$

But since $1 - x < e^{-x}$ for $x < 1$ we have

$$1 - \frac{1}{n} < e^{-1/n} \qquad 1 - \frac{2}{n} < e^{-2/n} \qquad \ldots \qquad 1 - \frac{n-1}{n} < e^{-(n-1)/n}$$

and therefore

$$S(p,n) < n^p \left(1 + e^{-p/n} + e^{-2p/n} + \ldots + e^{-(n-1)p/n}\right) = n^p \left(\frac{1-e^{-p}}{1-e^{-p/n}}\right).$$

We shall also need an estimate for the sum of the series: $\sum_{r=2}^{\infty} \frac{1}{r^{1+p}}$. Now

$$\frac{1}{2^{1+p}} + \frac{1}{3^{1+p}} + \frac{1}{4^{1+p}} + \ldots = \frac{1}{2^p}\left(\frac{1}{2} + \frac{1}{3}(\frac{2}{3})^p + \frac{1}{4}(\frac{2}{4})^p + \ldots\right)$$

But $(\frac{2}{n})^p < \frac{1}{n}$ if $p > 4$ and $n \geq 3$ so we easily deduce

$$\sum_{r=2}^{\infty} \frac{1}{r^{1+p}} < (\zeta(2) - \frac{3}{4})2^{-p} \qquad p > 4$$

Finally, we will also make particular use of Stirling's inequality for the gamma function:

$$(2\pi p)^{\frac{1}{2}} p^p e^{-p} < \Gamma(1+p) < (2\pi p)^{\frac{1}{2}} p^p e^{-p} e^{\frac{1}{12p}} \qquad p \gg 1$$

Theorem 1. *Let $\sigma = -p, p \geq 0$ and $0 < a < \alpha p$ for some positive α then the Hurwitz zeta-function satisfies the inequality*

$$\left|\frac{\zeta(-p,a)}{Q(-p)} - \sin(2\pi a - \frac{1}{2}\pi p)\right| < C_1 p^{-1/2}(2\pi e\alpha)^p + C_2 2^{-p}, \tag{3}$$

where C_1, C_2 are constants, which do not depend on p. In particular, on the interval $0 < a < \frac{1}{2\pi e}p$ we have the asymptotic behaviour

$$\frac{\zeta(-p,a)}{Q(-p)} = \sin(2\pi a - \frac{1}{2}\pi p) + o(1) \qquad \text{when} \quad p \to \infty.$$

Proof

Let us represent a as $n + b$, $0 < b \leq 1$ and with n integer. It follows from the functional relation (1) that

$$\left|\frac{\zeta(\sigma,a)}{Q(\sigma)} - \frac{\zeta(\sigma,b)}{Q(\sigma)}\right| = \left|\frac{\zeta(\sigma,n+b)}{Q(\sigma)} - \frac{\zeta(\sigma,b)}{Q(\sigma)}\right| \leq \frac{1}{|Q(\sigma)|}\sum_{r=0}^{n-1}(r+b)^{-\sigma}$$

Since $0 < b \leq 1$ we obviously have

$$\sum_{r=0}^{n-1}(r+b)^{-\sigma} \leq S(p,n) \qquad 0 < p = -\sigma$$

and, as we obtained above,

$$S(p,n) < n^p \left(\frac{1-e^{-p}}{1-e^{-p/n}}\right).$$

Also, from the Stirling formula for the Γ-function we have the following asymptotically exact inequality $\Gamma(1+p) > (2\pi p)^{1/2} p^p e^{-p}$ and therefore

$$\frac{1}{Q(-p)} = \frac{(2\pi)^{1+p}}{2\Gamma(1+p)} < \{\frac{2\pi e}{p}\}^p \frac{\pi}{\sqrt{2\pi p}}$$

