On integrability of (2+1)-dimensional quasilinear systems

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Abstract

A (2+1)-dimensional quasilinear system is said to be 'integrable' if it can be decoupled in infinitely many ways into a pair of compatible n-component one-dimensional systems in Riemann invariants. Exact solutions described by these reductions, known as nonlinear interactions of planar simple waves, can be viewed as natural dispersionless analogs of n-gap solutions. It is demonstrated that the requirement of the existence of 'sufficiently many' n-component reductions provides the effective classification criterion. As an example of this approach we classify integrable (2+1)-dimensional systems of conservation laws possessing a convex quadratic entropy.

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1 Introduction

In this paper we address the problem of integrability of $(2+1)$-dimensional quasilinear systems

$$\mathbf{u}_t + A(\mathbf{u})\mathbf{u}_x + B(\mathbf{u})\mathbf{u}_y = 0 \tag{1}$$

where t, x, y are independent variables, $\mathbf{u}$ is an m-component column vector and $A(\mathbf{u}), B(\mathbf{u})$ are $m \times m$ matrices. Systems of this type describe many physical phenomena. In particular, important examples occur in gas dynamics, shallow water theory, combustion theory, nonlinear elasticity, magneto-fluid dynamics, etc. [28]. Although many interesting systems of the form (1) arise as dispersionless limits of multidimensional soliton equations [39] or within the R-matrix approach [3] and, therefore, should be regarded as integrable, no intrinsic definition of the integrability for multidimensional quasilinear systems has been proposed until recently. In particular, the standard symmetry approach [32, 33], which proves to be extremely effective in the case of higher order evolution equations and systems, does not seem to work in this context.

The key element of our construction are exact solutions of the system (1) of the form $\mathbf{u} = \mathbf{u}(R^1, ..., R^n)$ where the *Riemann invariants* $R^1, ..., R^n$ solve a pair of commuting diagonal systems

$$R^i_t = \lambda^i(R)\ R^i_x, \qquad R^i_y = \mu^i(R)\ R^i_x; \tag{2}$$

notice that the number of Riemann invariants is allowed to be arbitrary! Thus, the original $2+1$-dimensional system (1) is decoupled into a pair of diagonal $1+1$-dimensional systems. Solutions of this type, known as nonlinear interactions of n planar simple waves, were extensively investigated in gas dynamics and magnetohydrodynamics in a series of publications [5, 6, 7, 36, 37, 9, 20]. Later, they appeared in the context of the dispersionless KP hierarchy [15, 16, 17, 18, 21, 31, 29, 30] and the theory of integrable hydrodynamic-type chains [34, 35]. We will call a multidimensional system *integrable* if it possesses 'sufficiently many' n-component reductions of the form (2) for arbitrary n (the precise definition follows). The corresponding nonlinear n-wave interactions can be viewed as dispersionless analogs of 'n-gap' solutions.

We recall, see [38], that the requirement of the commutativity of the flows (2) is equivalent to the following restrictions on their characteristic speeds:

$$\frac{\partial_j \lambda^i}{\lambda^j - \lambda^i} = \frac{\partial_j \mu^i}{\mu^j - \mu^i}, \quad i \neq j, \quad \partial_j = \partial/\partial_{R^j}; \tag{3}$$

(no summation!). Once these conditions are met, the general solution of (2) is given by the implicit 'generalized hodograph' formula [38]

$$v^i(R) = x + \lambda^i(R)\ t + \mu^i(R)\ y, \quad i = 1, ..., n, \tag{4}$$

where $v^i(R)$ are characteristic speeds of the general flow commuting with (2), that is, the general solution of the linear system

$$\frac{\partial_j v^i}{v^j - v^i} = \frac{\partial_j \lambda^i}{\lambda^j - \lambda^i} = \frac{\partial_j \mu^i}{\mu^j - \mu^i}, \quad i \neq j. \tag{5}$$

Substituting $\mathbf{u}(R^1, ..., R^n)$ into (1) and using (2), one readily arrives at the equations

$$(A + \mu^i B + \lambda^i I_m)\ \partial_i \mathbf{u} = 0, \qquad i = 1, ..., n, \tag{6}$$

implying that both λ^i and μ^i satisfy the dispersion relation

$$\det(A + \mu B + \lambda I_m) = 0. \tag{7}$$

Thus, the construction of nonlinear interactions of n planar simple waves consists of two steps:
(1) Reduce the initial system (1) to a pair of commuting flows (2) by solving the equations (3), (6) for $\mathbf{u}(R)$, $\lambda^i(R)$, $\mu^i(R)$ as functions of the Riemann invariants $R^1, ..., R^n$. For $n \geq 3$ these equations are highly overdetermined and do not possess solutions in general. However, once a particular reduction of the form (2) is constructed, the second step is fairly straightforward:
(2) Solve the linear system (5) for $v^i(R)$ and determine $R^1, ..., R^n$ as functions of t, x, y from the implicit hodograph formula (4).

Remark 1. For $n = 1$ we have $\mathbf{u} = \mathbf{u}(R)$, where the scalar variable $R = R^1$ solves a pair of first order PDEs

$$R_t = \lambda(R)\ R_x, \qquad R_y = \mu(R)\ R_x$$

which, in one-component situation, are automatically commuting. We recall that in the scalar case the hodograph formula (4) takes the form

$$f(R) = x + \lambda(R)t + \mu(R)y \tag{8}$$

where $f(R)$ is arbitrary. This formula shows that, in coordinates t, x, y, the surfaces $R = const$ are planes. Hence, $\mathbf{u} = \mathbf{u}(R)$ is constant along a one-parameter family of planes. Solutions of this type, known as planar simple waves, exist for all multidimensional quasilinear systems and, therefore, cannot detect the integrability.

Similarly, for $n = 2$, we have $\mathbf{u} = \mathbf{u}(R^1, R^2)$ where R^1, R^2 satisfy the system (2) whose general solution is given by the generalized hodograph formula

$$v^1(R) = x + \lambda^1(R)t + \mu^1(R)y, \quad v^2(R) = x + \lambda^2(R)t + \mu^2(R)y, \quad R = (R^1, R^2). \tag{9}$$

Setting $R^1 = const$, $R^2 = const$, one obtains a two-parameter family of lines (or, in the geometric language, a line congruence) in the 3-space of independent variables t, x, y. Therefore, the solution $\mathbf{u} = \mathbf{u}(R^1, R^2)$ is constant along the lines of a two-parameter family. The requirement of the existence of solutions of this type, known as nonlinear interactions of two planar simple waves, is also not very restrictive. For instance, for $m = 2$, any system (1) possesses infinitely many 2-component reductions of the form (2) parametrized by two arbitrary functions of a single argument (see examples below).

