Dynamic Inverse Problem for a hyperbolic equation and continuation of boundary data.

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Abstract. *We consider an inverse problem for a second order hyperbolic initial boundary value problem on a compact Riemannian manifold M with boundary. Assume that we know ∂M and the Cauchy data on $\partial M \times [0, T]$ of the solutions with vanishing initial data. In the paper we consider two problems. Firstly, when T is sufficiently large and the Riemannian manifold satisfies an additional geometrical condition, we show that we can continue the data on $\partial M \times \mathbb{R}_+$ without solving the inverse problem. Secondly, we show that it is possible to determine manifold M and the wave operator to within the group of the generalized gauge transformations.*

1. Introduction and main result.

In the paper we study an inverse problem for the hyperbolic initial boundary value problem

$$u_{tt} + bu_t + a(x, D)u = 0 \text{ in } M \times \mathbb{R}_+ \tag{1.1}$$

$$u|_{\partial M\times\mathbb{R}_+} = f; \quad u|_{t=0} = u_t|_{t=0} = 0; \quad f \in H_0^1(\partial M \times \mathbb{R}_+) \tag{1.2}$$

on a compact connected C^∞-Riemannian manifold $M, \dim M = m \geq 1$, with metric $g = (g^{jl})_{j,l=1}^m$ and non-empty boundary ∂M. The operator $a(x, D)$ is a first order perturbation of the Laplace Beltrami operator $-\Delta_g$,

$$a(x, D) = -\Delta_g + P + q. \tag{1.3}$$

Here in local coordinates $P = P^l \partial_l$ is a complex valued C^∞-vector field and q and b are complex valued C^∞-functions on M. The symbol $a(x, D)$ is, in general, not formally symmetric. Later in the paper we refer to the case (1.1), (1.2) with $b(x) \neq 0$ and $a(x, D)$ of form (1.3) as to a "generic case".

Remark. Any uniformly elliptic symbol on a differentiable manifold can be written in form (1.3).

In the paper we also study separately the "selfadjoint case",

$$a(x, D) = -\Delta_g + q, \quad b(x) = 0, \tag{1.4}$$

where q is a real-valued function.

By $H^s(A)$ we denote the Sobolev space of the functions on A and by $H_0^s(\partial M \times [0, t])$ the space of $u \in H^s(\partial M \times \mathbb{R})$ for which supp $u \in \partial M \times [0, t]$. We denote by ν the unit normal vector to ∂M with respect to g. We define the boundary operator $Bu = \partial_\nu u - P_\nu u|_{\partial M\times[0,T]}$, where ∂_ν and $P_\nu = (\nu, P)_g$ are the normal derivative and the normal component of P, correspondingly.

Definition 1. *Let $T > 0$. We define the response operator $R^T : H_0^1(\partial M \times [0,T]) \to L^2(\partial M \times [0,T])$ by*

$$R^T(f) = \partial_\nu u^f - P_\nu u^f|_{\partial M \times [0,T]} \quad = Bu^f|_{\partial M \times [0,T]},$$

where u^f is the solution of the problem (1.1), (1.2).

In the paper we consider two problems:

Problem I. *Let ∂M and R^T be given with some $T > 0$. Can we reconstruct from these data the operator R^t with any $t > T$ without solving the inverse problem?*

Problem II. *Let ∂M and R^T be given with some $T > 0$. Do these data determine $(M, a(x,D), b)$ uniquely?*

We give answers to Problems I-II assuming in the generic case that some geometric conditions are posed upon (M, g).

In the following we call the pair $\{\partial M, R^T\}$ the dynamical boundary data and abbreviate it by DBD.

To answer positively to problem I we have to assume that the waves sent at time $t = 0$ from boundary can reach all points in M and return back before time $t = T$. Hence in the selfadjoint case we assume that $T > 2r$ where $r = max\{\text{dist}(x, \partial M) : x \in M\}$ is the geodesic radius of M. In the generic case we pose the following geometrical condition (for details see [1]) which generalizes the condition that the rays of the geometrical optics hit the boundary transversally.

Definition 2. *(M, g) satisfies the Bardos-Lebeau-Rauch condition if there is $t_* > 0$ and an open conic neighborhood $\mathcal{O}$ of the set of the not-nondiffractive points $(x, t, \xi, \omega) \in T^*(M \times [0, t_*])$, $x \in \partial M$ such that any generalized bicharacteristic of the wave operator $\partial_t^2 - \Delta_g$ passes through a point of $(x, t, \xi, \omega) \in T^*(M \times [0, t_*]) \setminus \mathcal{O}$, $x \in \partial M$.*

Before stating our main results we discuss shortly Problem II. It is well known that Problem II can not have a positive answer since the generalized gauge transformations preserve the boundary data. This means that by replacing $a(x,D)$ by $a_\kappa(x,D)$,

$$a_\kappa(x,D) = \kappa a(x,D)\kappa^{-1}, \tag{1.5}$$

where $\kappa|_{\partial M} = 1$, $\kappa \neq 0$ on M we do not change R^T. Thus the best we can hope to recover is the equivalence class of $a(x,D)$ with respect to the generalized gauge transformations, namely the set

$$[a(x,D)] := \{\kappa a(x,D)\kappa^{-1} : \ \kappa \in C^\infty(M;\mathbb{C}),\ \kappa|_{\partial M} = 1,\ \kappa \neq 0 \text{ on } M\}.$$

The above hyperbolic inverse problem and its analogs were considered in several papers. Paper [14] considered the inverse problem in $M \subset \mathbb{R}^m$ with Euclidean metric $g^{jl} = \delta^{jl}$. The corresponding inverse boundary spectral problem was studied in [11]. A local variant of the dynamic inverse problem with data prescribed only on a part of the boundary was considered in [5] where is was assumed that $g^{ij} = \delta^{ij}$. In [13] the uniqueness of the reconstruction of the conformally Euclidean metric in $M \subset \mathbb{R}^m$ and the lower order terms (with some restrictions upon these terms) was proven for the geodesically regular domains M. The present work is based on paper [9] of the authors where an analogous problem was studied for the Gel'fand inverse boundary spectral problem.

In an anisotropic case an analogous inverse problem was considered in [6], [7] for the self-adjoint case and in [8] for the non-selfadjoint case, $a^*(x,D) \neq a(x,D)$ where is was, however, assumed that $b = 0$.

This paper is based on the Boundary Control method introduced in [2] (see also [3]). Particularly, we use here the geometrical formulation of the Boundary Control method (see [7]) and exact controllability results [1].

