The finite dock problem

by

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Abstract

The scattering of water waves by a dock of finite width and infinite length in water of finite depth is solved using the modified residue calculus technique. The problem is formulated for obliquely incident waves and the case of normal incidence is recovered by taking an appropriate limit. Exciting forces and pitching moments are calculated as well as reflection and transmission coefficients. The method presented in this paper takes account of the known solution for the scattering by a semi-infinite dock to produce new and extremely accurate approximations for the reflection and transmission coefficients as well as a highly efficient numerical procedure for the solution to the full linear problem.

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1 Introduction

Problems concerning the interaction of water waves with a rigid plate of finite width and infinite length lying in the free surface have a long history. Such problems are interesting for many reasons. First, the simple geometry allows considerable mathematical progress to be made and thus dock problems can be used as model problems against which to test new techniques or numerical results. Secondly they can be used as the first approximation in a perturbation analysis of wave interactions with shallow-draft ships, as was done in [15] where an integral equation technique was used to examine the heave radiation problem in deep water and in [6] where a similar method was used to study the oblique scattering problem, again for infinitely deep water.

If the rigid plate is semi-infinite in extent rather than of finite width, then many interaction problems yield explicit solutions. Thus the Wiener-Hopf technique was used in [7] to solve the interaction of oblique waves (generated by a line source along the edge of the plate) with a semi-infinite dock in water of finite depth. The case of normally incident waves is recovered by taking an appropriate limit. The method used is also applicable to infinite depth, but in that case the method breaks down in the limit corresponding to normally incident waves. However, this

latter problem can be solved using complex variable theory and a method related to Laplace transforms, see [5]. This method is reproduced in both [8] and [11].

Most work on finite dock problems has concentrated on the infinite depth case and utilized an integral equation approach. A standard application of Green's theorem using a fundamental Green's function for two-dimensional wave motions leads to an inhomogeneous Fredholm integral equation of the second kind for the velocity potential on the dock, in which the kernel is the vertical derivative of the Green's function. For normal incidence, Rubin [21] showed how this integral equation could be transformed into one with a much simpler kernel (though a more complicated right-hand side) and used this new formulation to prove the existence and uniqueness of a solution. The transformed integral equation was used in [22] and [15] to produce numerical solutions. For the case of oblique waves, the transformation can still be made (see [14]), but results were computed in [6] from the more complicated integral equation. The difficult subject of short wave asymptotics for finite dock problems in infinite depth was also the subject of a series of papers; [8, 9, 11, 12, 13], the last of these also including some results for finite water depths.

The first results for the finite depth case were based on shallow water theory [23] and then numerical calculations of the reflection and transmission coefficients were presented in [18] for the scattering problem based on the full linear theory. Only the case of normal incidence was considered in [18], though the matched eigenfunction technique that was employed easily generalizes to the oblique incidence case. A general numerical scheme, based on the finite element method, for the solution of oblique scattering problems by infinite cylinders of constant cross-section lying in the free surface in water of finite depth was developed in [2]. In particular, Bai considered the diffraction of waves by a cylinder with rectangular cross-section, including the case of zero draft, which corresponds to the finite dock problem. Recently the finite depth and oblique incidence problem has been attacked using the complicated machinery of dual integral transforms [3]. The solution given is extremely complicated and no numerical results are presented.

The purpose of this paper is to show that this problem can in fact be solved in an elegant manner, one which has the advantage over Mei and Black's solution procedure in that it takes into account the fact that the semi-infinite dock problem possesses an explicit solution. Another advantage of the method described below over that used in [18] is that it accurately and explicitly models the known singularity in the derivative of the potential at the plate edge.

The technique that we use is based on a combination of matched eigenfunction expansions and residue calculus theory (the latter being a technique developed for solving electromagnetic waveguide problems, see [19]). This is quite a technical procedure, but the resulting formulas for the reflection and transmission coefficients, R and T, are simple and make it straightforward to evaluate these quantities. To obtain numerical values it is necessary to first compute the equivalent quantities for

the semi-infinite dock (for which there are explicit formulas) and then to compute finite-width corrections by solving two infinite systems of real algebraic equations. These systems are exponentially convergent with increasing dock width and converge extremely rapidly provided the ratio of plate width to water depth $(2a/h)$ is not too small. As well as providing an efficient method for accurately computing R and T, the formulation also enables us to derive an approximation based on a 1×1 truncation of the infinite systems, an approximation which leads to new simple formulas for R and T that are extremely accurate except for very small values of a/h.

2 Formulation

Cartesian coordinates are chosen with the (x, y)-plane corresponding to the undisturbed free surface and z pointing vertically upwards. We consider the diffraction of an incident plane wave making an angle θ_I with the positive x-axis by a rigid dock which occupies $z = 0$, $-a \le x \le a$, $-\infty < y < \infty$ in water of uniform depth h. If we seek solutions which are time harmonic with angular frequency ω then, under the usual assumptions of linear water wave theory, the solution can be represented by a velocity potential $Re\left\{\Phi(x, y, z)\mathrm{e}^{-\mathrm{i}\omega t}\right\}$ where $\Phi(x, y, z)$ satisfies Laplace's equation.

We begin by defining an orthogonal set of functions which are in fact the appropriate depth eigenfunctions for this problem. Thus

$$\psi_n(z) = N_n^{-1} \cos k_n(z + h), \qquad N_n^2 = \frac{1}{2}\left(1 + \frac{\sin 2k_n h}{2k_n h}\right), \qquad (2.1)$$

where $u = \pm k_n$, $n \ge 0$ are the solutions to the dispersion relation $K + u \tan uh = 0$ in which $K = \omega^2/g$, g being the acceleration due to gravity. Here $k_0 = -\mathrm{i}k$ $(k > 0)$ is purely imaginary and k_n, $n \ge 1$ are real and positive and the depth eigenfunctions form an orthogonal set since

$$\frac{1}{h}\int_{-h}^0 \psi_n(z)\psi_m(z) = \delta_{mn}. \qquad (2.2)$$

The incident wave can be represented by the potential $\exp[\mathrm{i}\alpha(x + a) + \mathrm{i}\ell y]\psi_0(z)$ where $\ell = k \sin\theta_I$, $\alpha = k \cos\theta_I = (k^2 - \ell^2)^{1/2}$ and the oscillatory time dependence has been suppressed. The total velocity potential for the scattering problem can similarly be written $\phi(x, z)\exp(\mathrm{i}\ell y)$ where $\phi(x, z)$ satisfies

