On a class of three-dimensional integrable Lagrangians

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Abstract

We characterize non-degenerate Lagrangians of the form

$$\int f(u_x, u_y, u_t)\, dx\, dy\, dt$$

such that the corresponding Euler-Lagrange equations $(f_{u_x})_x + (f_{u_y})_y + (f_{u_t})_t = 0$ are integrable by the method of hydrodynamic reductions. The integrability conditions constitute an over-determined system of fourth order PDEs for the Lagrangian density f, which is in involution. The moduli space of integrable Lagrangians, factorized by the action of a natural equivalence group, is three-dimensional. Familiar examples include the dispersionless Kadomtsev-Petviashvili (dKP) and the Boyer-Finley Lagrangians, $f = u_x^3/3 + u_y^2 - u_x u_t$ and $f = u_x^2 + u_y^2 - 2e^{u_t}$, respectively. A complete description of integrable cubic and quartic Lagrangians is obtained. Up to the equivalence transformations, the list of integrable cubic Lagrangians reduces to three examples,

$$f = u_x u_y u_t, \quad f = u_x^2 u_y + u_y u_t \quad \text{and} \quad f = u_x^3/3 + u_y^2 - u_x u_t \ \text{(dKP)}.$$

There exists a unique integrable quartic Lagrangian,

$$f = u_x^4 + 2u_x^2 u_t - u_x u_y - u_t^2.$$

We conjecture that these examples exhaust the list of integrable polynomial Lagrangians which are essentially three-dimensional (it was verified that there exist no polynomial integrable Lagrangians of degree five).

We prove that the Euler-Lagrange equations are integrable by the method of hydrodynamic reductions if and only if they possess a scalar pseudopotential playing the role of a dispersionless 'Lax pair'.

MSC: 35Q58, 37K05, 37K10, 37K25.

Keywords: Multi-dimensional Dispersionless Integrable Systems, Hydrodynamic Reductions, Pseudopotentials.

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†SPT acknowledges partial financial support from the RFBR grant 04-01-00130.

1 Introduction

The method of hydrodynamic reductions, originally proposed for the dKP equation in [12, 13, 14], applies to a broad class of multi-dimensional dispersionless PDEs $F(u, u_s, \ldots) = 0$ where u is a (vector-) function of d independent variables $s = (x, y, t, \ldots)$. The main idea is to look for solutions in the form $u = u(R^1, \ldots, R^n)$ where the Riemann invariants $R^1(x, y, t, \ldots), \ldots, R^n(x, y, t, \ldots)$ are arbitrary solutions of $(d-1)$ commuting diagonal systems

$$R^i_y = \mu^i(R)\ R^i_x, \quad R^i_t = \lambda^i(R)\ R^i_x,$$

etc. We recall, see [31], that the commutativity conditions imply the following restrictions on the characteristic speeds:

$$\frac{\partial_j \mu^i}{\mu^j - \mu^i} = \frac{\partial_j \lambda^i}{\lambda^j - \lambda^i}, \quad i \neq j, \quad \partial_i = \partial/\partial_{R^i}. \tag{1}$$

Thus, the original multi-dimensional equation is decoupled into a collection of commuting n-component $(1+1)$-dimensional systems in Riemann invariants which can then be solved by the generalized hodograph method [31]. Solutions arising within this approach, known as nonlinear interactions of n planar simple waves, were extensively investigated in gas dynamics (simple waves, double waves, etc, [29]) and the theory of dispersionless $(2+1)$-dimensional hierarchies of the dKP type [12, 13, 14, 16, 21, 19, 22, 20, 5, 25, 18, 27].

It was suggested in [7, 9] to call a multi-dimensional equation *integrable* if it possesses 'sufficiently many' n-component hydrodynamic reductions (parametrized by $(d-2)n$ arbitrary functions of a single variable). It turned out that this requirement is very strong and provides the effective classification criterion. Partial classification results for $(2+1)$-dimensional integrable systems of hydrodynamic type,

$$\mathbf{u}_t + A(\mathbf{u})\mathbf{u}_x + B(\mathbf{u})\mathbf{u}_y = 0,$$

(here $\mathbf{u}$ is m-component vector, A and B are $m \times m$ matrices) were obtained in [7, 8]. It was observed that the class of PDEs amenable to the method of hydrodynamic reductions extends beyond the class of hydrodynamic type systems. For instance, the method applies to scalar second order PDEs

$$F(u_{xx}, u_{xy}, u_{xt}, u_{yy}, u_{yt}, u_{tt}) = 0,$$

see e.g. [28], [10] for the classification of integrable equations of the form $u_{tt} = f(u_{xx}, u_{xt}, u_{xy})$. Hydrodynamic reductions of multi-dimensional dispersionless integrable systems in dimensions greater then three were discussed in [6, 9].