The main results of the paper are:

Theorem 1.1. *Assume that*

i. In the generic case the Riemannian manifold (M,g) *satisfies the Bardos-Lebeau-Rauch condition with* t_* *and* R^T *is known for* $T > 2t_*$*;*

ii. In the self-adjoint case R^T *is known for* $T > 2r$*.*

Then these data determine uniquely R^t *for any* $t > 0$*.*

In the selfadjoint case (1.4) it is known (see e.g. [7]) that BSD determines (M,g) and q uniquely. We give a dynamic version of this result which is valid in generic case:

Theorem 1.2. *In generic case let the Riemannian manifold* (M,g) *satisfies the Bardos-Lebeau-Rauch condition with* t_**. Let* ∂M *and* R^T *be given and* $T > 2t_*$*. Then these data determine* M*,* b *and the equivalence class* $[a(x,D)]$ *uniquely.*

Before stating the proofs, we explain what we mean by the reconstruction of a Riemannian manifold (M,g). Since a manifold is an 'abstract' collection of coordinate patches we construct a representative of an equivalence class of the isometric Riemannian manifolds or a metric space X which is isometric to (M,g). After constructing X one can take any coordinatisation and construct the vector field P and the potential q in local coordinates.

2. Continuation of data in the selfadjoint case.

In this section we consider Problem I for the initial boundary value problem

$$u^f_{tt} - \Delta_g u^f + qu^f = 0 \ \text{ in } M \times \mathbb{R}_+,$$

$$u^f|_{\partial M \times \mathbb{R}_+} = f; \quad u^f|_{t=0} = u^f_t|_{t=0} = 0,$$

where q is a real valued function. We point out that we do not assume that the Bardos-Lebeau-Rauch condition is valid.

By λ_j and ϕ_j we denote the Dirichlet eigenvalues and the normalized eigenfunctions of the operator $-\Delta_g + q$. In this section all spaces $L^2(M)$ etc are spaces of real valued functions.

We start with a well-known result of appriximate controllability.

Lemma 2.1. *The pairs* $(u^f(2r), u^f_t(2r))$*,* $f \in C_0^\infty(\partial M \times [0,2r])$ *are dense in* $H^1_0(M) \times L^2(M)$*.*

Proof. Assume that a pair

$$(\psi, -\phi) \in (H^1_0(M) \times L^2(M))' = H^{-1}(M) \times L^2(M)$$

satisfy the duality

$$(u^f(2r),\psi)_{H_0^1,H^{-1}}+(u_t^f(2r),-\phi)_{L^2}=0$$

for all $f\in C_0^\infty(\partial M\times[0,2r])$. Let

$$e_{tt}-\Delta_g e+q\,e=0 \text{ in } M\times[0,2r], \tag{2.1}$$

$$e|_{\partial M}=0;\quad e|_{t=2r}=\phi,\ e_t|_{t=2r}=\psi.$$

By part integration

$$0=\int_{M\times[0,2r]}[(e_{tt}-\Delta_g e+qe)u^f-(u_{tt}^f-\Delta_g u^f+qu^f)e]\,dx\,dt\ =$$

$$=\int_M(u_t^f(2r)\,\phi-u^f(2r)\,\psi)\,dx\ +\int_{\partial M}\int_0^{2r}f\,\partial_\nu e\,dS_x\,dt\ =\int_{\partial M}\int_0^{2r}f\,\partial_\nu e\,dS_x\,dt$$

for all $f\in C_0^\infty(\partial M\times[0,2r])$. This yields that

$$e|_{\partial M\times[0,2r]}=\partial_\nu e|_{\partial M\times[0,2r]}=0.$$

Since by (2.1) $e\in\mathcal{D}'(]0,2r[,H_0^1(M))$ Tataru's Holmgren-John uniqueness theorem [15] is applicable and we obtain $e(r)=e_t(r)=0$. Hence $e=0$ identically on $M\times[0,2r]$ and thus $\phi=\psi=0$. □

Consider a bilinear form

$$E(u^f,u^g,t)=\int_M[(\nabla u^f(t),\nabla u^g(t))_g+u_t^f(t)\,u_t^g(t)+q\,u^f(t)\,u^g(t)]\,dx$$

and denote $E(u^f,t)=E(u^f,u^f,t)$.

Lemma 2.2. *Operator R^t determines $E(u^f,u^g,t)$ for $f,g\in C_0^\infty(\partial M\times[0,t])$.*

Proof. By part integration

$$\frac{\partial}{\partial t}E(u^f,t)=2\int_M[(\nabla u_t^f(t),\nabla u^f(t))_g+u_{tt}^f(t)\,u_t^f(t)+q\,u^f(t)\,u_t^f(t)]\,dx\ =$$

$$2\int_M[-\Delta_g u^f(t)+u_{tt}^f(t)+qu^f(t)]\,u_t^f(t)\,dx\ +2\int_{\partial M}u_t^f(t)\partial_\nu u^f(t)\,dS_x$$

$$=2\int_{\partial M}f_t(t)\,R^tf(t)\,dS_x.$$

Since $E(u^f,0)=0$ we can determine $E(u^f,t)$. Since $4E(u^f,u^g,t)=E(u^{f+g},t)-E(u^{f-g},t)$, this proves the assertion. □

Next we show that we can continue data without solving the inverse problem.

Proof. (of Theorem 1.1. in the selfadjoint case) It is sufficient to show that R^T determines R^tf for any $f\in C_0^\infty(\partial M\times[0,2r])$.

Let $\varepsilon = (T-2r)/2$ and $t_0 = 2r+\varepsilon$. By Lemma 2.1 there are $f_n \in C_0^\infty(\partial M \times [0,2r])$ such that

$$\lim_{n\to\infty} (u^{f_n}(2r), u_t^{f_n}(2r)) = (u^f(t_0), u_t^f(t_0)) \tag{2.2}$$

in $H_0^1(M)\times L^2(M)$-topology. We want to show that (2.2) is valid if and only if for every $h \in C_0^\infty(\partial M \times [0,2r])$

$$\lim_{n\to\infty} E(u^{g_n}, t_0) = 0, \tag{2.3}$$

$$\lim_{n\to\infty} E(u^{g_n}, u^h, t_0) = 0, \tag{2.4}$$

$$\lim_{n\to\infty} ||R^{t_0+\varepsilon} g_n||_{L^2(\partial M\times[t_0,t_0+\varepsilon])} = 0, \tag{2.5}$$

where $g_n(t) = f(t) - f_n(t-\varepsilon)$. Since the direct problem depends continuously on initial data [10], we see that (2.2) obviously yields (2.3)-(2.5). Thus assume that (2.3)-(2.5) are valid. We use the eigenfunction expansion $u^{g_n}(t_0) = \sum_j a_j^n \phi_j$ and $u^h(t_0) = \sum_j b_j \phi_j$. Then by (2.3)

$$\lim_{n\to\infty} \left(\sum_{j=0}^{\infty} \lambda_j (a_j^n)^2 + ||u_t^{g_n}||_{L^2}^2 \right) = 0. \tag{2.6}$$

