

Renormalization analysis of correlation properties in a quasiperiodically forced two-level system

B. D. Mestel

School of Mathematical Sciences

University of Exeter

Exeter

EX4 4QE, UK

B.D.Mestel@ex.ac.uk

A. H. Osbaldestin

Department of Mathematical Sciences

Loughborough University

Loughborough

LE11 3TU, UK

A.H.Osbaldestin@lboro.ac.uk

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Abstract

We give a rigorous renormalization analysis of the self-similarity of correlation functions in a quasiperiodically forced two-level system. More precisely the system considered is a quantum two-level system in a time-dependent field consisting of periodic kicks with amplitude given by a discontinuous modulation function driven in a quasiperiodic manner at golden mean frequency. Mathematically, our analysis consists of a description of all piecewise-constant periodic orbits of an additive functional recurrence. We further establish a criterion for such orbits to be globally bounded functions. In a particular example, previously only treated numerically, we further calculate explicitly the asymptotic height of the main peaks in the correlation function.

1 Introduction

A number of authors have investigated the possibility of the existence of dynamics with singular continuous spectrum in quasiperiodically forced two-level quantum systems [2], [5], [10]. In these works, such a spectrum—suggestive of a form of weak mixing—is observed in the presence of piecewise-constant discontinuous forcing. Moreover, for the forcing frequencies considered, the autocorrelation function is observed not only neither to decay to zero nor to return to unity repeatedly, but also to possess an asymptotic self-similar structure. (We shall confine our description of the form of the dynamics to one of the nature of the autocorrelation function.) This self-similarity suggests that a renormalization analysis is appropriate to help understand this phenomenon, and this is indeed the content of the analysis of Feudel *et al* in [5]. There have, in addition, been many other studies of the response of two-level systems to quasiperiodic forcing, but we shall concentrate on the self-similarity aspects here. (See for instance the references in [5], and also [1], [3], [15] to mention but a few.)

As is implicit in some of the works cited above (and is explicitly acknowledged in [5]), there is much in common between the response of these quantum systems and the nature of strange nonchaotic attractors. Consequently our work here will have important implications in that context too.

In this paper we give a rigorous renormalization analysis explaining and generalising the numerical results in [5]. Because of its number-theoretic simplicity, most analysis has been concerned with the case of forcing at golden mean frequency, and this paper will be no exception. The self-similar structure is then explained by means of the additive functional recurrence

$$Z_n(x) = Z_{n-1}(-\omega x) + Z_{n-2}(\omega^2 x + \omega), \quad (1.1)$$

where $\omega = (\sqrt{5} - 1)/2$ is the golden mean.

The multiplicative version of (1.1) has been previously studied in some detail. It naturally appears in seemingly disparate contexts. Firstly it arises in the analysis of the self-similar fluctuations of the localized eigenstates of the Harper equation [7]. Secondly it helps explain universal characteristics present in the birth of a strange nonchaotic attractor [9]. In [8] these two phenomena are indeed linked. It further arises in the analysis of the autocorrelation function of the orbit on a strange nonchaotic attractor [4]. (In each of these contexts the recurrence arises when the frequency parameter ω is taken to be the golden mean.)

In [11] we proved that there exists a fixed point of the multiplicative version of (1.1) of the type numerically found in [7]. (To do this at times we considered the additive recurrence (1.1) in our proof.) In [12] we gave a description of all of its piecewise-constant periodic orbits, thereby providing a mathematical understanding of, and generalising, the numerical observations in [4]. In a forthcoming work [13] we give an analysis of its analytic periodic orbits, thereby explaining the beautiful “orchid” picture of Ketoja and Satija [7] arising in a generalised Harper equation. We expect this analysis can be adapted to the birth of a strange nonchaotic attractor scenario [9].

In this paper we describe all piecewise-constant periodic orbits of (1.1). Further, we characterise those periodic orbits which are globally bounded. As a consequence, it follows that the the autocorrelation functions are indeed asymptotically self-similar for a wide class of piecewise-constant forcing functions. The precise locations of the discontinuities of these functions are, however, crucial. See Figure 1 for an example of such an autocorrelation function. Much of our analysis of the multiplicative recurrence in [12] can be adapted to the additive case (1.1), and we begin by doing this in Section 2. There are some important differences to be taken into account too. The piecewise-constant periodic orbits of the multiplicative problem consist of functions taking values ± 1 only, and, as a consequence, an analysis of the discontinuities suffices to determine periodicity. In the additive problem there is no such restriction. In Subsection 2.6 we establish a criterion to guarantee periodicity in this case. Moreover, it is quite simple to find piecewise-constant periodic orbits of the additive recurrence that are spatially unbounded. We identify the nature of the locations of the discontinuities of Z_n on the whole of \mathbb{R} in Section 3, and use this information, in Section 4, to establish a criterion to distinguish those periodic orbits that are spatially bounded. In Section 5 we look in detail at the particular example studied numerically by Feudel *et al* in [5] resulting in the period-6 orbit shown in Figure 2. For this example we calculate the averages of the function Z_n which give the asymptotic height of the main peaks in the corresponding correlation function, which is shown in Figure 1. As a consequence, we show that the asymptotic height of the peaks in Figure 1 is $1 - 1/\sqrt{5} = 0.552786\dots$

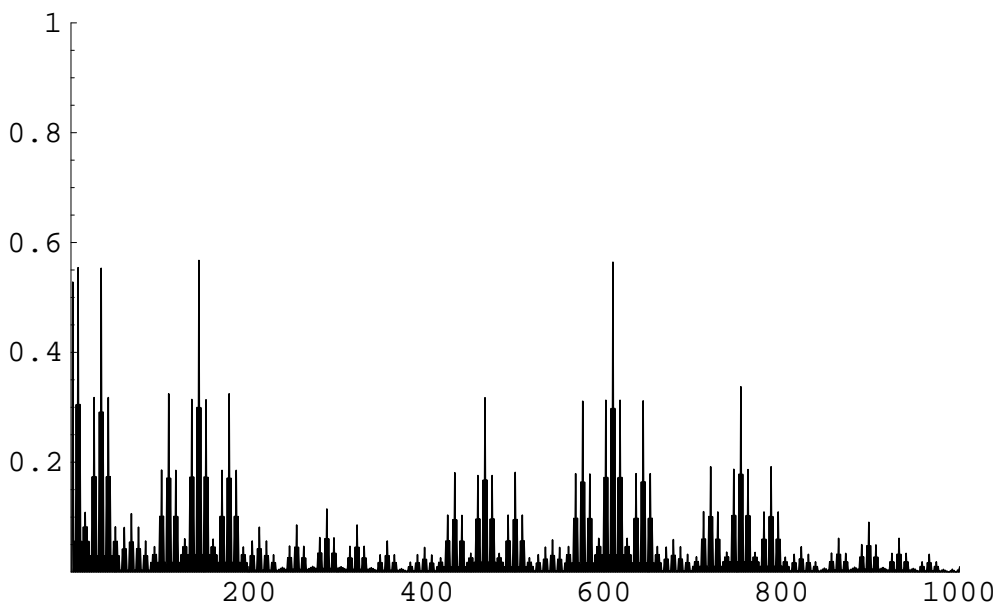


Figure 1: Autocorrelation function K_{P_y} for modulation function (1.29) and $\kappa = \pi/2$.

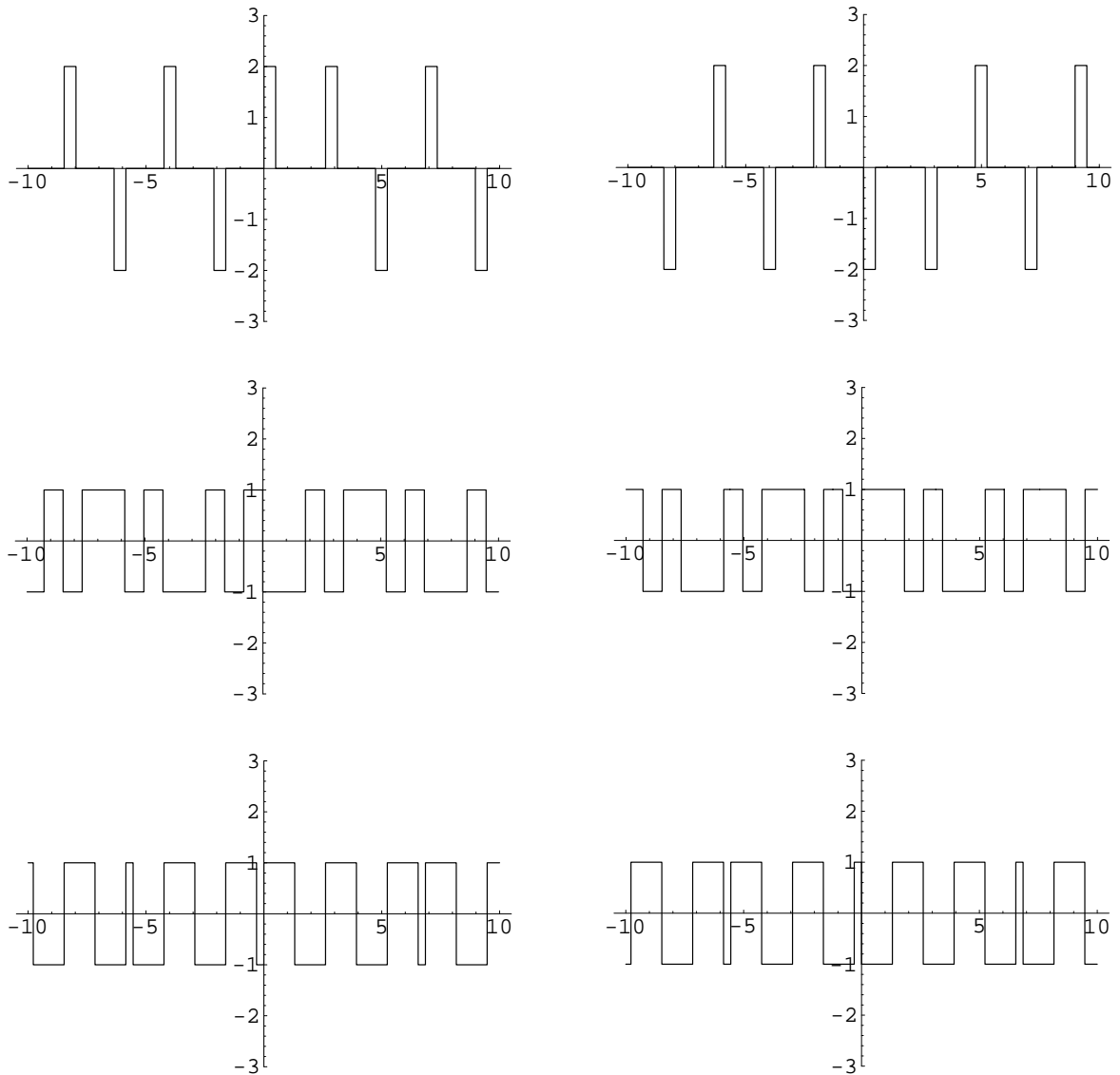


Figure 2: Period-6 orbit of the recurrence (1.1). Left column Z_0, Z_1, Z_2 reading downwards, right column Z_3, Z_4, Z_5 reading downwards.

In the remainder of this introduction we describe the system under consideration more precisely, and, following [5], indicate how the recurrence (1.1) arises in the analysis of its autocorrelation functions.