In this paper we apply the method of hydrodynamic reductions to three-dimensional equations of the form

$$(f_{u_x})_x + (f_{u_y})_y + (f_{u_t})_t = 0, \tag{2}$$

which are the Euler-Lagrange equations corresponding to first order Lagrangian densities $f(u_x, u_y, u_t)$. We begin with two illustrative examples.

Example 1. Consider the linear wave equation $u_{tt} = u_{xx} + u_{yy}$ corresponding to the quadratic Lagrangian density $f = u_x^2 + u_y^2 - u_t^2$. Setting $a = u_x,\ b = u_y,\ c = u_t$ we can rewrite it in the first order form

$$a_y = b_x, \quad a_t = c_x, \quad b_t = c_y, \quad a_x + b_y - c_t = 0. \tag{3}$$

Let us seek solutions in the form $a = a(R^1, \ldots, R^n),\ b = b(R^1, \ldots, R^n),\ c = c(R^1, \ldots, R^n)$ where the Riemann invariants $R^1(x, y, t), \ldots, R^n(x, y, t)$ are *arbitrary* solutions of a pair of commuting hydrodynamic type systems

$$R^i_y = \mu^i(R)\ R^i_t, \qquad R^i_x = \lambda^i(R)\ R^i_t;$$

(for the sake of symmetry we have chosen t as a distinguished variable). Substituting this ansatz into (3) one obtains the equations $\partial_i b = \mu^i \partial_i c,\ \partial_i a = \lambda^i \partial_i c$ along with the dispersion relation

$$(\lambda^i)^2 + (\mu^i)^2 = 1.$$

Setting $\lambda^i = \cos\varphi^i,\ \mu^i = \sin\varphi^i$ and taking into account the commutativity conditions (1) one obtains $\partial_j \varphi^i = 0$ for any $j \neq i$. Thus, $\varphi^i = \varphi^i(R^i)$ so that both systems of hydrodynamic type assume decoupled forms

$$R^i_y = \sin\varphi^i(R^i)\ R^i_t, \qquad R^i_x = \cos\varphi^i(R^i)\ R^i_t,$$

with the general solution $R^i(x, y, t)$ given by the implicit relation

$$g^i(R^i) = t + y\sin\varphi^i(R^i) + x\cos\varphi^i(R^i); \tag{4}$$

here $g^i(R^i)$ are yet another n arbitrary functions. Finally, the consistency conditions for the equations $\partial_i b = \mu^i \partial_i c,\ \partial_i a = \lambda^i \partial_i c$ imply $\partial_i \partial_j c = 0$ so that one can set $c = R^1 + ... + R^n$ (notice that one has a reparametrization freedom $R^i \to f^i(R^i)$). Thus, we arrive at solutions of the wave equation given by the formulae

$$u_t = c = R^1 + ... + R^n,$$

$$u_y = b = \int \sin\varphi^1(R^1) dR^1 + ... + \int \sin\varphi^n(R^n) dR^n,$$

$$u_x = a = \int \cos\varphi^1(R^1) dR^1 + ... + \int \cos\varphi^n(R^n) dR^n,$$

where $R^i(x, y, t)$ is defined by the implicit relation (4). These solutions depend on $2n$ arbitrary functions of a single argument and can be viewed as linear superpositions of n elementary solutions of the form

$$u_t = R, \quad u_y = \int \sin\varphi(R) dR, \quad u_x = \int \cos\varphi(R) dR$$

where $R(x, y, t)$ is defined by the implicit relation $g(R) = t + y\sin\varphi(R) + x\cos\varphi(R)$. This relation implies that the level sets $R = const$ are null-planes so that the gradient of u is constant along a one-parameter family of null-planes. Solutions of this type are known as planar simple waves.