Let j_0 be such that $\lambda_j > 0$ for $j > j_0$ and $\lambda_j \le 0$ for $j \le j_0$ and let P be the orthogonal projection in $H_0^1(M)$ onto the space of the eigenfunctions corresponding $\lambda_j = 0$. Using these notations we rewrite (2.6) in the following form

$$\sum_{j\le j_0} -\lambda_j (a_j^n)^2 = \sum_{j>j_0} \lambda_j (a_j^n)^2 + ||u_t^{g_n}(t_0)||_{L^2(M)}^2 + o(1) \tag{2.7}$$

where $o(1)$ goes to zero when $n\to\infty$.

First we show that $a_j^n \to 0$ for j satisfying $\lambda_j < 0$. Indeed, assume that there is k with $\lambda_k < 0$ such that $a_k^n \not\to 0$. By choosing a subsequence, the sign of a_j^n depends only upon j. Moreover, without loss of generality we can assume that $a_j^n \ge 0$.

Since $(u^g(t_0), u_t^g(t_0))$ are dense in $H_0^1(M)\times L^2(M)$ we can choose h such that its Fourier coefficients (b_j) satisfy $b_j = \delta_{j\le j_0} + c_j$ where $||(\lambda_j c_j)||_{\ell^2} < \varepsilon$ and $||u_t^g(t_0)||_{L^2(M)} < \varepsilon$, $\varepsilon \in]0, \frac{1}{2}[$. Then (2.4) yields that

$$\sum_{j\le j_0} -\lambda_j a_j^n (1+c_j) = \sum_{j>j_0} \lambda_j a_j^n c_j + (u_t^{g_n}(t_0), u_t^g(t_0))_{L^2(M)} + o(1).$$

Hence by (2.7)

$$\sum_{j\le j_0} -\lambda_j a_j^n (1+c_j) \le \Big(\sum_{j>j_0} \lambda_j (a_j^n)^2\Big)^{\frac{1}{2}} \Big(\sum_{j>j_0} \lambda_j (c_j)^2\Big)^{\frac{1}{2}} + ||u_t^{g_n}(t_0)||_{L^2} ||u_t^g(t_0)||_{L^2} + o(1)$$

$$\le \varepsilon \Big(\sum_{j>j_0} \lambda_j (a_j^n)^2\Big)^{\frac{1}{2}} + \varepsilon ||u_t^{g_n}(t_0)||_{L^2} + o(1) = \varepsilon \Big(\sum_{j\le j_0} -\lambda_j (a_j^n)^2\Big)^{1/2} + o(1). \tag{2.8}$$

On the other hand, there is $C>0$ independent of ε such that

$$\sum_{j\leq j_0} -\lambda_j a_j^n(1+c_j) \geq C\big(\sum_{j\leq j_0} -\lambda_j (a_j^n)^2\big)^{\frac{1}{2}}.$$

But for some k with $\lambda_k<0$ $a_k^n \not\to 0$. This leads to a contradiction with (2.8).

Thus we have proven that $a_j^n \to 0$ for all j satisfying $\lambda_j<0$. By (2.6), this implies that

$$\sum_{j=0}^{\infty} |\lambda_j|(a_j^n)^2 < C$$

uniformly for all n. Thus the pairs $((1-P)u^{g_n}, u_t^{g_n})$ are uniformly bounded in $H_0^1(M)\times L^2(M)$. By (2.4), this implies that $E(u^{g_n},a,t_0)\to 0$ for any $a\in H_0^1(M)\times L^2(M)$. Hence $(1-P)u^{g_n}(t_0)\to 0$ in $H_0^1(M)$ and $u_t^{g_n}(t_0)\to 0$ in $L^2(M)$.

It remains to show that $|a_j^n|\to 0$ when $\lambda_j=0$ and g_n satisfy (2.3)-(2.5). Since the solution of the direct problem depends continuously on the data,

$$\lim_{n\to\infty} ||R^T g_n - \sum_{\lambda_j=0} a_j^n \partial_\nu \phi_j|_{\partial M}||_{L^2(\partial M\times[t_0,T])} = 0.$$

Since $\partial_\nu\phi_j|_{\partial M}$ are linearly independent, (2.5) can be valid only if $a_j^n\to 0$.

Thus (2.2) and (2.3)-(2.5) are equivalent.

We can use Lemma 2.2 to construct f_n which satisfy conditions (2.2). The functions $y_n(t)=u^{f_n}(t)$ for $t\in[2r,T]$ are the solutions of the initial value problem

$$y_{tt}^n - \Delta_g y^n = 0 \text{ in } M\times[2r,T]$$

$$y^n|_{\partial M\times[2r,T]}=0;\quad y^n|_{t=2r}=u^{f_n}(2r),\ y_t^n|_{t=2r}=u_t^{f_n}(2r).$$

However, $y(t)=u^f(t+\varepsilon)$ satisfies the same equation with initial data $y|_{t=2r}=u^f(t_0),\ y_t|_{t=2r}=u_t^f(t_0)$. Then it follows from (2.2) and continuous dependence of solutions on the initial data (see e.g [10]) that

$$\lim_{n\to\infty} \partial_\nu y^n|_{\partial M\times[2r,T]} = \partial_\nu y|_{\partial M\times[2r,T]}$$

in L^2-topology. Since we know $y_n(t)|_{\partial M\times[2r,T]}=(R^Tf_n)(t),\ t\in[2r,T]$ we can determine $R^{T+\varepsilon}f$.