On the contrary, the requirement of the existence of nontrivial 3-component reductions is already sufficiently restrictive and, in particular, implies the existence of n-component reductions for arbitrary n. This phenomenon is similar to the well-known three-soliton condition in the Hirota bilinear approach [23, 24, 25] (recall that two-soliton solutions exist for arbitrary PDEs transformable to Hirota's bilinear form and, therefore, cannot detect the integrability), and the condition of three-dimensional consistency in the classification of discrete integrable systems on quad-graphs [1]. One can show, by analyzing equations (3), (6), that the maximum number of n-component reductions the system (1) may possess is parametrized, modulo changes of variables $R^i \to f^i(R^i)$, by n arbitrary functions of a single argument (notice that this number does not depend on m). Therefore, we propose the following

Definition. *A $(2+1)$-dimensional quasilinear system is said to be integrable if it possesses n-component reductions of the form (2) parametrized by n arbitrary functions of a single argument.*

Remark 2. This definition of integrability implies that the matrices A and B in (1) are *not* commuting (if we want the system (1) to be nonlinear and coupled). Therefore, our definition is by no means exhaustive: there exist examples of multidimensional quasilinear systems which are integrable (in the inverse scattering sense) although do not possess n-component reductions. In all these examples the corresponding matrices A and B commute: $[A, B] = 0$.

Remark 3. The anzatz somewhat similar to the key element of our construction was used for finding formal solutions of nonlinear evolution equations in the form of exponential series, see [22] and references therein, where solutions were sought as functions of real exponentials solving the linearized equation. The possibility to distinguish between integrable and nonintegrable equations in the framework of this approach was pointed out in [27].

In section 2 we discuss explicit examples which demonstrate that the above definition is indeed very effective for detecting the integrability.

In section 3 we classify integrable systems of conservation laws in Godunov's form [19],

$$v_t + (f_v)_x + (g_v)_y = 0, \quad w_t + (f_w)_x + (g_w)_y = 0;$$

notice that systems of this type automatically possess one extra convex quadratic entropy

$$\frac{1}{2}(v^2 + w^2)_t + (vf_v + wf_w - f)_x + (vg_v + wg_w - g)_y = 0.$$

The integrability conditions constitute a complicated overdetermined system (41) of fourth order PDEs for the potentials f and g. The analysis of this system leads to two possibilities.

Quadratic case. There exists a linear combination of f and g which is quadratic in v, w. Without any loss of generality one can assume that g is quadratic, say, $g = (v^2 - w^2)/2$ (one has a freedom of Euclidean isometries of the (v, w)-plane to bring g to the canonical form). In this case our equations take the form

$$v_t + (f_v)_x + v_y = 0, \quad w_t + (f_w)_x - w_y = 0; \tag{10}$$

the corresponding integrability conditions reduce to the system (44) of fourth order PDEs for the potential f which can be solved explicitly (see Sect. 3). Notice that in the new independent variables $\xi = -(t+y)/2,\ \eta = -(t-y)/2$ the system (10) takes the form

$$(f_v)_x = v_\xi, \quad (f_w)_x = w_\eta,$$

which is manifestly Hamiltonian in the new variables $V = f_v,\ W = f_w$:

$$V_x = (F_V)_\xi, \quad W_x = (F_W)_\eta.$$

Here F is the Legendre transform of f, $F = vf_v + wf_w - f$. Thus, we obtain a complete description of the class of integrable two-component (2+1)-dimensional Hamiltonian systems of hydrodynamic type. An independent treatment of the Hamiltonian case is given in Sect. 4 where, in particular, the integrability conditions in terms of the Hamiltonian density F are derived (formulae (56)).

Harmonic case. Here both f and g are harmonic functions. Further analysis shows that there exists a unique system of this type, with $f = Re(z\ln z - z),\ g = Im(z\ln z - z),\ z = v + iw$. The corresponding equations are

$$v_t + \frac{vv_x + ww_x}{v^2+w^2} + \frac{vw_y - wv_y}{v^2+w^2} = 0, \quad w_t + \frac{wv_x - vw_x}{v^2+w^2} + \frac{vv_y + ww_y}{v^2+w^2} = 0,$$

or, in conservative form,

$$v_t + (\ln\sqrt{v^2+w^2})_x + (arctg\frac{w}{v})_y = 0, \quad w_t - (arctg\frac{w}{v})_x + (\ln\sqrt{v^2+w^2})_y = 0.$$

Remarkably, this two-component system is a disguised form of the nonlinear wave equation. To see this we change to polar coordinates $v = r\cos\theta,\ w = r\sin\theta$:

$$\theta_y = -\frac{r_x}{r} + r\sin\theta\ \theta_t - \cos\theta\ r_t, \quad \theta_x = \frac{r_y}{r} + r\cos\theta\ \theta_t + \sin\theta\ r_t; \tag{11}$$

these equations lead, upon cross-differentiation, to the nonlinear wave equation $(r^2)_{tt} = \triangle\ln(r^2)$, known also as the Boyer-Finley equation [4]. Solving equations (11) for r_x and r_y and cross-differentiating, one obtains another second order equation $(r^2\theta_t)_t = \triangle\theta$ which can be viewed as a linear wave equation for θ. It is not clear whether the wave equation with a general nonlinearity can be written as a two-component first order system; however, as demonstrated in [26], it always possesses a simple three-component representation. Hydrodynamic reductions of the Boyer-Finley equation were recently studied in [13], see also Sect. 3.

2 Examples

In this section we list some of the known examples of $(2+1)$-dimensional systems of hydrodynamic type which are integrable in the above sense. All these examples turn out to be conservative and, moreover, possess exactly one 'extra' conservation law which is the necessary ingredient of the theory of weak solutions. This makes the systems below a possible venue for developing and testing mathematical theory (existence, uniqueness, weak solutions, etc.) of multidimensional conservation laws which, currently, remain terra incognita [8].

Example 1. The dispersionless KP equation,

$$(u_t - uu_x)_x = u_{yy},$$

plays an important role in nonlinear acoustics and differential geometry. Introducing the potential $u = \varphi_x$ we obtain the equation $\varphi_{xt} - \varphi_x\varphi_{xx} = \varphi_{yy}$ which takes the quasilinear form

$$v_y = w_x, \qquad w_y = v_t - vv_x \tag{12}$$

in the variables $v = \varphi_x,\ w = \varphi_y$. Looking for reductions $v = v(R^1, ..., R^n),\ w = w(R^1, ..., R^n)$, where the Riemann invariants R^i satisfy (2), one readily obtains

$$\partial_i w = \mu^i \partial_i v, \qquad \lambda^i = v + (\mu^i)^2. \tag{13}$$

The compatibility condition $\partial_i\partial_j w = \partial_j\partial_i w$ implies

$$\partial_i\partial_j v = \frac{\partial_j \mu^i}{\mu^j - \mu^i}\partial_i v + \frac{\partial_i \mu^j}{\mu^i - \mu^j}\partial_j v, \tag{14}$$

while the commutativity condition (3) results in

$$\partial_j \mu^i = \frac{\partial_j v}{\mu^j - \mu^i}. \tag{15}$$

The substitution of (15) into (14) implies the Gibbons-Tsarev system for $v(R)$ and $\mu^i(R)$,

$$\partial_j \mu^i = \frac{\partial_j v}{\mu^j - \mu^i}, \quad \partial_i\partial_j v = 2\frac{\partial_i v \partial_j v}{(\mu^j - \mu^i)^2}, \tag{16}$$

$i \neq j$, which was first derived in [17, 18] in the theory of hydrodynamic reductions of Benney's moment equations, see also [21, 29, 30] for further discussion. For any solution μ^i, v of the system (16) one can reconstruct λ^i and w by virtue of (13). In the two-component case this system takes the form

$$\partial_2 \mu^1 = \frac{\partial_2 v}{\mu^2 - \mu^1}, \quad \partial_1 \mu^2 = \frac{\partial_1 v}{\mu^1 - \mu^2}, \quad \partial_1\partial_2 v = 2\frac{\partial_1 v \partial_2 v}{(\mu^2 - \mu^1)^2}. \tag{17}$$

