Quasiinvariants of Coxeter groups and m-harmonic polynomials

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Abstract

The space of m-harmonic polynomials related to a Coxeter group G and a multiplicity function m on its root system is defined as the joint kernel of the properly gauged invariant integrals of the corresponding generalised quantum Calogero-Moser problem. The relation between this space and the ring of all quantum integrals of this system (which is isomorphic to the ring of corresponding quasiinvariants) is investigated.

1 Introduction

Let G be any Coxeter group, i.e. a finite group generated by reflections with respect to some hyperplanes in a Euclidean space V of dimension n. Let Σ be a set of the hyperplanes $\Pi_\alpha : (\alpha, x) = 0$ corresponding to all the reflections $s_\alpha \in G$ and let A be a set of the corresponding (arbitrarily chosen) normals α. Let us also consider a G-invariant $\mathbb{Z}_{\geq 0}$ -valued function m on Σ which will be called *multiplicity*. In other words to any hyperplane $\Pi_\alpha \in \Sigma$ we prescribe a nonnegative integer m_α such that if $\Pi_\alpha = g(\Pi_\beta)$, $g \in G$ then $m_\alpha = m_\beta$.

Let $S = S(V)$ be the ring of all polynomials on V, S^G be the subring of G-invariant polynomials. According to classical Chevalley result [1] S^G is freely generated by some homogeneous polynomials $\sigma_1, \dots, \sigma_n$.

The main object of our investigation is the following subring $Q_m = Q_m(\Sigma) \subset S(V)$. It consists of the polynomials q which are invariant up to order $2m_\alpha$ with respect to any reflection s_α:

$$q(s_\alpha(x)) = q(x) + o\left((\alpha, x)^{2m_\alpha}\right) \quad (1)$$

near the hyperplane $(\alpha, x) = 0$ for any $\alpha \in A$. Equivalently, for any $\alpha \in A$ the normal derivatives $\partial_\alpha^s q = (\alpha, \frac{\partial}{\partial x})^s q$ must vanish on Π_α for $s = 1, 3, 5, \dots, 2m_\alpha - 1$:

$$\partial_\alpha^s q|_{\Pi_\alpha} = 0.$$

We will call these polynomials m*–quasiinvariants* of the Coxeter group G or simply *quasiinvariants.*

The rings Q_m have been introduced in the theory of quantum Calogero–Moser systems by O.Chalykh and one of the authors [2]. It has been shown [2, 3] that for any Coxeter group G and any integer-valued multiplicity function m there exists a homomorphism

$$\varphi_m : Q_m \to D_\Sigma(V),$$

where $D_\Sigma(V)$ is the ring of all differential operators in V with rational coefficients from the algebra generated by $(\alpha, x)^{-1}, \alpha \in A$ and constant functions (see the next section for the details). In particular, for $q = x^2$ (which is obviously invariant and therefore quasiinvariant) the corresponding operator $\varphi_m(q)$ is the generalised Calogero–Moser operator

$$L = \Delta - \sum_{\alpha \in A} \frac{m_\alpha(m_\alpha + 1)(\alpha, \alpha)}{(\alpha, x)^2} \tag{2}$$

first introduced by Olshanetsky and Perelomov [4]. It will be more convenient for us to use the gauge transformation $\mathcal{L} = \hat{g} L \hat{g}^{-1}$, where $\hat{g}$ is the operator of multiplication by $g = \prod (\alpha, x)^{m_\alpha}$, after which the operator L takes the form

$$\mathcal{L} = \Delta - \sum_{\alpha \in A} \frac{2m_\alpha}{(\alpha, x)} \partial_\alpha. \tag{3}$$

This gauge is natural from the point of view of the theory of symmetric spaces where such operators appear as the radial parts of the Laplace–Beltrami operators (see e.g. [5]). Let

$$\chi_m : Q_m \to D_\Sigma(V)$$

be the corresponding gauged version of the homomorphism φ_m:

$$\chi_m(q) = \hat{g} \varphi_m(q) \hat{g}^{-1}.$$

We should mention that for a generic (not necessary integer-valued) multiplicity function there exists an isomorphism (sometimes called as Harish-Chandra isomorphism, see [6, 7]) between S^G and the ring D_m^G of G-invariant quantum integrals of the Calogero-Moser problem (2):

$$\gamma_m : S^G \cong D_m^G,$$

where
$$D_m^G = \{\mathcal{D} \in D_\Sigma(V) : [\mathcal{L}, \mathcal{D}] = 0, g(\mathcal{D}) = \mathcal{D} \text{ for all } g \in G\}.$$
As a corollary we have usual integrability for (3) with n commuting quantum integrals $\mathcal{L}_1 = \mathcal{L}, \mathcal{L}_2, ..., \mathcal{L}_n$ corresponding to some basic invariants $\sigma_1 = x^2, \sigma_2, ..., \sigma_n$.

An important novelty of [2] was the possibility of the extension of γ_m to a much bigger ring Q_m in the case when all m_α are integer, which implies the algebraic integrability of the corresponding quantum Calogero-Moser problem (see [3] for details).

The primary goal of the present paper is to explain the relation between these additional quantum integrals of Calogero-Moser problem and the joint kernel of its standard invariant integrals $\mathcal{L}_1, ..., \mathcal{L}_n$. Let us define the space H_m as the solutions of the following system
$$\begin{cases} \mathcal{L}_1 \psi = 0 \\ \cdots\cdots\cdots \\ \mathcal{L}_n \psi = 0 \end{cases} \tag{4}$$
We will show that all the solutions of (4) are polynomial and that all these polynomials are m-quasiinvariant. We will call these polynomials *m-harmonic*. In the case $m = 0$ we have the space of the usual harmonic polynomials related to Coxeter group G (see e.g [5]).

The following linear map π_m from Q_m to H_m will play the central role in our considerations. Let us introduce m-discriminant
$$w_m = \prod_{\alpha \in A} (\alpha, x)^{2m_\alpha + 1}$$
which obviously is m-quasiinvariant. We will show that w_m is also m–harmonic. Let q be any quasiinvariant, $\mathcal{L}_q = \chi_m(q)$ be the corresponding differential operator. The map π_m is defined by the formula
$$\pi_m(q) = \mathcal{L}_q(w_m).$$
Since $\mathcal{L}_q$ commutes with $\mathcal{L}_i$ it preserves the space H_m, so $\pi_m(q) \in H_m$. The question now is what is the kernel of π_m. It is easy to show that the kernel of π_m contains the ideal $I_m \subset Q_m$ generated by the invariants $\sigma_1, \ldots, \sigma_n$.

In the preprint [8] we have conjectured that the following statements are true for any Coxeter group G and multiplicity function m.

Conjecture 1 *Kernel of π_m coincides with the ideal I_m.*

Consider the restriction of the map π_m onto the subspace $H_m \subset Q_m$.

Conjecture 2 *The linear map*

$$\pi_m|_{H_m} : H_m \to H_m$$

is an isomorphism.

As a corollary we have the following isomorphism

$$Q_m/I_m \cong H_m,$$

and the fact that Q_m is generated by H_m over S^G.

Conjecture 3 *The ring Q_m is a free module over S^G generated by any basis in H_m.*

In algebraic terminology this implies that Q_m is a Cohen-Macauley ring. We have conjectured also the following stronger version of Conjecture 2, which implies that Q_m is also Gorenstein.