$$\nabla_{xz}^2 \phi - \ell^2 \phi = 0 \quad \text{in } -h < z < 0, \ -\infty < x < \infty, \qquad (2.3)$$

$$\phi_z = 0 \quad \text{on } z = -h, \qquad (2.4)$$

$$K\phi = \phi_z \quad \text{on } z = 0, \ |x| > a, \qquad (2.5)$$

$$\phi_z = 0 \quad \text{on } z = 0, \ |x| \le a \qquad (2.6)$$

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and we choose to define the reflection and transmission coefficients R and T through the far-field behaviour

$$\phi \sim (e^{i\alpha(x+a)} + Re^{-i\alpha(x+a)})\psi_0(z) \quad \text{as} \quad x \to -\infty, \tag{2.7}$$

$$\phi \sim Te^{i\alpha(x-a)}\psi_0(z) \qquad \qquad \text{as} \quad x \to \infty. \tag{2.8}$$

The problem is linear so we can of course multiply the solution by an arbitrary constant. The incident wave in (2.7) corresponds to a wave with free surface given by $\eta(x)\exp(i\ell y)$ where $\eta = (i\omega/gN_0)e^{i\alpha(x+a)}\cosh kh$.

One final condition needs to be applied and that is a condition which specifies the nature of the solution near the plate edge, $(x, z) = (a, 0)$. If we insist that ϕ is regular at this point then [24], Theorem 3.3, shows that $\phi \sim P(r, r \ln r)$ as $r = [(x-a)^2 + z^2]^{1/2} \to 0$, where P is some polynomial. It then follows that

$$\frac{\partial \phi}{\partial r} \sim A \ln r \quad \text{as} \quad r \to 0, \tag{2.9}$$

for some constant A.

To make the solution procedure simpler we use the fact that

$$R = \frac{1}{2}(R^+ + R^-), \qquad T = \frac{1}{2}(R^+ - R^-), \tag{2.10}$$

where R^+ and R^- are the reflection coefficients for symmetric and antisymmetric problems in $x < 0$ (with potentials ϕ^+ and ϕ^-) and with the boundary conditions

$$\phi_x^+ = 0 \quad \text{on} \quad x = 0, \tag{2.11}$$

$$\phi^- = 0 \quad \text{on} \quad x = 0. \tag{2.12}$$

In $x < -a$ we can expand the potentials as eigenfunction series as follows:

$$\phi^\pm = (e^{-\alpha_0(x+a)} + R^\pm e^{\alpha_0(x+a)})\psi_0(z) + \sum_{n=1}^{\infty} A_n^\pm e^{\alpha_n(x+a)}\psi_n(z), \tag{2.13}$$

where $\alpha_0 = -i\alpha = -i(k^2 - \ell^2)^{1/2}$, $\alpha_n = (k_n^2 + \ell^2)^{1/2}$, $n \geq 1$, whereas in $-a < x < 0$ we expand $\phi^\pm$ as

$$\phi^\pm = \sum_{n=0}^{\infty} \frac{\epsilon_n}{2} B_n^\pm \left(e^{\beta_n x} \pm e^{-\beta_n x}\right) \cos \lambda_n z, \tag{2.14}$$

where $\beta_n = (\lambda_n^2 + \ell^2)^{1/2}$ and $\lambda_n = n\pi/h$. The edge condition (2.9) enables us to determine the behaviour of the coefficients A_n and B_n for large n. We use the result that $\sum_{n=1}^{\infty} n^{-1} \exp(-nx) \sim -\ln x$ as $x \to 0+$ (this can be derived using

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Mellin transforms, see e.g. [16]). It follows that we must have $nA_n = O(n^{-1})$, $nB_n \exp(\beta_n a) = O(n^{-1})$ and hence that

$$A_n = O(n^{-2}), \quad B_n e^{\beta_n a} = O(n^{-2}) \quad \text{as} \quad n \to \infty. \tag{2.15}$$

Equations (2.13)–(2.15) ensure that $\phi^{\pm}$ satisfies all the appropriate equations and boundary conditions, provided $\phi^{\pm}$ and $\phi_x^{\pm}$ are continuous across $x = -a$. Note from (2.14) that there will be significant differences depending on whether or not $\ell = 0$. In particular, if $\ell = 0$ the $n = 0$ term will not contribute to ϕ^- or ϕ_x^+. Below, we will assume that $\ell \neq 0$ and then recover results for the $\ell = 0$ case (i.e. normal incidence) by letting $\ell \to 0$.

The continuity of $\phi^{\pm}$ and the orthogonality of the functions $\cos \lambda_m z$ on $(-h, 0)$ can be used to show that

$$2c_{0m} + \sum_{n=0}^{\infty} A_n^{\pm} c_{nm} = \pm \frac{1}{2} B_m^{\pm} \left(e^{\beta_m a} \pm e^{-\beta_m a} \right) \qquad m \geq 0, \tag{2.16}$$

where

$$c_{nm} = \frac{1}{h} \int_{-h}^{0} \psi_n(z) \cos \lambda_m z \, \mathrm{d}z = \frac{k_n \sin k_n h}{h N_n (k_n^2 - \lambda_m^2)}. \tag{2.17}$$

Similarly, the equation

$$\sum_{n=0}^{\infty} \alpha_n A_n^{\pm} c_{nm} = \mp \frac{\beta_m}{2} B_m^{\pm} \left(e^{\beta_m a} \mp e^{-\beta_m a} \right) \qquad m \geq 0, \tag{2.18}$$

results from the continuity in velocity. If we eliminate $B_m^{\pm}$ between (2.16) and (2.18) we obtain

$$\sum_{n=0}^{\infty} \alpha_n A_n^{\pm} c_{nm} \left(\frac{1}{\alpha_n} + \frac{e^{\beta_m a} \pm e^{-\beta_m a}}{\beta_m (e^{\beta_m a} \mp e^{-\beta_m a})} \right) = -2c_{0m} \qquad m \geq 0. \tag{2.19}$$