1.1 Formulation of the equations of motion

The Hamiltonian of a two-level system in a time-dependent magnetic field $B(t)$ takes the form $H = B(t) \cdot \sigma$, where $\sigma = (\sigma_x, \sigma_y, \sigma_z)$ consists of the Pauli spin matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (1.2)$$

Following the previous authors, we consider the special case $B(t) = (S(t)/2, 0, k/2)$, which gives the Hamiltonian

$$H(t) = \frac{1}{2}k\sigma_z + \frac{1}{2}S(t)\sigma_x. \quad (1.3)$$

Schrödinger's equation for the spinor $\psi = (\psi_1, \psi_2)$ is then

$$i\dot{\psi}_1 = \frac{1}{2}k\psi_1 + \frac{1}{2}S(t)\psi_2, \quad i\dot{\psi}_2 = -\frac{1}{2}k\psi_2 + \frac{1}{2}S(t)\psi_1. \quad (1.4)$$

(We take $\hbar = 1$.) This is conveniently expressed in terms the components of the polarization vector $P = \psi^* \sigma \psi$, the so-called Bloch variables,

$$P_x = \psi_2 \psi_1^* + \psi_1 \psi_2^*, \quad P_y = i(\psi_1 \psi_2^* - \psi_2 \psi_1^*), \quad P_z = \psi_1 \psi_1^* - \psi_2 \psi_2^*, \quad (1.5)$$

as the first-order linear time-dependent system

$$\dot{P}_x = -kP_y, \quad \dot{P}_y = kP_x - S(t)P_z, \quad \dot{P}_z = S(t)P_y, \quad (1.6)$$

which, from the normalization $\psi_1 \psi_1^* + \psi_2 \psi_2^* = 1$, satisfies the constraint

$$P_x^2 + P_y^2 + P_z^2 = 1. \quad (1.7)$$

We suppose that the forcing consists of period- T δ -function kicks, so that we have

$$S(t) = \sum_{n=-\infty}^{\infty} R_n \delta(t - nT), \quad (1.8)$$

with variable amplitude R_n . Between kicks we have a rotation in the (P_x, P_y) plane, and at kicks a rotation in the (P_y, P_z) plane resulting in the linear kick-to-kick mapping

$$\begin{pmatrix} P_{x,n+1} \\ P_{y,n+1} \\ P_{z,n+1} \end{pmatrix} = \begin{pmatrix} \cos kT & -\sin kT \cos R_n & \sin kT \sin R_n \\ \sin kT & \cos kT \cos R_n & -\cos kT \sin R_n \\ 0 & \sin R_n & \cos R_n \end{pmatrix} \begin{pmatrix} P_{x,n} \\ P_{y,n} \\ P_{z,n} \end{pmatrix} \quad (1.9)$$

where $(P_{x,n}, P_{y,n}, P_{z,n})$ denotes the value of the polarization vector at time step n .

Although the kicks occur periodically they do so with variable amplitude R_n which we now take to be determined in a quasiperiodic manner governed by the rotation

$$\phi_{n+1} = \phi_n + \omega \pmod{1}, \quad (1.10)$$

with the rotation number $\omega \notin \mathbb{Q}$. The amplitude R_n is then defined via a period-1 *modulation function* Φ to be

$$R_n = \kappa \Phi(\phi_n), \quad (1.11)$$

where κ is an amplitude. The precise form of the modulation function Φ is crucial for the resulting dynamics. It is the case of piecewise-constant discontinuous modulation function, where a singular continuous spectrum is observed, that will concern us in this paper.

To simplify matters we assume that the time between kicks, T , is commensurate with the fundamental frequency k , setting

$$kT = 2\pi m, \quad m \in \mathbb{Z}, \quad (1.12)$$

thereby decoupling the variable $P_{x,n}$ so that the resulting dynamics is merely a rotation in the (P_y, P_z) plane. (Numerical work in [5] indicates that this simplification may not be essential to witness the singular continuous spectrum.) Because of the constraint (1.7), without loss of generality, we may set $P_{x,n} = 0$ (otherwise we can simply rescale the remaining variables). Thus, writing

$$P_{y,n} = \cos \theta_n, \quad P_{z,n} = \sin \theta_n, \quad (1.13)$$

we arrive at the skew-product system

$$\phi_{n+1} = \phi_n + \omega \pmod{1}, \quad (1.14)$$

$$\theta_{n+1} = \theta_n + \kappa \Phi(\phi_n). \quad (1.15)$$

We remark that it is in such systems that early work on the appearance of strange nonchaotic attractors was undertaken [6], and that, in some sense, such attractors lie intermediate between regular and chaotic

dynamics. In particular, the presence of a singular continuous spectrum may be mooted as a candidate for their characterization.

Of course it is straightforward to “solve” the system (1.14), (1.15):

$$\phi_n = \phi_0 + n\omega \pmod{1}, \quad (1.16)$$

$$\theta_n = \theta_0 + \kappa \sum_{\ell=0}^{n-1} \Phi(\phi_0 + \ell\omega), \quad (1.17)$$

but this does not help illuminate the behaviour of the variable θ , i.e., (P_y, P_z) . To this end we turn to an analysis of correlations.

1.2 Renormalization analysis of the autocorrelation function

For completeness we recall here the renormalization analysis of the autocorrelation function from Feudel *et al* [5]. The autocorrelation function of the (zero mean) observable P_y is

$$K_{P_y}(t) = \frac{\langle P_{y,n} P_{y,n+t} \rangle}{\langle P_{y,n}^2 \rangle}, \quad (1.18)$$

where we have defined the average

$$\langle f(n) \rangle = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N f(n). \quad (1.19)$$

Using the trigonometric identity $2 \cos A \cos B = \cos(A + B) + \cos(A - B)$, we have

$$\langle P_{y,n} P_{y,n+t} \rangle = \langle \cos \theta_n \cos \theta_{n+t} \rangle = \frac{1}{2} \langle \cos(\theta_{n+t} - \theta_n) \rangle = \frac{1}{2} \langle \cos \kappa Q_t(\phi_n) \rangle, \quad (1.20)$$

where we have averaged over the initial phase θ_0 to remove the average of $\cos(\theta_{n+t} + \theta_n)$, and then defined

$$Q_t(\phi) = \sum_{\ell=0}^{t-1} \Phi(\phi + \ell\omega), \quad Q_0(\phi) = 0. \quad (1.21)$$

Thus

$$K_{P_y}(t) = \langle \cos \kappa Q_t(\phi_n) \rangle = \int_0^1 \cos \kappa Q_t(\phi) d\phi, \quad (1.22)$$

where we have used the fact that for irrational ω the rotation (1.14) is ergodic with respect to Lebesgue measure to replace the time average by a space average.

We now specialise to the case of golden mean rotation number, setting $\omega = (\sqrt{5} - 1)/2$, and accordingly consider Fibonacci times only. Then Q_n satisfies the recurrence relation

$$Q_{F_n}(\phi) = Q_{F_{n-1}}(\phi) + Q_{F_{n-2}}(\phi + F_{n-1}\omega), \quad (1.23)$$

where the Fibonacci numbers (F_n) are defined by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Using the identity

$$F_{n-1}\omega = F_{n-2} - (-\omega)^{n-1}, \quad (1.24)$$

and defining the rescaled variables

$$Z_n(y) = Q_{F_n}(y(-\omega)^n), \quad (1.25)$$

results in the iterative functional recurrence

$$Z_n(x) = Z_{n-1}(-\omega x) + Z_{n-2}(\omega^2 x + \omega), \quad (1.26)$$

with initial conditions

$$Z_0(x) = 0, \quad Z_1(x) = \Phi(-\omega x). \quad (1.27)$$

This recurrence is the object of study in this paper.

In terms of the functions Z_n , the autocorrelation function K_{P_y} at Fibonacci times is

$$K_{P_y}(F_n) = \frac{1}{(-\omega)^{-n}} \int_0^{(-\omega)^{-n}} \cos \kappa Z_n(y) dy. \quad (1.28)$$

For the particular choice of modulation function

$$\Phi(\phi) = \begin{cases} +1; & 0 \leq \{\phi\} < 1/2, \\ -1; & 1/2 \leq \{\phi\} < 1, \end{cases} \quad (1.29)$$

where $\{\phi\}$ denotes $\phi \pmod{1}$, with the initial conditions given by (1.27), Feudel *et al* [5] observe that iteration of (1.1) leads to a period-6 orbit as shown in Figure 2. The corresponding autocorrelation function K_{P_y} (with $\kappa = \pi/2$) is shown in Figure 1. Note that the height of the largest peaks is approximately 0.55 and is numerically calculated in [5] from an average of $|Z_n|$ with $n \equiv 0 \pmod{3}$, i.e., either of the top two figures in Figure 2, to be asymptotically approximately 0.55279. By carefully describing the locations of the discontinuities in Figure 2, in Section 5 we shall show that this value is in fact $1 - 1/\sqrt{5} = 0.552786\dots$

Our analysis shows that self-similarity of the autocorrelation function occurs for many other choices of modulation function in addition to (1.29). However the locations of the discontinuities of the modulation function must be preperiodic points of a map of the interval to be introduced in the next section.

2 Periodic orbits and their discontinuities

We begin this section by adapting our previous analysis [12] of the multiplicative version of (1.1) to the problem at hand. We introduce an expanding piecewise-linear map of an interval whose periodic orbits correspond to the discontinuities of the piecewise-constant periodic orbits of (1.1), and show that the dynamics of this interval map ‘drives’ the global behaviour of period orbits of (1.1). We also give a detailed analysis of the periodicity of the discontinuities. We then give a necessary and sufficient criterion for the piecewise-constant orbits of (1.1) to be periodic.

2.1 Iterated function system and the inverse map F

Defining

$$\phi_1(x) = -\omega x, \quad \phi_2(x) = \omega^2 x + \omega, \quad (2.1)$$

we may write equation (1.1) in the form

$$Z_n(x) = Z_{n-1}(\phi_1(x)) + Z_{n-2}(\phi_2(x)), \quad (2.2)$$

where $\omega = (\sqrt{5} - 1)/2$ is the golden mean,

The iterated function system (IFS) on \mathbb{R} given by ϕ_1, ϕ_2 has the following properties:

1. ϕ_1 and ϕ_2 are linear contractions with fixed points 0 and 1 respectively, and with $\phi_1'(x) = -\omega$ and $\phi_2'(x) = \omega^2$.
2. The *fundamental interval* $I = [-\omega, 1]$ is the fixed point set for the IFS. Indeed $\phi_1(I) = [-\omega, \omega^2]$, $\phi_2(I) = [\omega^2, 1]$, so that $\phi_1(I) \cup \phi_2(I) = I$.
3. The fundamental interval I is the attractor for the IFS. Indeed given any compact subset $K \subset \mathbb{R}$ and any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for any $k \geq N$ and any choice $i_1, \dots, i_k \in \{1, 2\}$ we have

$$\phi_{i_1} \circ \dots \circ \phi_{i_k}(x) \in [-\omega - \varepsilon, 1 + \varepsilon] \quad (2.3)$$

for any $x \in K$. This property will be important when we consider the behaviour of equation (2.2) outside the fundamental interval.

Let $F : I \rightarrow I$ be defined by

$$F(x) = \begin{cases} \phi_1^{-1}(x) = -\omega^{-1}x, & x \in [-\omega, \omega^2]; \\ \phi_2^{-1}(x) = \omega^{-2}x - \omega^{-1}, & x \in [\omega^2, 1], \end{cases} \quad (2.4)$$

as drawn in Figure 3. We shall see below that periodic points of F correspond to discontinuities of the periodic solutions of (2.2).

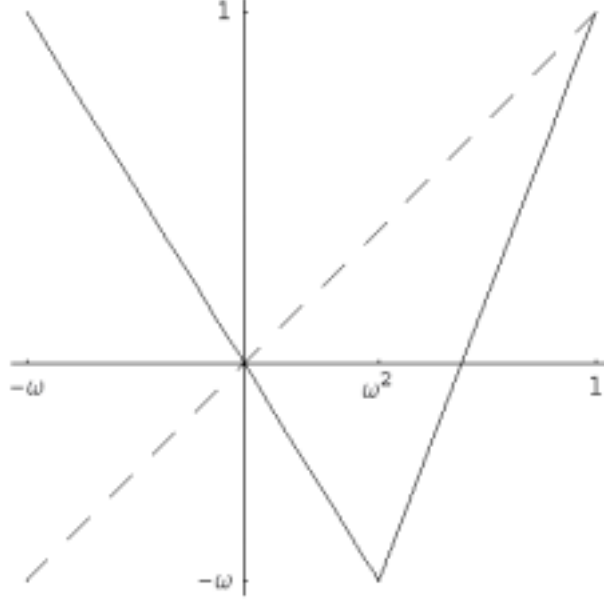


Figure 3: The function F .

We note that for any periodic point $y \in I$, precisely one of $\phi_1(y)$, $\phi_2(y)$ is also a periodic point of F . This follows from the fact that each period point has two preimages, exactly one of which is a periodic point on the same orbit. (Note that ω^2 is not a periodic point of F .)