Let us introduce the following bilinear form on the space H_m:

$$< p, q >= (\mathcal{L}_p \mathcal{L}_q w_m)(0),$$

where $\mathcal{L}_p = \chi_m(p), \mathcal{L}_q = \chi_m(q)$.

Conjecture 2* *The form $<,>$ on H_m is non-degenerate.*

This implies Conjecture 2. Indeed, if $q \in Ker \pi_m|_{H_m}$ then by definition $\mathcal{L}_q(w_m) = 0$ and therefore $< q, p >= 0$ for all $p \in H_m$. Conjecture 2^* also implies that the dimensions h_k of the spaces of m-harmonics of degree k satisfy the duality relation

$$h_k = h_{\mathcal{N}-k},$$

where $\mathcal{N} = \sum(2m_\alpha + 1)$ is the degree of w_m.

A month later after our preprint had appeared on the web P.Etingof and V.Ginzburg made a substantial progress in this direction [9]. They have proved that Conjecture 1 is indeed true for any Coxeter group and multiplicity function m and showed that Q_m is Cohen-Macauley and Gorenstein in general case. Conjectures 2 and 3 in general case turned out to be wrong. The counterexample found in [9] is related to the Coxeter group G of type B_6 with the multiplicity function being equal to 0 on the long roots and to 1 on the short roots. In this case the m-harmonic polynomials are dependent over S^G and the space H_m has a nontrivial intersection with the ideal I_m.

However we believe that such cases are very exceptional and in particular all the conjectures are true if *all the multiplicities are equal.* In this paper which is a slightly revised version of [8] we prove this for all the Coxeter groups of rank 2 (i.e. for the dihedral groups $I_2(N)$). In this case we describe all the m-harmonic polynomials explicitly.

As a corollary we show that the Poincare series for the quasiinvariants of dihedral group $I_2(N)$ is given by

$$p(Q_m^{I_2(N)}, t) = \frac{1 + 2t^{(mN+1)} + \ldots + 2t^{(mN+N-1)} + t^{(2m+1)N}}{(1-t^2)(1-t^N)}.$$

Notice that the Poincare series $p(Q_m, t)$ of the quasiinvariants of any Coxeter group and Poincare polynomial of corresponding m-harmonics $P(H_m, t) = \sum_{k=0}^{\mathcal{N}} h_k t^k$ are related by the following formula conjectured in [8] and proved in [9]:

$$p(Q_m, t) = \frac{P(H_m, t)}{\prod_{i=1}^{n}(1 - t^{d_i})},$$

where $d_i = \deg \sigma_i$ are the degrees of the Coxeter group. Some formulas for the polynomials $P(H_m, t)$ have been recently found in [10][1]. In particular, they are always palindromic:

$$P(H_m, t^{-1}) = t^{-\mathcal{N}} P(H_m, t),$$

which is related to Gorenstein property of Q_m.

2 Quasiinvariants and quantum integrals of Calogero–Moser systems

Let us first discuss the homomorphism χ_m in more details. We will need some facts from the theory of multidimensional Baker–Akhiezer functions related to a Coxeter configuration of hyperplanes (see [2], [3], [13]). Let us remind that we are using the gauge which is different from the chosen in these papers by $g(x) = \prod(\alpha, x)^{m_\alpha}$.

For any Coxeter group G and multiplicity function m there exists a Baker–Akhiezer function (BA function) of the form

$$\psi = P(k, x) e^{(k,x)} \tag{5}$$

with the following properties:

- $P(k, x)$ is a polynomial in $k \in V$ and $x \in V$ with the highest term

$$g(k)g(x) = \prod_{\alpha \in A} (\alpha, k)^{m_\alpha} (\alpha, x)^{m_\alpha}.$$

[1] As we have recently learnt from I. Cherednik such formulas can be extracted from Opdam's papers [11, 12].

- ψ satisfies the quasiinvariance conditions in k-space

$$\psi(s_\alpha(k)) - \psi(k) = o((\alpha, k)^{2m_\alpha}) \text{ near } (\alpha, k) = 0.$$

It is known that such function does exist, and is unique and symmetric with respect to x and k:

$$\psi(k, x) = \psi(x, k)$$

(see [2], [3]). As it has been explained in [2] for any quasiinvariant $q \in Q_m$ there exists a differential operator $\chi_m(q) = \mathcal{L}_q(x, \frac{\partial}{\partial x})$ such that

$$\mathcal{L}_q(x, \frac{\partial}{\partial x})\psi(x, k) = q(k)\psi(x, k).$$

The procedure of finding $\mathcal{L}_q$ is effective provided the formula for ψ is given. Since for $q = k^2$ we have the (gauged) Calogero-Moser operator (3) we have the following

Theorem 1 [2, 3] *For any Coxeter group G and integer-valued multiplicity m there exists a homomorphism $\chi_m : Q_m \to D_\Sigma(V)$ mapping the algebra of quasiinvariants Q_m into the commutative algebra of quantum integrals of generalised Calogero–Moser problem.*

One can write down the following explicit formula for this homomorphism suggested by Yu. Berest [14]:

$$\chi_m(q) = c(ad_{\mathcal{L}})^{d(q)}\hat{q}. \tag{6}$$

Here $\mathcal{L}$ is the gauged Calogero-Moser operator (3), $ad_L A = LA - AL$, $\hat{q}$ is the operator of multiplication by q, $d(q)$ is degree of polynomial q and the constant $c = c(q) = (2^{d(q)}d(q)!)^{-1}$.

Indeed because of the symmetry of ψ with respect to x and k we have

$$\mathcal{L}(k, \frac{\partial}{\partial k})\psi(x, k) = x^2\psi(x, k).$$

Thus ψ satisfies the so-called bispectral problem in the sense of Duistermaat and Grünbaum and one can use the general identity (1.8) from their paper [15] which states that

$$(ad_{\mathcal{L}})^r(q)[\psi] = (ad_{\hat{x}^2})^r(\mathcal{L}_q)[\psi].$$

For $r = \deg q$ we arrive at the formula (6).

As we have mentioned in the Introduction the restriction χ_m onto the subring of invariants S^G gives an isomorphism γ_m between S^G and the ring D_m^G of invariant integrals of the Calogero-Moser quantum problem in the gauge (3). The following result shows that the map χ_m defined in the Theorem 1 is in a certain sense the maximal extension of this map.

Let D_m be the maximal commutative ring of differential operators on V with rational coefficients which contains D_m^G as a subring.

Theorem 2 *The map χ_m is an isomorphism between the ring of m-quasiinvariants Q_m and the ring D_m.*

The proof follows from the following lemma. Let $\sigma_1 = k^2, \sigma_2, ..., \sigma_n$ be some generators of S^G, and let $\mathcal{L}_1 = \chi_m(\sigma_1) = \mathcal{L}, \ldots, \mathcal{L}_n = \chi_m(\sigma_n)$ be the corresponding invariant integrals of the Calogero-Moser problem.