If we now define $V_0^{\pm} = A_0^{\pm} + 1 = R^{\pm}$ and $V_n^{\pm} = A_n^{\pm} N_0 k_n \sin k_n h / N_n k_0 \sin k_0 h$ and note that $\alpha_n^2 - \beta_m^2 = k_n^2 - \lambda_m^2$, we can rewrite this system of equations in the form

$$\sum_{n=0}^{\infty} V_n^{\pm} \left(\frac{1}{\alpha_n - \beta_m} \pm \frac{e^{-2\beta_m a}}{\alpha_n + \beta_m} \right) = \frac{1}{\alpha_0 + \beta_m} \pm \frac{e^{-2\beta_m a}}{\alpha_0 - \beta_m} \qquad m \geq 0 \tag{2.20}$$

and, from (2.15), we require $V_n^{\pm} = O(n^{-1})$ as $n \to \infty$. The method that we use to solve this system is the modified residue calculus technique originally devised in [20] and described in [10], §2.12, which takes advantage of the fact that the terms $\exp(-2\beta_m a)$ all tend to zero rapidly as a/h gets large.

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Consider the function $f^\pm(z) = G^\pm g(z) h^\pm(z)$, where

$$g(z) = \frac{1}{z + \alpha_0} \prod_{n=0}^{\infty} \frac{1 - z/\beta_n}{1 - z/\alpha_n}, \tag{2.21}$$

$$h^\pm(z) = 1 + \sum_{n=0}^{\infty} \frac{C_n^\pm}{z - \beta_n} \tag{2.22}$$

and $G^\pm$, $C_n^\pm$, $n \geq 0$, are constants to be determined. It is possible to show that $g(z) = O(z^{-1})$ as $|z| \to \infty$ provided we avoid a discrete set of real, positive values (see Appendix A). We then consider the numbers

$$I_m = \lim_{N \to \infty} \frac{1}{2\pi i} \int_{C_N} f^\pm(z) \left(\frac{1}{z - \beta_m} \pm \frac{e^{-2\beta_m a}}{z + \beta_m} \right) dz \qquad m \geq 0, \tag{2.23}$$

where C_N are contours chosen to avoid the discrete set of points mentioned above and on which $|z| \to \infty$ as $N \to \infty$. The behaviour of g for large z implies that $I_m = 0$ and then Cauchy's residue theorem gives

$$f^\pm(\beta_m) \pm e^{-2\beta_m a} f^\pm(-\beta_m) + R(f^\pm : -\alpha_0) \left(\frac{1}{-\alpha_0 - \beta_m} \pm \frac{e^{-2\beta_m a}}{-\alpha_0 + \beta_m} \right)$$

$$+ \sum_{n=0}^{\infty} R(f^\pm : \alpha_n) \left(\frac{1}{\alpha_n - \beta_m} \pm \frac{e^{-2\beta_m a}}{\alpha_n + \beta_m} \right) = 0, \quad (2.24)$$

for each $m \geq 0$, where $R(f : z_0)$ means the residue of $f(z)$ at $z = z_0$. A comparison with (2.20) shows that our solution will be given by $V_n^\pm = R(f^\pm : \alpha_n)$ provided $R(f^\pm : \alpha_n) = O(n^{-1})$ as $n \to \infty$, $G^\pm$ is chosen so that $R(f^\pm : -\alpha_0) = 1$ and

$$f^\pm(\beta_m) \pm e^{-2\beta_m a} f^\pm(-\beta_m) = 0 \qquad m \geq 0. \tag{2.25}$$

The fact that $R(f^\pm : \alpha_n) = O(n^{-1})$ as $n \to \infty$ can easily be demonstrated using the fact that $f^\pm(z) = O(z^{-1})$ as $z \to \infty$ (see, e.g., Appendix C of [4]). We thus define

$$G^\pm = \frac{1}{h^\pm(-\alpha_0)} \prod_{n=0}^{\infty} \frac{1 + \alpha_0/\alpha_n}{1 + \alpha_0/\beta_n} \tag{2.26}$$

and then provided (2.25) is satisfied we have

$$R^\pm = R(f^\pm : \alpha_0) = \frac{h^\pm(-i\alpha)}{h^\pm(i\alpha)} R_\infty, \tag{2.27}$$

where

$$R_\infty = e^{-2i\theta_I} \prod_{n=1}^{\infty} \frac{(1 - i\alpha/\alpha_n)(1 + i\alpha/\beta_n)}{(1 + i\alpha/\alpha_n)(1 - i\alpha/\beta_n)} = e^{-2i\theta_I} e^{2i\delta_\infty}, \tag{2.28}$$

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$$\delta_\infty = \sum_{n=1}^{\infty} \left(\tan^{-1}(\alpha/\beta_n) - \tan^{-1}(\alpha/\alpha_n) \right). \tag{2.29}$$

Note that as $a/h \to \infty$, (2.25) reduces to $f^\pm(\beta_m) = 0$, which in turn implies that $C_m^\pm = 0$, $m \geq 0$ and hence that $h^\pm(z) \equiv 1$. Thus the functions $h^\pm(z)$ can be thought of as accounting for the finite length of the dock. In particular, R_∞ is the reflection coefficient for the semi-infinite problem.

In order to satisfy (2.25), the coefficients $C_m^\pm$, $m \geq 0$ are the solutions to the infinite systems of real equations

$$C_m^\pm \pm D_m \sum_{n=0}^{\infty} \frac{C_n^\pm}{\beta_m + \beta_n} = \pm D_m \qquad m \geq 0, \tag{2.30}$$

where

$$D_0 = 2\ell e^{-2\ell a} \prod_{n=1}^{\infty} \frac{(1 - \ell/\alpha_n)(1 + \ell/\beta_n)}{(1 + \ell/\alpha_n)(1 - \ell/\beta_n)}, \tag{2.31}$$

and for $m \geq 1$,

$$D_m = \frac{2\beta_m(\ell + \beta_m)(\alpha_m - \beta_m)}{(\ell - \beta_m)(\alpha_m + \beta_m)} e^{-2\beta_m a} \prod_{\substack{n=1 \\ n \neq m}}^{\infty} \frac{(1 - \beta_m/\alpha_n)(1 + \beta_m/\beta_n)}{(1 + \beta_m/\alpha_n)(1 - \beta_m/\beta_n)}. \tag{2.32}$$

Because of the presence of the factor $\exp(-2\beta_m a)$ in the expression (2.32) for D_m, the systems of equations (2.30) converge very quickly provided a/h is not too small and provide an extremely efficient method for computing the unknowns $C_m^\pm$. Moreover, it is possible to prove (using the method described in Appendix B of [4]) that, for sufficiently large a/h, (2.30) has a unique solution with $\sum_{m=1}^{\infty} (C_m^\pm)^2 < \infty$.