Thus for a large p

$$\left|\frac{\zeta(-p,a)}{Q(-p)}-\frac{\zeta(-p,b)}{Q(-p)}\right|<\left\{\frac{2\pi en}{p}\right\}^{p}\frac{\pi}{\sqrt{2\pi p}}\left(\frac{1-e^{-p}}{1-e^{-p/n}}\right)$$

$$<\left\{\frac{2\pi en}{p}\right\}^{p}\frac{\pi}{\sqrt{2\pi p}}\frac{1}{(1-e^{-\frac{1}{\alpha}})}\qquad\text{if}\quad\frac{n}{p}<\alpha \tag{4}$$

which is true since $\frac{n}{p}<\frac{a}{p}<\alpha$ by assumption.
However, from Hurwitz' formula (2) it follows that

$$\frac{\zeta(-p,b)}{Q(-p)}=\sum_{r=1}^{\infty}\frac{\sin(2\pi rb-\frac{1}{2}\pi p)}{r^{1+p}}=\sin(2\pi b-\frac{1}{2}\pi p)+\frac{\sin(4\pi b-\frac{1}{2}\pi p)}{2^{1+p}}+\frac{\sin(6\pi b-\frac{1}{2}\pi p)}{3^{1+p}}+....$$

Therefore

$$\left|\frac{\zeta(-p,b)}{Q(-p)}-\sin(2\pi b-\frac{1}{2}\pi p)\right|=\left|\frac{\zeta(-p,b)}{Q(-p)}-\sin(2\pi a-\frac{1}{2}\pi p)\right|<\sum_{r=2}^{\infty}r^{-p-1}<(\zeta(2)-\frac{3}{4})2^{-p} \tag{5}$$

if $p>4$. The estimates (4) and (5) imply the theorem.
By a slight modification of the previous arguments we can prove the following result.
Theorem 2 *In the region*

$$a>\frac{p}{2\pi e}+\frac{1}{4\pi e}\log p+1$$

$\zeta(-p,a)$ *is negative if* p *is sufficiently large.*
Proof From the same functional relation (1)

$$\frac{\zeta(-p,a)}{Q(-p)}<\frac{\zeta(-p,b)}{Q(-p)}-\frac{(a-1)^{p}}{Q(-p)}.$$

Now assuming that p is sufficiently large we can use Stirling's inequality $\Gamma(1+p)<\sqrt{2\pi p}\left(\frac{p}{e}\right)^{p}e^{\frac{1}{12p}}$, so

$$Q(-p)=\frac{2\Gamma(1+p)}{(2\pi)^{1+p}}<\frac{\sqrt{2\pi p}}{\pi}\left(\frac{p}{2\pi e}\right)^{p}e^{\frac{1}{12p}}$$

However, we know from the Hurwitz formula, that when $p\to\infty$

$$\frac{\zeta(-p,b)}{Q(-p)}=\sin(2\pi b-\frac{1}{2}\pi p)+o(1)$$

Therefore if $\frac{(a-1)^{p}}{Q(-p)}>1$ and p large enough then $\zeta(-p,a)<0$. But as we have shown

$$\frac{(a-1)^{p}}{Q(-p)}>\left(\frac{2\pi e(a-1)}{p}\right)^{p}\sqrt{\frac{\pi}{2p}}e^{-\frac{1}{12p}}$$

which is greater than 1 if

$$a-1>\frac{p}{2\pi e}\left(\frac{2p}{\pi}\right)^{\frac{1}{2p}}e^{\frac{1}{12p^{2}}}=\frac{p}{2\pi e}e^{\{\frac{(\log 2p-\log\pi)}{2p}+\frac{1}{12p^{2}}\}}$$

Now, using the inequality $e^{x}<1+x+x^{2}$ for sufficiently small x we find that to guarantee that $\frac{(a-1)^{p}}{Q(-p)}>1$ it is enough to demand that

$$a>\frac{p}{2\pi e}+\frac{1}{4\pi e}\log p+1.$$

This implies the theorem.

3 The Real Zeroes of the Hurwitz Zeta-function and Bernoulli Polynomials.

Let us now fix $\sigma = -p$ and consider $\zeta(-p, a)$ as a function of a. It follows from theorem 2 that the zeroes of this function for large p are located in the interval $0 < a < \frac{p}{2\pi e} + \frac{1}{4\pi e}\log p + 1$. For given p let $N(p)$ be the number of real zeroes of $\zeta(-p, a)$, and $A(p)$ be the largest of these zeroes.