Example 2. Let us apply the same approach to the *nonlinear* wave equation $e^{u_t} u_{tt} = u_{xx} + u_{yy}$ (also known as the Boyer-Finley equation) which corresponds to the Lagrangian density $f = u_x^2 + u_y^2 - 2e^{u_t}$. Setting $a = u_x,\ b = u_y,\ c = u_t$ we can rewrite it as follows:

$$a_y = b_x, \quad a_t = c_x, \quad b_t = c_y, \quad a_x + b_y - e^c c_t = 0. \tag{5}$$

We again seek solutions in the form $a = a(R^1, \dots, R^n),\ b = b(R^1, \dots, R^n),\ c = c(R^1, \dots, R^n)$ where the Riemann invariants solve hydrodynamic type systems

$$R^i_y = \mu^i(R)\ R^i_t, \qquad R^i_x = \lambda^i(R)\ R^i_t.$$

The substitution into (5) implies

$$\partial_i b = \mu^i \partial_i c, \quad \partial_i a = \lambda^i \partial_i c \tag{6}$$

along with the dispersion relation

$$(\lambda^i)^2 + (\mu^i)^2 = e^c.$$

Setting $\lambda^i = e^{c/2}\cos\varphi^i,\ \mu^i = e^{c/2}\sin\varphi^i$ and taking into account the commutativity conditions (1) one obtains the expression for $\partial_j \varphi^i$:

$$\partial_j \varphi^i = -\frac{1}{2}\cot\frac{\varphi^i - \varphi^j}{2}\ \partial_j c, \quad i \neq j. \tag{7}$$

The consistency conditions for the equations (6) imply

$$\partial_i\partial_j c = \frac{\partial_j\lambda^i}{\lambda^j-\lambda^i}\partial_i c + \frac{\partial_i\lambda^j}{\lambda^i-\lambda^j}\partial_j c, \quad i\neq j.$$

Substituting here $\lambda^i = e^{c/2}\cos\varphi^i$ and taking into account (7) we obtain the following system which governs hydrodynamic reductions of the nonlinear wave equation:

$$\partial_j\varphi^i = -\frac{1}{2}\cot\frac{\varphi^i-\varphi^j}{2}\,\partial_j c, \quad \partial_j\partial_i c = \frac{\partial_j c\partial_i c}{2\sin^2\frac{\varphi^i-\varphi^j}{2}}; \tag{8}$$

notice that this system is a nonlinear analog of the system $\partial_j\varphi^i=0,\ \partial_j\partial_i c=0$ derived in Example 1 for the linear wave equation. One can show that the system (8) is in involution (that is, the compatibility conditions $\partial_k(\partial_j\varphi^i)=\partial_j(\partial_k\varphi^i)$ and $\partial_k(\partial_j\partial_i c)=\partial_j(\partial_k\partial_i c)$ are satisfied identically) and its general solution depends, modulo reparametrizations $R^i\to f^i(R^i)$, on n arbitrary functions of a single argument. Once a solution to the system (8) is found, one can reconstruct $b(R)$ and $a(R)$ from the equations (6) which are consistent by construction. After that one has to solve hydrodynamic type systems (where the characteristic speeds $\mu^i(R)$ and $\lambda^i(R)$ are known) which can be done by the generalized hodograph method [31]. This gives some more n arbitrary functions. Thus, solutions arising from n-component reductions of the nonlinear wave equation depend on $2n$ arbitrary functions of a single argument. For one-component reductions, equations (8) become vacuous and, setting $c=R$, one has

$$u_t=c=R, \quad u_y=b=\int e^{R/2}\sin\varphi(R)dR, \quad u_x=a=\int e^{R/2}\cos\varphi(R)dR$$

where the function $R(x,y,t)$ is defined by the implicit relation $g(R)=t+ye^{R/2}\sin\varphi(R)+xe^{R/2}\cos\varphi(R)$. This relation implies that the level sets $R=const$ are characteristic planes. Solutions of this type are known as planar simple waves. Thus, solutions governed by the system (8) can be interpreted as *nonlinear interactions* of planar simple waves. We refer to [5, 6, 22, 19] for further discussion and explicit examples.