Since the coefficients $C_m^\pm$ are real it immediately follows from (2.27) that, as one would expect, $|R_\infty| = |R^\pm| = 1$. Combining the symmetric and antisymmetric solutions using (2.10) we thus obtain

$$R = \frac{1}{2}(e^{2i\delta^+} + e^{2i\delta^-})e^{-2i\theta_I}e^{2i\delta_\infty}, \qquad T = \frac{1}{2}(e^{2i\delta^+} - e^{2i\delta^-})e^{-2i\theta_I}e^{2i\delta_\infty}, \tag{2.33}$$

where

$$\delta^\pm = \arg\left(h^\pm(-i\alpha)\right) = \arg\left(1 - \sum_{n=0}^{\infty} \frac{C_n^\pm}{i\alpha + \beta_n}\right) \tag{2.34}$$

and δ_∞ is given by (2.29).

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Results for normal incidence can be extracted from the above analysis by taking the limit as $\theta_{\mathrm{I}} \to 0$ (i.e. $\ell \to 0$ for fixed k). First we note that in this limit

$$\ln \prod_{n=1}^{\infty} \frac{(1 - \ell/\alpha_n)(1 + \ell/\beta_n)}{(1 + \ell/\alpha_n)(1 - \ell/\beta_n)} = \sum_{n=1}^{\infty} \ln \frac{(1 - \ell/\alpha_n)(1 + \ell/\beta_n)}{(1 + \ell/\alpha_n)(1 - \ell/\beta_n)} \sim 2\ell\sigma + O(\ell^3),$$

$$(2.35)$$

where

$$\sigma = \sum_{n=1}^{\infty} \left(\frac{1}{\lambda_n} - \frac{1}{k_n} \right) \tag{2.36}$$

and hence

$$D_0 \sim 2\ell + 4\ell^2(\sigma - a) + O(\ell^3). \tag{2.37}$$

The $m = 0$ equation in (2.30) with the plus sign then becomes $C_0^+ = 0$ and we only need to solve the system for $m \geq 1$. The leading order behaviour of the $m = 0$ equation in (2.30) with the minus sign is more complicated; we find that

$$(\sigma - a)C_0^- + \sum_{n=1}^{\infty} \frac{C_n^-}{\beta_n} = 1. \tag{2.38}$$

Apart from these changes, we can simply set $\ell = 0$ in the general expressions for R and T.

It is also possible to examine the long wave limit, i.e. $Kh \to 0$, for fixed θ_{I}. In this limit $kh \sim (Kh)^{1/2}$ and an analysis of (2.30) reveals that

$$\delta^+ \sim \theta_{\mathrm{I}} - ka\sin\theta_{\mathrm{I}}\tan\theta_{\mathrm{I}}, \qquad \delta^- \sim \theta_{\mathrm{I}} - \frac{\pi}{2} + ka\cos\theta_{\mathrm{I}} \tag{2.39}$$

and hence that

$$R \sim -\mathrm{i}ka\sec\theta_{\mathrm{I}}, \qquad T \sim 1 + \mathrm{i}ka\sec\theta_{\mathrm{I}}\cos 2\theta_{\mathrm{I}}. \tag{2.40}$$

For $\theta_{\mathrm{I}} = 0$, these results agree with those in [17] after taking account the different definitions of T used in that paper.

3 Forces

The non-dimensionalized vertical exciting force on the dock due to an incident wave of unit amplitude is given by

$$F = \frac{N_0 \mathrm{e}^{-\mathrm{i}\alpha a}}{a\cosh kh} \int_{-a}^{a} \phi\big|_{z=0}\,\mathrm{d}x = \frac{N_0 \mathrm{e}^{-\mathrm{i}\alpha a}}{a\cosh kh} \int_{0}^{a} \phi^+\big|_{z=0}\,\mathrm{d}x$$

$$= \frac{N_0 \mathrm{e}^{-\mathrm{i}\alpha a}}{\cosh kh} \sum_{m=0}^{\infty} \frac{\epsilon_m B_m^+}{\beta_m a} \sinh\beta_m a \tag{3.1}$$

and the non-dimensionalized pitching moment by

$$M = \frac{N_0 \mathrm{e}^{-\mathrm{i}\alpha a}}{a^2 \cosh kh} \int_{-a}^{a} x\phi\big|_{z=0}\,\mathrm{d}x = \frac{N_0 \mathrm{e}^{-\mathrm{i}\alpha a}}{a^2 \cosh kh} \int_{0}^{a} x\phi^-\big|_{z=0}\,\mathrm{d}x$$

$$= \frac{N_0 \mathrm{e}^{-\mathrm{i}\alpha a}}{\cosh kh} \sum_{m=0}^{\infty} \frac{\epsilon_m B_m^-}{\beta_m^2 a^2}(\beta_m a \cosh \beta_m a - \sinh \beta_m a). \tag{3.2}$$

We could calculate $B^\pm$ directly from (2.16) or (2.18) but it is more convenient to proceed as follows. We rewrite equations (2.16) and (2.18) in the form

$$\frac{2\beta_m c_{0m}}{\mathrm{e}^{\beta_m a} \pm \mathrm{e}^{-\beta_m a}} + \sum_{n=0}^{\infty} \frac{\beta_m A_n^\pm c_{nm}}{\mathrm{e}^{\beta_m a} \pm \mathrm{e}^{-\beta_m a}} = \pm \frac{\beta_m B_m^\pm}{2} \qquad m \geq 0, \tag{3.3}$$

$$\sum_{n=0}^{\infty} \frac{\alpha_n A_n^\pm c_{nm}}{\mathrm{e}^{\beta_m a} \mp \mathrm{e}^{-\beta_m a}} = \mp \frac{\beta_m B_m^\pm}{2} \qquad m \geq 0, \tag{3.4}$$

and then subtract one from the other.