Theorem 3 *For p sufficiently large*

$$\frac{p-1}{2\pi e} - \frac{1}{2} < A(p) < \frac{p}{2\pi e} + \frac{1}{4\pi e}\log p + 1 \tag{6}$$

$$\frac{p-1}{\pi e} - 1 < N(p) < \frac{p-1}{\pi e} + \frac{1}{2}\log p + 2\pi e + 2 \tag{7}$$

The zeroes of $\zeta(-p, a)$ on the interval $0 < a < \frac{p-1}{2\pi e}$ are simple and close to the half-integer lattice: $a = \frac{p}{4} + \frac{l}{2}, \quad l \in \mathbf{Z}$.

Proof Let us introduce the function $Z_p(a) = \frac{\zeta(-p,a)}{Q(-p)}$. From theorem 1 it follows that

$$Z_p(a) = \sin(2\pi a - \frac{1}{2}\pi p) + o(1)$$

on the interval $I_p : 0 < a < \frac{p}{2\pi e}$ when $p \to \infty$. Actually this is true also for the k^{th} derivative of $Z_p(a)$ but on a smaller interval I_{p-k}. Indeed, from the definition of the Hurwitz zeta-function we see that

$$\frac{\partial}{\partial a}\zeta(s, a) = (-s)\zeta(s + 1, a).$$

From the property of the Γ-function $\Gamma(p + 1) = p\Gamma(p)$ it follows that

$$Q(-p) = \frac{2\Gamma(1 + p)}{(2\pi)^{1+p}} = \frac{p}{2\pi}Q(-p + 1).$$

Thus the derivative of $Z_p(a)$ is equal to

$$Z_p'(a) = 2\pi Z_{p-1}(a) = 2\pi \sin(2\pi a - \frac{1}{2}\pi(p - 1)) + o(1) = 2\pi \cos(2\pi a - \frac{1}{2}\pi p) + o(1)$$

on the interval I_{p-1}. Similarly we have for the k^{th} derivative of $Z_p(a)$

$$Z_p^{(k)}(a) = \sin^{(k)}(2\pi a - \frac{1}{2}\pi p) + o(1)$$

on the interval I_{p-k}.

In particular, on the interval I_{p-1} the function $Z_p(a)$ (and its derivative) tend to $\sin(2\pi a - \frac{1}{2}\pi p)$ (and its derivative) when $p \to \infty$ which ensures that for large p all the roots of $\zeta(-p, a)$ on this interval are simple and located near the points $a = \frac{p}{4} + \frac{l}{2} \quad l \in \mathbf{Z}$. This implies the last statement of the theorem and the lower estimates of (6) and (7).

The upper estimates for $A(p)$ follows directly from theorem 2. To prove the upper estimates for $N(p)$ we need the following simple lemma.

Lemma If a function $f(x)$ (with a continuous n^{th} derivative) on some interval (a, b) has the property that the sign of the n^{th} derivative is constant throughout the interval then f has no more than n roots on this interval.

Now we apply this lemma to the function $Z_p(a)$ on the interval J_p : $(\frac{p-1}{2\pi e}, \frac{p}{2\pi e} + \frac{1}{4\pi e}\log p + 1)$ to estimate the number of roots there. The idea of this calculation is clear from Figure 2 (in which $\kappa \equiv \frac{1}{2\pi e}$).