Lemma 1 *Let A be a differential operator commuting with all $\mathcal{L}_i, i = 1, ..., n$. Then $A = \mathcal{L}_q = \chi_m(q)$ for some quasiinvariant $q \in Q_m$.*

To prove this let us notice that since A commutes with all $\mathcal{L}_i$ it preserves their joint eigenspace $V(k)$ consisting of the solutions of the system

$$\begin{cases} \mathcal{L}_1\psi = \sigma_1(k)\psi \\ \ldots\ldots\ldots\ldots\ldots \\ \mathcal{L}_n\psi = \sigma_n(k)\psi \end{cases} \tag{7}$$

where $k \in V$ is a "spectral" parameter. For generic k this space is spanned by the Baker-Akhiezer functions $\psi(x, g(k)), g \in G$. From the form (5) of this function it follows that ψ itself must be an eigenvector of A:

$$A\psi(x, k) = a(k)\psi(x, k)$$

for some polynomial $a(k)$. To show that $a(k)$ is a quasiinvariant let us notice that the left hand side of the last formula satisfies the quasiinvariance conditions in k (see the properties of the BA function) and therefore must the right hand side. A simple analysis shows that $a(k)$ must be a quasiinvariant in that case.

Another relation between quasiinvariants and quantum integrals $\mathcal{L}_q$ is given by the following

Theorem 3 *The space Q_m of all m-quasiinvariants is invariant under the action of all the operators $\mathcal{L}_q$, $q \in Q_m$.*

For the operator $\mathcal{L}$ (3) this can be proven by direct local considerations (c.f. [16] where a similar observation has been first made). The fact that the same is true for any $\mathcal{L}_q$ now follows from Berest's formula (6).

3 m-harmonic polynomials

Consider again the space $V(k)$ of the solutions of compatible system of equations (7). Let us put now k to be zero, i.e. consider the system

$$\begin{cases} \mathcal{L}_1\psi = 0 \\ \ldots\ldots\ldots \\ \mathcal{L}_n\psi = 0 \end{cases} \tag{8}$$

We claim that all the solutions of the system (8) are polynomials in x. More precisely we have the following

Theorem 4 *For any Coxeter group G and multiplicity function m all the solutions of the system (8) are polynomial. They form the space of dimension $|G|$ where the natural action of G is its regular representation.*

When all the multiplicities are zero this is the classical result (see [17, 5]) and the corresponding polynomials are called harmonic. For the general multiplicity m we will call the corresponding solutions of (8) as *m-harmonic polynomials* and denote the space $V(0)$ as H_m.

To prove the theorem let us consider first the general system (7). Heckman and Opdam [6] showed that it is equivalent to a holonomic system of the first order of rank $|G|$. Components of this system are $\phi = (\phi_i)$, $\phi_i = q_i(\partial)\psi$, where q_i, $i = 1, \dots, |G|$ is a basis of harmonic polynomials of G (see [6]).[2]

For generic k (more precisely, if $\prod_\alpha (k, \alpha) \neq 0$) we can choose the functions $\psi_\sigma = \psi(\sigma(k), x), \sigma \in G$, where ψ is the Baker–Akhiezer function (5) from the previous section, as a basis of the correspondent space $V(k)$. Since BA function is regular everywhere as a function of x, the same is true for the solutions of (7) if $\prod_\alpha (k, \alpha) \neq 0$.

To prove that this is true for any k, in particular for $k = 0$, one can argue as follows. Let us consider the natural complex version of the system (7) by assuming simply that $x \in V^{\mathbb{C}}$ and ψ takes values in $\mathbb{C}$. Consider a point x_0 such that $\prod_\alpha (x, \alpha) \neq 0$ and fix the solution of (7) $\psi(k, x; x_0, a)$, $a \in \mathbb{C}^{|G|}$ by fixing the initial data in the corresponding holonomic system $\phi_i(x_0) = a_i$. Since the system (7) (and the corresponding holonomic system) is regular in k everywhere $\psi(k, x; x_0, a)$ is analytic in k everywhere for any x such that $\prod_\alpha (x, \alpha) \neq 0$. By Hartogs theorem (see e.g. [18]), $\psi(k, x; x_0, a)$ is analytic in k and x everywhere. In particular, $\psi(0, x; x_0, a)$ is analytic in x at $x = 0$. Since the system (8) is homogeneous, any component in the Taylor expansion of this function at $x = 0$ of a given degree d is as well a solution of this system. This proves that all the solutions of (8) are polynomial.

To prove the second statement of the theorem let us notice that a natural action of the group G on the space $V(k)$ for a generic k is regular. This immediately follows from the formula for a basis

$$\psi_\sigma(x, k) = \psi(x, \sigma(k)), \ \sigma \in G$$

in terms of BA function. Indeed, for any $\tau \in G$

$$\psi_\sigma(\tau^{-1}(x), k) = \psi(\tau^{-1}(x), \sigma(k)) = \psi(x, (\tau \circ \sigma)(k)) = \psi_{\tau\sigma}(x, k)$$

[2] Strictly speaking the authors of [6] considered the trigonometric analogues of (7), (8) related to Weyl groups; the systems (7), (8) for all Coxeter groups (with generic parameters m_α) have been discussed in details later by E. Opdam in [11]. Corresponding holonomic systems have been recently rewritten in explicit way as a version of Knizhnik–Zamolodchikov equation in [10].

since $\psi(\tau x, \tau k) = \psi(x, k)$. For arbitrary k this follows now by standard continuation arguments.

4 Quasiinvariants and m-harmonic polynomials

The first relation between the space H_m of m-harmonic polynomials and the space Q_m of m-quasiinvariants is given by the following

Theorem 5 *Any m-harmonic polynomial is m-quasiinvariant: $H_m \subset Q_m$.*

We will prove actually the following more general statement.

Proposition 1 *Any polynomial $p(x)$ belonging to the kernel of the operator*

$$\mathcal{L} = \Delta - \sum_{\alpha \in A} \frac{2m_\alpha}{(\alpha, x)} \partial_\alpha$$

is a quasiinvariant.

Let us deduce the quasiinvariance condition (1) for polynomial $p(x)$ at the hyperplane $(\alpha, x) = 0$. Choose an orthogonal coordinate system $(t, y_1, \ldots, y_{n-1})$ such that the first axis is normal to the hyperplane. Then the operator $\mathcal{L}$ can be represented as

$$\mathcal{L} = \partial_t^2 + \Delta_y - \left(\frac{2m_\alpha}{t} + t f(t^2, y) \right) \partial_t + \sum_{i=1}^{n-1} g_i(t^2, y) \partial_{y_i},$$

where $\Delta_y = \partial_{y_1}^2 + \ldots + \partial_{y_{n-1}}^2$. The functions f and g_i are analytic at $t = 0$ and invariant under reflection $t \to -t$ with respect to $(\alpha, x) = 0$ due to invariance of the operator $\mathcal{L}$ (c.f. [3]). For a polynomial $p(x)$ we also have a similar expansion

$$p = \sum_{i=0}^{\deg p} p_i(y) t^i.$$

Substituting this into the equation $\mathcal{L}p = 0$ we have

$$\left(\partial_t^2 - \frac{2m_\alpha}{t} \partial_t + \Delta_y - t f(t^2, y) \partial_t + \sum_{i=1}^{n-1} g_i(t^2, y) \partial_{y_i} \right) \left(\sum_{i=0}^{\deg p} p_i(y) t^i \right) = 0.$$

Considering all possible terms at t^{-1} in the lefthand side we conclude that $p_1 \equiv 0$. Considering now the terms at t we come to

$$(6 - 6m_\alpha) p_3 \equiv 0$$

which implies that $p_3 \equiv 0$ if $m_\alpha > 1$. Continuing in this way we obtain

$$p_1 = p_3 = \ldots = p_{2m_\alpha - 1} \equiv 0$$

or equivalently

$$\partial_\alpha^{2s-1} p(x)|_{(\alpha,x)=0} = 0 \text{ for } 1 \leq s \leq m_\alpha$$

which are the quasiinvariance conditions. Thus the proposition (and therefore the theorem) is proven.