The main object of our study are nonlinear Lagrangian PDEs (of the type discussed in Example 2) which are integrable in the above sense, that is, possess infinitely many hydrodynamic reductions. This requirement imposes strong restrictions on the Lagrangian density f. Our first main result is the system of partial differential equations for the density $f(a,b,c)$ providing the necessary and sufficient conditions for the integrability (we use the notation $a=u_x,\ b=u_y,\ c=u_t$). These conditions can be represented in a remarkable compact form:

Theorem 1 *For non-degenerate Lagrangians, the Euler-Lagrange equations (2) are integrable by the method of hydrodynamic reductions if and only if the density f satisfies the identity*

$$d^4f = d^3f\frac{dH}{H}+\frac{3}{H}det(dM). \tag{9}$$

Here d^3f and d^4f are the symmetric differentials of f which appear in the standard Taylor expansion $f(s+ds)-f(s)=df+d^2f/2!+d^3f/3!+\dots$ for a function $f(s)$ of three variables $s=(a,b,c)$. The Hessian H and the 4×4 matrix M are defined as follows:

$$H = det\begin{pmatrix} f_{aa} & f_{ab} & f_{ac}\\ f_{ab} & f_{bb} & f_{bc}\\ f_{ac} & f_{bc} & f_{cc}\end{pmatrix}, \quad M=\begin{pmatrix} 0 & f_a & f_b & f_c\\ f_a & f_{aa} & f_{ab} & f_{ac}\\ f_b & f_{ab} & f_{bb} & f_{bc}\\ f_c & f_{ac} & f_{bc} & f_{cc}\end{pmatrix}. \tag{10}$$

The differential $dM = M_a da + M_b db + M_c dc$ is a matrix-valued one-form

$$\begin{pmatrix} 0 & f_{aa} & f_{ab} & f_{ac} \\ f_{aa} & f_{aaa} & f_{aab} & f_{aac} \\ f_{ab} & f_{aab} & f_{abb} & f_{abc} \\ f_{ac} & f_{aac} & f_{abc} & f_{acc} \end{pmatrix} da + \begin{pmatrix} 0 & f_{ab} & f_{bb} & f_{bc} \\ f_{ab} & f_{aab} & f_{abb} & f_{abc} \\ f_{bb} & f_{abb} & f_{bbb} & f_{bbc} \\ f_{bc} & f_{abc} & f_{bbc} & f_{bcc} \end{pmatrix} db + \begin{pmatrix} 0 & f_{ac} & f_{bc} & f_{cc} \\ f_{ac} & f_{aac} & f_{abc} & f_{acc} \\ f_{bc} & f_{abc} & f_{bbc} & f_{bcc} \\ f_{cc} & f_{acc} & f_{bcc} & f_{ccc} \end{pmatrix} dc.$$

A Lagrangian is said to be non-degenerate iff $H \neq 0$.

Both sides of the equation (9) are homogeneous quartic forms in da, db, dc. Equating similar terms we obtain expressions for *all* fourth order partial derivatives of f in terms of its second and third order derivatives (15 equations altogether). The resulting over-determined system for f is in involution, and its solution space is 20-dimensional (indeed, the values of the derivatives of f up to order 3 at a given point (a_0, b_0, c_0) amount to $1 + 3 + 6 + 10 = 20$ arbitrary constants). Factorised by the action of a natural equivalence group of dimension 17 (generated by arbitrary affine transformations of the independent variables a, b, c plus transformations $f \to \mu f + \alpha a + \beta b + \gamma c + \delta$, see the end of Sect. 2 for the discussion of the origin of these symmetries) this provides a three-dimensional moduli space of integrable Lagrangians. Details of the derivation of the integrability conditions (9) are given in Sect. 2. Notice that these conditions arise in a somewhat different form, namely, as explicit formulae for the fourth order derivatives of f (which are very complicated). It is a truly remarkable fact that they compactify into a single expression (9).

Example 3. For the dKP Lagrangian ($f = a^3/3 + b^2 - ac$) one has $d^4 f = 0,\ H = -2,\ \det(dM) = 0$. Similarly, for the Boyer-Finley Lagrangian ($f = a^2 + b^2 - 2e^c$) one has $d^4 f = -2e^c(dc)^4,\ d^3 f = -2e^c(dc)^3,\ H = -2e^c,\ \det(dM) = 0$. In both cases the identity (9) is obviously satisfied.