The resulting system of equations can be reduced to

$$\sum_{n=0}^{\infty} V_n^\pm \left(\frac{1}{\alpha_n + \beta_m} \pm \frac{\mathrm{e}^{-2\beta_m a}}{\alpha_n - \beta_m} \right)$$

$$= \frac{1}{\alpha_0 - \beta_m} \pm \frac{\mathrm{e}^{-2\beta_m a}}{\alpha_0 + \beta_m} \mp \frac{2hN_0}{k_0 \sin k_0 h}\beta_m B_m^\pm \mathrm{e}^{-\beta_m a}\sinh 2\beta_m a \qquad m \geq 0. \tag{3.5}$$

We then consider

$$I_m = \lim_{N\to\infty} \frac{1}{2\pi\mathrm{i}} \int_{C_N} f^\pm(z)\left(\frac{1}{z + \beta_m} \pm \frac{\mathrm{e}^{-2\beta_m a}}{z - \beta_m} \right)\mathrm{d}z \qquad m \geq 0 \tag{3.6}$$

and obtain, since $I_m = 0$,

$$\sum_{n=0}^{\infty} V_n^\pm \left(\frac{1}{\alpha_n + \beta_m} \pm \frac{\mathrm{e}^{-2\beta_m a}}{\alpha_n - \beta_m} \right)$$

$$= \frac{1}{\alpha_0 - \beta_m} \pm \frac{\mathrm{e}^{-2\beta_m a}}{\alpha_0 + \beta_m} - f^\pm(-\beta_m) \mp \mathrm{e}^{-2\beta_m a}f^\pm(\beta_m) \qquad m \geq 0. \tag{3.7}$$

It follows that

$$\pm\frac{hN_0}{k_0 \sin k_0 h}\beta_m B_m^\pm \mathrm{e}^{\beta_m a}(1 - \mathrm{e}^{-4\beta_m a}) = f^\pm(-\beta_m) \pm \mathrm{e}^{-2\beta_m a}f^\pm(\beta_m) \tag{3.8}$$

and if we use (2.25) we obtain

$$B_m^\pm = \pm\frac{k_0 \sin k_0 h}{hN_0\beta_m}f^\pm(-\beta_m)\mathrm{e}^{-\beta_m a}. \tag{3.9}$$

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It is straightforward to show that $B_m^{\pm} \exp(\beta_m a) = O(m^{-2})$ as $m \to \infty$ as required. Hence

$$F = -\frac{Ke^{-i\alpha a}}{2ah} \sum_{m=0}^{\infty} \frac{\epsilon_m}{\beta_m^2}(1 - e^{-2\beta_m a})f^+(-\beta_m) \tag{3.10}$$

$$M = \frac{Ke^{-i\alpha a}}{2a^2 h} \sum_{m=0}^{\infty} \frac{\epsilon_m}{\beta_m^3} \left(\beta_m a - 1 + (\beta_m a + 1)e^{-2\beta_m a}\right) f^-(-\beta_m), \tag{3.11}$$

where, from (2.21) and (2.26),

$$f^{\pm}(-\beta_m) = \frac{2i\alpha(\ell + \beta_m)(\ell + i\alpha)}{k^2(k^2 + \lambda_m^2)} \frac{h^{\pm}(-\beta_m)}{h^{\pm}(i\alpha)} \prod_{n=1}^{\infty} \frac{(1 - i\alpha/\alpha_n)(1 + \beta_m/\beta_n)}{(1 - i\alpha/\beta_n)(1 + \beta_m/\alpha_n)}. \tag{3.12}$$

The long wave asymptotics of F and M are readily obtained from these expressions. To leading order we have that as $Kh \to 0$, for fixed θ_{I},

$$F \to 2, \qquad M \sim \frac{2i}{3} ka \cos \theta_{\mathrm{I}}. \tag{3.13}$$

4 Results

Equations (2.33)–(2.32) provide a numerically straightforward way of computing the reflection and transmission coefficients for the finite dock problem in finite depth. The infinite systems of equations that need to be solved converge extremely rapidly and the sums and products that need to be evaluated cause no difficulty. For example, the terms in the summation in the definition of δ_{∞}, equation (2.29), are $O(n^{-3})$ as $n \to \infty$. This is computationally acceptable, but the series is easily accelerated. By subtracting off the leading order asymptotics of the summand we can derive the expression

$$\delta_{\infty} = -\frac{\alpha h K h}{\pi^3} \zeta(3) + \sum_{n=1}^{\infty} \left(\tan^{-1}(\alpha/\beta_n) - \tan^{-1}(\alpha/\alpha_n) + \frac{\alpha h K h}{n^3 \pi^3}\right), \tag{4.1}$$

in which ζ is the Riemann zeta function and the terms are $O(n^{-5})$ as $n \to \infty$. All the infinite products can be accelerated in the same way after first taking their logarithms.

To demonstrate the rapid convergence of the infinite systems of equations we can consider the case of a 1×1 truncation. If we only include one term from the summation in (2.30), solve for $C_0^{\pm}$ and substitute into (2.34) we obtain

$$\tan \delta^{\pm} \approx \frac{\pm \sin 2\theta_{\mathrm{I}}}{b^{-1} \pm \cos 2\theta_{\mathrm{I}}}, \tag{4.2}$$

| | a/h | | | |
θ_{I}	0.25	0.5	0.75	1
0°	8.6	0.94	0.13	0.021
40°	5.2	0.43	0.046	0.0056
80°	0.61	0.026	0.0020	0.00020

Table 1: Maximum percentage error (over all frequencies) when computing $|R|$ from the approximations (4.3) and (4.4).

where $b = D_0/2\ell$. If we substitute this expression into (2.33) we obtain, after some lengthy algebra, the approximations

$$R \approx \mathrm{e}^{2\mathrm{i}\delta_\infty}\mathrm{e}^{2\mathrm{i}\theta_{\mathrm{I}}}\left(\frac{b^2-1}{b^2-\mathrm{e}^{4\mathrm{i}\theta_{\mathrm{I}}}}\right), \qquad T \approx b\,\mathrm{e}^{2\mathrm{i}\delta_\infty}\left(\frac{1-\mathrm{e}^{4\mathrm{i}\theta_{\mathrm{I}}}}{b^2-\mathrm{e}^{4\mathrm{i}\theta_{\mathrm{I}}}}\right). \qquad (4.3)$$