We analyse the dynamics of F in terms of a code of a point $x \in I$. As in [12] we let the interval $[-\omega, \omega^2)$ be encoded with the symbol 1 and $(\omega^2, 1]$ with the symbol 2. We define the code of $x \in I$ to be the sequence $(a_n)_{n \geq 0}$ in $\{1, 2\}^{\mathbb{N}_0}$ given by

$$a_n = \begin{cases} 1, & F^n(x) \in [-\omega, \omega^2); \\ 2, & F^n(x) \in (\omega^2, 1], \end{cases} \quad (2.5)$$

and ignore the (countable) set of (nonperiodic) points whose orbits under F include the point ω^2 . Hence the codes are all infinite sequences. In terms of the code $a_0 a_1 a_2 \dots$ of a point $x \in I$, we have

$$F(x) = (-\omega^{-1})^{a_0} x - (a_0 - 1)\omega^{-1}. \quad (2.6)$$

Since F is uniformly expanding ($|F'(x)| \geq \omega^{-1}$) every point $x \in I$ corresponds to a unique code and vice versa. In particular, periodic orbits of F correspond to periodic codes in $\{1, 2\}^{\mathbb{N}_0}$ under the shift map σ :

$$\sigma(a_0 a_1 a_2 \dots) = a_1 a_2 \dots \quad (2.7)$$

A periodic orbit y_0, y_1, \dots, y_{k-1} of period k of F is given uniquely by a periodic code $\overline{a_0 a_1 \dots a_{k-1}}$, which we henceforth denote as just $a_0 a_1 \dots a_{k-1}$. Indeed, given a code $a_0 a_1 \dots a_{k-1}$, it is straightforward to calculate the corresponding periodic orbit y_0, y_1, \dots, y_{k-1} of F . We have $\phi_{a_{k-1}}^{-1} \circ \dots \circ \phi_{a_0}^{-1}(y_0) = y_0$, or, equivalently, $\phi_{a_0} \circ \dots \circ \phi_{a_{k-1}}(y_0) = y_0$, whose (unique) solution is readily calculated to be

$$y_0 = \frac{-\sum_{j=0}^{k-1} (a_j - 1)(-\omega)^{1 + \sum_{i=0}^{j-1} a_i}}{1 - (-\omega)^{\sum a_j}}, \quad (2.8)$$

where empty sums are defined to be zero. The other points of the orbit may be calculated by applying this formula with the code $a_0 a_1 \dots a_{k-1}$ cyclically permuted.

For example, $F(y)$ has two fixed points: $y = 0$ with code 1, and $y = 1$ with code 2. The period-two orbit with code 21 is given by $y_0 = 1/2$ and $y_1 = -\omega/2$. It is the fixed point $y = 0$ and this period-two orbit that are the discontinuity points in the fundamental interval of the period-6 orbit shown in Figure 2.

2.2 Reduction of Z_n on \mathbb{R} to the fundamental interval

In this section we consider equation (2.2) outside the fundamental interval $I = [-\omega, 1]$, i.e., on the whole of \mathbb{R} . The fact that I is the attractor for the IFS leads to the conclusion that the global behaviour of (2.2) is ‘driven’ by its behaviour in I .

The following result is a straightforward variation of the corresponding result in [12].

Lemma 1. *Let Z_0, Z_1 be initial conditions for (2.2) on \mathbb{R} and let $\varepsilon > 0$ be such that $Z_0(x) = Z_1(x) = 0$ for all $x \in [-\omega - \varepsilon, 1 + \varepsilon]$. Then for each $L > 1$, there exists $N > 0$ (depending only on L) such that $Z_n(x) = 0$ for all $x \in [-L, L]$ and all $n > N$.*

In other words, if the initial conditions on, and just outside, the fundamental interval are zero, then the value of Z_n at all points eventually becomes zero.

From the lemma we may prove the following proposition whose proof again follows mutatis mutandis the corresponding result in [12].

Proposition 1. *Let Z_n be a piecewise-constant periodic orbit of (2.2) of period p on \mathbb{R} with $Z_n(1+) = Z_n(1)$. Then Z_n is periodic with period p on the fundamental interval I . Conversely, suppose that Z_n is periodic with period p on I . Then there is a unique extension \tilde{Z}_n of Z_n to \mathbb{R} such that \tilde{Z}_n is periodic with period p on \mathbb{R} .*

Moreover Lemma 1 further implies the following.

Proposition 2. *Let Z_0, Z_1 be piecewise-constant initial conditions for (2.2) on \mathbb{R} with $Z_0(1+) = Z_0(1)$, $Z_1(1+) = Z_1(1)$. Suppose Z_n is periodic of period p on the fundamental interval I . Then the sequence Z_n converges to the unique periodic extension \tilde{Z}_n given by proposition 1, i.e., for all integers $r \geq 0$ we have $Z_{r+np}(x) \rightarrow \tilde{Z}_r(x)$ as $n \rightarrow \infty$.*

In other words, initial data on the fundamental interval resulting in periodic behaviour uniquely determines an asymptotic (right continuous at 1) global periodic behaviour. It is important to realise however that such globally defined periodic orbits will not in general consist of bounded functions. We shall address this issue in Section 4.

2.3 Analysis of the discontinuities

In order to study the piecewise-constant periodic orbits of the recurrence (1.1) we firstly consider the dynamics of the discontinuities. We define, for each $x \in \mathbb{R}$ and $n \geq 0$,

$$D_n(x) = Z_n(x+) - Z_n(x-), \quad (2.9)$$

the difference of the right-hand limit at x to the left-hand limit at x , so that $D_n(x) \neq 0$ if and only if Z_n has a discontinuity at x . The recurrence (2.2) for Z_n naturally induces a recurrence for D_n :

$$\begin{aligned} D_n(x) &= Z_n(x+) - Z_n(x-) \\ &= Z_{n-1}(\phi_1(x+)) + Z_{n-2}(\phi_2(x+)) - Z_{n-1}(\phi_1(x-)) - Z_{n-2}(\phi_2(x-)) \\ &= Z_{n-1}(\phi_1(x)-) - Z_{n-1}(\phi_1(x)+) + Z_{n-2}(\phi_2(x)+) - Z_{n-2}(\phi_2(x)-) \\ &= -D_{n-1}(\phi_1(x)) + D_{n-2}(\phi_2(x)). \end{aligned} \quad (2.10)$$

Clearly if Z_n is periodic with period p , then D_n is also periodic with period m dividing p . Our task is to determine the possible periods m of D_n and relate m to p , the period of Z_n .

From now on we assume that Z_n is periodic with period p and that D_n is periodic with period m , and, in view of proposition 1, we only consider the behaviour of Z_n and D_n on the fundamental interval $I = [-\omega, 1]$. We define the *restricted discontinuity set*

$$D = \{x \in I : D_n(x) \neq 0 \text{ for some } n \geq 0\}. \quad (2.11)$$

Then D is the finite set of points in I for which Z_n has a discontinuity for at least one $n \geq 0$.

As in our analysis of the multiplicative recurrence [12], the restricted discontinuity set D consists of finitely many periodic orbits of the map F as we now show.

Proposition 3. *Let Z_n be a piecewise-constant periodic orbit of (2.2), and let D be the associated restricted discontinuity set. Then D consists of a finite collection of periodic orbits of the map F .*

Proof. Suppose $y \in D$. Then $D_n(y) \neq 0$ for some $n \geq 0$. From (2.10) we have that $D_{n-i_1}(\phi_{i_1}(y)) \neq 0$ for some $i_1 \in \{1, 2\}$. We therefore have $\phi_{i_1}(y) \in D$. Continuing in this way, we obtain a sequence $i_1, i_2, \dots \in \{1, 2\}$ such that $\phi_{i_k} \circ \dots \circ \phi_{i_1}(y) \in D$ for each $k \in \mathbb{N}$. Since D is finite there exist $r, r' \in \mathbb{N}$ with $r > r'$ and $\phi_{i_r} \circ \dots \circ \phi_{i_1}(y) = \phi_{i_{r'}} \circ \dots \circ \phi_{i_1}(y)$. Applying F^r to this equation gives $F^{r-r'}(y) = y$, so that y is a periodic point of F of period j dividing $r - r'$.

Now let $y_0 = y, y_1, \dots, y_{k-1}$ be the points on the orbit of y_0 under F with $y_{i+1 \pmod k} = F(y_i)$ for $i = 0, 1, \dots, k-1$, and let $a_0 a_1 \dots a_{k-1}$ be the code of the orbit. Then for $0 \leq i \leq k-1$ we have

$$\phi_{a_i}^{-1}(y_i) = y_{i+1}, \quad \text{or, equivalently,} \quad \phi_{a_{i-1}}(y_i) = y_{i-1}, \quad (2.12)$$

where here, and in what follows, we assume that expressions relating to the periodic orbit y_0, y_1, \dots, y_{k-1} are reduced modulo k .

Moreover, as we noted earlier, precisely one of $\phi_1(y_i), \phi_2(y_i)$ is periodic. Thus, if $a_{i-1} = 1$ then $\phi_2(y_i) \notin D$ and so $D_n(\phi_2(y_i)) = 0$ for all n , whilst if $a_{i-1} = 2$ then $\phi_1(y_i) \notin D$ and so $D_{n-1}(\phi_1(y_i)) = 0$ for all n . The recurrence (2.10) may be therefore be written

$$D_n(y_i) = \begin{cases} -D_{n-1}(y_{i-1}), & a_{i-1} = 1; \\ D_{n-2}(y_{i-1}), & a_{i-1} = 2, \end{cases} \quad (2.13)$$

which can be written as

$$D_n(y_i) = (-1)^{a_i-1} D_{n-a_{i-1}}(y_{i-1}). \quad (2.14)$$

Thus $D_{n+a_0+\dots+a_{i-1}}(y_i) = (-1)^{a_0+\dots+a_{i-1}} D_n(y_0)$, so that if $D_n(y_0) \neq 0$ then $D_{n+a_0+\dots+a_{i-1}}(y_i) \neq 0$, i.e., if $y_0 \in D$ then $y_i \in D$. We conclude that not only must every point $y \in D$ be periodic point of F , but that every point on the periodic orbit of y also lies in D , so that D consists of complete orbits of F . \square

2.4 Period of the discontinuities for a single periodic orbit of F

From (2.14) we see that over a complete periodic orbit with code $a_0 a_1 \dots a_{k-1}$ the index n decreases

$$\ell = \sum_{i=0}^{k-1} a_i, \quad (2.15)$$

and moreover we have $D_n(y_i) = (-1)^\ell D_{n-\ell}(y_i)$, for $0 \leq i \leq k-1$. We therefore deduce the following result.

Proposition 4. *Let m be the period of the discontinuity function D_n restricted to a periodic orbit y_0, \dots, y_{k-1} of F and let ℓ be the sum of the code over the orbit of F . Then m divides $\text{lcm}\{2, \ell\}$, i.e., if ℓ is even m divides ℓ , whilst if ℓ is odd m divides 2ℓ .*

We define for any integer m

$$\hat{m} = \text{lcm}\{2, m\} = \begin{cases} m, & m \text{ even}; \\ 2m, & m \text{ odd}. \end{cases} \quad (2.16)$$

A convenient representation of the discontinuity data is in terms of an $\hat{\ell} \times k$ matrix M , the *discontinuity matrix*, with entries defined by

$$M_{n,i} = D_n(y_i), \quad (2.17)$$

for $0 \leq n \leq \hat{\ell} - 1$, $0 \leq i \leq k - 1$, so that the entry in row n and column i is the value of D_n at the point y_i on the orbit y_0, y_1, \dots, y_{k-1} .

The relation (2.14) above gives a special structure to the matrix M . Indeed (2.14) translates to

$$M_{n,i} = (-1)^{a_{i-1}} M_{n-a_{i-1}, i-1}, \quad (2.18)$$

where here, and in what follows, indices referring to the periodicity of D_n are reduced modulo $\hat{\ell}$.