In the classical case the space H_0 of usual harmonic polynomials can be effectively described as the image of the following homomorphism π:

$$\pi : p \to p(\partial)w,$$

where $p \in S(V), w = \prod_\alpha (\alpha, x)$ (see [17]).

Following the same route let us define the map

$$\pi_m : Q_m \to H_m \tag{9}$$

by the formula

$$\pi_m(q) = \mathcal{L}_q(w_m) \tag{10}$$

where $w_m = \prod_\alpha (\alpha, x)^{2m_\alpha + 1}$ and $\mathcal{L}_q = \chi_m(q)$ is defined by (6). To prove that $\mathcal{L}_q(w_m) \in H_m$ we will need the following lemma. Let $A_m \subset Q_m$ be the subspace of antiinvariants, i.e. the quasiinvariants q satisfying the property

$$q(s_\alpha(x)) = -q(x)$$

for any reflection $s_\alpha \in G$.

Lemma 2 *A_m is a one-dimensional module over S^G generated by w_m.*

It is easy to show that such an antiinvariant is divisible by w_m. Since the quotient is G-invariant this implies the lemma.

Lemma 3 *The quasiinvariant w_m is m-harmonic.*

Indeed, since all $\mathcal{L}_i$ are G-invariant and preserve the space Q_m (see theorem 3 above) the polynomials $\mathcal{L}_i(w_m)$ belong to A_m. Since they have degree less than the degree of w_m they must be zero.

Lemma 4 *The space H_m is invariant under the action of $\mathcal{L}_q$ for any $q \in Q_m$.*

This follows from commutativity of $\mathcal{L}_q$ and $\mathcal{L}_i$, $i = 1, \ldots, n$. All this implies

Theorem 6 *The formula*

$$\pi_m(q) = \mathcal{L}_q(w_m)$$

defines a linear map from Q_m *to* H_m.

Let us discuss the properties of the map π_m.

Theorem 7 *The kernel of* π_m *contains the ideal* I_m.

To prove this let us represent any element $q \in I_m$ as

$$q = \sum_s q_s p_s$$

where $q_s \in Q_m$, $p_s \in S^G$. We have

$$\mathcal{L}_q(w_m) = \sum_s \mathcal{L}_{q_s} \mathcal{L}_{p_s}(w_m) = 0$$

since $\mathcal{L}_{p_s}(w_m) = 0$ due to lemma 3.

Our first two conjectures (see the Introduction) claim that like in the classical situation $m = 0$ the kernel of π_m coincides with I_m and that the restriction of π_m onto H_m:

$$\pi_m|_{H_m} : H_m \to H_m$$

is an isomorphism. This implies that

$$Q_m/I_m \approx H_m$$

and that Q_m is generated by H_m as a module over S^G. Our third conjecture says that this module is actually free.

As it was shown by Etingof and Ginzburg [9] the kernel of π_m indeed coincides with I_m and the image of π_m coincides with H_m but the restriction of π_m onto H_m may not be isomorphism. This happens if H_m has a non-trivial intersection with I_m. In fact the map π_m is an isomorphism between any complement T to the ideal I_m in Q_m and H_m, and Q_m is freely generated over S^G by T (see [9]).

The question for which groups and multiplicity functions the space H_m is transversal to the ideal I_m in Q_m is still open. We believe that this is true in most of the cases, in particular if the multiplicity function is constant. In the next section we prove this for all two-dimensional Coxeter groups. Some of the statements now are already known to be true in the general case due to [9] but our original proofs seem to be more straightforward and effective.

5 Proofs for the dihedral groups

Consider the dihedral group $G = I_2(N)$ which is a symmetry group of a regular N-gon. It is generated by the plane reflections with respect to the lines $\alpha_1^j x_1 + \alpha_2^j x_2 = 0$, $\alpha_1^j = -\sin\frac{2\pi j}{N}$, $\alpha_2^j = \cos\frac{2\pi j}{N}$, $j = 0, 1, \dots N-1$. We suppose that the multiplicity function m_{α^j} is equal to $m \in \mathbb{Z}_{\geq 0}$ for all $j = 0, \dots, N-1$. It will be convenient for us to use the complex coordinates $z = x_1 + ix_2, \bar{z} = x_1 - ix_2$. The generators of the ring of invariants can be chosen as $\sigma_1 = z\bar{z}, \sigma_2 = z^N + \bar{z}^N$.

We are going first to investigate the ring Q_m of m-quasiinvariants. For $m = 1$ this ring has been studied in the paper [19] where in particular the multiplicative generators of Q_1 have been found. We will use some observations from that paper to describe the ring Q_m with general m.

Let us consider the following $2(N-1)$ polynomials:

$$q_j = a_{j0} z^{mN+j} + a_{j1} \bar{z}^N z^{(m-1)N+j} + \ldots + a_{jm} \bar{z}^{mN} z^j, \tag{11}$$

$$\bar{q}_j = \bar{a}_{j0} \bar{z}^{mN+j} + \bar{a}_{j1} z^N \bar{z}^{(m-1)N+j} + \ldots + \bar{a}_{jm} z^{mN} \bar{z}^j, \tag{12}$$

$j = 1, \dots, N-1$, where the coefficients a_{js} are chosen to satisfy the system of equations

$$\begin{cases} (mN+j)a_{j0} + ((m-2)N+j)a_{j1} + \ldots + (-mN+j)a_{jm} = 0 \\ (mN+j)^3 a_{j0} + ((m-2)N+j)^3 a_{j1} + \ldots + (-mN+j)^3 a_{jm} = 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ (mN+j)^{2m-1} a_{j0} + ((m-2)N+j)^{2m-1} a_{j1} + \ldots + (-mN+j)^{2m-1} a_{jm} = 0 \end{cases}$$

One can easily see that rank of this system is equal to m so for any j the coefficients a_{js} are defined uniquely up to proportionality.

Explicitly polynomial q_j can be given as determinant

$$q_j = det Z_j \tag{13}$$

of the following matrix

$$Z_j = \begin{pmatrix} (mN+j) & ((m-2)N+j) & \dots & (-mN+j) \\ (mN+j)^3 & ((m-2)N+j)^3 & \dots & (-mN+j)^3 \\ \vdots & \vdots & \ddots & \vdots \\ (mN+j)^{2m-1} & ((m-2)N+j)^{2m-1} & \dots & (-mN+j)^{2m-1} \\ z^{mN+j} & \bar{z}^N z^{(m-1)N+j} & \dots & \bar{z}^{mN} z^j \end{pmatrix}.$$

Proposition 2 *The polynomials $q_j, \bar{q}_j$ belong to the space Q_m of quasiinvariants.*

Proof. Let us introduce the polar coordinates $z = re^{i\varphi}$, $\bar{z} = re^{-i\varphi}$. Then from the system of equations defining coefficients a_{js} it obviously follows that $\partial_\varphi^{2s-1} q_j|_{\varphi=\frac{\pi k}{N}} = 0$, $\partial_\varphi^{2s-1} \bar{q}_j|_{\varphi=\frac{\pi k}{N}} = 0$, $k = 0, \ldots, N-1$, $s = 1, \ldots, m$. Now the statement follows from the following lemma.