The general solution of this system is parametrized by four arbitrary functions of a single argument, indeed, one can arbitrarily prescribe the Goursat data $v(R^1),\ \mu^1(R^1)$ and $v(R^2),\ \mu^2(R^2)$ on the characteristics $R^2 = 0$ and $R^1 = 0$, respectively. Moreover, the system (17) is invariant under the reparametrizations $R^1 \to f^1(R^1),\ R^2 \to f^2(R^2)$ where f^1, f^2 are arbitrary functions of their arguments. Since reparametrizations of this type do not effect the corresponding solutions, one concludes that two-component reductions are parametrized by two arbitrary functions of a single argument. A remarkable feature of the system (17) is its multidimensional compatibility, that is, the compatibility of the system (16) which is obtained by 'gluing together' several identical copies of the system

(17) for each pair of Riemann invariants R^i, R^j. Indeed, calculating $\partial_k(\partial_j\mu^i)$ by virtue of (16) one obtains

$$\partial_k(\partial_j\mu^i) = \frac{\partial_j v\partial_k v(\mu^j+\mu^k-2\mu^i)}{(\mu^j-\mu^k)^2(\mu^j-\mu^i)(\mu^k-\mu^i)},$$

which is manifestly symmetric in j, k. Therefore, the first compatibility condition $\partial_k(\partial_j\mu^i) = \partial_j(\partial_k\mu^i)$ is satisfied. Similarly, the computation of $\partial_k(\partial_i\partial_j v)$ results in

$$\partial_k(\partial_i\partial_j v) = 4\frac{\partial_i v\partial_j v\partial_k v((\mu^i)^2+(\mu^j)^2+(\mu^k)^2-\mu^i\mu^j-\mu^i\mu^k-\mu^j\mu^k)}{(\mu^i-\mu^j)^2(\mu^i-\mu^k)^2(\mu^j-\mu^k)^2},$$

which is totally symmetric in i, j, k. Therefore, the second compatibility condition $\partial_k(\partial_i\partial_j v) = \partial_j(\partial_i\partial_k v)$ is also satisfied. The general solution of the system (16) depends, modulo trivial symmetries $R^i \to f^i(R^i)$, on n arbitrary functions of a single argument.

We just mention that the system (12) possesses exactly three conservation laws of hydrodynamic type:

$$v_y = w_x,$$

$$w_y = v_t - (v^2/2)_x,$$

$$(vw)_y = (v^2/2)_t + (w^2/2 - v^3/3)_x.$$

Example 2. The Boyer-Finley equation,

$$u_{xy} = (e^u)_{tt},$$

(notice that the signature here is different from the one discussed in the introduction, which makes the analysis much easier) is descriptive of self-dual Einstein spaces with a Killing vector [4]. Introducing the potential $u = \varphi_t$, one obtains the equation $\varphi_{xy} = (e^{\varphi_t})_t$ which takes the form

$$v_t = w_x/w, \qquad w_t = v_y \tag{18}$$

in the new variables $v = \varphi_x$, $w = e^{\varphi_t}$. Looking for reductions in the form $v = v(R^1, ..., R^n)$, $w = w(R^1, ..., R^n)$, where the Riemann invariants R^i satisfy (2), one readily obtains

$$\partial_i w = w\lambda^i\partial_i v, \quad \mu^i = w(\lambda^i)^2. \tag{19}$$

The compatibility condition $\partial_i\partial_j w = \partial_j\partial_i w$ implies

$$\partial_i\partial_j v = \frac{\partial_j\lambda^i}{\lambda^j-\lambda^i}\partial_i v + \frac{\partial_i\lambda^j}{\lambda^i-\lambda^j}\partial_j v, \tag{20}$$

while the commutativity condition (3) results in

$$\partial_j\lambda^i = \frac{(\lambda^i)^2\lambda^j}{\lambda^j-\lambda^i}\partial_j v. \tag{21}$$

The substitution of (21) into (20) implies the system for $v(R)$ and $\lambda^i(R)$,

$$\partial_j\lambda^i = \frac{(\lambda^i)^2\lambda^j}{\lambda^j-\lambda^i}\partial_j v, \quad \partial_i\partial_j v = \frac{\lambda^i\lambda^j(\lambda^i+\lambda^j)}{(\lambda^j-\lambda^i)^2}\partial_i v\partial_j v, \tag{22}$$

which, in a somewhat different form, was first discussed in [13]. For any solution λ^i, v of the system (22) one can reconstruct μ^i, w by virtue of (19). The system (22) is compatible, with the general solution depending on n arbitrary functions of a single argument (modulo trivial symmetries $R^i \to f^i(R^i)$).

The system (18) possesses three conservation laws of hydrodynamic type:

$$w_t = v_y,$$

$$v_t = (\ln w)_x,$$

$$(vw)_t = w_x + (v^2/2)_y.$$

Example 3. An interesting integrable modification of the Boyer-Finley equation is the PDE

$$u_{xy} = (\partial_t^2 - c\partial_x^2)(e^u),$$

$c = const$ (see [34, 35] for further examples of this type). Introducing the potential $e^u = \varphi_x$, one obtains the equation $\varphi_{tt} - c\varphi_{xx} = (\ln \varphi_x)_y$ which takes the form

$$v_t = cw_x + w_y/w, \qquad w_t = v_x \tag{23}$$

in the new variables $v = \varphi_t,\ w = \varphi_x$. Looking for reductions in the form $v = v(R^1, ..., R^n)$, $w = w(R^1, ..., R^n)$, where the Riemann invariants R^i satisfy (2), one obtains

$$\partial_i v = \lambda^i \partial_i w, \quad \mu^i = w((\lambda^i)^2 - c). \tag{24}$$

The compatibility condition $\partial_i\partial_j v = \partial_j\partial_i v$ implies

$$\partial_i\partial_j w = \frac{\partial_j \lambda^i}{\lambda^j - \lambda^i}\partial_i w + \frac{\partial_i \lambda^j}{\lambda^i - \lambda^j}\partial_j w, \tag{25}$$

while the commutativity condition (3) results in

$$\partial_j \lambda^i = \frac{(\lambda^i)^2 - c}{\lambda^j - \lambda^i}\frac{\partial_j w}{w}. \tag{26}$$

The substitution of (26) into (25) implies the following system for $w(R)$ and $\lambda^i(R)$,

$$\partial_j \lambda^i = \frac{(\lambda^i)^2 - c}{\lambda^j - \lambda^i}\frac{\partial_j w}{w}, \quad \partial_i\partial_j w = \frac{(\lambda^i)^2 + (\lambda^j)^2 - 2c}{w(\lambda^j - \lambda^i)^2}\partial_i w \partial_j w. \tag{27}$$

For any solution λ^i, w of the system (27) one can reconstruct μ^i, v by virtue of (24). The system (27) is compatible, with the general solution depending on n arbitrary functions of a single argument (modulo symmetries $R^i \to f^i(R^i)$).