The structure (2.18) can be more easily understood as follows. Column i of the matrix M is simply column $(i - 1)$ cyclically permuted downwards by a_{i-1} single cyclic permutations with a change of sign if $a_{i-1} = 1$. This observation also holds when $i = 0$, for then (2.18) becomes

$$M_{n,0} = (-1)^{a_{k-1}} M_{n-a_{k-1}, k-1}. \quad (2.19)$$

Let us denote the first column of M by $(X_0, X_1, \dots, X_{\hat{\ell}-1})$, i.e., $M_{n,0} = X_n$ for $0 \leq n \leq \hat{\ell} - 1$. Then the relation (2.18) tells us that

$$M_{n,1} = (-1)^{a_0} M_{n-a_0,0} = (-1)^{a_0} X_{n-a_0}, \quad (2.20)$$

and, in general,

$$M_{n,i} = (-1)^{\sum_{j=0}^{i-1} a_j} M_{n-\sum_{j=0}^{i-1} a_j, 0} = (-1)^{\sum_{j=0}^{i-1} a_j} X_{n-\sum_{j=0}^{i-1} a_j}, \quad (2.21)$$

so that each column of M is simply a cyclic permutation with an appropriate sign change of the first column of M . In the case of odd ℓ , in order to satisfy (2.19), the first column of M must take the form $(X_0, X_1, \dots, X_{\ell-1}, -X_0, -X_1, \dots, -X_{\ell-1})$.

Example 1. The period-two orbit $\{1/2, -\omega/2\}$ has code 21, so $k = 2$, $\ell = 3$, $\hat{\ell} = 6$ and

$$M = \begin{pmatrix} X_0 & -X_1 \\ X_1 & -X_2 \\ X_2 & X_0 \\ -X_0 & X_1 \\ -X_1 & X_2 \\ -X_2 & -X_0 \end{pmatrix},$$

in which, since $a_0 = 2$, the second column is the first column shifted down two rows without sign change. Utilising $a_1 = 1$ on the second column reproduces the first column by shifting down one and changing sign.

Example 2. The fixed point 0 of F has code 1 so here $k = \ell = 1$, $\hat{\ell} = 2$ and

$$M = \begin{pmatrix} X_0 \\ -X_0 \end{pmatrix}.$$

The combination of Examples 1 and 2 was our initial motivation for studying this problem, and enables us to give a precise description of the numerical results of Feudel *et al* [5]. We shall return again to this example later on this paper, after discussing how to treat discontinuity sets containing more than periodic orbit of F in the next subsection.

Example 3. The fixed point 1 of F has code 2 so here $k = 1$, $\ell = \hat{\ell} = 2$ and

$$M = \begin{pmatrix} X_0 \\ X_1 \end{pmatrix}.$$

Example 4. When the code is 2112 we have $k = 4$, $\ell = \ell' = 6$ and

$$M = \begin{pmatrix} X_0 & X_4 & -X_3 & X_2 \\ X_1 & X_5 & -X_4 & X_3 \\ X_2 & X_0 & -X_5 & X_4 \\ X_3 & X_1 & -X_0 & X_5 \\ X_4 & X_2 & -X_1 & X_0 \\ X_5 & X_3 & -X_2 & X_1 \end{pmatrix}.$$

When ℓ is even, by a suitable choice of first column, i.e., of the ℓ numbers $X_0, X_1, \dots, X_{\ell-1}$, we may arrange for the discontinuities to have any period dividing ℓ . When ℓ is odd, by suitably choosing $X_0, X_1, \dots, X_{\ell-1}$, any period twice any divisor of ℓ can be obtained (or the trivial period-1 case of no discontinuities). Moreover, in this latter case, the discontinuities satisfy $X_{n+\ell} = -X_n$.

We emphasise that our freedom of choice is in choosing the values of the discontinuities at each point of a periodic orbit of F . It is *not* the case that one may arbitrarily select the discontinuities of the initial conditions Z_0, Z_1 . This would correspond to specifying the first two *rows* of the discontinuity matrix. As our analysis shows, however, the rows are not independent. However, we remark that each row of the discontinuity matrix contains each X_i at most once, and that any two adjacent rows together contain each X_i at least once. Indeed, given one row, it is clear that those entries which appear in the following row are simply those in the columns corresponding to y_i with code 1. Thus an alternative (equivalent) method of specifying a periodic discontinuity orbit may be given directly in terms of the discontinuities of Z_0 and Z_1 at each y_i . Namely, the discontinuities of Z_1 , i.e., $D_1(y_i)$ may be chosen arbitrarily, as may the the discontinuities $D_0(y_i)$ of Z_0 at each y_i with code 2. (This exhausts ℓ freedoms.) The discontinuities of Z_0 at each y_i with code 1, however are given as $D_0(y_i) = -D_1(y_{i+1})$ (with periodic k on the index). Whilst useful for ensuring that initial conditions result in periodic behaviour, this alternative description is less useful for determining the precise period. This description will however be useful in Section 4 when we wish to analyse the global behaviour of recurrence (2.2).

2.5 Multiple periodic orbits in D

In general, the restricted discontinuity set D will be composed of points of more than one periodic orbit of F . Let t be the number of periodic orbits of F in D . For $0 \leq s \leq t - 1$, we consider the periodic orbit s of F in D . We make the general convention that superscript s refers to the orbit s . Now, from the additive structure of (2.10), we have that a sum of solutions is again a solution of the equation. Moreover, because the periodic orbits in D are distinct, and are never mapped to each other under the two maps ϕ_1, ϕ_2 , we have that the dynamics of D_n on each of the periodic orbits in D are independent. Indeed, we may write

$$D_n(x) = \sum_{s=0}^{t-1} D_n^s(x), \quad (2.22)$$

where D_n^s is the restriction of D_n to the periodic orbit s , i.e.,

$$D_n^s(x) = \begin{cases} D_n(x), & x \in \{y_0^s, \dots, y_{k^s-1}^s\}; \\ 0, & \text{otherwise.} \end{cases} \quad (2.23)$$

We may then apply the analysis of the previous subsection to each of the functions D_n^s . This is because $D_n^s(x) = 0$, except when x is one of the points on the periodic orbit $y_0^s \dots y_{k^s-1}^s$ of F .

The theory for D_n that we discussed above carries over in a straightforward manner to the function D_n^s . In particular, for each orbit in D we can formulate an $\hat{\ell}^s \times k^s$ discontinuity matrix M^s , defined by $M_{n,i}^s = D_n^s(y_i^s)$ for $0 \leq n \leq \hat{\ell}^s - 1$ and $0 \leq i \leq k^s - 1$. To simplify notation, we adopt the convention that matrix indices are reduced modulo k^s when dealing with expressions relating to the periodic orbit $y_0^s, \dots, y_{k^s-1}^s$ of F , whilst those relating to the periodicities of D_n are reduced modulo $\hat{\ell}^s$. These matrices are independent of each other since the dynamics of D_n on each periodic orbit in D are independent.

Thus, as in (2.18), we have

$$M_{n,i}^s = (-1)^{a_{i-1}^s} M_{n-a_{i-1}^s, i-1}^s, \quad (2.24)$$

for $0 \leq n \leq \hat{\ell}^s - 1$ and $0 \leq i \leq k^s - 1$, and the matrix M^s is determined by its first column $(X_0^s, X_1^s, \dots, X_{\hat{\ell}^s-1}^s)$. (When ℓ^s is odd this column takes the form $(X_0^s, X_1^s, \dots, X_{\ell^s-1}^s, -X_0^s, -X_1^s, \dots, -X_{\ell^s-1}^s)$.)

Indeed, as in (2.21),

$$M_{n,i}^s = (-1)^{\sum_{j=0}^{i-1} a_j^s} X_{n-\sum_{j=0}^{i-1} a_j^s}^s, \quad (2.25)$$

and the period m^s of the first column is precisely the row period of M^s . We also have $m^s \mid \hat{\ell}^s$.

Conversely, let $\hat{\ell} = \text{lcm}(\hat{\ell}^0, \dots, \hat{\ell}^{t-1})$. Then for any $m \mid \hat{\ell}$ we define $m^s = \text{gcd}(m, \hat{\ell}^s)$. Then $m^s \mid \hat{\ell}^s$ and by appropriate choices of $(X_0^s, X_1^s, \dots, X_{\hat{\ell}^s-1}^s)$ we may construct a matrix M^s with row period m^s if ℓ^s is even, or \hat{m}^s if ℓ^s is odd, and, extending periodically to all $n \geq 0$, we have that D_n has period m^s if ℓ^s is even, or \hat{m}^s if ℓ^s is odd, when restricted to the orbit $y_0^s, \dots, y_{k^s-1}^s$.

We therefore have the following proposition.

Proposition 5. *Let Z_n be a piecewise-constant periodic orbit of (2.2). Then the period m of the discontinuity function D_n is given by*

$$m = \text{lcm}(m^0, \dots, m^{t-1}), \quad (2.26)$$

where m^s is the period of the function D_n^s , given by the period of $(X_0^s, X_1^s, \dots, X_{\hat{\ell}^s-1}^s)$, i.e., the first column of the discontinuity matrix M^s . Furthermore, m divides

$$\hat{\ell} = \text{lcm}(\hat{\ell}^0, \dots, \hat{\ell}^{t-1}). \quad (2.27)$$

Moreover, by appropriate choices of $(X_0^s, X_1^s, \dots, X_{\hat{\ell}^s-1}^s)$, for any m dividing $\hat{\ell}$ we may construct a periodic orbit of D_n with period \hat{m} (and, if all ℓ^s are even, we may construct a periodic orbit of odd period m).

2.6 The criterion for orbits to be periodic

In the previous subsection we dealt quite extensively with the periodicity of the discontinuities on the fundamental interval $I = [-\omega, 1]$. It is perfectly feasible for the sequence of discontinuities D_n to be periodic whilst the sequence (Z_n) itself is not. The simple example $Z_0(x) = 0$, $Z_1(x) = 1$ for all x , with no discontinuities at all, generates the Fibonacci numbers as the values for Z_n . This sequence is clearly not periodic, growing without bound.

In order to ensure that (Z_n) is itself periodic it suffices to ensure that the values at a single point are periodic. It will be convenient to choose an end point of I for this purpose. We consider then the values $Z_n(1)$. As we saw in Subsection 2.2, it is not just the fundamental interval, but also the right hand limit at the end point 1 of the interval that dictates the global behaviour. With this in mind we take Z_n to be right continuous at 1 and write $Z_n^{1+} = Z_n(1) = Z_n(1+)$.

Iteration (2.2) gives

$$Z_n(1+) = Z_{n-1}(\phi_1(1+)) + Z_{n-2}(\phi_2(1+)). \quad (2.28)$$

Now ϕ_2 is increasing and $\phi_2(1) = 1$, so $\phi_2(1+) = 1+$, and thus $Z_{n-2}(\phi_2(1+)) = Z_{n-2}(1+)$. On the other hand ϕ_1 is decreasing and $\phi_1(1) = -\omega$, so now $\phi_1(1+) = -\omega-$. To relate $Z_{n-1}(\phi_1(1+))$ to $Z_{n-1}(1+)$ we therefore need to pass from $-\omega-$ to $1+$, which means that we need to add in the effect of all of the discontinuities of Z_{n-1} in I . Let us therefore write

$$Z_{n-1}(1+) = Z_{n-1}(-\omega-) + \Sigma_{n-1}, \quad (2.29)$$

where we denote

$$\Sigma_{n-1} = \sum_{y \in [-\omega, 1]} D_{n-1}(y). \quad (2.30)$$

We thus have the following recurrence relation for Z_n^{1+} :

$$Z_n^{1+} = Z_{n-1}^{1+} + Z_{n-2}^{1+} - \Sigma_{n-1}. \quad (2.31)$$

This is a second-order inhomogeneous recurrence relation of a Fibonacci type. Its solution is

$$Z_n^{1+} = F_n Z_1^{1+} + F_{n-1} Z_0^{1+} - \sum_{i=2}^n F_{n+1-i} \Sigma_{i-1}. \quad (2.32)$$

If we now require Z_n to be a periodic orbit of period p of (2.2) then in particular we require Z_n^{1+} to have period p . The inhomogeneity is merely the sum of the discontinuities of Z_n and therefore this too has period p . Thus imposing periodicity of period p we can arrive at two simultaneous linear equations for Z_0^{1+} and Z_1^{1+} , namely

$$Z_0^{1+} = Z_p^{1+} = F_p Z_1^{1+} + F_{p-1} Z_0^{1+} - \sum_{i=2}^p F_{p+1-i} \Sigma_{i-1} \quad (2.33)$$

$$Z_1^{1+} = Z_{p+1}^{1+} = F_{p+1} Z_1^{1+} + F_p Z_0^{1+} - \sum_{i=2}^{p+1} F_{p+2-i} \Sigma_{i-1}. \quad (2.34)$$

When written in matrix form, the determinant the matrix of coefficients of this system is $1 + (-1)^p - F_{p+1} - F_{p-1} \neq 0$, thus these equations always possess a unique solution. The conclusion is that we may arbitrarily select the discontinuities over a set of periodic orbits of F on the fundamental interval (i.e., select the first columns of the discontinuity matrices), and then define Z_0^{1+} , Z_1^{1+} , i.e., $Z_0(1)$, $Z_1(1)$, by solving this system of linear equations, and the resulting orbit is periodic on the fundamental interval. By the results of Subsection 2.2 this local data determines the asymptotic (right continuous at 1) globally periodic orbit.