Lemma 5 *For any polynomial $p(x_1, x_2)$, any vector $\alpha = (-\sin\varphi_0, \cos\varphi_0)$ and for arbitrary $m \in \mathbb{Z}_+$ the conditions*

$$\partial_\alpha^{2s-1} p|_{(\alpha,x)=0} = 0, \quad s = 1, \ldots, m$$

are satisfied if and only if the following conditions in polar coordinates hold:

$$\partial_\varphi^{2s-1} p|_{\varphi=\varphi_0} = 0, \quad s = 1, \ldots, m$$

Now let us introduce two more quasiinvariants

$$q_0 = 1, \; q_N = (z^N - \bar{z}^N)^{2m+1}. \tag{14}$$

The last polynomial q_N is actually a basic antiinvariant quasiinvariant (proportional to w_m).

Theorem 8 *The ring Q_m is a free finitely generated module over its subring $S^G \subset Q_m$ of invariant polynomials. One can choose the polynomials $q_0, q_1, \ldots, q_{N-1}$, $\bar{q}_1, \ldots, \bar{q}_{N-1}, q_N$ as a basis of Q_m over S^G.*

Proof. At first let us show that the polynomials $q_0, q_1, \ldots, q_{N-1}, \bar{q}_1, \ldots, \bar{q}_{N-1}, q_N$ do generate Q_m over S^G. To prove this we will use induction on a degree of a polynomial $q \in Q_m$. If $\deg q = 0$ then $q = const = c$, thus $q = cq_0$ so we have checked the base of induction. Suppose now that $\deg q = d$ and $q = Az^d + B\bar{z}^d + z\bar{z}p_{d-2}$ is an arbitrary quasiinavariant, $q \notin S^G$. We will use further the following two lemmas.

Lemma 6 *Any m-quasiinvariant of degree $d \leq mN$ is actually invariant.*

Proof. It is enough to prove the lemma for an arbitrary homogeneous polynomial. Let

$$q = a_0 z^{mN-\sigma} + a_1 z^{mN-\sigma-1}\bar{z} + a_2 z^{mN-\sigma-2}\bar{z}^2 + \ldots + a_{mN-\sigma}\bar{z}^{mN-\sigma} \in Q_m \tag{15}$$

for some $\sigma \geq 0$. According to lemma 5 the conditions of quasiinvariance in polar coordinates are $\partial_\varphi^{2s-1} q = 0$ for $\varphi = \frac{\pi k}{N}$, $0 \leq k \leq N-1$. We have

$$(mN-\sigma)^{2s-1} a_0 e^{i\frac{\pi}{N}(mN-\sigma)k} + (mN-\sigma-2)^{2s-1} a_1 e^{i\frac{\pi}{N}(mN-\sigma-2)k} +$$

$$(mN-\sigma-4)^{2s-1}a_2 e^{i\frac{\pi}{N}(mN-\sigma-4)k}+\ldots+(-mN+\sigma)^{2s-1}a_{mN-\sigma}e^{i\frac{\pi}{N}(-mN+\sigma)k}=0$$

Collecting the terms in this sum with equal exponents, we get

$$\sum_{j=0}^{N-1}\sum_{\substack{t\geq 0\\ j+Nt\leq mN-\sigma}}(mN-\sigma-2(j+Nt))^{2s-1}a_{j+Nt}e^{i\frac{\pi}{N}(mN-\sigma-2(j+Nt))k}=$$

$$\sum_{j=0}^{N-1}e^{i\frac{\pi}{N}(mN-\sigma-2j)k}\sum_{\substack{t\geq 0\\ j+Nt\leq mN-\sigma}}(mN-\sigma-2(j+Nt))^{2s-1}a_{j+Nt}=0$$

Now let us consider these conditions for all possible $k=0,1,\ldots,N-1$. We arrive at the Vandermonde-type system with different exponents $e^{i\frac{\pi}{N}(mN-\sigma-2j)}$, $0\leq j\leq N-1$. Hence, for all $j=0,\ldots,N-1$ the following property is satisfied

$$\sum_{\substack{t\geq 0\\ j+Nt\leq mN-\sigma}}(mN-\sigma-2(j+Nt))^{2s-1}a_{j+Nt}=0 \tag{16}$$

Let us analyze the conditions (16) for all possible s, $1\leq s\leq m$. We again have a system of Vandermonde type with the exponents $(mN-\sigma-2(j+Nt))^2$, $0\leq t\leq [\frac{mN-\sigma-j}{N}]\leq m$. Notice that the exponents corresponding to different t may coincide only in pairs. The condition for that is

$$mN-\sigma-(j+Nt_1)=j+Nt_2. \tag{17}$$

In the case $j=\sigma=0$ the number of equations in (16) is less than number of unknown coefficients a_{j+Nt}. But after collecting the terms in (16) corresponding to equal exponents the number of equations becomes not less than the number of unknowns. We conclude that all nonzero coefficients a_{j+Nt} can be divided into pairs so that $a_{j+Nt_1}-a_{j+Nt_2}=0$ and also the condition (17) is satisfied. In terms of polynomial q this means that it can be represented in the form

$$q=\sum_{j=0}^{N-1}\sum_{(t_1,t_2)}a_{j+Nt_1}z^{mN-\sigma-(j+Nt_1)}\bar{z}^{j+Nt_1}+a_{j+Nt_2}z^{mN-\sigma-(j+Nt_2)}\bar{z}^{j+Nt_2}=$$

$$\sum_{j=0}^{N-1}\sum_{(t_1,t_2)}a_{j+Nt_1}(z\bar{z})^{j+Nt_2}(z^{N(t_1-t_2)}+\bar{z}^{N(t_1-t_2)}),$$

where the pairs (t_1,t_2) satisfy (17) and also we suppose that $t_1>t_2$. Hence polynomial q is an invariant.

Lemma 7 *If* $q = Az^d + B\bar{z}^d + z\bar{z}p_{d-2}$ *is* m*-quasiinavariant of degree* $d = mN + lN$, $1 \le l \le m$ *then* $A = B$.

Proof. We will use the notations and scheme of proof of lemma 6. Let us consider q of the form (15), now we have $\sigma = -lN$. As above in lemma 6 the conditions (16) for $j = 0, \ldots, N-1$ should be satisfied. Let us fix $j = 0$, we get

$$\sum_{t=0}^{m+l} ((m+l-2t)N)^{2s-1} a_{Nt} = 0$$

or equivalently

$$\sum_{t=0}^{[\frac{m+l}{2}]} ((m+l-2t)N)^{2s-1} (a_{Nt} - a_{N(m+l-t)}) = 0.$$

We have got a system of Vandermonde type with different exponents. For $l \le m$ from that it follows that $a_{Nt} = a_{N(m+l-t)}$. If $t = 0$ we get $a_0 = a_{N(m+l)}$ which completes the proof of the lemma.