Remark. Notice that in two dimensions any Euler-Lagrange equation of the form $(f_{u_x})_x + (f_{u_y})_y = 0$ is automatically integrable. Indeed, in the new variables $a = u_x,\ b = u_y$ it takes the form of a two-component quasilinear system $a_y = b_x,\ (f_a)_x + (f_b)_y = 0$ which linearises under the hodograph transformation interchanging dependent and independent variables. This trick, however, does not work in more than two dimensions.

Sect. 3 is devoted to polynomial Lagrangians. Our first result is a simple Lemma stating that, for non-degenerate homogeneous solutions of the equation (9), the degree of homogeneity can take only one of the three values $0, 2$ or 3 (recall that Lagrangians of homogeneity one are automatically degenerate). This observation is particularly useful for the classification of polynomial Lagrangians implying that the 'leading' homogeneous part of a polynomial solution f must be either of degree 3 or degenerate (polynomials of degree two give rise to linear Euler-Lagrange equations). In Sect. 3.1 we obtain a complete list of integrable cubic Lagrangians with the densities

$$f = C(u_x, u_y, u_t) + \alpha u_x^2 + \beta u_y^2 + \gamma u_t^2 + \mu u_x u_y + \nu u_x u_t + \eta u_y u_t;$$

here C is a homogeneous cubic form. The substitution into the integrability conditions (9) implies that C must be totally reducible, leading to three essentially different possibilities:

(1) C consists of three lines in a general position. Further analysis allows one to eliminate quadratic terms, leading to a unique Lagrangian density

$$f = u_x u_y u_t$$

with the corresponding Euler-Lagrange equation $u_t u_{xy} + u_y u_{xt} + u_x u_{yt} = 0$. Although this example looks deceptively simple, the corresponding dispersionless Lax pair is quite non-trivial, see formula (12). To the best of our knowledge this Lagrangian has not been recorded before.

(2) C contains a double line. Up to equivalence transformations this case reduces to the density

$$f = u_x^2 u_y + u_y u_t$$

which generates a particular flow of the so-called r-Dym hierarchy.
(3) C is a triple line. This case reduces to the dKP Lagrangian density

$$f = u_x^3/3 + u_y^2 - u_x u_t.$$

Quartic Lagrangians are classified in Sect. 3.2. Here the leading part of the density f must necessarily be degenerate, leading to a unique integrable example

$$f = u_x^4 + 2u_x^2 u_t - u_x u_y - u_t^2$$

which corresponds to a particular flow of the so-called r-dKP hierarchy.

We have verified that there exist no polynomial solutions to the system (9) of degree five. It is tempting to conjecture that the four examples listed above exhaust the list of polynomial integrable Lagrangians.

Homogeneous Lagrangians can be obtained by setting $f(a,b,c) = a^k g(\xi,\eta),\ \xi = b/a,\ \eta = c/a$ (recall that the degree of homogeneity k can take the values $0, 2, 3$ only). Substituting this ansatz into the integrability conditions (9) and analysing the resulting equations for $g(\xi,\eta)$ one can show that the case $k = 3$ leads to the cubic Lagrangian $f = u_x u_y u_t$ and, thus, gives no new examples. The case $k = 2$ reduces to functions g which are quadratic in ξ, η and, hence, generate quadratic Lagrangians with linear Euler-Lagrange equations. The last case $k = 0$ turned out to be quite nontrivial. The substitution of $f(a,b,c) = g(\xi,\eta)$ into (9) results in a system of five equations expressing all fourth order partial derivatives of g in terms of lower order derivatives. In symbolic form, this system can be represented as follows:

$$d^4 g = d^3 g \frac{dh}{h} + 6\frac{dg}{h} det(dm) + 3\frac{(dg)^2}{h} det(dn). \tag{11}$$