In summary we have the following

Theorem 1. *A necessary and sufficient condition for a piecewise-constant, right-continuous at 1, orbit of (2.2) to be periodic (on the whole of \mathbb{R}) with period p is that its discontinuities have period p and that Z_0^{1+} , Z_1^{1+} satisfy equations (2.33) and (2.34).*

For example, consider Example 3 above with arbitrary discontinuities X_0 , X_1 at 1 of Z_0 , Z_1 respectively. Solving (2.33-2.34) with $p = \ell = 2$ gives $Z_0^{1+} = X_0$, $Z_1^{1+} = X_1$, i.e., that both Z_0 and Z_1 must be zero to the left of 1 on I . Figure 4 shows the case $X_0 = 1$, $X_1 = -2$.

For the case of discontinuity set given by the union of Examples 1 and 2, $D = \{1/2, -\omega/2\} \cup \{0\}$, with discontinuities X_0^0 , $-X_1^0$, X_0^1 , and X_1^1 , $-X_2^0$, $-X_0^1$ of Z_0 and Z_1 at $1/2$, $-\omega/2$, 0 respectively, for the period-6 orbit of the discontinuities to be a period-6 orbit of (2.2) we solve (2.33-2.34) with $p = 6$, and thereby specify

$$Z_0^{1+} = (2X_0^0 - X_1^0 + X_2^0 + 2X_0^1)/2, \quad Z_1^{1+} = (-X_0^0 + 2X_1^0 - X_2^0 - 2X_0^1)/2. \quad (2.35)$$

3 Discontinuities on \mathbb{R}

Thus far we have a necessary and sufficient condition for orbits of (2.2) to be periodic on the whole of \mathbb{R} . We further wish to address the problem of the spatial boundedness of such orbits. The example shown in Figure 4 demonstrates that orbits can be periodic (in time) but unbounded (in space). To find conditions for spatial boundedness we must understand both the locations and the sizes of the discontinuities of Z_n on the whole of \mathbb{R} , and that is the purpose of this section. The sizes of the discontinuities will be straightforward to calculate, but identifying their locations presents some difficulty. Since recurrence (2.2) is linear, we may consider the contribution of each periodic orbit, and, indeed, each discontinuity of each periodic orbit, to the global discontinuity set of Z_n separately.

We shall return to the issue of spatial boundedness in Section 4. A precise identification of the locations of the discontinuities of Z_n will also enable us to calculate the autocorrelation functions. This we shall do in Section 5.

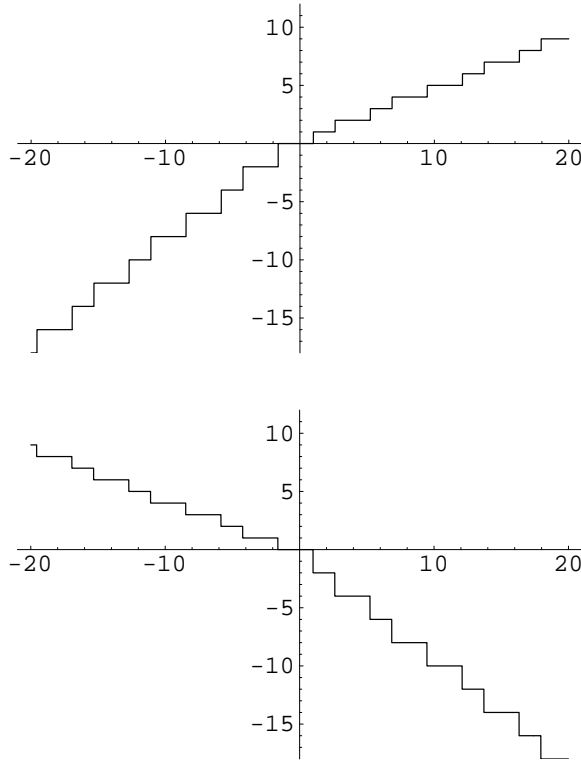


Figure 4: Period-2 orbit of the recurrence (2.2).

3.1 Discontinuities and the field $\mathbb{Q}(\omega)$

Let L_n denote the set of locations of the discontinuities of Z_n , i.e.,

$$L_n = \{x \in \mathbb{R} : D_n(x) \neq 0\}. \quad (3.1)$$

From (2.2) it is clear that

$$L_n = \phi_1^{-1}(L_{n-1}) \cup \phi_2^{-1}(L_{n-2}), \quad (3.2)$$

unless there is a cancellation of discontinuities (which, as we shall see below, may occur if both 1 and 0 are discontinuities). As we saw in Subsection 2.2, the global discontinuities of Z_n are generated from those in the fundamental interval I , and the latter consist of elements of periodic orbits of the map F . It is clear from (2.8) that such periodic orbits must be composed of elements of the field

$$\mathbb{Q}(\omega) = \{a + b\omega : a, b \in \mathbb{Q}\}. \quad (3.3)$$

As a consequence, the sets L_n consist of elements of $\mathbb{Q}(\omega)$, since the maps ϕ_1^{-1} , ϕ_2^{-1} act on $\mathbb{Q}(\omega)$ as

$$\phi_1^{-1}(a + b\omega) = -(a + b) - a\omega, \quad (3.4)$$

$$\phi_2^{-1}(a + b\omega) = 2a + b - 1 + (a + b - 1)\omega. \quad (3.5)$$

Rather than consider a periodic orbit itself, we shall consider initially an orbit asymptotic to it generated from discontinuity data on the fundamental interval only. By the results of Subsection 2.2 this orbit is eventually periodic and identical to the desired periodic orbit on any bounded subset of \mathbb{R} .

Consider then the case of Z_0 having a single discontinuity of size X at $a + b\omega \in (-\omega, 1]$, and Z_1 having discontinuity only at $\phi_1^{-1}(a + b\omega)$. This discontinuity will have size $-X$. If $a + b\omega \in (-\omega, \omega^2)$ (i.e., $a + b\omega$ has code 1) then $\phi_1^{-1}(a + b\omega) = F(a + b\omega) \in (-\omega, 1)$, otherwise ($a + b\omega$ has code 2) $\phi_1^{-1}(a + b\omega) \notin (-\omega, 1]$.

The first few discontinuity location sets are

$$\begin{aligned}
L_0 &= \{a + b\omega\} = a + b\omega + \{0 + 0\omega\} \\
L_1 &= \{-(a + b) - a\omega\} = -(a + b) - a\omega + \{0 + 0\omega\} \\
L_2 &= 2a + b + (a + b)\omega + \{-1 - \omega, 0 + 0\omega\} \\
L_3 &= -(3a + 2b) - (2a + b)\omega + \{-1 - \omega, 0 + 0\omega, 2 + \omega\} \\
L_4 &= 5a + 3b + (3a + 2b)\omega + \{-4 - 3\omega, -3 - 2\omega, -1 - \omega, 0 + 0\omega, 2 + \omega\} \\
L_5 &= -(8a + 5b) - (5a + 3b)\omega + \{-4 - 3\omega, -3 - 2\omega, -1 - \omega, 0 + 0\omega, 2 + \omega, 4 + 2\omega, 5 + 3\omega, 7 + 4\omega\},
\end{aligned} \tag{3.6}$$

where, when x is a number and S is a set of numbers, $x + S$ denotes the set $\{x + s : s \in S\}$. We shall write these sets in the form

$$L_n = c^{(n)} + d^{(n)}\omega + M_n. \tag{3.7}$$

We observe that the numbers $c^{(n)}$, $d^{(n)}$ obey a simple Fibonacci recurrence, and that the sets (M_n) grow in an alternating manner, appending successively on the right and left of the existing list, and that M_n contains F_{n+1} elements. Moreover M_n consists of numbers of the form $a + b\omega$ in which b increases uniformly by 1 from $-F_{n-1}$ to $F_n - 1$ when n is odd, and from $-F_n$ to $F_{n-1} - 1$ when n is even. However the component a has increments given by a sequence $1, 2, 1, 2, 1, 2, \dots$, ranging from $-(F_n - 1)$ to $F_{n+1} - 1$ when n is odd, and from $-(F_{n+1} - 1)$ to $F_n - 1$ when n is even. The size of each discontinuity in L_n is $(-1)^n X$.

We shall now identify the precise form of the set M_n , and then prove that the above observations hold for all n .

3.2 Discontinuities arising from a single discontinuity in I

We now show state and prove the precise form of the sets L_n based on our preliminary observation in the previous subsection.

For $x \in \mathbb{R}$, we let $\lceil x \rceil$ denote the ceiling of x , namely $\min\{n \in \mathbb{Z} : n \geq x\}$, and $\lfloor x \rfloor$ denote the floor of x , namely $\max\{n \in \mathbb{Z} : n \leq x\}$. Firstly, we have

Proposition 6. *The discontinuity location sets (L_n) arising from applying recurrence (2.2) to initial conditions in which Z_0 has a single discontinuity of size X at $a + b\omega \in I$, and Z_1 has a single discontinuity of size $-X$ at $\phi_1^{-1}(a + b\omega)$ are $L_0 = \{a + b\omega\}$ and for $n \geq 1$*

$$L_n = c^{(n)} + d^{(n)}\omega + M_n, \tag{3.8}$$

where

$$M_n = \{\lceil i/\omega \rceil + i\omega : i = \ell_n, \dots, r_n\}, \tag{3.9}$$

with

$$\ell_n, r_n = \begin{cases} -F_{n-1}, F_n - 1, & n \text{ odd}; \\ -F_n, F_{n-1} - 1, & n \text{ even}, \end{cases} \tag{3.10}$$

and where

$$c^{(n)} = (-1)^n (F_{n+1}a + F_n b), \quad d^{(n)} = (-1)^n (F_n a + F_{n-1} b). \tag{3.11}$$

Moreover, each discontinuity of L_n is of size $(-1)^n X$.

Proof. The proof is by induction with the base case being clear. Suppose now that L_{n-1} and L_{n-2} are given as in the statement of the proposition. As we noted earlier (3.2) we have $L_n = \phi_1^{-1}(L_{n-1}) \cup \phi_2^{-1}(L_{n-2})$. We readily calculate that

$$\phi_1^{-1}(L_{n-1}) = c^{(n)} + d^{(n)}\omega + \phi_1^{-1}(M_{n-1}), \tag{3.12}$$

$$\phi_2^{-1}(L_{n-2}) = c^{(n)} + d^{(n)}\omega + \phi_2^{-1}(M_{n-2}), \tag{3.13}$$

so that we need only establish that $M_n = \phi_1^{-1}(M_{n-1}) \cup \phi_2^{-1}(M_{n-2})$ with M_n as given by (3.9). To do this it suffices to show that each element of M_n is in either $\phi_1^{-1}(M_{n-1})$ or $\phi_2^{-1}(M_{n-2})$, since M_n has F_{n+1} elements and $\phi_1^{-1}(M_{n-1})$, $\phi_2^{-1}(M_{n-2})$ have F_n , F_{n-1} elements respectively.

Suppose that n is even and let $i \in \{-F_n, \dots, F_{n-1}-1\}$. Considering (3.4) and (3.5), we must show there exists either $j \in \{-F_{n-2}, \dots, F_{n-1}-1\}$ such that $i = -\lceil j/\omega \rceil$ and $\lceil i/\omega \rceil = -\lceil j/\omega \rceil - j$ or $k \in \{-F_{n-2}, \dots, F_{n-3}-1\}$ such that $i = \lceil k/\omega \rceil + k - 1$ and $\lceil i/\omega \rceil = 2\lceil k/\omega \rceil + k - 1$.