The system (23) possesses three conservation laws of hydrodynamic type:

$$v_t = cw_x + (\ln w)_y,$$

$$w_t = v_x,$$

$$(vw)_t = (v^2/2 + w^2/(2c))_x + w_y.$$

Notice that when $c > 0$ the x-flux of the third conservation law is a convex function of the previous two x-fluxes (convex entropy). This example can be of particular interest for the general theory of multidimensional strictly hyperbolic conservation laws.

Example 4. We are going to demonstrate that the method of hydrodynamic reductions is in fact the effective classification criterion. As an illustration of our approach we will classify integrable nonlinear wave equations of the form

$$u_{xy} = (f(u))_{tt};$$

it will follow that $f(u) = e^u$ is the only nontrivial possibility. Introducing the potential $u = \varphi_t$, one obtains the equation $\varphi_{xy} = (f(\varphi_t))_t$ which takes the form

$$v_t = w_x, \qquad f'(w)w_t = v_y \tag{28}$$

in the new variables $v = \varphi_x,\ w = \varphi_t$. Looking for reductions in the form $v = v(R^1, ..., R^n)$, $w = w(R^1, ..., R^n)$ where the Riemann invariants R^i satisfy (2) one readily obtains

$$\partial_i v = \partial_i w/\lambda^i, \quad \mu^i = f'(w)(\lambda^i)^2. \tag{29}$$

The compatibility condition $\partial_i\partial_j v = \partial_j\partial_i v$ implies

$$\partial_i\partial_j w = \frac{\lambda^j\partial_j\lambda^i}{(\lambda^j - \lambda^i)\lambda^i}\partial_i w + \frac{\lambda^i\partial_i\lambda^j}{(\lambda^i - \lambda^j)\lambda^j}\partial_j w, \tag{30}$$

while the commutativity condition (3) results in

$$\partial_j\lambda^i = \frac{f''}{f'}\frac{(\lambda^i)^2}{\lambda^j - \lambda^i}\partial_j w. \tag{31}$$

The substitution of (21) into (20) implies the system for $w(R)$ and $\lambda^i(R)$,

$$\partial_j\lambda^i = \frac{f''}{f'}\frac{(\lambda^i)^2}{\lambda^j - \lambda^i}\partial_j w, \quad \partial_i\partial_j w = 2\frac{f''}{f'}\frac{\lambda^i\lambda^j}{(\lambda^j - \lambda^i)^2}\partial_i w\partial_j w. \tag{32}$$

A direct computation of $\partial_k(\partial_j\lambda^i)$ implies

$$\partial_k(\partial_j\lambda^i) = \left(\frac{f''}{f'}\right)' \frac{(\lambda^i)^2}{\lambda^j-\lambda^i}\partial_j w\partial_k w +$$

$$\left(\frac{f''}{f'}\right)^2 \frac{(\lambda^i)^2(\lambda^i((\lambda^j)^2+(\lambda^k)^2)+\lambda^j\lambda^k(\lambda^j+\lambda^k)-4\lambda^i\lambda^j\lambda^k)}{(\lambda^j-\lambda^k)^2(\lambda^j-\lambda^i)(\lambda^k-\lambda^i)}\partial_j w\partial_k w.$$

The compatibility condition $\partial_k(\partial_j\lambda^i) = \partial_j(\partial_k\lambda^i)$ is equivalent to the requirement that the above expression is symmetric in j, k. Since the second term is manifestly symmetric, one has to require

$$\left(\frac{f''}{f'}\right)' = 0$$

to ensure the compatibility. This implies $f = ae^{bu} + c$, which is essentially the case of Example 2.

Example 5. There exist remarkable examples with a fairly simple structure of hydrodynamic reductions. Let us consider the system [34]

$$v_t = w_x, \qquad w_t = wv_x - vw_x + v_y. \tag{33}$$

Looking for reductions in the form $v = v(R^1, ..., R^n)$, $w = w(R^1, ..., R^n)$ where the Riemann invariants R^i satisfy (2) one obtains

$$\partial_i w = \lambda^i \partial_i v, \quad \mu^i = (\lambda^i)^2 + v\lambda^i - w,$$

so that

$$\partial_i\partial_j v = \frac{\partial_j \lambda^i}{\lambda^j - \lambda^i}\partial_i v + \frac{\partial_i \lambda^j}{\lambda^i - \lambda^j}\partial_j v.$$

The commutativity condition (3) implies

$$\partial_j \lambda^i = -\partial_j v, \quad \partial_i\partial_j v = 0.$$

Hence,

$$v = \sum_k f^k(R^k), \quad \lambda^i = \varphi^i(R^i) - \sum_k f^k(R^k),$$

where $f^i(R^i)$ and $\varphi^i(R^i)$ are arbitrary functions of a single argument. Further properties of these reductions were investigated in [34]. We just mention that the system (33) possesses three conservation laws of hydrodynamic type:

$$v_t = w_x,$$

$$(w + v^2)_t = (vw)_x + v_y,$$

$$(2vw + v^3)_t = (v^2 w + w^2)_x + (v^2)_y.$$

Example 6. The method of hydrodynamic reductions carries over to multicomponent situation in a straightforward way. Here we give details of calculations for the 3-component system first proposed in [39], see also [21]:

$$a_t + (av)_x = 0, \quad v_t + vv_x + w_x = 0, \quad w_y + a_x = 0. \tag{34}$$

Looking for reductions in the form $a = a(R^1, ..., R^n)$, $b = b(R^1, ..., R^n)$, $w = w(R^1, ..., R^n)$ where the Riemann invariants R^i satisfy (2) one obtains the relations

$$\partial_i w = -(\lambda^i + v)\partial_i v, \quad \partial_i a = \mu^i(\lambda^i + v)\partial_i v, \quad \mu^i = -\frac{a}{(\lambda^i + v)^2}. \tag{35}$$

The compatibility condition $\partial_i\partial_j w = \partial_j\partial_i w$ implies

$$\partial_i\partial_j v = \frac{\partial_j \lambda^i}{\lambda^j - \lambda^i}\partial_i v + \frac{\partial_i \lambda^j}{\lambda^i - \lambda^j}\partial_j v, \tag{36}$$

while the commutativity condition (3) results in

$$\partial_j \lambda^i = \frac{\lambda^j + v}{\lambda^i - \lambda^j} \partial_j v. \tag{37}$$

The substitution of (37) into (36) implies the system for $v(R)$ and $\lambda^i(R)$,

$$\partial_j \lambda^i = \frac{\lambda^j + v}{\lambda^i - \lambda^j} \partial_j v, \quad \partial_i \partial_j v = -\frac{\lambda^i + \lambda^j + 2v}{(\lambda^j - \lambda^i)^2} \partial_i v \partial_j v. \tag{38}$$

One can verify that the remaining compatibility conditions $\partial_i \partial_j a = \partial_j \partial_i a$ are satisfied identically. For any solution λ^i, v of the system (38) one can reconstruct w, a, μ^i by virtue of (35). The system (38) is compatible, with the general solution depending on n arbitrary functions of a single argument (modulo symmetries $R^i \to f^i(R^i)$).

Notice that the system (34) possesses four conservation laws of hydrodynamic type:

$$a_t + (av)_x = 0,$$

$$v_t + (v^2/2 + w)_x = 0,$$

$$w_y + a_x = 0,$$

$$(aw + av^2)_x + (w^2/2)_y + (av)_t = 0.$$

Equations (34) can be generalized as follows:

$$a_t + (av)_x = 0, \quad v_t + vv_x + w_x = 0, \quad w_y + p(a)_x = 0.$$

(so that one obtains isentropic gas dynamics in the limit $x = y$). It can be shown that this system passes the integrability test if and only if $p'' = 0$, which leads to (34).