These approximations preserve the limiting behaviour in long waves given by (2.40). If we take the limit of (4.3) as $\theta_{\mathrm{I}} \to 0$ using (2.37) we obtain the following approximations for the normal incidence case:

$$R \approx \frac{k(\sigma-a)\mathrm{e}^{2\mathrm{i}\delta_\infty}}{k(\sigma-a)-\mathrm{i}}, \qquad T \approx \frac{-\mathrm{i}\mathrm{e}^{2\mathrm{i}\delta_\infty}}{k(\sigma-a)-\mathrm{i}}. \qquad (4.4)$$

The accuracy of these approximations, which appear to be new, depends strongly on the value of a/h and to a lesser extent on the value of θ_{I}, with larger values of either parameter resulting in greater accuracy. This is illustrated in Table 1 which shows the errors that result from using these approximations to compute $|R|$. For each value of a/h and θ_{I} the table gives the maximum percentage error in the computed value of $|R|$ as K varies over the entire frequency range. The table shows that 1% accuracy is achieved for all values of $a/h \geq 0.5$, with the accuracy increasing rapidly as a/h increases.

Table 2 shows a sample set of results for $\theta_{\mathrm{I}} = 45°$ and $a/h = 1$, based on a 2×2 truncation of the system of equations (2.30). All the digits displayed are believed to be accurate. Results for the same parameter values are displayed graphically in Figure 1. Figure 2 shows how the quantities $|R|$, $|T|$, $|F|$ and $|M|$ vary with θ_{I} for the case $a/h = 1$, $Kh = 0.4$ and Figure 3 shows the same quantities plotted against a/h for the case $\theta_{\mathrm{I}} = 60°$ and $Kh = 0.5$.

5 Conclusion

The classical problem of surface wave scattering by a finite dock in water of finite depth has been solved using the modified residue calculus technique. This method

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Kh	$|R|$	$|T|$	$|F|$	$|M|$
0.2	0.5781	0.8160	1.7336	0.1903
0.4	0.7506	0.6608	1.5010	0.2428
0.6	0.8479	0.5302	1.2987	0.2682
0.8	0.9070	0.4211	1.1238	0.2793
1.0	0.9437	0.3307	0.9736	0.2816
1.2	0.9665	0.2566	0.8454	0.2784
1.4	0.9805	0.1966	0.7370	0.2719
1.6	0.9889	0.1489	0.6458	0.2633
1.8	0.9938	0.1115	0.5695	0.2537
2.0	0.9966	0.0826	0.5060	0.2438

Table 2: Values of $|R|$, $|T|$, $|F|$ and $|M|$ for different values of Kh when $\theta_{\mathrm{I}} = 45°$ and $a/h = 1$.

appears to be superior to other known methods of solution because it takes into account the known exact results for the equivalent semi-infinite dock problem. It also explicitly includes the correct form for the potential near to the edge of the dock. Application of the method to the finite dock problem leads to a modification of the semi-infinite dock solution by a term involving a set of unknown real coefficients. These coefficients are the solution to an infinite system of real equations which can be solved numerically by truncation. Crucially, this system is exponentially convergent with increasing dock width and so only a very few equations are required to obtain accurate numerical results for the various hydrodynamic quantities of interest. Furthermore, if a 1×1 truncation is used, analytic manipulation yields approximations for the reflection and transmission coefficients which are highly accurate over the entire frequency range.

A   Asymptotics of $g(z)$ as $z \to \infty$

In this appendix we will determine the asymptotics for large z of the function

$$g(z) = \frac{1}{z + \alpha_0} \prod_{n=0}^{\infty} \frac{1 - z/\beta_n}{1 - z/\alpha_n} = \frac{1 - z/\beta_0}{(z + \alpha_0)(1 - z/\alpha_0)} \frac{g_1(z)}{g_2(z)}, \qquad \text{(A.1)}$$

where

$$g_1(z) = \prod_{n=1}^{\infty} (1 - z/\beta_n)\mathrm{e}^{z/\lambda_n}, \qquad g_2(z) = \prod_{n=1}^{\infty} (1 - z/\alpha_n)\mathrm{e}^{z/\lambda_n}. \qquad \text{(A.2)}$$
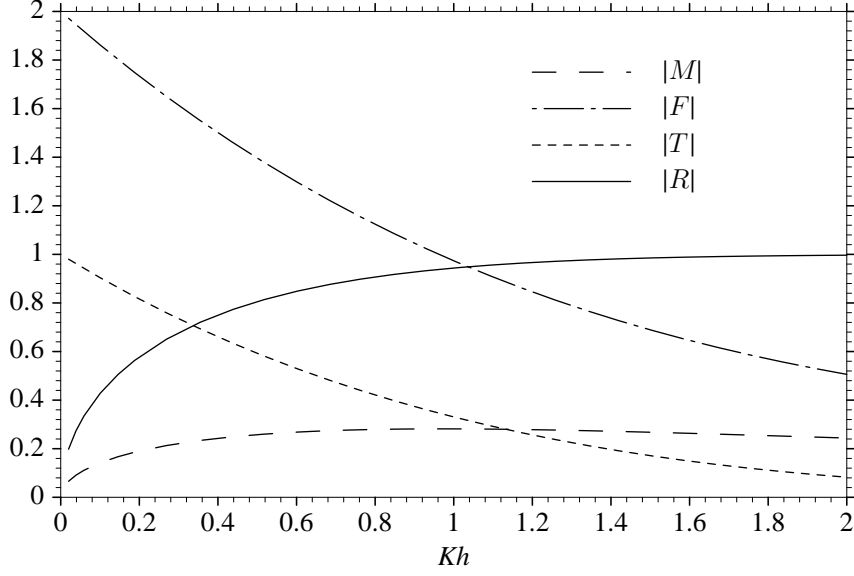
Figure 1: $|R|$, $|T|$, $|F|$ and $|M|$ plotted against Kh when $\theta_{\mathrm{I}} = 45°$ and $a/h = 1$.