Set $j = \lfloor -i\omega \rfloor$ so that $j + \varepsilon_1 = -i\omega$, i.e.,

$$j/\omega + \varepsilon_1/\omega = -i \quad (3.14)$$

for some $\varepsilon_1 \in (0, 1)$. We note that $j \in \{\lfloor -(F_{n-1}-1)\omega \rfloor, \dots, \lfloor F_n\omega \rfloor\}$, for which the identity (1.24) (recalling that n is even) gives $\lfloor F_n\omega \rfloor = \lfloor F_{n-1}-\omega^n \rfloor = F_{n-1}-1$, and $\lfloor -(F_{n-1}-1)\omega \rfloor = \lfloor -F_{n-2} + \omega - \omega^{n-1} \rfloor = -F_{n-2}$, so that j is in the desired range.

Now (case 1) if $\varepsilon_1 \in (0, \omega)$ then $\varepsilon_1/\omega \in (0, 1)$ so that (3.14) gives $\lceil j/\omega \rceil = -i$, i.e., $i = -\lceil j/\omega \rceil$. We further have $i/\omega = i + i\omega = i - j - \varepsilon_1 = -\lceil j/\omega \rceil - j - \varepsilon_1$, and thus $\lceil i/\omega \rceil = -\lceil j/\omega \rceil - j$ as desired.

On the other hand (case 2) if $\varepsilon_1 \in (\omega, 1)$ then $\varepsilon_1/\omega \in (1, \omega^{-1})$ and we deduce that $\lceil j/\omega \rceil = -i - 1$, so that $i \neq -\lceil j/\omega \rceil$, and we thus need to define a suitable k . (Note that the case $\varepsilon_1 = \omega$ is clearly impossible since ω is irrational. For the same reason we have $\varepsilon_1 \neq 0$.) We set $k = \lfloor -j\omega \rfloor$ so that there exists $\varepsilon_2 \in (0, 1)$ such that

$$-j\omega = k + \varepsilon_2. \quad (3.15)$$

Since $j \in \{-F_{n-2}, \dots, F_{n-1}-1\}$, by an argument similar to that above, we deduce, as desired, that $k \in \{-F_{n-2}, \dots, F_{n-3}-1\}$. Addition of (3.14) and (3.15) gives $j + \varepsilon_1/\omega = -i + k + \varepsilon_2$ so that, since $\varepsilon_1/\omega \in (1, \omega^{-1})$ and $\varepsilon_2 \in (0, 1)$, we have $\varepsilon_2 = \varepsilon_1/\omega - 1$, and thus $\varepsilon_2 \in (0, \omega)$, i.e., $\varepsilon_2/\omega \in (0, 1)$. Equation (3.15) is $-j = k/\omega + \varepsilon_2/\omega$ which thus gives

$$-j = \lceil k/\omega \rceil. \quad (3.16)$$

Now addition of (3.15) and (3.16) gives $-j - j\omega - \varepsilon_2 = k + \lceil k/\omega \rceil$, i.e., $-k - \lceil k/\omega \rceil = j/\omega + \varepsilon_2 = \lceil j/\omega \rceil$ and so, since $\lceil j/\omega \rceil = -i - 1$, we deduce that $i = \lceil k/\omega \rceil + k - 1$. From $j + \varepsilon_1 = -i\omega$ and $j = -\lceil k/\omega \rceil$, it immediately follows that $i/\omega = i + i\omega = i + \lceil k/\omega \rceil - \varepsilon_1 = 2\lceil k/\omega \rceil + k - 1 - \varepsilon_1$, and so $\lceil i/\omega \rceil = 2\lceil k/\omega \rceil + k - 1$ as desired.

The case n odd is established in a similar manner.

That each discontinuity of L_n is of size $(-1)^n X$ follows immediately from the facts that ϕ_1^{-1} is orientation reversing and ϕ_2^{-1} is orientation preserving. \square

We have thus established the locations and sizes of the discontinuities of Z_n that arise from the presence of a discontinuity in Z_0 at $a + b\omega \in I$.

We must also consider the case in which Z_0 has no discontinuities and Z_1 has a single discontinuity in I . However, this situation is clearly identical to the case just considered, but with a reduction of one in all subscripts. More precisely we have

Proposition 7. *The discontinuity location sets (L_n) arising from applying recurrence (2.2) to initial conditions in which Z_0 has no discontinuities and Z_1 has a single discontinuity of size $-X$ at $a + b\omega \in I$ are $L_0 = \emptyset$, $L_1 = \{a + b\omega\}$, and for $n \geq 2$*

$$L_n = c^{(n-1)} + d^{(n-1)}\omega + M_{n-1}, \quad (3.17)$$

with $c^{(n)}$, $d^{(n)}$, M_n as in the statement of Proposition 6. Moreover, each discontinuity of L_n is of size $(-1)^n X$.

The proof of Proposition 7 is analogous to that of Proposition 6.

Thus far we have considered functions possessing only a finite number of discontinuities. A globally periodic orbit however possesses an infinite number of discontinuities, being the limit (defined by pointwise

convergence) of the sets (L_n) considered thus far. It is clear from Proposition 6 what the discontinuity set generated by a single discontinuity at the fixed point zero is, namely

$$\{[i/\omega] + i\omega : i \in \mathbb{Z}\}, \quad (3.18)$$

this being the limit of the corresponding sets L_n with $a = b = 0$, so that $c^{(n)} = d^{(n)} = 0$ for all n . However care is needed in the determination of these sets for other periodic orbits. We shall return to this issue in Section 5 where we consider the period-6 orbit of (2.2) generated by the period-2 orbit $\{1/2, -\omega/2\}$ of F .

We remark that a symbolic formulation of the discontinuity locations is possible in terms of “words” with letters drawn from the “alphabet” $\{1, 2\}$. More precisely a sequence of Fibonacci words with first few elements

$$\begin{aligned} &1 \\ &12 \\ &1212 \\ &1212212 \\ &121221212212 \end{aligned} \quad (3.19)$$

may be used to describe the increments in the elements of the discontinuity sets L_n . These words may be obtained by the following rules. We denote by \diamond the empty word, by S the substitution operator $1 \mapsto 2$, $2 \mapsto 12$, and by R the reversal operator $w_1 \dots w_k \mapsto w_k \dots w_1$. The n th Fibonacci word w^n may be written as the concatenation $u^n v^n$ with the rule $u^1 = v^1 = \diamond$, and, for $n \geq 1$, $u^{n+1} = R(S(v^n)1)$, $v^{n+1} = R(S(u^n))$. We index the “letters” in w^n by setting w_0^n to be the last letter of u^n and setting w_1^n to be the first letter of v^n . Then the words w^n are the words given in (3.19). As $n \rightarrow \infty$, $w^n \rightarrow w^*$, the biinfinite Fibonacci word. One may write the set (3.18) as $\{x_i + i\omega\}$ where $x_0 = 0$ and $x_i - x_{i-1} = w_i^*$.

4 Global boundedness

In the previous section we determined the global effect of a single discontinuity in the fundamental interval. Each of the ℓ variables $X_0, X_1, \dots, X_{\ell-1}$ in the discontinuity matrix of a periodic orbit gives rise to such a discontinuity in the initial conditions Z_0, Z_1 in the form of one of the three following cases.

1. Z_0 has a single discontinuity in I at $x \in (-\omega, \omega^2)$ and Z_1 has a single discontinuity at $F(x) \in I - \{1\}$.
2. Z_0 has a single discontinuity in $(\omega^2, 1]$ and Z_1 has no discontinuities in I .
3. Z_0 has no discontinuities in I and Z_1 has a single discontinuity in I .

Case 1 corresponds to a discontinuity of Z_0 at a point y_i with code 1, so that y_{i+1} is a discontinuity location of Z_1 (having the same size, but opposite sign). Case 2 corresponds to a discontinuity location y_i of Z_0 with code 2. Case 3 corresponds to a discontinuity location y_i of Z_1 such that y_{i-1} has code 2. (It is illustrative to look at a discontinuity matrix such as that of Example 4 to understand these three cases.) Cases 1 and 2 are handled by Proposition 6, whilst Proposition 7 treats Case 3.

The important thing to note from Propositions 6 and 7 is that each discontinuity generates an ordered set of discontinuities of the same size and sign in which the elements of each set have identical relative displacements: the elements of each set are separated from each other *by the same amount* ($1 + \omega$ or $2 + \omega$) *in the same order*. Moreover, with the exception of the fixed point 1, each nonzero discontinuity causes an unbounded monotonic growth as we move further from the fundamental interval on both sides.

From Proposition 6 we see that if Z_0 has a discontinuity at 1 of size X (case 2) then this discontinuity gives rise to discontinuities in Z_n of size X at locations ≥ 1 for even n and of size $-X$ at locations $\leq -1 - \omega$ for odd n . Proposition 7 shows that if Z_1 has a discontinuity at 1 of size $-X'$ (case 3) then *this* discontinuity

gives rise to discontinuities in Z_n of size X' at locations $\leq -1 - \omega$ for *even* n and of size $-X'$ at locations ≥ 1 for *odd* n . This may be seen in Figure 4, in which the discontinuities in the first figure (even n) are of size 1 on the right (corresponding to the discontinuity in Z_0), whilst those on the left are of size 2 (corresponding to the discontinuity in Z_1). The situation is reversed in the second figure (where n is odd).

For global boundedness we require there to be cumulative growth on neither the left nor the right hand sides of the fundamental interval.

Consider the case of even n . For boundedness on the right we must include the effect of 1 being a discontinuity location for Z_0 . For boundedness on the left we must include the effect of 1 being a discontinuity location for Z_1 .

We firstly look at the cumulative effect of the discontinuities contributing to growth to the right of the fundamental interval. We must consider all discontinuities from cases 1 and 2, and those of case 3 excluding the fixed point 1. The sum over all discontinuities of all periodic orbits of the type of case 2 is

$$\sum_{y \in (\omega^2, 1]} D_0(y). \quad (4.1)$$

The combined contribution of all discontinuities of all periodic orbits of the type of cases 1 and 3 is

$$- \sum_{y \in (-\omega, 1)} D_1(y), \quad (4.2)$$

the contributions from case 1 being of the form $+D_0(y_{i-1}) = -D_1(y_i)$, and from case 3 of the form $-D_1(y_i)$. Note that 1 is excluded from the range of this sum since, as we noted above, discontinuities associated with 1 being a discontinuity of the type of case 3 are to the left of the fundamental interval for even n . Because of the same relative spacing of discontinuity locations, as we increase to the right the contributions from all three cases grow at the same rate. For boundedness we therefore require the sum of (4.1) and (4.2) to be zero, i.e., we require

$$\sum_{y \in (\omega^2, 1]} D_0(y) = \sum_{y \in (-\omega, 1)} D_1(y). \quad (4.3)$$

Now look to the left of the fundamental interval (with n still even). The sum over all discontinuities of all periodic orbits of the type of case 2 is now

$$\sum_{y \in (\omega^2, 1)} D_0(y), \quad (4.4)$$

and the sum over all discontinuities of all periodic orbits of the type of cases 1 and 3 is

$$- \sum_{y \in (-\omega, 1]} D_1(y), \quad (4.5)$$

where 1 is now excluded from the first sum since discontinuities associated with a 1 being a discontinuity of the type of case 2 are to the right of the fundamental interval for even n . For the same reasons as above, we require the sum of (4.4) and (4.5) to be zero, i.e., we require

$$\sum_{y \in (\omega^2, 1)} D_0(y) = \sum_{y \in (-\omega, 1]} D_1(y). \quad (4.6)$$

For even n , it is clear that if condition (4.3) is violated there will be unbounded growth on the right, whilst if condition (4.6) is violated there will be unbounded growth on the left. Thus our combined conditions are necessary and sufficient. (Consideration of boundedness for odd n results in the same two conditions (4.3) and (4.6).) We thus have the following

Theorem 2. *A necessary and sufficient criterion for a piecewise-constant periodic orbit of (2.2) to be globally bounded is*

$$\sum_{y \in (-\omega, 1]} D_1(y) = \sum_{y \in (\omega^2, 1)} D_0(y) \quad \text{and} \quad \sum_{y \in (-\omega, 1)} D_1(y) = \sum_{y \in (\omega^2, 1]} D_0(y). \quad (4.7)$$

Note that in general the inclusion or exclusion of 1 from the intervals in these sums is crucial. (However, the points $-\omega$ and ω^2 are never discontinuities.) Further note that when 1 is not a discontinuity location these two conditions are identical. For instance, for the period-6 orbit resulting from Examples 1 and 2 the condition (4.7) is the single constraint $X_0^0 - X_1^0 + X_2^0 + X_0^1 = 0$.