3 Classification of integrable systems of conservation laws with a convex quadratic entropy

In this section we discuss systems of conservation laws in Godunov's form [19],

$$v_t + (f_v)_x + (g_v)_y = 0, \quad w_t + (f_w)_x + (g_w)_y = 0; \tag{39}$$

here $f(v, w)$ and $g(v, w)$ are given potentials. Systems of this type automatically possess one extra convex quadratic entropy

$$\frac{1}{2}(v^2 + w^2)_t + (vf_v + wf_w - f)_x + (vg_v + wg_w - g)_y = 0.$$

Equations (39) can be written in the matrix form (1) with

$$\mathbf{u} = \begin{pmatrix} v \\ w \end{pmatrix}, \quad A = \begin{pmatrix} f_{vv} & f_{vw} \\ f_{vw} & f_{ww} \end{pmatrix}, \quad B = \begin{pmatrix} g_{vv} & g_{vw} \\ g_{vw} & g_{ww} \end{pmatrix};$$

in what follows we assume that the commutator

$$[A, B] = \begin{pmatrix} 0 & s \\ -s & 0 \end{pmatrix}$$

is nonzero, that is, $s = g_{vw}(f_{vv} - f_{ww}) - f_{vw}(g_{vv} - g_{ww}) \neq 0$; otherwise, the system possesses no nontrivial n-component hydrodynamic reductions. The integrability conditions lead to an overdetermined system of fourth order PDEs for f and g. A careful analysis of this system shows that there exist two essentially different possibilities:
– g is quadratic in v, w (quadratic case) or
– both f and g are harmonic functions (harmonic case).
Remarkably, in both cases the equations for f and g can be solved in a closed form. We hope that these examples would provide a good venue for developing and testing the general theory of multidimensional conservation laws (breakdown of solutions, weak solutions, etc).

The integrability conditions can be derived in the standard way. Looking for reductions of the system (25) in the form $v = v(R^1, ..., R^n)$, $w = w(R^1, ..., R^n)$ where the Riemann invariants satisfy equations (2), and substituting into (25), one arrives at

$$(\lambda^i + f_{vv} + \mu^i g_{vv})\partial_i v + (f_{vw} + \mu^i g_{vw})\partial_i w = 0, \quad (f_{vw} + \mu^i g_{vw})\partial_i v + (\lambda^i + f_{ww} + \mu^i g_{ww})\partial_i w = 0,$$

(no summation!) so that λ^i and μ^i satisfy the dispersion relation

$$(\lambda^i + f_{vv} + \mu^i g_{vv})(\lambda^i + f_{ww} + \mu^i g_{ww}) = (f_{vw} + \mu^i g_{vw})^2.$$

Setting $\partial_i v = \varphi^i \partial_i w$ one obtains the following expressions for λ^i and μ^i in terms of φ^i,

$$\lambda^i = \frac{(f_{vv}g_{vw} - f_{vw}g_{vv})(\varphi^i)^2 + (f_{vv}g_{ww} - f_{ww}g_{vv})\varphi^i + (f_{vw}g_{ww} - f_{ww}g_{vw})}{g_{vw}(1 - (\varphi^i)^2) + (g_{vv} - g_{ww})\varphi^i},$$

$$\mu^i = -\frac{f_{vw}(1 - (\varphi^i)^2) + (f_{vv} - f_{ww})\varphi^i}{g_{vw}(1 - (\varphi^i)^2) + (g_{vv} - g_{ww})\varphi^i},$$

which define a rational parametrization of the dispersion relation. The compatibility conditions of the equations $\partial_i v = \varphi^i \partial_i w$ imply

$$\partial_i \partial_j w = \frac{\partial_j \varphi^i}{\varphi^j - \varphi^i}\partial_i w + \frac{\partial_i \varphi^j}{\varphi^i - \varphi^j}\partial_j w, \tag{40}$$

while the commutativity conditions (3) lead to the expressions for $\partial_j \varphi^i$ in the form $\partial_j \varphi^i = (...)\partial_j w$. Here dots denote a rational expression in φ^i, φ^j whose coefficients are functions of the second and third derivatives of f and g. We do not write them down explicitly due to their complexity. To manipulate with these expressions we used symbolic computations. One can see that the compatibility condition $\partial_k \partial_j \varphi^i - \partial_j \partial_k \varphi^i = 0$ is of the form $P \partial_j w \partial_k w = 0$, where P is a complicated rational expression in $\varphi^i, \varphi^j, \varphi^k$ whose coefficients depend on partial derivatives of f and g up to fourth order. Requiring that P vanishes identically we obtain the overdetermined system of fourth order PDEs for f and g. This system yields the following expressions for the fourth derivatives of f:

$$
\begin{array}{rcl}
sf_{vvvv} & = & f_{vvv}[2(g_{ww}-g_{vv})f_{vvw}+3(f_{vv}-f_{ww})g_{vvw}+ \\
& & 2g_{vw}(f_{vvv}+2f_{vww})-2f_{vw}g_{vww}]+ \\
& & g_{vvv}[(f_{ww}-f_{vv})f_{vvw}-2f_{vw}(f_{vww}+f_{vvv})]+ \\
& & 6f_{vvw}(f_{vw}g_{vvw}-g_{vw}f_{vvw}), \\
-sf_{vvvw} & = & f_{vvw}(f_{vw}g_{vvv}+3g_{vw}f_{vww}-3f_{vw}g_{vww})+f_{vvv}[2(f_{ww}-f_{vv})g_{vww}- \\
& & 2(g_{ww}-g_{vv})f_{vww}+f_{vw}(g_{www}+g_{vvw})-g_{vw}(f_{www}+2f_{vvw})], \\
sf_{vvww} & = & (g_{ww}-g_{vv})(f_{vww}f_{vvw}+f_{vvv}f_{www})+ \\
& & (f_{vv}-f_{ww})(f_{vww}g_{vvw}+g_{vvv}f_{www})+ \\
& & 2g_{vw}(f_{vvw}^2-f_{vww}^2)+2f_{vw}(f_{vww}g_{vww}-f_{vvw}g_{vvw}), \\
sf_{vwww} & = & f_{vww}(f_{vw}g_{www}+3g_{vw}f_{vvw}-3f_{vw}g_{vvw})+f_{www}[2(f_{vv}-f_{ww})g_{vvw}- \\
& & 2(g_{vv}-g_{ww})f_{vvw}+f_{vw}(g_{vvv}+g_{vww})-g_{vw}(f_{vvv}+2f_{vww})], \\
-sf_{wwww} & = & f_{www}[2(g_{vv}-g_{ww})f_{vww}+3(f_{ww}-f_{vv})g_{vww}+ \\
& & 2g_{vw}(f_{www}+2f_{vvw})-2f_{vw}g_{vvw}]+ \\
& & g_{www}[(f_{vv}-f_{ww})f_{vww}-2f_{vw}(f_{vvw}+f_{www})]+ \\
& & 6f_{vww}(f_{vw}g_{vww}-g_{vw}f_{vww});
\end{array}
\tag{41}
$$

here $s=g_{vw}(f_{vv}-f_{ww})-f_{vw}(g_{vv}-g_{ww})\neq 0$. Notice that equations $(41)_4$ and $(41)_5$ can be obtained from $(41)_2$ and $(41)_1$ by interchanging v and w. Analogous expressions for the fourth derivatives of g can be obtained from (41) by interchanging f and g. Moreover, one has five quadratic relations among the third derivatives of f and g:

$$
\begin{array}{c}
f_{vvv}g_{www}-f_{www}g_{vvv}+f_{vww}g_{www}-f_{www}g_{vww}=0, \\
f_{vvv}g_{www}-f_{www}g_{vvv}+f_{vvv}g_{vvw}-f_{vvw}g_{vvv}=0, \\
f_{vvv}g_{www}-f_{www}g_{vvv}+f_{vvw}g_{vww}-f_{vww}g_{vvw}=0, \\
f_{vvv}g_{vww}-f_{vww}g_{vvv}=0, \\
f_{vvw}g_{www}-f_{www}g_{vvw}=0.
\end{array}
\tag{42}
$$