By iterating the above consideration, we get the assertion. □

3. Continuation of data and uniqueness results in the non-selfadjoint case.

In this section we study the inverse problem foir the initial-boundary value problem in generic case

$$u_{tt}^f + bu_t^f + a(x,D)u^f = 0 \text{ in } M\times\mathbb{R}_+ \tag{3.1}$$

$$u^f|_{\partial M\times\mathbb{R}_+}=f;\quad u|_{t=0}=u_t|_{t=0}=0;\quad f\in H_0^1(\partial M\times\mathbb{R}_+), \tag{3.2}$$

where $a(x,D)$ is of form (1.3) and (M,g) satisfies the Bardos-Lebeau-Rauch condition. We use the notations

$$U^f(t) := \begin{pmatrix} u^f(x,t) \\ u_t^f(x,t)\end{pmatrix} \in L^2(M)^2,\quad J\begin{pmatrix} u^1\\ u^2\end{pmatrix} = \begin{pmatrix} u^2+bu^1 \\ u^1\end{pmatrix}. \tag{3.3}$$

and denote the inner product in $L^2(M)^2$ by $(\cdot,\cdot)$.

3.1 Adjoint equation.

Let $v^g(x,s)$ be the solution to the adjoint initial-boundary value problem,

$$v^g_{tt} + \bar{b} v^g_t + a^*(x,D) v^g = 0 \text{ in } M \times \mathbb{R}_+ \tag{3.4}$$

$$v^g|_{\partial M \times \mathbb{R}_+} = g, \quad v^g|_{t=0} = v^g_t|_{t=0} = 0. \tag{3.5}$$

We denote

$$V^g(t) = \begin{pmatrix} v^g(x,t) \\ v^g_t(x,t) \end{pmatrix}. \tag{3.6}$$

For the adjoint equation we define the response operator $R^T_* : H^1_0(\partial M \times [0,T]) \to L^2(\partial M \times [0,T])$,

$$R^T_*(g) = B^* v^g, \quad B^* v := \partial_\nu v|_{\partial M \times [0,T]}. \tag{3.7}$$

Lemma 3.1. *For any $t_0 > 0$ R^{t_0} determines $R^{t_0}_*$.*

Proof. Let $f, h \in H^1_0(\partial M \times [0,t_0])$ and let e^h be the solution of the backward wave equation

$$e^h_{tt} - \bar{b} e^h_t + a^*(x,D) e^h = 0 \text{ in } M \times [0,t_0], \tag{3.8}$$

$$e^h|_{\partial M \times [0,t_0]} = h; \quad e^h|_{t=t_0} = e^h_t|_{t=t_0} = 0. \tag{3.9}$$

Notice that for $h(t) = g(t_0 - t)$ we have $e^h(t) = v^g(t_0 - t)$. Part integration together with initial and final conditions (3.2), (3.9) yield that

$$0 = \int_0^{t_0} \int_M \left((u^f_{tt} + b u^f_t + a(x,D) u^f) \bar{e}^h - u^f \left(\overline{e^h_{tt} - \bar{b} e^h_t + a^*(x,D) e^h} \right) \right) dx dt$$

$$= \int_0^{t_0} \int_{\partial M} \left(B u^f \, \overline{e^h} - u^f \, \overline{B^* e^h} \right) dS_x dt = \int_0^{t_0} \int_{\partial M} \left(R^{t_0} f \overline{h} - f \, \overline{B^* e^h} \right) dS_x dt.$$

Since f is arbitrary and $R^{t_0} f$ is known, we can determine $B^* e^h|_{\partial M \times [0,t_0]}$ for each $h \in H^1_0(\partial M \times [0,t_0])$, i.e. to find $R^{t_0}_*$. □

3.2 Controllability results and continuation of R^T.

In the following we denote by $\mathcal{L}^s, s \in \mathbb{R}$ the subspace of functions in $H^{s+1}_0(M) \times H^s(M)$ which satisfy the natural boundary compatibility conditions for the hyperbolic problem (3.1), (3.2) for $t \notin \operatorname{supp} f$ (see e.g [12]) and by $\mathcal{L}^s_{\rm ad}$ the analogous subspace for (3.4), (3.5).

We use the following exact controllability result.

Theorem 3.2. [1] *Assume that (M,g) satisfies the Bardos-Lebeau-Rauch condition. Then*

$$\{U^f(t_1) : \; f \in H^{s+1}_0(\partial M \times [0,t_0])\} = \mathcal{L}^s, \quad t_1 \geq t_0 > t_*, s \geq 0.$$

The analogous result is valid for the adjoint equation.

Lemma 3.3. *Assume that we know R^T. For given $f, g \in H_0^1(\partial M\times[0,T])$, $t+s\leq T$ we can evaluate*

$$(J\,U^f(t), V^g(s)) =$$

$$= \int_M [u_t^f(x,t)\overline{v^g(x,s)} + u^f(t)\overline{v_s^g(x,s)} + b(x)u^f(x,t)\overline{v^g(x,s)}]dx.$$

Proof. By part integration

$$(\partial_t - \partial_s)\big(JU^f(t), V^g(s)\big) =$$

$$= \int_M [(u_{tt}^f + bu_t^f\,\overline{v^g} - u^f)(\overline{v_{tt}^g + \bar{b}v_t^g})]\,dx\, =$$

$$= -\int_{\partial M}[(u^f(t)\overline{B^*v^g(s)} - Bu^f(t)\overline{v^g(s)})]\,dS_x\, = \int_{\partial M}[R^T f(t)\,\overline{g(s)} - f(t)\overline{R_*^T g(s)}]\,dS_x. \tag{3.9'}$$

As R^T and R_*^T are known, all the functions in the last integral are known. Hence (3.9') is a differential equation along the characteristic $t+s=$ const. Furthermore,

$$\big(JU^f(0), V^g(s)\big) = \big(JU^f(t), V^g(0)\big) = 0$$

due to the initial conditions (3.2), (3.5). Equation (3.9') together with the above initial condition indicates the possibility to find $(J\,U^f(t), V^g(s))$. □

Next we prove that, in the generic case, R^t can be determined for all $t>0$.

Proof. (of Theorem 1.1) Let $\varepsilon < T/2 - t_*$, $T_0 = T/2$. We will first prove that when R^T and R_*^T are given, it is possible to find $R^{T+\varepsilon}$ and $R_*^{T+\varepsilon}$.

Clearly it is sufficient to determine $R^{T+\varepsilon}f$ for any $f \in H_0^1(\partial M \times [0,T_0])$. As $T_0 - \varepsilon > t_*$ then by Theorem 3.2 there is $\tilde{f} \in H_0^1(\partial M \times [0, T_0-\varepsilon])$ for which

$$U^f(T_0) = U^{\tilde{f}}(T_0 - \varepsilon).$$

Moreover, this function can be found by choosing $\tilde{f}$ which satisfies the following equation

$$(J\,U^f(T_0), V^g(T_0)) = (J\,U^{\tilde{f}}(T_0-\varepsilon), V^g(T_0))$$

for all $g \in H_0^1(\partial M \times [0,T_0])$.