We remark that we may produce orbits that are unbounded, but *relatively globally bounded* by ensuring condition of Theorem 2 is satisfied but the condition for periodicity of Subsection 2.6 (Theorem 1) is violated. (We define the orbit (Z_n) to be relatively globally bounded if there exists a constant K such that $|Z_n(x) - Z_n(0)| \leq K$ for all $x \in \mathbb{R}$ and for all $n \in \mathbb{N}$.) The simplest example of such an orbit is that generated by the constant functions $Z_0(x) = 0$, $Z_1(x) = 1$ in which there are no discontinuities.

From the physical point of view we deduce that we may choose many forms for the modulation function in (1.11), and, provided the conditions for periodicity (Theorem 1) and global boundedness (Theorem 2) are satisfied, the resulting autocorrelation function will display asymptotic self-similarity. More precisely, provided the function Φ is chosen so that the initial conditions $Z_0(x) = 0$, $Z_1(x) = \Phi(-\omega x)$ are periodic or preperiodic on the fundamental interval $[-\omega, 1]$, and that two successive functions on this periodic orbit satisfy the periodicity and global boundedness conditions, then there is asymptotic self-similarity of the autocorrelation function.

We look in detail at the particular choice (1.29) in the next section.

5 The period-6 orbit

As a further application of our work in Section 3, we now look in detail at the particular example of Feudel *et al* [5] briefly mentioned in the introduction. The choice of modulation function (1.29) results in the period-6 orbit shown in Figure 2. However it is important to note that the discontinuities of the initial conditions (1.27) do not all form discontinuities of the period-6 orbit. In particular, the function $Z_1(x) = \Phi(-\omega x)$ has a discontinuity at $x = \omega^{-1}/2$ which is not periodic for F , but its image under F is $1/2$ which is part of the period-two orbit $\{1/2, -\omega/2\}$. The other discontinuity of Z_1 in the fundamental interval $[-\omega, 1]$ is the fixed point 0.

Henceforth, when we refer to the orbit (Z_n) we mean the global periodic orbit, and not the orbit asymptotic to it that we generate from the periodic orbit on the fundamental interval. This global period-6 orbit is generated from the restricted discontinuity set equal to the union of Examples 1 and 2, namely $\{1/2, -\omega/2\} \cup \{0\}$ with discontinuity values $X_0^0 = -2$, $X_1^0 = 0$, $X_2^0 = 0$, corresponding to $\{1/2, -\omega/2\}$, and $X_0^1 = 2$ corresponding to $\{0\}$. This data means that Z_0 has discontinuities $X_0^0 = -2$ at $1/2$, $-X_1^0 = 0$ at $-\omega/2$, and $X_0^1 = 2$ at 0, whilst Z_1 has discontinuities $X_1^0 = 0$ at $1/2$, $-X_2^0 = 0$ at $-\omega/2$, and $-X_0^1 = -2$ at 0. The point 1 is not a discontinuity point, and to guarantee periodicity we specify $Z_0^{1+} = Z_0(1) = 0$, and $Z_1^{1+} = Z_1(1) = -1$ according to (2.35). We further note that this data satisfies the condition $X_0^0 - X_1^0 + X_2^0 + X_0^1 = 0$, which is the criterion of the previous section (Theorem 2) guaranteeing a globally bounded periodic orbit.

The value of the corresponding autocorrelation function K_{P_y} at Fibonacci times F_n will be calculated explicitly in terms of the amplitude κ . Figure 1 shows K_{P_y} for the case $\kappa = \pi/2$. In the cases $n \equiv 1, 2 \pmod 3$, since (as we shall show) $Z_n = \pm 1$, equation (1.28) gives us $K_{P_y}(F_n) = \cos \kappa$ ($= 0$ when $\kappa = \pi/2$). In the case $n \equiv 0 \pmod 3$ where now (again, as we shall show) $Z_n = 0$, ± 2 we have $K_{P_y}(F_n) = 1 - \alpha + \alpha \cos 2\kappa$ ($= 1 - 2\alpha$ when $\kappa = \pi/2$) where 2α is the asymptotic average value of $|Z_n|$. In [5] it is numerically calculated that $\alpha \approx 0.2236$. In fact this constant is $1/(2\sqrt{5})$ as we shall show in this section. As a consequence, when $\kappa = \pi/2$, for $n \equiv 0 \pmod 3$ we have $K_{P_y}(F_n) \rightarrow 1 - 1/\sqrt{5} = 0.552786\dots$ as $n \rightarrow \infty$.

5.1 The discontinuity location sets

5.1.1 Discontinuities arising from the fixed point 0

We have that Z_0 and Z_1 both have a single discontinuity at 0 ($= 0 + 0\omega$) only, and so, taking the limit $n \rightarrow \infty$ in Proposition 6, the discontinuity location set associated with the fixed point 0 is

$$L^1 = \{[i/\omega] + i\omega : i \in \mathbb{Z}\}. \quad (5.1)$$

5.1.2 Discontinuities arising from the period-two orbit $\{1/2, -\omega/2\}$

The discontinuity location sets associated with the period-two orbit $\{1/2, -\omega/2\}$ are generated from Z_0 having a discontinuity at $1/2$ and Z_1 having no discontinuities in fundamental interval $[-\omega, 1]$. (Of course, Z_1 has a discontinuity at $\phi_1^{-1}(1/2) = -\omega^{-1}/2$, but this is outside the fundamental interval.) The result is period three behaviour. Let us define the sets

$$L^{0,0} = \{[i/\omega + 1/2] - 1/2 + i\omega : i \in \mathbb{Z}\}, \quad (5.2)$$

$$L^{0,1} = \{[(i - 1/2)/\omega + 1/2] - 1/2 + (i - 1/2)\omega : i \in \mathbb{Z}\}, \quad (5.3)$$

$$L^{0,2} = \{[(i - 1/2)/\omega] + (i - 1/2)\omega : i \in \mathbb{Z}\}. \quad (5.4)$$

Then we have the following

Proposition 8. *The discontinuity location sets L_n generated by the period-two orbit $\{1/2, -\omega/2\}$ satisfy*

$$L^{0,0} \cap L_n = \{[i/\omega + 1/2] - 1/2 + i\omega : i = \ell_n^0, \dots, r_n^0\}, \quad n \equiv 0 \pmod{3}; \quad (5.5)$$

$$L^{0,1} \cap L_n = \{[(i - 1/2)/\omega + 1/2] - 1/2 + (i - 1/2)\omega : i = \ell_n^1, \dots, r_n^1\}, \quad n \equiv 1 \pmod{3}; \quad (5.6)$$

$$L^{0,2} \cap L_n = \{[(i - 1/2)/\omega] + (i - 1/2)\omega : i = \ell_n^2, \dots, r_n^2\}, \quad n \equiv 2 \pmod{3}, \quad (5.7)$$

where, as $n \rightarrow \infty$, $\ell_n^k \rightarrow -\infty$, $r_n^k \rightarrow \infty$ for $k = 0, 1, 2$, so that the sets $L^{0,0}$, $L^{0,1}$, $L^{0,2}$ are the limits as $m \rightarrow \infty$ of the sequences (L_{3m}) , (L_{3m+1}) , (L_{3m+2}) respectively.

Proof. We must consider the three residue classes modulo 3 separately. We shall use readily established fact that the Fibonacci numbers have the following parity structure: $F_n \equiv 0 \pmod{2}$ when $n \equiv 0 \pmod{3}$, and $F_n \equiv 1 \pmod{2}$ when $n \equiv 1, 2 \pmod{3}$.

a) $n \equiv 0 \pmod{3}$. We treat the two cases $n \equiv 0 \pmod{6}$, $n \equiv 3 \pmod{6}$ separately.

In the case $n \equiv 0 \pmod{6}$, so that n is even, Proposition 6 gives

$$L_n = \{F_{n+1}/2 + [i/\omega] + (F_n/2 + i)\omega : i = -F_n, \dots, F_{n-1} - 1\}, \quad (5.8)$$

which by relabelling is

$$L_n = \{F_{n+1}/2 + [(i - F_n/2)/\omega] + i\omega : i = -F_n/2, \dots, F_n/2 + F_{n-1} - 1\}. \quad (5.9)$$

(Note that F_n is even.) To establish the result we must show that

$$F_{n+1}/2 + [(i - F_n/2)/\omega] = [i/\omega + 1/2] - 1/2 \quad (5.10)$$

for a suitable range of i . Now, since F_{n+1} is odd,

$$\begin{aligned} 1/2 + F_{n+1}/2 + [(i - F_n/2)/\omega] &= [1/2 + F_{n+1}/2 + (i - F_n/2)/\omega] \\ &= [1/2 + (i + (F_{n+1}\omega - F_n)/2)/\omega] \\ &= [1/2 + (i + \omega^{n+1}/2)/\omega] \\ &= [i/\omega + 1/2 + \omega^n/2], \end{aligned} \quad (5.11)$$

where we have used the identity (1.24). We must show that for suitable i this equals $\lceil i/\omega + 1/2 \rceil$. Since $i/\omega = i + i\omega$ our desired equality may be written as $\lceil i\omega + 1/2 + \omega^n/2 \rceil = \lceil i\omega + 1/2 \rceil$, which is true if, and only if, $\lceil i\omega + 1/2 \rceil - (i\omega + 1/2) \geq \omega^n/2$, in which we note that $\lceil i\omega + 1/2 \rceil - (i\omega + 1/2) = |i\omega + 1/2 - r|$, where $r = \lceil i\omega + 1/2 \rceil$.

Now since the ratios of successive Fibonacci numbers are the continued fraction approximants to ω we have that

$$|q\omega - p| \geq |F_n\omega - F_{n-1}| = \omega^n, \quad (5.12)$$

for all $p \in \mathbb{Z}$ and all integers q with $|q| \leq F_n$.

We thus have $|2i\omega + 1 - 2r| \geq \omega^n$ for all integers i such that $2|i| \leq F_n$, i.e., $|i\omega + 1/2 - r| \geq \omega^n/2$, for all integers i such that $|i| \leq F_n/2$. This establishes our desired equality for a range of i of the desired type.

The case $n \equiv 3 \pmod{6}$ is similar, with minor modifications to take into account the oddness of n . Such modifications may be seen in the proof of the first part of the next case.

b) $n \equiv 1 \pmod{3}$. We consider separately the two cases $n \equiv 1 \pmod{6}$, $n \equiv 4 \pmod{6}$.

In the case $n \equiv 1 \pmod{6}$, so that n is odd, Proposition 6 gives

$$L_n = \{-F_{n+1}/2 + \lceil i/\omega \rceil + (i - F_n/2)\omega : i = -F_{n-1}, \dots, F_n - 1\}, \quad (5.13)$$

which by relabelling is

$$L_n = \{-F_{n+1}/2 + \lceil (i + (F_n - 1)/2)/\omega \rceil + (i - 1/2)\omega : i = -F_{n-1} - (F_n - 1)/2, \dots, (F_n - 1)/2\}. \quad (5.14)$$

(Note that F_n is now odd.) To establish the result we must show that

$$-F_{n+1}/2 + \lceil (i + (F_n - 1)/2)/\omega \rceil = \lceil (i - 1/2)/\omega + 1/2 \rceil - 1/2 \quad (5.15)$$

for a suitable range of i . We now proceed as in case a). Firstly we have

$$1/2 - F_{n+1}/2 + \lceil (i + (F_n - 1)/2)/\omega \rceil = \lceil i/\omega - \omega/2 - \omega^n/2 \rceil, \quad (5.16)$$

and so we must show that $\lceil i\omega - \omega/2 - \omega^n/2 \rceil = \lceil i\omega - \omega/2 \rceil$ for suitable i , which is true if, and only if, $i\omega - \omega/2 - \lceil i\omega - \omega/2 \rceil + 1 > \omega^n/2$. Now $i\omega - \omega/2 - \lceil i\omega - \omega/2 \rceil + 1 = |i\omega - \omega/2 - r|$, where $r = \lceil i\omega - \omega/2 \rceil - 1$, and we have $|2i\omega - \omega - 2r| = |(2i - 1)\omega - 2r| \geq \omega^n$ for all integers i such that $|2i - 1| < F_n$. This is just $|i\omega - \omega/2 - r| > \omega^n/2$ for such a range of i as desired.