To continue the proof of the theorem let us represent d in the form $d = mN + jN + k$, where $0 \le k < N$. We have to consider few different cases.
a) If $k \ne 0$ then $q - \frac{A}{a_{k0}} q_k (z^N + \bar{z}^N)^j - \frac{B}{\bar{a}_{k0}} \bar{q}_k (z^N + \bar{z}^N)^j = z\bar{z}p_1$ for some polynomial $p_1 \in Q_m$, where constants $a_{k0}, \bar{a}_{k0}$ are the same as in (11), (12). Since $\deg p_1 = d - 2 < d$ we have done the induction step.
b) If $k = 0$, $1 \le j \le m$ then lemma 7 states that $A = B$, hence $q - A(z^N + \bar{z}^N)^{m+j} = z\bar{z}p_2$, where $p_2 \in Q_m$ and it can be represented as a linear combination of the polynomials $q_i, \bar{q}_i$ with invariant coefficients.
c) If $k = 0$, $j \ge m+1$ then $q - \frac{A-B}{2} q_N (z^N + \bar{z}^N)^{j-m-1} - \frac{A+B}{2} (z^N + \bar{z}^N)^{m+j} = z\bar{z}p_3$, where $p_3 \in Q_m$ and one can apply the induction hypothesis.

Thus we have proved that polynomials $q_i, \bar{q}_i$ generate Q_m as an S^G–module. Now we are going to show that this module is free.

To see this let us consider arbitrary nontrivial combination of the polynomials $q_i, \bar{q}_i$ with invariant coefficients. Since the ring of invariants for the dihedral group is the ring freely generated by two polynomials $\sigma_1 = z\bar{z}$ and $\sigma_2 = z^N + \bar{z}^N$, the linear combination takes the form

$$p_0^1(\sigma_1,\sigma_2)q_0 + p_1^1(\sigma_1,\sigma_2)q_1 + p_1^2(\sigma_1,\sigma_2)\bar{q}_1 + \ldots +$$
$$p_{N-1}^1(\sigma_1,\sigma_2)q_{N-1} + p_{N-1}^2(\sigma_1,\sigma_2)\bar{q}_{N-1} + p_N^1(\sigma_1,\sigma_2)q_N = 0,$$

where for some s, ϵ we have $p_s^\epsilon \ne 0$. Also we can suppose that p_s^ϵ is not divisible by σ_2. Further, let us represent polynomials p_j as combinations of monomials in σ_1, σ_2 and let us move monomials containing σ_1 into righthand side. We have then that

$$r_0^1(\sigma_2)q_0 + r_1^1(\sigma_2)q_1 + r_1^2(\sigma_2)\bar{q}_1 + \ldots\ldots + r_{N-1}^1(\sigma_2)q_{N-1} + r_{N-1}^2(\sigma_2)\bar{q}_{N-1} + r_N^1(\sigma_2)q_N$$

is divisible by σ_1 and some polynomial $r_s^\epsilon \neq 0$. Let us consider monomials having degree which is equal to s modulo N. If $1 \leq s \leq N-1$ then $r_s^1(\sigma_2)q_s + r_s^2(\sigma_2)\bar{q}_s$ must be divisible by $z\bar{z}$, which is impossible as $r_s^1(\sigma_2)q_s$ contains monomial of the form $\lambda_1 z^{\mu_1}$ and does not contain degrees of $\bar{z}$, and $r_s^2(\sigma_2)\bar{q}_s$ contains monomial of the form $\lambda_2 \bar{z}^{\mu_2}$ and does not contain degrees of z. If $s = 0$ or $s = N$ then $r_0^1(\sigma_2) + r_N^1(\sigma_2)(z^{(2m+1)N} - \bar{z}^{(2m+1)N})$ must be divisible by $z\bar{z}$, which is possible only if $r_0^1 = r_N^1 = 0$ but this is not the case. Thus the theorem is proven.

Corollary *The Poincare series for the m-quasiinvariants of dihedral group $I_2(N)$ is*

$$p(Q_m, t) = \frac{1 + 2t^{(mN+1)} + \ldots + 2t^{(mN+N-1)} + t^{(2m+1)N}}{(1-t^2)(1-t^N)}.$$

Now we are going to show that polynomials $q_i, \bar{q}_i$ (11), (12), (14) are actually m-harmonic. This will complete the proof of conjecture 3.

First let us rewrite the operator $\mathcal{L}$ in the complex coordinates. The set of vectors α for the group $I_2(N)$ has the form $\alpha = (-\sin\varphi_k, \cos\varphi_k)$, where $\varphi_k = \frac{\pi k}{N}$, $k = 0, \ldots, N-1$. Substituting $\partial_x = \partial_z + \partial_{\bar{z}}$, $\partial_y = i(\partial_z - \partial_{\bar{z}})$ we get

$$\mathcal{L} = \Delta - 2m \sum_\alpha \frac{\partial_\alpha}{(\alpha, x)} = \Delta - 2m \sum_\alpha \frac{-\sin\varphi_k \partial_x + \cos\varphi_k \partial_y}{-\sin\varphi_k x + \cos\varphi_k y} =$$

$$4\partial_z\partial_{\bar{z}} - 2m \sum_{k=0}^{N-1} \frac{(-\sin\varphi_k + i\cos\varphi_k)\partial_z + (-i\cos\varphi_k - \sin\varphi_k)\partial_{\bar{z}}}{\frac{1}{2}z(-i\cos\varphi_k - \sin\varphi_k) + \frac{1}{2}\bar{z}(i\cos\varphi_k - \sin\varphi_k)} =$$

$$4\left(\partial_z\partial_{\bar{z}} - m\sum_{k=0}^{N-1} \frac{e^{i\varphi_k}\partial_z - e^{-i\varphi_k}\partial_{\bar{z}}}{-e^{-i\varphi_k}z + e^{i\varphi_k}\bar{z}}\right).$$

The operator $\mathcal{L} = \mathcal{L}_1$ has a commuting operator $\mathcal{L}_2$ which is also invariant under dihedral group, it is homogeneous of degree N and has the form $\mathcal{L}_2 = \partial_z^N + \partial_{\bar{z}}^N +$ lower order terms.

Theorem 9 *The polynomials* (11), (12), (14) *belong to the common kernel of the operators $\mathcal{L}_1$ and $\mathcal{L}_2$, i.e. they are m-harmonic.*

Proof. From theorem 3 and degree consideration it follows immediately that $\mathcal{L}_1(q_0) = \mathcal{L}_2(q_0) = 0$. As q_N is an antiinvariant quasiinvariant of the smallest possible degree and due to invariance of the operators $\mathcal{L}_1, \mathcal{L}_2$, we have $\mathcal{L}_1(q_N) = \mathcal{L}_2(q_N) = 0$.