Here $\lambda_n = n\pi/h$, $\alpha_n = (k_n^2 + \ell^2)^{1/2}$, $\beta_n = (\lambda_n^2 + \ell^2)^{1/2}$ and as $n \to \infty$,

$$\alpha_n = \lambda_n + O(n^{-1}), \qquad \beta_n = \lambda_n + O(n^{-1}). \tag{A.3}$$

The asymptotics of g_1 and g_2 can be determined by making use of the identity (see [1], eqn 6.1.3)

$$\prod_{n=1}^{\infty}(1 - z/\lambda_n)\mathrm{e}^{z/\lambda_n} = \frac{\mathrm{e}^{\gamma zh/\pi}}{\Gamma(1 - zh/\pi)} \tag{A.4}$$

in which $\Gamma(\cdot)$ is the Gamma function and $\gamma \approx 0.5772$ is Euler's constant. Thus

$$g_1(z) = \frac{\mathrm{e}^{\gamma zh/\pi}}{\Gamma(1 - zh/\pi)} \prod_{n=1}^{\infty} \frac{\lambda_n}{\beta_n} \prod_{n=1}^{\infty} \frac{z - \beta_n}{z - \lambda_n}. \tag{A.5}$$

Now

$$\prod_{n=1}^{\infty} \frac{\lambda_n}{\beta_n} = \prod_{n=1}^{\infty} \left(1 + \ell^2/\lambda_n^2\right)^{-1/2} = \left(\frac{\sinh \ell h}{\ell h}\right)^{-1/2}, \tag{A.6}$$
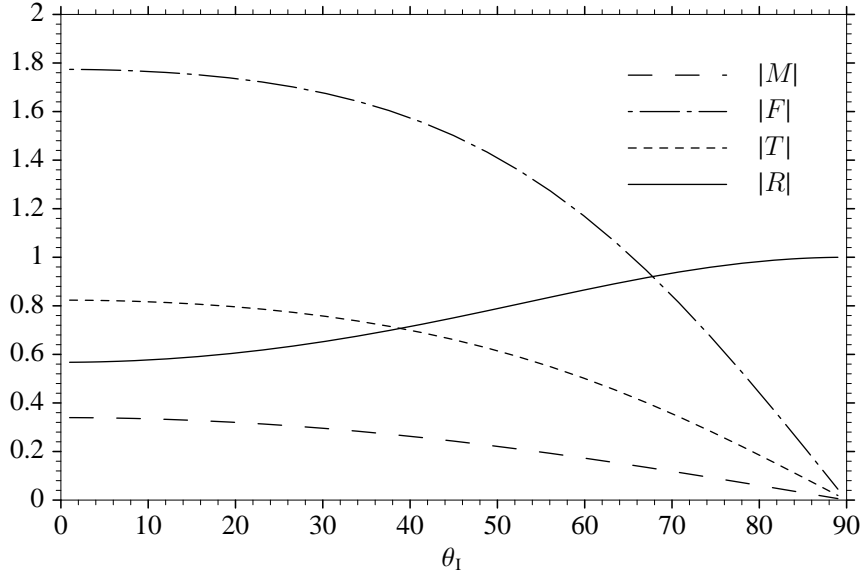
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Figure 2: $|R|$, $|T|$, $|F|$ and $|M|$ plotted against θ_{I} when $Kh = 0.4$ and $a/h = 1$.

and

$$\prod_{n=1}^{\infty} \frac{z - \beta_n}{z - \lambda_n} = \prod_{n=1}^{\infty} \left(1 + \frac{\lambda_n - \beta_n}{z - \lambda_n} \right) = \prod_{n=1}^{\infty} \left(1 - \frac{\ell^2}{(z - \lambda_n)(\lambda_n + \beta_n)} \right). \qquad (A.7)$$

It follows that as $z \to \infty$,

$$\prod_{n=1}^{\infty} \frac{\lambda_n}{\beta_n} \prod_{n=1}^{\infty} \frac{z - \beta_n}{z - \lambda_n} \sim \left(\frac{\sinh \ell h}{\ell h} \right)^{-1/2}, \qquad (A.8)$$

provided z is not real and positive. Using Stirling's formula, [1], eqn 6.1.37, we thus have

$$g_1(z) \sim \left(\frac{\ell}{-2z \sinh \ell h} \right)^{1/2} e(z), \qquad (A.9)$$

where

$$e(z) = \exp \left[\frac{zh}{\pi} \left(\gamma - 1 + \ln \frac{-zh}{\pi} \right) \right], \qquad (A.10)$$

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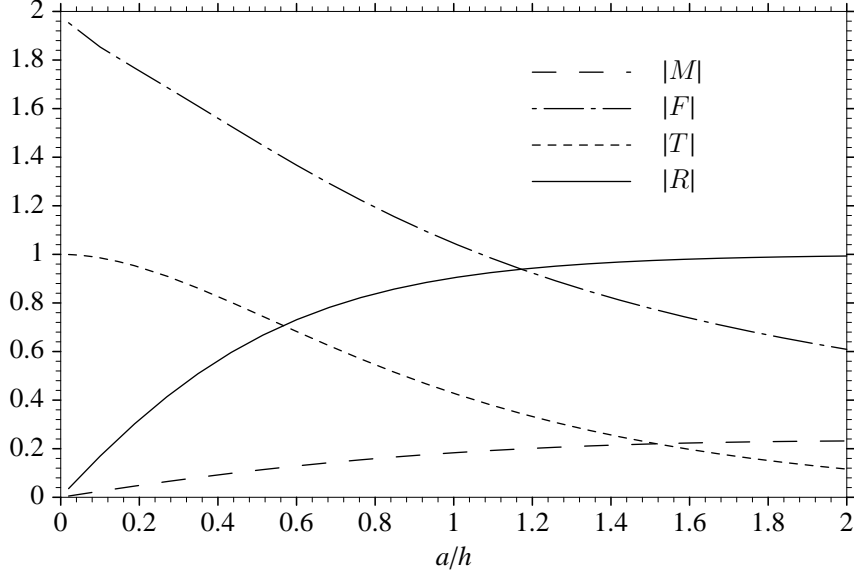Figure 3: $|R|$, $|T|$, $|F|$ and $|M|$ plotted against a/h when $Kh = 0.5$ and $\theta_{\mathrm{I}} = 60°$.

provided z is not real and positive.