The case $n \equiv 4 \pmod{6}$ is similar.

c) $n \equiv 2 \pmod{3}$. We omit the detail for this case which is similar to those above. \square

5.2 Combining the discontinuities

Note that

$$L^{0,n} \subset \begin{cases} (\mathbb{Z}/2 - \mathbb{Z}) + \omega\mathbb{Z}, & n = 0; \\ (\mathbb{Z}/2 - \mathbb{Z}) + \omega(\mathbb{Z}/2 - \mathbb{Z}), & n = 1; \\ \mathbb{Z} + \omega(\mathbb{Z}/2 - \mathbb{Z}), & n = 2, \end{cases} \quad (5.17)$$

whilst

$$L^1 \subset \mathbb{Z} + \omega\mathbb{Z}. \quad (5.18)$$

It follows that the discontinuity location sets generated from the period-two orbit $\{1/2, -\omega/2\}$ are disjoint from those associated with the fixed point 0. Thus, there can be no cancellation, or even modification, of the discontinuities generated by one periodic orbit ($\{1/2, -\omega/2\}$ or $\{0\}$) by the other. We further recall that the elements of each of these sets form an increasing sequence with separation either $1 + \omega$ or $2 + \omega$.

The initial conditions for the discontinuities are $X_0^0 = -2$, $X_1^0 = X_2^0 = 0$, and $X_0^1 = 2$. It follows that each discontinuity of Z_n due to the period-two orbit $\{1/2, -\omega/2\}$ is of size -2 when n is even, and of size 2 when

n is odd. Each discontinuity of Z_n due to the fixed point 0 is of size 2 when n is even, and of size -2 when n is odd.

As well as specifying the initial discontinuity data, we have also specified the initial values $Z_0(1) = 0$ and $Z_1(1) = -1$, thereby giving us initial conditions asymptotic to a globally bounded periodic orbit of period-6. It is straightforward to check that $Z_2(1) = 1$, $Z_3(1) = 0$, $Z_4(1) = 1$, $Z_5(1) = -1$, so that, knowing from Section 2 that the discontinuities have period-6 satisfying $D_{n+3} = -D_n$, the periodic orbit (Z_n) itself satisfies $Z_{n+3} = -Z_n$.

5.3 Calculation of the autocorrelation function

Given this detailed knowledge of the locations and sizes of all discontinuities we can calculate the autocorrelation function (1.28). We denote the discontinuity locations associated with the fixed point 0 given in (5.1) as $a_i^1 + b_i^1\omega$, and those associated with the period-two orbit $\{1/2, -\omega/2\}$ given in (5.2–5.4) as $a_i^0 + b_i^0\omega$.

We must consider the three residue classes modulo 3 separately.

5.3.1 $n \equiv 0 \pmod{3}$

When $n \equiv 0 \pmod{3}$ we see from (5.1) and (5.2) that discontinuities may be matched by the coefficients of ω . The separation of matched discontinuities is then

$$\begin{aligned} a_i^1 + b_i^1\omega - (a_i^0 + b_i^0\omega) &= [i/\omega] + i\omega - ([i/\omega + 1/2] - 1/2 + i\omega) \\ &= 1/2 + [i/\omega] - [i/\omega + 1/2] = \pm 1/2, \end{aligned} \quad (5.19)$$

and these discontinuities occur in strict alternation since the separation of elements in each sequence is either $1 + \omega$ or $2 + \omega$.

Now when, in addition, n is even so that $n \equiv 0 \pmod{6}$, each discontinuity from $\{1/2, -\omega/2\}$ has size $X_0^0 = -2$, whilst each discontinuity from $\{0\}$ has size $X_0^1 = 2$. Moreover, we have that $Z_n(1) = 0$, so that the matched discontinuities combine to give an interval of height ± 2 and width $1/2$ in the graph of Z_n , as in the top two graphs in Figure 2. For instance, the first pair after 1 is $2 + \omega$ with discontinuity 2, and $5/2 + \omega$ with discontinuity -2 . Thus the graph of Z_n consists of intervals at height zero punctuated by excursions of width $1/2$ to a height ± 2 . The analysis for odd n (i.e., so that now $n \equiv 3 \pmod{6}$) is identical, but with all signs reversed.

The autocorrelation function values $K_{P_y}(F_n)$ may now be calculated.

Theorem 3. *Let (Z_n) be the globally bounded period-6 orbit of (2.2) generated by discontinuity set $\{1/2, -\omega/2, 0\}$ with initial data $D_0(1/2) = -2$, $D_0(-\omega/2) = 0$, $D_0(0) = 2$, $D_1(1/2) = 0$, $D_1(-\omega/2) = 0$, $D_1(0) = -2$, with $Z_0^{1+} = Z_0(1) = 0$, and $Z_1^{1+} = Z_1(1) = -1$. Then, for $n \equiv 0 \pmod{3}$,*

$$\lim_{n \rightarrow \infty} K_{P_y}(F_n) = 1 - \alpha + \alpha \cos 2\kappa, \quad (5.20)$$

where

$$\alpha = \frac{1}{2\sqrt{5}}. \quad (5.21)$$

For the special case $\kappa = \pi/2$ shown in Figure 1 we thus have $K_{P_y}(F_n) \rightarrow 1 - 1/\sqrt{5} = 0.552786\dots$ as $n \rightarrow \infty$ with $n \equiv 0 \pmod{3}$.

Proof. We have already seen that Z_n takes values 0, ± 2 . Note that

$$\cos \kappa Z_n(y) = \begin{cases} 1, & Z_n(y) = 0; \\ \cos 2\kappa, & |Z_n(y)| = 2. \end{cases} \quad (5.22)$$

Thus, defining

$$\alpha_n = \frac{1}{(-\omega)^{-n}} \int_0^{(-\omega)^{-n}} \frac{|Z_n(y)|}{2} dy, \quad (5.23)$$

as the proportion of the interval $[0, \omega^{-n}]$ for even n , $[-\omega^{-n}, 0]$ for odd n , for which $|Z_n(y)| = 2$, (1.28) becomes

$$K_{P_y}(F_n) = 1 - \alpha_n + \alpha_n \cos 2\kappa. \quad (5.24)$$

Now

$$F_n = \frac{\omega^{-n}}{\sqrt{5}} + O(1), \quad (5.25)$$

from which it is straightforward to deduce that

$$\omega^{-n} = F_{n+1} + F_n \omega + O(1). \quad (5.26)$$

We now recognise that $F_{n+1} + F_n \omega$, respectively $-(F_{n+1} + F_n \omega)$, is a discontinuity point generated by the fixed point 0, and thus the interval $[0, \omega^{-n}]$, respectively $[-\omega^{-n}, 0]$, contains $F_n + O(1)$ intervals of length $1/2$ with $|Z_n| = 2$, and thus

$$\alpha_n = \frac{F_n + O(1)}{2\omega^{-n}} \rightarrow \frac{1}{2\sqrt{5}}. \quad (5.27)$$

Hence the result. \square

5.3.2 $n \equiv 1 \pmod{3}$

When $n \equiv 1 \pmod{3}$ the discontinuity location sets are given by (5.1) and (5.3). We have

$$\begin{aligned} a_i^1 + b_i^1 \omega - (a_i^0 + b_i^0 \omega) &= \lceil i/\omega \rceil + i\omega - (\lceil (i-1/2)/\omega + 1/2 \rceil - 1/2 + (i-1/2)\omega) \\ &= \omega^{-1}/2 + \lceil i/\omega \rceil - \lceil i/\omega - \omega/2 \rceil > 0 \end{aligned} \quad (5.28)$$

and

$$\begin{aligned} a_{i+1}^0 + b_{i+1}^0 \omega - (a_i^1 + b_i^1 \omega) &= \lceil (i+1/2)/\omega + 1/2 \rceil - 1/2 + (i+1/2)\omega - (\lceil i/\omega \rceil + i\omega) \\ &= \lceil i/\omega + \omega/2 \rceil - \lceil i/\omega \rceil + \omega^{-1}/2 > 0, \end{aligned} \quad (5.29)$$

so that the two sets of discontinuities alternate in a strict manner. (Note that the distance between discontinuities may be $\omega^{-1}/2$ or $1 + \omega^{-1}/2$. See the second row of Figure 2.)

When, in addition, n is odd, i.e., $n \equiv 1 \pmod{6}$, each discontinuity from $\{1/2, -\omega/2\}$ has size $-X_0^0 = 2$, whilst each discontinuity from $\{0\}$ has size $-X_0^1 = -2$. We further have $Z_n(1) = -1$ in this case, and the first discontinuity after 1 is $3/2 + \omega/2$ at which Z_n increases to $+1$. Thus Z_n oscillates between values ± 1 with discontinuity locations as calculated above.

The situation for the subcase in which n is even (i.e., $n \equiv 4 \pmod{6}$) follows similarly, or more simply from the fact that $Z_{n+3} = -Z_n$.

It is a simple consequence of the fact that $Z_n = \pm 1$ that the autocorrelation function value $K_{P_y}(F_n)$ in equation (1.28) equals $\cos \kappa$.

5.3.3 $n \equiv 2 \pmod{3}$

When $n \equiv 2 \pmod{3}$ the discontinuity location sets are given by (5.1) and (5.4). We have

$$\begin{aligned} a_i^1 + b_i^1 \omega - (a_i^0 + b_i^0 \omega) &= \lceil i/\omega \rceil + i\omega - (\lceil (i-1/2)/\omega \rceil + (i-1/2)\omega) \\ &= \omega/2 + \lceil i/\omega \rceil - \lceil i/\omega - \omega^{-1}/2 \rceil > 0 \end{aligned} \quad (5.30)$$

and

$$\begin{aligned} a_{i+1}^0 + b_{i+1}^0 \omega - (a_i^1 + b_i^1 \omega) &= \lceil (i+1/2)/\omega \rceil + (i+1/2)\omega - (\lceil i/\omega \rceil + i\omega) \\ &= \lceil i/\omega + \omega^{-1}/2 \rceil - \lceil i/\omega \rceil + \omega/2 > 0. \end{aligned} \quad (5.31)$$

(Note now that the distance between discontinuities may be $\omega/2$ or $1 + \omega/2$. See the third row of Figure 2.) For both even and odd cases here, as in the previous case, we deduce that the two sets of discontinuities alternate in a strict manner, and, since $Z_{6\ell+2}(1) = -Z_{6\ell+5}(1) = 1$, that for all $n \equiv 2 \pmod 3$, Z_n oscillates between values ± 1 with discontinuity locations as calculated above.

Again, we immediately deduce that in this case (1.28) is simply $K_{P_y}(F_n) = \cos \kappa$.

6 Conclusion

We have verified and generalised rigorously the numerical results in [5] concerning the asymptotic self-similarity of the autocorrelation function in a quasiperiodically forced two-level quantum system. As in studies of the self-similar fluctuations of the localized eigenstates of the Harper equation [7], the birth of a strange nonchaotic attractor [9], and of the autocorrelation function of a strange nonchaotic attractor [4], this self-similarity is explained by means of a functional recurrence, the key to the understanding of which is the dynamics of a simple piecewise-linear map of the interval [11, 12, 13].

To accomplish this task, a description of the piecewise-constant periodic orbits of the additive recurrence (1.1) has been completed. Moreover a necessary and sufficient criterion for such orbits to be spatially bounded has been derived. As a consequence, provided the locations of its discontinuities are carefully chosen, a piecewise-constant modulation function gives rise to asymptotic self-similarity of the autocorrelation function. It seems likely that, as in the case of the generalised Harper equation with next-nearest-neighbour interactions [7, 13], there will be an underlying strange set on which these orbits lie. If so, an understanding of the form of the autocorrelation function in the presence of a general piecewise-constant modulation function will follow.

Our work has been in the case of golden mean frequency only, but numerical work in [5] for this problem indicates that the singular continuous spectrum is present for a wider class of irrationals. For the parallel problem of the autocorrelation function in strange nonchaotic attractors, there is evidence in [14] that, at least for certain quadratic irrational frequencies, the autocorrelation function displays self-similarity of the type studied here. Periodic orbits of a generalisation of the functional recurrence (1.1) provide an explanation of this phenomenon in both settings, and will be the subject of a forthcoming paper.

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