Let now $F \in H_0^1(\partial M \times [0,T])$ be the function

$$F(x,t) = \tilde{f}(x,t) \text{ for } t\in[0,T_0-\varepsilon],\quad F(x,t) = 0 \text{ for } t\in]T_0-\varepsilon, T].$$

Let $\phi = u^f|_{t=T_0}$ and $\psi = u_t^f|_{t=T_0}$. Since u^f solves the equation

$$u_{tt}^f + bu_t^f + a(x,D)u^f = 0 \text{ in } M\times[T_0, T+\varepsilon],$$

$$u^f|_{\partial M\times[T_0,T+\varepsilon]} = 0,$$

$$u^f|_{t=T_0} = \phi,\; u_t^f|_{t=T_0} = \psi$$

and u^F solves the equation

$$u_{tt}^F + bu_t^F + a(x,D)u^F = 0 \text{ in } M \times [T_0 - \varepsilon, T]$$

$$u^F|_{\partial M \times [T_0 - \varepsilon, T]} = 0,$$

$$u^F|_{t=T_0-\varepsilon} = \phi, \ u_t^F|_{t=T_0-\varepsilon} = \psi$$

we see that

$$u^f(t+\varepsilon) = u^F(t) \text{ for } t \in [T_0 - \varepsilon, T].$$

Hence we get

$$R^{T+\varepsilon} f(\cdot\,,t) = R^T F(\cdot\,,t-\varepsilon) \text{ for } t \in [T_0, T+\varepsilon].$$

Since by assertion $(R^T F)(\cdot\,,t)$ for $t \le T$ is known, we reconstruct $R^{T+\varepsilon}$. The claim follows similarly for $R_*^{T+\varepsilon}$.

By iterating the above procedure with fixed T_0 we reconstruct $R^{T+n\varepsilon}$, $n = 0, 1, 2, \dots$. This proves Theorem 1.1. □

Analogously to Lemma 3.3, we obtain

Corollary 3.4. *Assume that DBD is given for $T > 2t_*$. Then for given $f, g \in H_0^1(\partial M \times \mathbb{R}_+)$ and $t, s > 0$ we can evaluate $(JU^f(t), V^g(s))$.*

3.4 Construction of the boundary distance functions.

Let $r_x(y), x \in M$ be the boundary distance functions

$$r_x(y) = d(x,y), \quad y \in \partial M.$$

We define a mapping $\mathcal{R} : M \to L^\infty(\partial M)$ by setting

$$\mathcal{R}(x) = r_x.$$

We are going to show that we can reconstruct the set $\mathcal{R}(M) = \{r_x : \ x \in M\}$.

In the standard Boundary Control method one constructs the projections to the spaces of the Fourier coefficients of the functions $L^2(A)$, $A \subset M$. Inspirated by this we define the following spaces.

Definition 3. *Let $H \subset \mathcal{L}^s$ be a lineal, $s \ge 0$. We define the control sets $\mathcal{H}^s(H)$ for H by*

$$\mathcal{H}^s(H) = \{ \ f \in H_0^{s+1}(\partial M \times [0, T/2]) : \ U^f(T) \in H\},$$

$$\mathcal{H}_{\text{ad}}^s(H) = \{g \in H_0^{s+1}(\partial M \times [0, T/2]) : \ V^g(T) \in H\}.$$

Let $\Gamma \subset M$ be open, $t_0 \ge 0$. Denote

$$M(\Gamma, t_0) = \{x \in M : d(x,\Gamma) \le t_0\}. \tag{3.10}$$

Definition 4. *For $s \geq 0$ let*

$$\mathcal{L}^s(\Gamma, t_0) = \{U \in \mathcal{L}^s : \text{ supp } U \subset cl(M(\Gamma, t_0))\},$$
$$[\mathcal{L}^s(\Gamma, t_0)]^c = \{U \in \mathcal{L}^s : \text{ supp } U \subset cl(M \setminus M(\Gamma, t_0))\}$$

and analogous sets $\mathcal{L}^s_{\rm ad}(\Gamma, t_0), [\mathcal{L}^s_{\rm ad}(\Gamma, t_0)]^c$.

Our next goal is to find the control sets for the above subsets of $\mathcal{L}^s$.
In the following let m_g be the Riemannian measure on (M, g).

Lemma 3.5. *Let $f \in H^{s+1}_0(\partial M \times [0, T/2]), s \geq 0$. Then for any $\Gamma \subset \partial M, t_0 \in [0, T/2]$ DBD determine whether*

$$m_g(\text{supp } U^f(T) \ \cap M(\Gamma, t_0)) = 0$$

or not. Analogous statement takes place for the adjoint solutions $V^g(T)$.

Proof. Note that $f(x, t) = 0$ for $t > T/2$. If

$$m_g(\text{supp } U^f(T) \cap M(\Gamma, t_0)) = 0$$

then by the finite velocity of the wave propagation

$$Bu^f|_{\Gamma \times [T - t_0, T + t_0]} = 0 \text{ and } f|_{\Gamma \times [T - t_0, T + t_0]} = 0.$$

On the other hand, by Tataru's Holmgren-John theorem [15] the converse is also true. By Theorem 1.1 $Bu^f|_{\partial M \times [0, 3T/2]} = R^{3T/2} f$ is known and hence the statement follows. The claim for adjoint solutions follows from Lemma 3.1. □

Corollary 3.6. *Let $\Gamma \subset \partial M, t_0 \geq 0$ and $s \geq 0$. Then DBD determine lineals $\mathcal{H}^s(\mathcal{L}^s(\Gamma, t_0)), \mathcal{H}^s([\mathcal{L}^s(\Gamma, t_0)]^c)$ and $\mathcal{H}^s_{\rm ad}(\mathcal{L}^s_{\rm ad}(\Gamma, t_0)), \mathcal{H}^s_{\rm ad}([\mathcal{L}^s_{\rm ad}(\Gamma, t_0)]^c)$.*