In order to obtain an asymptotic formula valid for real and positive z we note that

$$\prod_{n=1}^{\infty}(1 - z/\beta_n)(1 + z/\beta_n) = \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{\lambda_n^2 + \ell^2}\right) = \prod_{n=1}^{\infty}\frac{1 - (\zeta/\lambda_n)^2}{1 + (\ell/\lambda_n)^2} = \frac{\ell \sin \zeta h}{\zeta \sinh \ell h}, \tag{A.11}$$

where $\zeta = (z^2 - \ell^2)^{1/2}$. It follows that

$$g_1(z) = \frac{\ell \sin \zeta h}{\zeta \sinh \ell h} \bigg/ \prod_{n=1}^{\infty}\left(1 + \frac{z}{\beta_n}\right)\mathrm{e}^{-z/\lambda_n} \tag{A.12}$$

and thus using (A.9) with z replaced by $-z$ that as $z \to \infty$ through real positive values,

$$g_1(z) \sim \frac{\sin \zeta h}{\zeta e(-z)}\left(\frac{2z\ell}{\sinh \ell h}\right)^{1/2}. \tag{A.13}$$

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The asymptotics of g_2 are obtained in a similar fashion. We being by writing

$$g_2(z) = \frac{e^{\gamma zh/\pi}}{\Gamma(1 - zh/\pi)} \prod_{n=1}^{\infty} \frac{k_n}{\alpha_n} \prod_{n=1}^{\infty} \frac{\lambda_n(z - k_n)}{k_n(z - \lambda_n)} \prod_{n=1}^{\infty} \frac{z - \alpha_n}{z - k_n} \qquad (A.14)$$

and note that

$$\prod_{n=1}^{\infty} \frac{k_n}{\alpha_n} = \prod_{n=1}^{\infty} \left(1 + \ell^2/k_n^2\right)^{-1/2} = \left(\frac{K(1 - \ell^2/k^2)}{K \cosh \ell h - \ell \sinh \ell h}\right)^{1/2}, \qquad (A.15)$$

where the last step follows from the identity

$$K \cosh zh - z \sinh zh = K \prod_{n=0}^{\infty}(1 + z^2/k_n^2). \qquad (A.16)$$

Next

$$\prod_{n=1}^{\infty} \frac{\lambda_n(z - k_n)}{k_n(z - \lambda_n)} = \prod_{n=1}^{\infty} \left(1 + \frac{z(\lambda_n - k_n)}{k_n(z - \lambda_n)}\right) \qquad (A.17)$$

and

$$\prod_{n=1}^{\infty} \frac{z - \alpha_n}{z - k_n} = \prod_{n=1}^{\infty} \left(1 - \frac{\ell^2}{(z - k_n)(\alpha_n + k_n)}\right) \qquad (A.18)$$

which (since $0 < \lambda_n - k_n < \pi/2h$) shows that both infinite products are absolutely convergent and therefore that as $z \to \infty$, avoiding real, positive values,

$$\prod_{n=1}^{\infty} \frac{\lambda_n(z - k_n)}{k_n(z - \lambda_n)} \to \prod_{n=1}^{\infty} \frac{\lambda_n}{k_n}, \qquad \prod_{n=1}^{\infty} \frac{z - \alpha_n}{z - k_n} \to 1. \qquad (A.19)$$

Stirling's formula then shows that

$$g_2(z) \sim e(z) \left(\frac{-K(1 - \ell^2/k^2)}{2zh(K \cosh \ell h - \ell \sinh \ell h)}\right)^{1/2} \prod_{n=1}^{\infty} \frac{\lambda_n}{k_n} \qquad (A.20)$$

provided z is not real and positive. Proceeding as in (A.11) we obtain

$$\prod_{n=1}^{\infty} (1 - z/\alpha_n)(1 + z/\alpha_n) = \frac{(K \cos \zeta h - \zeta \sin \zeta h)(1 - \ell^2/k^2)}{(K \cosh \ell h - \ell \sinh \ell h)(1 + \zeta^2/k^2)} \qquad (A.21)$$

and hence we have that as $z \to \infty$ through real positive values

$$g_2(z) \sim \left(\frac{2zh(1 - \ell^2/k^2)}{K(K \cosh \ell h - \ell \sinh \ell h)}\right)^{1/2} \frac{K \cos \zeta h - \zeta \sin \zeta h}{(1 + \zeta^2/k^2)e(-z)} \prod_{n=1}^{\infty} \frac{\lambda_n}{k_n}. \qquad (A.22)$$

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From (A.9) and (A.20) we thus have that as $z \to \infty$

$$\frac{g_1(z)}{g_2(z)} \sim \left(\frac{\ell h(K \cosh \ell h - \ell \sinh \ell h)}{K \sinh \ell h (1 - \ell^2/k^2)} \right)^{1/2} \prod_{n=1}^{\infty} \frac{k_n}{\lambda_n} \qquad (A.23)$$

provided z is not real and positive, and from (A.13) and (A.22) we have

$$\frac{g_1(z)}{g_2(z)} \sim \left(\frac{K\ell(K \cosh \ell h - \ell \sinh \ell h)}{h \sinh \ell h (1 - \ell^2/k^2)} \right)^{1/2} \frac{(1 + \zeta^2/k^2) \sin \zeta h}{\zeta(K \cos \zeta h - \zeta \sin \zeta h)} \prod_{n=1}^{\infty} \frac{k_n}{\lambda_n} \quad (A.24)$$

as $z \to \infty$ through real positive values.

The objective of the above analysis is to show that $g_1(z)/g_2(z) = O(1)$ (and hence that $g(z) = O(z^{-1})$) as $|z| \to \infty$ through an appropriate set of values. Clearly we must avoid points at which $\zeta \tan \zeta h = K$, which are the points $z = \alpha_n$. Hence if we define C_N to be circles centred on the origin with radius $(N + \frac{1}{2})\pi/h$ then we will have the desired behaviour as $|z| \to \infty$ on C_N as $N \to \infty$.

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