Let us show now that $\mathcal{L}_2(q_s) = 0$, $1 \leq s \leq N-1$. Let $\mathcal{L}_2(q_s) = r_s, \mathcal{L}_2(\bar{q}_s) = \bar{r}_s$. Let us assume for simplicity that N is odd. Then two dimensional space $V_s = < q_s, \bar{q}_s >$ is an irreducible representation for the group $G = I_2(N)$. Since operator $\mathcal{L}_2$ being invariant commutes with the action of G, then by Schur lemma the kernel of $\mathcal{L}_2|_{V_s}$ is either V_s or 0. In the last case the space $< r_s, \bar{r}_s >$ is an irreducible representation for G. But since $\deg r_s = \deg q_s - N < mN$ the polynomials $r_s, \bar{r}_s$ should be invariant according to lemma 6. The contradiction means that $\mathcal{L}_2|_{V_s} = 0$, i.e. $\mathcal{L}_2(q_s) = \mathcal{L}_2(\bar{q}_s) = 0$.

Now let us show that $\mathcal{L}_1(q_s) = \mathcal{L}_1(\bar{q}_s) = 0$. Let $p_s = \mathcal{L}_1(q_s), \bar{p}_s = \mathcal{L}_1(\bar{q}_s)$. As above by Schur lemma either $\mathcal{L}_1|_{V_s} = 0$ or $\mathcal{L}_1|_{V_s}$ is an isomorphism. In the second case representation $< p_s, \bar{p}_s >$ is isomorphic to irreducible representation V_s. It is easy to see from the formulas (11), (12) that among all the representations $V_t, 1 \le t \le N-1, t \ne s$ only V_{N-s} is isomorphic to V_s, so that q_{N-s} corresponds to $\bar{q}_s$, and $\bar{q}_{N-s}$ corresponds to q_s. This implies that $p_s = P(\sigma_1, \sigma_2)\bar{q}_{N-s}$, $\bar{p}_s = P(\sigma_1, \sigma_2) q_{N-s}$ for some polynomial $P(x, y)$. But since

$$p_s = 4\left(\partial_z \partial_{\bar{z}} - m \sum_{k=0}^{N-1} \frac{e^{i\varphi_k}\partial_z - e^{-i\varphi_k}\partial_{\bar{z}}}{-e^{-i\varphi_k} z + e^{i\varphi_k}\bar{z}}\right)(a_{s0} z^{mN+s} + a_{s1}\bar{z}^N z^{(m-1)N+s} + \ldots + a_{sm}\bar{z}^{mN} z^s),$$

the degree of p_s in $\bar{z}$ is less or equal then mN while $\deg_{\bar{z}} P(\sigma_1, \sigma_2)\bar{q}_{N-s} > mN$. This means that $\mathcal{L}_1|_{V_s} = 0$ so $\mathcal{L}_1(q_s) = \mathcal{L}_1(\bar{q}_s) = 0$. When N is even one should take into account that $V_{N/2}$ is reducible but the arguments are essentially the same. The theorem is proven.

The theorems 8 and 9 imply that our conjecture 3 is true for any dihedral group and constant multiplicity function. Let us now prove the first two conjectures.

The following lemma is true for any Coxeter group G and multiplicity function m. Recall that $w_m(x) = \prod_\alpha (\alpha, x)^{2m_\alpha+1}$ is m-discriminant.

Lemma 8

$$\mathcal{L}_{w_m} w_m \ne 0.$$

Proof of the lemma. Let us introduce

$$\tilde{w}_m(k, x) = \frac{\sum_{g\in G}(-1)^g \psi(g(k), x)}{w_m(k)}, \tag{18}$$

where $\psi(k, x)$ is the BA function (see Section 2). When $k \to 0$, $\tilde{w}_m(k, x)$ tends to $C w_m(x)$ where C is some constant. Indeed, the limit is an entire skew-symmetric function of the homogeneity $\mathcal{N} = \sum_\alpha (2m_\alpha + 1)$ since $\tilde{w}_m(\lambda^{-1}k, \lambda x) = \lambda^{\mathcal{N}} \tilde{w}_m(k, x)$. Now

$$\mathcal{L}_{w_m} \tilde{w}_m(k, x) = \sum_{g\in G} \frac{(-1)^g w_m(g(k))\psi(g(k), x)}{w_m(k)} = \sum_g \psi(g(k), x) \xrightarrow[k\to 0]{} |G|\psi(0, x).$$

On the other hand $\mathcal{L}_{w_m}\tilde{w}_m(k, x) \to C\mathcal{L}_{w_m} w_m$, so $C\mathcal{L}_{w_m} w_m = |G|\psi(0, x)$ but $\psi(0, x)$ is known to be non-zero constant (see [11]). Thus we see that both C and

$\mathcal{L}_{w_m} w_m$ are non-zero.

Remark In the preprint [8] we have shown that for the dihedral group $I_2(N)$ the constant $\mathcal{L}_{w_m} w_m$ is given by the following explicit formula

$$\mathcal{L}_{w_m}(w_m) = (2N)^{2m+1} \prod_{j=1}^{2m+1} (2j-2m-1) \prod_{\substack{d=1 \\ d \not\equiv 0 (mod N)}}^{(2m+1)N} (d - mN), \tag{19}$$

where all the roots are normalised such that $(\alpha, \alpha) = 2$. When $m = 0$ this formula claims that

$$(\prod_{\alpha} \partial_\alpha) \prod_{\alpha} (\alpha, x) = 2N! = 2!N!$$

which is a particular case of the following Macdonald identity:

$$(\prod_{\alpha} \partial_\alpha) \prod_{\alpha} (\alpha, x) = \prod_i d_i!$$

where d_i are the degrees of generators σ_i of the invariants S^G of a Coxeter group G and again all $(\alpha, \alpha) = 2$. It would be interesting to find the analogue of this formula for any m.

Now we are ready to prove Conjecture 1.

Theorem 10 *For any dihedral group $I_2(N)$*

$$Ker \pi_m = I_m,$$

where π_m is defined by (10) and I_m is the ideal in Q_m generated by basic invariants σ_1, σ_2.

Proof. Let us represent an arbitrary quasiinvariant in the form

$$q = s_0 q_0 + \sum_{j=1}^{N-1} (s_j q_j + \bar{s}_j \bar{q}_j) + s_N q_N \tag{20}$$

where $s_j, \bar{s}_j$ are invariants and $q_j, \bar{q}_j$ are defined by (11), (12). Suppose that $q \in Ker \pi_m$, that is $\mathcal{L}_q w_m = 0$. Since

$$\mathcal{L}_q = \mathcal{L}_{s_0} + \sum_{j=1}^{N-1} (\mathcal{L}_{s_j} \mathcal{L}_{q_j} + \mathcal{L}_{\bar{s}_j} \mathcal{L}_{\bar{q}_j}) + \mathcal{L}_{s_N} \mathcal{L}_{q_N},$$

the condition $\mathcal{L}_q w_m = 0$ is equivalent to $\mathcal{L}_{q^H} w_m = 0$, where $q^H \in H_m$ is defined by

$$q^H = s_0(0)q_0 + \sum_{j=1}^{N-1}(s_j(0)q_j + \bar{s}_j(0)\bar{q}_j) + s_N(0)q_N$$

Since $q - q^H \in I_m$ it is sufficient to prove that $\mathcal{L}_h w_m \neq 0$ for any $h \in H_m$.