This system of PDEs for f and g is not in involution; a careful analysis of the quadratic relations (42) leads to two essentially different possibilities:
Quadratic case. The third derivatives of g are proportional to the corresponding third derivatives of f,

$$g_{vvv}=\mu f_{vvv}, \quad g_{vvw}=\mu f_{vvw}, \quad g_{vww}=\mu f_{vww}, \quad g_{www}=\mu f_{www}.$$

Substituting these relations into the remaining integrability conditions one can show that μ must be constant. Therefore, $g-\mu f$ is at most quadratic in v, w. Without any loss of generality one can assume that, say, g is quadratic.
Harmonic case. Here

$$f_{vvv}+f_{vww}=0, \quad f_{www}+f_{vvw}=0, \quad g_{vvv}+g_{vww}=0, \quad g_{www}+g_{vvw}=0,$$

which imply that $\triangle f$ and $\triangle g$ are constants. Without any loss of generality one can assume that both f and g are harmonic, $\triangle f = \triangle g = 0$.

Remark 4. There exists an obvious group of equivalence transformations which preserve the integrability and leave equations (39) form-invariant. These are, first of all, orthogonal transformations of the (v, w)-plane, generated by translations and rotations. Secondly, these are linear changes of the independent variables in (39),

$$x \to a_{11}x + a_{12}y + a_{13}t, \quad y \to a_{21}x + a_{22}y + a_{23}t,$$

which induce the transformations

$$f \to a_{11}f + a_{12}g + a_{13}\frac{v^2+w^2}{2}, \quad g \to a_{21}f + a_{22}g + a_{23}\frac{v^2+w^2}{2}.$$

The classification below is carried out up to this natural equivalence.

3.1 Quadratic case

Setting $g(v, w) = av^2 + 2bvw + cw^2$ and introducing $C = b(f_{ww} - f_{vv}) + (a - c)f_{vw}$ one can rewrite equations (41) as follows:

$$\begin{aligned}
Cf_{vvvv} &= 2b(3f_{vvw}^2 - 2f_{vvv}f_{vww} - f_{vvv}^2) + 2(a-c)f_{vvv}f_{vvw},\\
Cf_{vvvw} &= b(3f_{vvw}f_{vww} - 2f_{vvv}f_{vvw} - f_{vvv}f_{www}) + 2(a-c)f_{vvv}f_{vww},\\
Cf_{vvww} &= 2b(f_{vww}^2 - f_{vvw}^2) + (a-c)(f_{vvw}f_{vww} + f_{vvv}f_{www}),\\
Cf_{vwww} &= -b(3f_{vww}f_{vvw} - 2f_{www}f_{vww} - f_{vvv}f_{www}) + 2(a-c)f_{vvw}f_{www},\\
Cf_{wwww} &= -2b(3f_{vww}^2 - 2f_{www}f_{vvw} - f_{www}^2) + 2(a-c)f_{vww}f_{www},
\end{aligned} \tag{43}$$

(the corresponding equations for g are satisfied identically). For any a, b, c this system is in involution and its solution space is 10-dimensional. Indeed, the values of f, its first, second and third derivatives can be choosen arbitrarily, while the fourth and higher derivatives are determined by virtue of (43). Diagonalizing the quadratic form g by a linear orthogonal change of variables v, w, one can set $b = 0$. In this case equations (43) simplify to

$$\begin{aligned}
f_{vw}f_{vvvv} &= 2f_{vvv}f_{vvw},\\
f_{vw}f_{vvvw} &= 2f_{vvv}f_{vww},\\
f_{vw}f_{vvww} &= f_{vvw}f_{vww} + f_{vvv}f_{www},\\
f_{vw}f_{vwww} &= 2f_{vvw}f_{www},\\
f_{vw}f_{wwww} &= 2f_{vww}f_{www}.
\end{aligned} \tag{44}$$

The first two equations imply that $f_{vvv}/f_{vw}^2 = \text{const}$. Similarly, the last two equations imply $f_{www}/f_{vw}^2 = \text{const}$. Setting $f_{vw} = e$ one can parametrise the third derivatives of f as follows:

$$f_{vvv} = \frac{1}{2}me^2, \quad f_{vvw} = e_v, \quad f_{vww} = e_w, \quad f_{www} = \frac{1}{2}ne^2; \tag{45}$$

here m, n are arbitrary constants. The compatibility conditions of these equations plus the equation $(44)_3$ result in the following overdetermined system for e:

$$(\ln e)_{vw} = \frac{mn}{4}e^2, \quad e_{vv} = mee_w, \quad e_{ww} = nee_v. \tag{46}$$

It is worth mentioning that the system (46) arises in a completely different context in projective differential geometry constituting the projective Gauss-Codazzi equations of the Roman surface of Steiner which is known as the only quartic in P^3 containing a two-parameter family of conics, see e.g. [12]. At the moment we have no explanation of this remarkable coincidence.
Solving the first (Liouville) equation for e in the form

$$e^2 = \frac{4}{mn}\frac{p'(v)q'(w)}{(p(v)+q(w))^2}$$

and setting

$$(p')^{3/2} = \sqrt{m}\ P(p), \quad (q')^{3/2} = \sqrt{n}\ Q(q), \tag{47}$$

(here $P(p)$ and $Q(q)$ are functions to be determined), one obtains from the last two equations (46) the following functional-differential equations for P and Q:

$$P''(p+q)^2 - 4P'(p+q) + 6P = 2Q'(p+q) - 6Q, \quad Q''(p+q)^2 - 4Q'(p+q) + 6Q = 2P'(p+q) - 6P;$$

these equations imply that both P and Q are cubic polynomials in p and q,

$$P = ap^3 + bp^2 + cp + d, \quad Q = aq^3 - bq^2 + cq - d,$$

where a, b, c, d are arbitrary constants. In the general case, using translations and scalings, equations (47) can be brought to the form $(y')^3 = (y^3 - 3\lambda y^2 + 3y)^2$ and solved in terms of elliptic functions [2]:

$$y = \frac{2}{\lambda - 3\wp'(z; 0, g_3)}, \quad g_3 = \frac{4 - 3\lambda^2}{27}.$$