Proof. By Theorem 3.2,

$$\{U^f(T) : \ f \in H^{s+1}_0(\partial M \times [0, T/2])\} = \mathcal{L}^s.$$

Thus by Lemma 3.5 DBD determine $\mathcal{H}^s[\mathcal{L}^s(\Gamma, t_0)]^c$ and $\mathcal{H}^s_{\rm ad}[\mathcal{L}^s_{\rm ad}(\Gamma, t_0)]^c$.
For $f \in H^{s+1}_0(\partial M \times [0, T/2])$ we have $f \in \mathcal{H}^s(\mathcal{L}^s(\Gamma, t_0))$ if and only if

$$(J\, U^f(T), V^g(T)) = 0$$

for all $g \in \mathcal{H}^s_{\rm ad}[\mathcal{L}^s_{\rm ad}(\Gamma, t_0)]^c$. Hence we can determine $\mathcal{H}^s(\mathcal{L}^s(\Gamma, t_0))$.
The case $\mathcal{H}^s_{\rm ad}(\mathcal{L}^s_{\rm ad}(\Gamma, t_0))$ can be considered analogously. □

Corollary 3.7. *Let $\Gamma_i \subset \partial M, t_i^+ > t_i^- \geq 0;\ i = 1, ..., I$. Denote by M_I the set*

$$M_I = \bigcap_{i=1}^{I} (M(\Gamma, t_i^+) \setminus M(\Gamma, t_i^-)). \tag{3.11}$$

Then DBD determine whether $m_g(M_I) = 0$ or not.

Proof. Using intersections of sets described in Corollary 3.6 we find whether $\mathcal{L}^s$ contains functions supported in the closure of M_I. That kind of functions exists if and only if $m_g(M_I) \neq 0$. □

Corollary 3.7 is the basic analytic tool in reconstruction of $\mathcal{R}(M)$.

Theorem 3.8. *DBD with $T > 2t_*$ determines $R(M)$ uniquely.*

Proof. For every $\varepsilon > 0$ we choose a collection $\Gamma_i \subset \partial M, i = 1, ..., I(\varepsilon)$ such that $\mathrm{diam}(\Gamma_i) \leq \varepsilon$, $\cup\Gamma_i = \partial M$. Let

$$p = (p_1, ..., p_{I(\varepsilon)}), \quad p_i \in \mathbb{Z}_+, \quad t_i^+ = (p_i + 1)\varepsilon, \quad t_i^- = (p_i - 1)\varepsilon. \tag{3.12}$$

Denote by $M(\varepsilon, p)$ the set M_I (see (3.11)) with $t_i^\pm$ of form (3.12). For every p we define a piecewise constant function $r_p \in L^\infty(\partial M)$ by setting $r_p(y) = p_i\varepsilon$ when $y \in \Gamma_i$. Using Corollary 3.7 we define whether $m_g(M(\epsilon, p)) > 0$ or not and introduce a set

$$\mathcal{R}_\varepsilon(M) = \{r_p : \ p \in \mathbb{Z}_+^{I(\varepsilon)} \text{ such that } m_g(M(\varepsilon, p)) > 0\} \subset L^\infty(\partial M).$$

As $||r_x - r_p|| < 2\varepsilon + \max \mathrm{diam}\ (\Gamma_i)$ when $x \in M(\varepsilon, p)$, then

$$\mathrm{dist}_H(\mathcal{R}_\varepsilon(M), \mathcal{R}(M)) \leq 3\varepsilon.$$

Here $\mathrm{dist}_H(\Omega, \widetilde{\Omega})$ is the Hausdorff distance between subsets $\Omega, \widetilde{\Omega} \in L^\infty(\partial M)$. When $\varepsilon \to 0$ we find the set $\mathcal{R}(M) \subset L^\infty(\partial M)$ as the limit of $\mathcal{R}_\varepsilon(M)$. ☐

Let $\mathcal{R}(M) \subset L^\infty(\partial M)$ be given. It is shown in [7] that then it is possible to uniquely define a Riemannian structure on $R(M)$ such that $\mathcal{R} : M \to \mathcal{R}(M)$ is an isometry. For the sake of completeness, we construct (M, g) explicitly. To this end we need the following result (see [7]).

Lemma 3.9. *$\mathcal{R}(M) \subset L^\infty(\partial M)$ is homeomorphic to M.*

Proof. Obviously $\mathcal{R}$ is continuous. Assume that $r_x = r_y,\ x, y \in M$. If $z \in \partial M$ is a nearest point to x, r_x achieves the minimum $h = r_x(z)$ at z. Thus x lies on the normal geodesic from z and $x = \exp_z(h\nu)$, exp being the standard exponential map on TM. The same holds for y and hence $\mathcal{R} : M \to \mathcal{R}(M)$ is one-to-one. By definition it is onto. Since M is compact, $\mathcal{R}$ is a homeomorphism. ☐

3.5. Reconstruction of the Riemannian metric and the operator.

Let $f, g \in H_0^{s+1}(\partial M \times [0, T/2]),\ s \geq 0$. We define a bilinear form

$$\langle f, g\rangle \ = (J\,U^f(T), V^g(T)).$$

Let

$$\mathcal{R}(\varepsilon, p) = \mathcal{R}(M(\varepsilon, p)),\ \ \varepsilon > 0,\, p \in \mathbb{Z}_+^{I(\varepsilon)}. \tag{3.13}$$

Here $M(\varepsilon, p)$ is defined as in the proof of Theorem 3.8, i.e. $\mathcal{R}(\varepsilon, p)$ is the set of all boundary distance functions r_x with $x \in M(\varepsilon, p) \subset M$.

Let $r_{x_0} \in \mathcal{R}(M \backslash \partial M)$. Then for any ε there exists $p_\varepsilon \in \mathbb{Z}_+^{I(\varepsilon)}$ such that $x_0 \in M(\varepsilon, p_\varepsilon)$ and

$$\mathcal{R}(\varepsilon, p_\varepsilon) \longrightarrow \{r_{x_0}\} \text{ when } \varepsilon \to 0,$$

i.e. the Hausdorff distance between the above sets goes to 0 when $\varepsilon \to 0$. By Lemma 3.9, this yields that

$$M(\varepsilon, p_\varepsilon) \longrightarrow \{x_0\} \text{ when } \varepsilon \to 0. \tag{3.14}$$

Denote by $g(\varepsilon),\ \varepsilon > 0$ a family of functions in $H^1_0(\partial M \times [0,T/2])$ such that

i. supp $V^{g(\varepsilon)}(T) \subset$ cl $(M(\varepsilon, p_\varepsilon))$.

ii. For any $f \in H^{s+1}_0(\partial M \times [0,T/2]),\ s < m/2 < s+1$ there exists a limit

$$\mathcal{W}^{x_0}(f) = \lim_{\varepsilon\to 0}\langle f, g(\varepsilon)\rangle.$$