It is sufficient to consider only the homogeneous h. If $h = const$ then the statement obviously holds. When $h = \mathrm{const}\, w_m$ it follows from lemma 8. Suppose now that $h = \lambda_1 q_j + \lambda_2 \bar{q}_j$ and $\mathcal{L}_h w_m = 0$. Let us consider

$$\mathcal{L}_{q_{N-j}} \mathcal{L}_h w_m = 0 = \mathcal{L}_{q_{N-j}h} w_m = \mathcal{L}_{\lambda_1 q_{N-j} q_j + \lambda_2 q_{N-j} \bar{q}_j} w_m$$

The formulas (11), (12) show that

$$\lambda_1 q_{N-j} q_j + \lambda_2 q_{N-j} \bar{q}_j = \lambda_1 a_{N-j0} a_{j0} z^{(2m+1)N} + (z\bar{z})p, \tag{21}$$

where p is some polynomial in $z, \bar{z}$. On the other hand, we should have a general representation (20) for some invariants $s_i, \bar{s}_i$

$$\lambda_1 q_{N-j} q_j + \lambda_2 q_{N-j} \bar{q}_j = \sum (s_i q_i + \bar{s}_i \bar{q}_i) + s_0 q_0 + s_N q_N \tag{22}$$

In the last expression the sum $\sum(s_i q_i + \bar{s}_i \bar{q}_i)$ cannot contain monomials $z^{(2m+1)N}$, $\bar{z}^{(2m+1)N}$ as $s_i, \bar{s}_i$ are nontrivial polynomials of $z\bar{z}$ which follows from degree considerations. Suppose that $\lambda_1 \neq 0$, then the lefthand side of (22) contains z^{2m+1} and it does not contain $\bar{z}^{(2m+1)N}$ (see (21)). Hence s_N must be a nonzero constant c. Now

$$\mathcal{L}_{q_{N-j}} \mathcal{L}_h w_m = \mathcal{L}_{\sum(s_i q_i + \bar{s}_i \bar{q}_i) + s_0} w_m + c\mathcal{L}_{q_N} w_m = c\mathcal{L}_{q_N} w_m$$

since $\sum(s_i q_i + \bar{s}_i \bar{q}_i) + s_0 \in I_m$. Due to lemma 8 $\mathcal{L}_{q_N} w_m \neq 0$ so $\mathcal{L}_{q_{N-j}} \mathcal{L}_h w_m \neq 0$ which contradicts the assumption that $\mathcal{L}_h w_m = 0$. This implies that $\lambda_1 = 0$. Similarly multiplying $\mathcal{L}_h w_m = 0$ by $\mathcal{L}_{\bar{q}_{N-j}}$ we derive that $\lambda_2 = 0$ which means that $h = 0$. This proves the theorem and conjecture 1 in this case.

Conjecture 2 now simply follows from the previous arguments. Indeed we have shown in the proof of the previous theorem that if $\mathcal{L}_h w_m = 0$ for some $h \in H_m$ then $h = 0$. This implies the following

Theorem 11 *For any dihedral group the linear map*

$$\pi_m : H_m \to H_m$$

is an isomorphism.

Let us finally show that conjecture 2^* also holds. For that we fix normalisation of basic quasiinvariants as

$$q_j = z^{mN+j} + z\bar{z}p_j,$$
$$\bar{q}_j = \bar{z}^{mN+j} + z\bar{z}\bar{p}_j.$$

Since $q_{j_1}\bar{q}_{j_2}$ is divisible by $z\bar{z}$ it is obvious that $< q_{j_1}, \bar{q}_{j_2} >= 0$. The consideration of the degrees shows that $< q_{j_1}, q_{j_2} >= 0$ if $j_1 + j_2 \neq N$. Let us calculate $< q_j, q_{N-j} >$. We have

$$q_j q_{N-j} = z^{(2m+1)N} + (z\bar{z})\hat{p}_j = \frac{1}{2}(q_N + \sigma_2^{2m+1}) + (z\bar{z})Q_j,$$

where $\hat{p}_j$ is some polynomial and Q_j is a quasiinvariant. Hence

$$< q_j, q_{N-j} >= (\mathcal{L}_{\frac{1}{2}q_N} + \mathcal{L}_{\frac{1}{2}\sigma_2^{2m+1}+z\bar{z}Q_j})w_m = \frac{1}{2}\mathcal{L}_{q_N}w_m$$

which is non-zero by lemma 8. This implies the nondegeneracy of the form $<,>$ and conjecture 2^*.

Remark For any Coxeter group the form $< P, Q >$ can be given by the formula

$$< P, Q >= \gamma \frac{\sum_g (-1)^g PQ(g(k))}{w_m(k)}|_{k=0}, \tag{23}$$

where $w_m(k) = \prod(\alpha, k)^{2m_\alpha+1}$ and γ is a non-zero constant. Indeed, let $\tilde{w}_m(k, x)$ be given by the formula (18) above. Then

$$\mathcal{L}_{PQ}\tilde{w}_m = \frac{\sum_g (-1)^g PQ(g(k))\psi(g(k), x)}{w_m(k)}.$$

Now letting $x, k \to 0$ we arrive to the formula (23), where constant γ is non-zero and given by the relation

$$\gamma = \frac{\mathcal{L}_{w_m}w_m}{|G|}$$

(cf. the proof of lemma 8).

6 Concluding remarks.

Comparing the results of this paper with Etingof–Ginzburg results [9] we see that the only obstacle for all our conjectures to be true is a possible non-transversality of the space H_m of m-harmonic polynomials and the ideal I_m. To understand when exactly this may happen is the main open question in this direction. A related question is to find a good candidate for a complement T to the ideal I_m in Q_m in case when H_m does not work.

We should mention that the space H_m itself needs a better description. In the classical case $m = 0$ this space can be described as the result of the differential operators with constant coefficients applied to the discriminant $w = \prod_{\alpha \in A}(\alpha, x)$ (see [17]). In the general case one can use the operators $\mathcal{L}_q$ which correspond to the quasiinvariants but we have no effective description of the quasiinvariants themselves.

One of the promising alternative ideas is to use the Matsuo–Cherednik isomorphism between the systems (7) and modified Knizhnik–Zamolodchikov equations [20], [21]. In the recent paper [10] this relation was used to find an explicit formula for the Poincare polynomials $P(H_m, t)$ and to compute some of the m-harmonic polynomials for the group $G = S_n$ but this certainly does not exhaust all the possibilities of this approach.

Another interesting direction is to develop a similar theory for the rings of integrals of other algebraically integrable quantum problems, in particular for the trigonometric and difference versions of Calogero-Moser problem related to root systems. In trigonometric case the corresponding polynomials satisfy certain difference relations which in the rational limit coincide with the quasiinvariance relations (1) (see [2],[3]). It would be also very interesting to investigate possible analogues of harmonic polynomials in relation with the ring of quantum integrals for the generalised Calogero-Moser problems related to the deformed root systems discovered in [22] (see also [13]).

Finally we would like to mention that for the Weyl groups G the space of usual harmonic polynomials can be interpreted as cohomology of the generalised flag varieties [23]. It would be interesting to look at the space of m-harmonic polynomials from this point of view. This is also related to an important question about the multiplication structure on H_m induced by the isomorphism with Q_m/I_m.

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