Here, as in (9), $d^s g$ are symmetric differentials of g (notice that g is now a function of two variables), the matrices m and n are defined as

$$m = \begin{pmatrix} 0 & g_\xi & g_\eta \\ g_\xi & g_{\xi\xi} & g_{\xi\eta} \\ g_\eta & g_{\xi\eta} & g_{\eta\eta} \end{pmatrix}, \qquad n = \begin{pmatrix} g_{\xi\xi} & g_{\xi\eta} \\ g_{\xi\eta} & g_{\eta\eta} \end{pmatrix},$$

and

$$h = -det(m) = g_\eta^2 g_{\xi\xi} - 2g_\xi g_\eta g_{\xi\eta} + g_\xi^2 g_{\eta\eta}.$$

The non-degeneracy of the Lagrangian density $f(a,b,c) = g(\xi,\eta)$ is equivalent to the condition $h \neq 0$. One can show that the over-determined system (11) is in involution and its solution space is 10-dimensional (indeed, the values of partial derivatives of g up to order 3 at a fixed point (ξ_0,η_0) amount to $1+2+3+4 = 10$ arbitrary constants). Although it is still difficult to integrate this system in general, some particular solutions can readily be constructed. For instance, any quadratic form $g(\xi,\eta) = \alpha\xi^2 + \beta\xi\eta + \gamma\eta^2 + \mu\xi + \nu\eta$ is a solution of (11), leading to integrable Lagrangians with the densities

$$f = \alpha\frac{u_y^2}{u_x^2} + \beta\frac{u_t u_y}{u_x^2} + \gamma\frac{u_t^2}{u_x^2} + \mu\frac{u_y}{u_x} + \nu\frac{u_t}{u_x}.$$

Up to the equivalence transformations any expression of this type can be reduced to either of the non-equivalent (over reals) canonical forms,

$$f = \frac{u_y^2 + u_t^2}{u_x^2}, \qquad f = \frac{u_y u_t}{u_x^2}, \qquad f = \frac{u_y^2}{u_x^2} + \frac{u_t}{u_x}.$$

In Sect. 4 we study scalar pseudopotentials for integrable Euler-Lagrange equations (also known as S-functions, dispersionless Lax pairs, etc). Examples thereof include the dispersionless Lax pair

$$S_t = \frac{1}{3}S_x^3 + u_x S_x + u_y, \quad S_y = \frac{1}{2}S_x^2 + u_x$$

which generates the dKP equation $u_{xt}-u_xu_{xx}=u_{yy}$. Similarly, the Boyer-Finley equation $u_{xy}=(e^{u_t})_t$ possesses the dispersionless Lax pair

$$S_t=u_t-\ln S_y, \quad S_x=u_x-\frac{e^{u_t}}{S_y}.$$

Further examples of integrable multi-dimensional equations possessing pseudopotentials of the above type can be found in [32, 28, 17]. It was proved in [8] that for two-component $(2+1)$-dimensional systems of hydrodynamic type the existence of dispersionless Lax pairs is necessary and sufficient for the integrability (that is, for the existence of sufficiently many hydrodynamic reductions). Dispersionless Lax pairs constitute a basis for the twistor and dispersionless $\bar{\partial}$-approaches to multi-dimensional dispersionless hierarchies [23, 17, 2].

Our second main result is the following

Theorem 2 *The Euler-Lagrange equation $(f_{u_x})_x+(f_{u_y})_y+(f_{u_t})_t=0$ is integrable by the method of hydrodynamic reductions if and only if it possesses a dispersionless Lax pair*

$$S_t=F(S_x,\ u_x,\ u_y,\ u_t), \quad S_y=G(S_x,\ u_x,\ u_y,\ u_t).$$

In some cases it seems to be more convenient to work with parametric Lax pairs

$$S_t=F(p,\ u_x,\ u_y,\ u_t), \quad S_y=G(p,\ u_x,\ u_y,\ u_t), \quad S_x=H(p,\ u_x,\ u_y,\ u_t),$$

which take the above form if one expresses the parameter p in terms of S_x from the third equation. For instance, the equation $u_tu_{xy}+u_yu_{xt}+u_xu_{yt}=0$ corresponding to the Lagrangian density $f=u_xu_yu_t$ possesses the parametric Lax pair

$$\frac{S_x}{u_x}=\zeta(p), \quad \frac{S_y}{u_y}=\zeta(p)+\frac{\wp'(p)+\lambda}{2\wp(p)}, \quad \frac{S_t}{u_t}=\zeta(p)+\frac{\wp'(p)-\lambda}{2\wp(p)}; \tag{12}$$

here $(\wp')^2=4\wp^3+\lambda^2$ and $\zeta'=-\wp$ (Weierstrass $\wp$ and ζ functions, see Sect. 3).