Such families exist, indeed it is sufficient to take $V^{g(\varepsilon)}$ to be C^∞_0-approximations to $(0, \delta(\cdot - x_0))$. On the other hand, assume that for every $f \in H^{s+1}_0(\partial M \times [0,T/2])$ the limit

$$\lim_{\varepsilon\to 0}\langle f, g(\varepsilon)\rangle = \lim_{\varepsilon\to 0}(J\,U^f(T), V^{g(\varepsilon)}(T))$$

exists. Then by Banach-Steinhaus theorem there is $W^{x_0} \in (\mathcal{L}^s)' \subset H^{s+1}_0(M)' \times H^s_0(M)'$ such that

$$\lim_{\varepsilon\to 0}\langle f, g(\varepsilon)\rangle = (J\,U^f(T), W^{x_0}),$$

where the right hand side is interpreted in the distribution sense. Assumption i. together with (3.14) imply that supp $(W^{x_0}) \subset \{x_0\}$. Since any distribution supported in a point is a finite sum of derivatives of the delta-distribution, and since $W^{x_0} \in H^s_0(M)' \times H^{s+1}_0(M)',\ s < m/2 < s+1$, it follows that there is a constant $\kappa(x_0)$ that

$$W^{x_0} = \begin{pmatrix} 0 \\ \kappa(x_0)\delta(\cdot - x_0) \end{pmatrix}.$$

Lemma 3.10. *Let DBD be given for $T > 2t_*$. Assume that (M,g) satisfies the Bardos-Lebeau-Rauch condition. Then it is possible to construct functions $g(\varepsilon)$ such that*

$$\mathcal{W}^{x_0}(f) = \kappa(x_0)u^f(x_0,t), \quad f \in H^{s+1}_0(\partial M \times [0,T/2]),\, t \geq 0,\, s < m/2 < s+1$$

and

$$\kappa \in C^0(M), \quad \kappa|_{\partial M} = 1, \quad \kappa \neq 0 \quad \text{on } M. \tag{3.15}$$

Proof. To prove the statement is sufficient to show that for any $r_{x_0} \in \mathcal{R}(\mathrm{int}(M))$ it is possible to find a family $g_{x_0}(\varepsilon), \varepsilon > 0$ such that the corresponding $\mathcal{W}^{x_0}$ satisfy the following conditions

iii. $\mathcal{W}^{x_0} \neq 0$ for any $x_0 \in M$.

iv. The function $r_{x_0} \mapsto \mathcal{W}^{x_0}(f)$ has a continuous extension to $\mathcal{R}(M)$ when $f \in C^\infty_0(\partial M \times [0,T])$.

v. For $f \in C^\infty_0(\partial M \times [0,T])$ and $x_1 \in \partial M$

$$\lim_{x_0\to x_1}\mathcal{W}^{x_0}(f) = f(x_1,T).$$

As we already know such sequence exists. Indeed, we can take functions $g_{x_0}(\varepsilon)$ such that $V^{g_{x_0}(\varepsilon)}(T)$ are smooth approximations to $(0, \delta(\cdot - x_0))^t$. On the other hand, Corollary 3.4 makes possible to algorithmically verify conditions iii.-v. □

Corollary 3.11. *Let DBD and $r_{x_0} \in \mathcal{R}(M)$ be given. These data determine $\kappa(x)u^f(x_0,t)$ for any $t>0$ and $f \in H_0^1(\partial M \times \mathbb{R}_+)$.*

Proof. The statement follows from Corollary 3.4 and Lemma 3.10. □

We want to emphasize that we do not know $\kappa(x)$ and, henceforth, can not reconstruct $u^f(x,t)$ using Lemma 3.10. However, we have the following

Theorem 3.12. *The DBD determines a metric E on $\mathcal{R}(M)$ such that $(\mathcal{R}(M),E)$ is isometric to (M,g).*

Proof. Let $r_x, r_y \in \mathcal{R}(int(M))$ and let $\mathcal{R}(\varepsilon, p_\varepsilon)$ (see (3.13)) be a sequence satisfying

$$\mathcal{R}(\varepsilon, p_\varepsilon) \to \{r_x\}$$

when $\varepsilon \to 0$. We denote $h_\varepsilon = \text{diam}\, M(\varepsilon, p_\varepsilon)$. By Corollary 3.7, we can construct the set

$$X(\varepsilon) = \{f \in H_0^{s+1}(\partial M \times [0,T/2]) :\ \text{supp}\, U^f(T) \subset M(\varepsilon, p_\varepsilon)\}. \tag{3.17}$$

Let $\tau > 0$. Assume that $d(x,y) > \tau$. Then due to finite velocity of the wave propagation and the fact that $h_\varepsilon \to 0$ there is ε_0 such that for $\varepsilon < \varepsilon_0$ we have:

(A) There is a neighborhood N of y such that for any $f \in X(\varepsilon)$

$$U^f|_{N\times]T,T+\tau[} = 0.$$

Using Lemma 3.5 we can check if the property (A) is satisfied.

Let now $s(r_x,r_y)$ be the supremum of all $\tau > 0$ for which the property (A) is satisfied with some $\varepsilon > 0$. Then

$$s(r_x,r_y) \geq d(x,y). \tag{3.16}$$

On the other hand, assume that x and y are so near to each other that $d(x,y) < d(x,\partial M)/2$ and there is an unique minimal geodesic $\gamma(t) = \exp_x(tv)$ from x to y. Let $\tau > d(x,y)$. Then for every $\varepsilon > 0$ there is a solution $(u^f(x,T),0)$, $f \in X(\varepsilon)$ such that $(x,T,v,1) \in T^*(M \times \mathbb{R}_+)$ is in the wavefront set of u^f. By standard theory of propagation of singularities,

$$\text{singsupp}\ u^f \cap \{y\}\times]T,T+\tau[\neq \emptyset.$$

Thus the function u^f can not vanish in any neighborhood of $y\times]T,T+\tau[$ and the property (A) is not satisfied with any ε. Thus $s(r_x,r_y) \leq d(x,y)$. Hence for y sufficiently close to x we have the equality in (3.16).