In the simplest case $m = n = 0$ equations (46) imply

$$e = (\alpha v + \beta)(\gamma w + \delta),$$

and the elementary integration of (45) results in

$$f(v, w) = \frac{\alpha\gamma}{4}v^2w^2 + \frac{\alpha\delta}{2}v^2w + \frac{\beta\gamma}{2}vw^2 + \beta\delta vw;$$

here $\alpha, \beta, \gamma, \delta$ are arbitrary constants. Using the equivalence transformations one can reduce f to either $f = v^2w^2$ (if both α and γ are nonzero) or $f = vw^2$ (if $\alpha = 0$). The corresponding equations (39) take the form

$$v_t + 2(vw^2)_x + v_y = 0, \quad w_t + 2(v^2w)_x - w_y = 0$$

and

$$v_t + (w^2)_x + v_y = 0, \quad w_t + 2(vw)_x - w_y = 0,$$

respectively (in both cases $g = (v^2 - w^2)/2$).
If $m = 0,\ n \neq 0$, equations (46) imply

$$e = (\alpha v + \beta)\varphi'(w)$$

where $\varphi''' = \alpha n(\varphi')^2,\ \alpha, \beta = const$. Therefore, $\varphi(w) = -\frac{6}{\alpha n}\zeta(w; 0, g_3) + \gamma$, where $\zeta'(z) = -\wp(z)$ and g_3, γ are constants. The elementary integration of the equations (45) gives

$$f = \frac{1}{2\alpha}(\alpha v + \beta)^2\varphi(w).$$

This reduces to the previous case if $n = 0$.

3.2 Harmonic case

We have $f_{vv} = -f_{ww},\ g_{vv} = -g_{ww}$, so that

$$f_{vvv} = -f_{vww}, \quad f_{www} = -f_{vvw}, \quad g_{vvv} = -g_{vww}, \quad g_{www} = -g_{vvw}.$$

Differentiating these relations by virtue of (41) (and the analogous equations for g) one arrives at the additional constraints

$$f_{vv}p_1 + f_{vw}p_2 + g_{ww}p_3 = 0, \quad -f_{vw}p_1 + f_{vv}p_2 + g_{vw}p_3 = 0,$$

$$-g_{vw}p_1 + g_{ww}p_2 + f_{vw}p_4 = 0, \quad g_{ww}p_1 + g_{vw}p_2 + f_{vv}p_4 = 0$$

where

$$p_1 = f_{vww}g_{vww} + f_{vvw}g_{vvw}, \quad p_2 = f_{vww}g_{vvw} - f_{vvw}g_{vww}, \quad p_3 = f_{vvw}^2 + f_{vww}^2, \quad p_4 = g_{vvw}^2 + g_{vww}^2.$$

Solving the linear system for p_i (notice that the corresponding 4×4 matrix has rank three) one obtains

$$\begin{aligned} p_1 &= f_{vww}g_{vww} + f_{vvw}g_{vvw} = \mu^2(f_{vw}g_{vw} - f_{vv}g_{ww}), \\ p_2 &= f_{vww}g_{vvw} - f_{vvw}g_{vww} = -\mu^2(f_{vv}g_{vw} + f_{vw}g_{ww}), \\ p_3 &= f_{vvw}^2 + f_{vww}^2 = \mu^2(f_{vw}^2 + f_{vv}^2), \\ p_4 &= g_{vvw}^2 + g_{vww}^2 = \mu^2(g_{vw}^2 + g_{ww}^2). \end{aligned} \tag{48}$$

Introducing the two-component vectors

$$e_1 = \begin{pmatrix} f_{vww} \\ f_{vvw} \end{pmatrix}, \quad e_2 = \begin{pmatrix} g_{vww} \\ g_{vvw} \end{pmatrix}, \quad s_1 = \begin{pmatrix} f_{vw} \\ f_{ww} \end{pmatrix}, \quad s_2 = \begin{pmatrix} g_{vw} \\ g_{ww} \end{pmatrix},$$

one can rewrite (48) in vector notation as follows

$$(e_1, e_1) = \mu^2(s_1, s_1), \quad (e_2, e_2) = \mu^2(s_2, s_2), \quad (e_1, e_2) = \mu^2(s_1, s_2), \quad e_1 \wedge e_2 = -\mu^2 s_1 \wedge s_2$$

where $(,)$ is the standard Euclidean scalar product. Therefore, the vectors e_1, e_2 and s_1, s_2 are related by a composition of a scaling, rotation and reflection, that is,

$$e_1 = \begin{pmatrix} X & Y \\ Y & -X \end{pmatrix} s_1, \quad e_2 = \begin{pmatrix} X & Y \\ Y & -X \end{pmatrix} s_2$$

or, explicitly,

$$\begin{aligned} f_{vww} &= X f_{vw} + Y f_{ww}, \quad & f_{vvw} &= Y f_{vw} - X f_{ww}, \\ g_{vww} &= X g_{vw} + Y g_{ww}, \quad & g_{vvw} &= Y g_{vw} - X g_{ww} \end{aligned} \tag{49}$$

(notice that equations for f and g coincide). The compatibility conditions of these equations imply the following equations for X and Y,

$$X_v = 2XY, \quad X_w = X^2 - Y^2, \quad Y_v = Y^2 - X^2, \quad Y_w = 2XY,$$

whose general solution (up to translations in v and w) is

$$X = -\frac{w}{v^2 + w^2}, \quad Y = -\frac{v}{v^2 + w^2}.$$

Substituting these expressions into (49) and integrating the corresponding linear system for the harmonic function f (this integration simplifies if one changes to the complex variables $z = v + iw,\ \bar{z} = v - iw$) one readily obtains that f must be a linear combination of the real and imaginary parts of the function $z \ln z - z$. The same result holds for g. Therefore, without any loss of generality one can set $f = Re(z \ln z - z),\ g = Im(z \ln z - z)$.

3.3 Reductions of the system (11)

Here we briefly discuss reductions of the system (11),

$$\theta_y = -\frac{r_x}{r} + r \sin\theta\ \theta_t - \cos\theta\ r_t, \quad \theta_x = \frac{r_y}{r} + r\cos\theta\ \theta_t + \sin\theta\ r_t.$$

Assuming $\theta = \theta(R^1, ..., R^n),\ r = r(R^1, ..., R^n)$ where the Riemann invariants satisfy the equations

$$R^i_x = \lambda^i(R)\ R^i_t, \qquad R^i_y = \mu^i(R)\ R^i_t$$

and substituting into (11) one arrives at

$$(\mu^i - r\sin\theta)\partial_i\theta + (\lambda^i/r + \cos\theta)\partial_i r = 0, \quad (\lambda^i - r\cos\theta)\partial_i\theta - (\mu^i/r + \sin\theta)\partial_i r = 0,$$

(no summation!) Hence, the characteristic speeds λ^i and μ^i satisfy the dispersion relation which takes a particularly simple form

$$(\lambda^i)^2 + (\mu^i)^2 = r^2.$$

Parametrising λ^i and μ^i in the form $\lambda^i = r\cos\varphi^i$, $\mu^i = r\sin\varphi^i$ we obtain

$$\partial_i\theta = \frac{\cos\theta + \cos\varphi^i}{\sin\theta - \sin\varphi^i}\ \partial_i U, \tag{50}$$

$U = \ln r$. The compatibility conditions of these equations together with the commutativity conditions (3) imply the system

$$\partial_i\partial_j U = -\frac{\partial_i U \partial_j U}{\sin^2 \frac{\varphi^i - \varphi^j}{2}}, \quad \partial_j\varphi^i = \cot\frac{\varphi^i - \varphi^j}{2}\ \partial_j U \tag{51}$$

which is a trigonometric version of the Gibbons-Tsarev system (16). As shown in [13] this system is, in fact, equivalent to (16), and its solutions can be constructed from the known solutions of the Gibbons-Tsarev system [17, 18]. Once the solution of (51) is known, the corresponding polar angle θ can be calculated from the equations (50) which are compatible by construction.