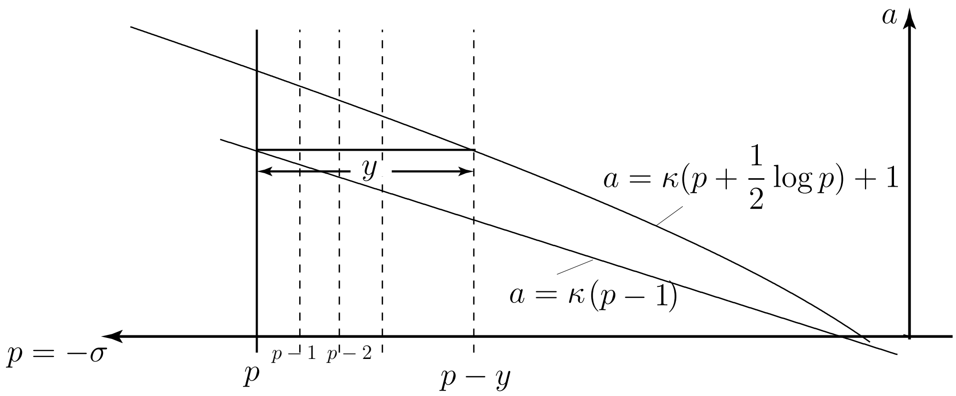

Figure 2

Using the fact that

$$Z_p^{(n)}(a) = (2\pi)^n Z_{p-n}(a)$$

we differentiate $Z_p(a)$ many times until we have a negative function and then apply the lemma. As one can see from Figure 2 if $n > [y]$ then $Z_p^{(n)}(a)$ will be negative in the interval J_p and, as such, cannot have more than n simple roots in this interval. Now y is the solution to the equation

$$\kappa\Big((p-y) + \frac{1}{2}\log(p-y)\Big) + 1 = \kappa(p-1)$$

or

$$-y + \frac{1}{2}\log(p-y) + (\kappa^{-1} + 1) = 0$$

We claim that the solution to this equation for large p satisfies the inequality

$$y < \frac{1}{2}\log p + 2\pi e + 1$$

Indeed the function $F(y) = -y + \frac{1}{2}\log(p-y) + (2\pi e + 1)$ is monotonically decreasing and

$$F(\frac{1}{2}\log p + 2\pi e + 1) = -\frac{1}{2}\log p - 2\pi e - 1 + \frac{1}{2}\log(p - \frac{1}{2}\log p - 2\pi e - 1) + (2\pi e + 1)$$
$$= \frac{1}{2}\log(1 - \frac{1}{2p}\log p - \frac{2\pi e + 1}{p}) < 0$$

for large p. Thus according to the lemma $Z_p(a)$ has no more than $\frac{1}{2}\log p + 2\pi e + 1$ roots on the interval $(\frac{p-1}{2\pi e}, \frac{p}{2\pi e} + \frac{1}{4\pi e}\log p + 1)$. Since on the interval $(0, \frac{p-1}{2\pi e}]$ we have no more than $\frac{p-1}{\pi e} + 1$ zeroes this proves theorem 3.

As we have already mentioned, when $p = -\sigma = m,\ m \in \mathbf{Z}^+$ the Hurwitz-zeta function reduces to certain polynomials related in a simple way to the Bernoulli polynomials:

$$\zeta(-m, a) = -\frac{B_{m+1}(a)}{m+1} \tag{8}$$

Theorem 3 applied to these special values of p gives some estimates on the real positive roots of the Bernoulli polynomials but because of their well-known symmetry properties:

$$B_m(1-a) = (-1)^m B_m(a) \tag{9}$$

we can immediately extend this result for all real roots of $B_m(a)$. In particular if $\mathbf{N}(m)$ is the number of all real roots of $B_m(a)$ and $\mathbf{A}(m)$ is the largest of these roots then from Theorem 3 it follows that for a large m

$$\frac{m}{2\pi e} - \left(\frac{1}{\pi e} + \frac{1}{2}\right) < \mathbf{A}(m) < \frac{m}{2\pi e} + \frac{1}{4\pi e}\log m + \left(1 - \frac{1}{2\pi e}\right) \tag{10}$$

$$\frac{2m}{\pi e} - \left(\frac{2}{\pi e} + 2\right) < \mathbf{N}(m) < \frac{2m}{\pi e} + \log m + \left(4\pi e + 3 - \frac{4}{\pi e}\right) \tag{11}$$

Corollary (*K. Inkeri*).

$$\lim_{m\to\infty} \frac{\mathbf{N}(m)}{m} = \frac{2}{\pi e}, \qquad \lim_{m\to\infty} \frac{\mathbf{A}(m)}{m} = \frac{1}{2\pi e}.$$

Remark. H. Delange in [6, 7] has found sharper estimates for $\mathbf{A}(m)$ and $\mathbf{N}(m)$. In particular he showed that the additional logarithmic terms exist in both upper and lower bounds. This should be true also for the real zeroes of the Hurwitz zeta-function but it does not follow from our elementary arguments.

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