Define the metric

$$E(r_x,r_y) := \inf\{\sum_{j=0}^{l} s(r_{y_j}, r_{y_{j+1}}) :\ x_0 = x,\ y_l = y,\ y_j \in \text{int}\ (M),\ l \geq 1\}.$$

For any curve $\gamma \subset \text{int}(M)$, we see that the E-length of $\mathcal{R}(\gamma)$ is equal to the length of γ. Hence $E(r_x,r_y) = d(x,y)$ for any $x,y \in \text{int}(M)$. By continuing E onto $\mathcal{R}(\partial M)$ we obtain $(\mathcal{R}(M),E)$ which is isometric to (M,g). □

Thus $(\mathcal{R}(M),E)$ can be identified with (M,g) as a metric space. In order to construct local coordinates on $\mathcal{R}(M)$, we start with constructing geodesics. By using triangular comparison theorems we can find the angles of intersecting geodesics. This defines normal coordinates near any $r_x \in \mathcal{R}(M)$ and, henceforth the differentiable structure on $\mathcal{R}(M)$.

Using this structure, we can go back to Lemma 3.10 and demand (see iv. in the proof) that $\kappa \in C^\infty(M)$.

Lemma 3.13. *The functions* $e^f(x,t) = \kappa(x)u^f(x,t)$, $x \in M$, $t \geq 0$ *with* $f \in H_0^{s+1}(\partial M \times [0,T/2])$ *and* $\kappa \in C^\infty(M)$ *of form (3.15) determine* $a_\kappa(x,D)$ *and* $b(x)$.

Proof. The functions $e^f(x,t) = \kappa(x)u^f(x,t)$ are the solutions of the initial boundary value problem (see (1.5))

$$e^f_{tt} + be^f_t + a_\kappa(x,D)e^f = 0, \tag{3.17}$$

$$e^f|_{\partial M\times\mathbb{R}_+} = f; \quad e^f|_{t=0} = e^f_t|_{t=0} = 0.$$

However, Theorem 3.2 implies that for any $x_0 \in \text{int}(M)$ the vectors

$$\left(e^f(x_0,T), \partial_j(e^f(x_0,T)), \partial_k\partial_l(e^f(x_0,T))), e^f_t(x_0,t)\right)_{j,k,l=1}^m$$

span the space $\mathbb{C}^{(m^2+3m+4)/2}$ when $f \in C_0^\infty(\partial M \times [0,T])$. Hence equation (3.17) may be used to determine b and $a_\kappa(x,D)$.

Theorem 1.2 is proven. □

4. Results for one measurement and further remarks..

In the first part of this section we analyse the possibility of the reconstruction of the response operator R^{t_0} using only one measurement.

Theorem 4.1. *For any* $t_0 > 0$ *there is* $f \in H^1_{loc}(\partial\Omega \times \mathbb{R}_+)$, $f|_{t=0} = 0$, *such that* $\partial_\nu u^f|_{\partial\Omega\times\mathbb{R}_+}$ *determines* R^{t_0}.

Proof. Our main tool is the consequence of energy inequality (see e.g. [10]),

$$||\partial_\nu u^f||_{L^2(\partial M\times[0,t])} \leq c_0 e^{c_1 t}||f||_{H^1_0(\partial M\times[0,t])}, \quad f \in H^1_0(\partial M \times [0,T]), \tag{4.1}$$

where c_0 and c_1 are independent of t.

For $t_0 > 0$ let $(f_j:\ j = 1,\dots)$ be an orthonormal basis of $H^1_0(\partial M \times [0,t_0])$. Let $g_n, n = 1,2,...$ be a sequence where each f_j occurs infinitely many times. Consider

$$f(x,t) = \sum_{n=1}^\infty e^{cn^2}\, g_n(x,t-nt_0)$$

with $c > c_1 t_0$ where c_1 is the constant in (4.1). Assume that $\partial_\nu u^f|_{\partial M\times\mathbb{R}_+}$ is known. By inequality (4.1) we see that

$$||e^{-cn^2}\partial_\nu u^f(x,t+nt_0)|_{\partial M\times[0,t_0]} - (R^{t_0}g_n)(x,t)||_{L^2} \leq c'\, ne^{-c_1 n t_0}.$$

As $ne^{-c_1 n t_0} \to 0$ when $n \to \infty$, this shows that we can determine all $R^{t_0}f_j$, $j = 1,2,\dots.$ □

Corollary 4.2. *Let, in generic case, (M, g) satisfy the Bardos-Lebeau-Rauch condition. There is $f \in H^1_{loc}(\partial\Omega \times \mathbb{R}_+)$, $f|_{t=0} = 0$, such that $\partial_\nu u^f|_{\partial\Omega\times\mathbb{R}_+}$ determines M, b and the equivalence class $[a(x, D)]$ uniquely.*

In the self-adjoint case the Bardos-Lebeau-Rauch condition is unnecessary.

We conclude the paper with several remarks:

i. The Bardos-Lebeau-Rauch condition is always satisfied for $M \subset \mathbb{R}^m$ with the metric $g^{jl} = \delta^{j,l}$ or its C^1-small perturbations (see e.g. [16]);

ii. In the case $b = 0$ but $a(x, D) \neq a^*(x, D)$ an analog of Theorem 1.1 states that given R^T for $T > t_*$ determines R^t for all t. Indeed, in this case we can use a sesquilinear form $u^f_t(t)\overline{v^g(t)} - u^f(t)\overline{v^g_t(t)}$. Then an analog of lemma 3.3 states that given R^T it is possible to find the value of this form for $t \leq T$. Further proof of Theorem 1.1 (with $T > t_*$ instead of $T > 2t_*$) follows as in §3.

iii. The present work remains open the question what is the minimum time T needed to reconstruct the manifold and the operator. Indeed, in the case $b = 0$, as we have just shown, $T > t_*$ is sufficient. In the selfadjoint case $T > 2r$ is sufficient where r is the geodesic radius of (M, g), $r \leq t_*/2$. Moreover, it is known that in the one-dimensional case when $2r = t_*$ the case $b \neq 0$ doesneed time $T > 2t_*$.

v. Clearly the considerations of the paper remain valid for (M, g) satisfying the Bardos-Rauch-Lebeau conditions for a part of boundary $\Gamma \subset \partial M$.

iv. Corollary 1.3 remains open in the question if there is $f \in H^1_0(\partial M \times \mathbb{R}_+)$, that is, a boundary source with finite energy which determines R^T. By modifying the proof of Corollary 1.3 we see that this is true if $c_1 < 0$ in inequality (4.1).

vi. Instead the boundary operator $B = \partial_\nu - P_\nu$ we can use $B = \partial_\nu - \beta$, where β is an arbitrary complex-valued C^∞-function on ∂M.

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