4 Classification of integrable Hamiltonian systems of hydrodynamic type in $2+1$ dimensions

In this section we classify Hamiltonian systems

$$v_t = (h_v)_x, \quad w_t = (h_w)_y \tag{52}$$

which possess infinitely many hydrodynamic reductions. Here $h(v, w)$ is the Hamiltonian density. Notice that any system of the form (52) possesses one extra conservation law

$$h_t = (h_v^2/2)_x + (h_w^2/2)_y.$$

As mentioned in the introduction, Hamiltonian systems are related to the quadratic case of Sect. 3 by virtue of the Legendre transform. However, we find it instructive to treat the Hamiltonian case independently to better illustrate our method.

Looking for reductions in the form $v = v(R^1, ..., R^n)$, $w = w(R^1, ..., R^n)$ where the Riemann invariants satisfy the equations

$$R^i_x = \lambda^i(R)\ R^i_t, \quad R^i_y = \mu^i(R)\ R^i_t \tag{53}$$

and substituting into (52) one arrives at the equations

$$(1 - \lambda^i h_{vv})\partial_i v = \lambda^i h_{vw}\partial_i w, \quad (1 - \mu^i h_{ww})\partial_i w = \mu^i h_{vw}\partial_i v,$$

(no summation!) so that λ^i and μ^i satisfy the dispersion relation

$$(1-\lambda^i h_{vv})(1-\mu^i h_{ww})=\lambda^i\mu^i h_{vw}^2.$$

We require that the dispersion relation is nondegenerate (as a conic), that is, $h_{vw}\neq 0,\ h_{vw}^2-h_{vv}h_{ww}\neq 0$. Setting $\partial_i v=\varphi^i\partial_i w$, we obtain the following expressions for λ^i and μ^i in terms of φ^i,

$$\lambda^i=\frac{\varphi^i}{h_{vw}+\varphi^i h_{vv}},\qquad \mu^i=\frac{1}{h_{ww}+\varphi^i h_{vw}},$$

which define a rational parametrization of the dispersion relation. The compatibility conditions of the equations $\partial_i v=\varphi^i\partial_i w$ imply

$$\partial_i\partial_j w=\frac{\partial_j\varphi^i}{\varphi^j-\varphi^i}\partial_i w+\frac{\partial_i\varphi^j}{\varphi^i-\varphi^j}\partial_j w \tag{54}$$

while the commutativity equations (3) lead to the following complicated expressions for φ^i:

$$\begin{gathered}\partial_j\varphi^i=\tfrac{\varphi^i(h_{vw}+\varphi^j h_{vv})(h_{ww}+\varphi^i h_{vw})\partial_j w}{h_{vw}(h_{vw}^2-h_{vv}h_{ww})(\varphi^i-\varphi^j)}\left[(h_{vvw}+\varphi^i h_{vvv})\varphi^j+h_{vww}+\varphi^i h_{vvw}\right]+\\ \tfrac{(h_{vw}+\varphi^i h_{vv})(h_{ww}+\varphi^j h_{vw})\partial_j w}{h_{vw}(h_{vw}^2-h_{vv}h_{ww})(\varphi^i-\varphi^j)}\left[(h_{vww}+\varphi^i h_{vvw})\varphi^j+h_{www}+\varphi^i h_{vww}\right].\end{gathered} \tag{55}$$

One can see that the compatibility condition $\partial_k\partial_j\varphi^i-\partial_j\partial_k\varphi^i=0$ is of the form $P\partial_j w\partial_k w=0$ where P is a rational expression in $\varphi^i,\varphi^j,\varphi^k$ whose coefficients depend on partial derivatives of the Hamiltonian density $h(v,w)$ up to fourth order. Requiring that P vanishes we obtain the overdetermined system of fourth order PDEs for the density h:

$$\begin{gathered}
h_{vw}(h_{vw}^2-h_{vv}h_{ww})h_{vvvv}=4h_{vw}h_{vvv}(h_{vw}h_{vvw}-h_{vv}h_{vww})\\
+3h_{vv}h_{vw}h_{vvw}^2-2h_{vv}h_{ww}h_{vvv}h_{vvw}-h_{vw}h_{ww}h_{vvv}^2,\\[2ex]
h_{vw}(h_{vw}^2-h_{vv}h_{ww})h_{vvvw}=-h_{vw}h_{vvv}(h_{vv}h_{www}+h_{ww}h_{vvw})\\
+3h_{vw}^2h_{vvw}^2-2h_{vv}h_{ww}h_{vvv}h_{vww}+h_{vw}^2h_{vvv}h_{vww},\\[2ex]
h_{vw}(h_{vw}^2-h_{vv}h_{ww})h_{vvww}=4h_{vw}^2h_{vvw}h_{vww}\\
-h_{vv}h_{vvw}(h_{vw}h_{www}+h_{ww}h_{vww})-h_{ww}h_{vvv}(h_{vw}h_{vww}+h_{vv}h_{www}),\\[2ex]
h_{vw}(h_{vw}^2-h_{vv}h_{ww})h_{vwww}=-h_{vw}h_{www}(h_{ww}h_{vvv}+h_{vv}h_{vww})\\
+3h_{vw}^2h_{vww}^2-2h_{vv}h_{ww}h_{www}h_{vvw}+h_{vw}^2h_{www}h_{vvw},\\[2ex]
h_{vw}(h_{vw}^2-h_{vv}h_{ww})h_{wwww}=4h_{vw}h_{www}(h_{vw}h_{vww}-h_{ww}h_{vvw})\\
+3h_{ww}h_{vw}h_{vww}^2-2h_{vv}h_{ww}h_{www}h_{vww}-h_{vw}h_{vv}h_{www}^2.
\end{gathered} \tag{56}$$

It was verified that this system is in involution and its solution space is 10-dimensional. As mentioned in the introduction, the Legendre transform identifies the systems (56) and (44).

5 Concluding remarks

We have demonstrated that the existence of hydrodynamic reductions describing nonlinear interactions of $n \geq 3$ planar simple waves can be viewed as the effective integrability criterion. The most natural problems arising in this context are the following:
1. Classify multicomponent $(2+1)$-dimensional integrable quasilinear systems. The main difference from the two-component case is that the dispersion relation (7) will no longer define a rational curve.
2. The recent publication [14] suggests that the method of hydrodynamic reductions carries over to $3+1$ dimensions. It would be extremely interesting to obtain further examples (classification results) of dispersionless integrable systems in many dimensions.
3. Nonlinear interactions of n simple waves can be viewed as a natural dispersionless analogue of 'n-gap' solutions. It would be desirable to obtain an alternative description of these solutions as the 'stationary points' of the appropriate 'higher symmetries'.

We hope to address these questions elsewhere.

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