Differential-geometric aspects of the integrability conditions (9) are investigated in Sect. 5. The main object associated with the Lagrangian density $f(a,b,c)$ is the Hessian metric d^2f. We show that, by virtue of (9), this metric is necessarily conformally flat.

Remark. Notice that any Euler-Lagrange equation $(f_{u_x})_x+(f_{u_y})_y+(f_{u_t})_t=0$ is manifestly conservative and, moreover, possesses three extra conservation laws

$$(u_xf_{u_x}-f)_x+(u_xf_{u_y})_y+(u_xf_{u_t})_t=0,$$

$$(u_yf_{u_x})_x+(u_yf_{u_y}-f)_y+(u_yf_{u_t})_t=0,$$

$$(u_tf_{u_x})_x+(u_tf_{u_y})_y+(u_tf_{u_t}-f)_t=0,$$

which are components of the energy-momentum tensor. In the dKP case, $f=u_x^3/3+u_y^2-u_xu_t$, these four conservation laws take the form

$$(u_x^2-u_t)_x+(2u_y)_y-(u_x)_t=0,$$

$$(2u_x^3/3-u_y^2)_x+(2u_xu_y)_y-(u_x^2)_t=0,$$

$$(u_x^2u_y-u_yu_t)_x+(u_xu_t+u_y^2-u_x^3/3)_y-(u_xu_y)_t=0,$$

$$(u_x^2u_t-u_t^2)_x+(2u_yu_t)_y-(u_y^2+u_x^3/3)_t=0,$$

respectively. One can show that the dKP equation possesses no extra conservation laws of the form

$$g(u_x, u_y, u_t)_x + h(u_x, u_y, u_t)_y + p(u_x, u_y, u_t)_t = 0.$$

The discussion of the corresponding hierarchies of *higher nonlocal* symmetries and conservation laws for integrable Euler-Lagrange equations is beyond the scope of this paper. Our primary goal is the characterization of integrable Lagrangians based on the method of hydrodynamic reductions.

2 Derivation of the integrability conditions: proof of Theorem 1

Introducing the variables $a = u_x,\ b = u_y,\ c = u_t$ one rewrites the Euler-Lagrange equation (2) in the first order form

$$a_y = b_x, \quad a_t = c_x, \quad b_t = c_y, \quad (f_a)_x + (f_b)_y + (f_c)_t = 0. \tag{13}$$

The idea of the method of hydrodynamic reductions is to look for solutions of the system (13) in the form $a = a(R^1, \dots, R^n),\ b = b(R^1, \dots, R^n),\ c = c(R^1, \dots, R^n)$ where the Riemann invariants $R^1(x, y, t), \dots, R^n(x, y, t)$ are *arbitrary* solutions of a pair of commuting hydrodynamic type flows

$$R^i_y = \mu^i(R)\ R^i_x, \qquad R^i_t = \lambda^i(R)\ R^i_x. \tag{14}$$

Substituting this ansatz into (13) one obtains the equations

$$\partial_i b = \mu^i \partial_i a, \quad \partial_i c = \lambda^i \partial_i a \tag{15}$$

(here $\partial_i = \partial/\partial_{R^i}$) along with the dispersion relation

$$D(\lambda^i, \mu^i) = f_{aa} + 2f_{ab}\mu^i + 2f_{ac}\lambda^i + f_{bb}(\mu^i)^2 + 2f_{bc}\mu^i\lambda^i + f_{cc}(\lambda^i)^2 = 0. \tag{16}$$

Hereafter, we assume the conic (16) to be irreducible. This condition is equivalent to the non-degeneracy of the Lagrangian or, equivalently, to the non-vanishing of the Hessian: $H \neq 0$, see (10). The consistency conditions of the equations (15) imply

$$\partial_i\partial_j a = \frac{\partial_j \lambda^i}{\lambda^j - \lambda^i}\partial_i a + \frac{\partial_i \lambda^j}{\lambda^i - \lambda^j}\partial_j a. \tag{17}$$

Differentiating the dispersion relation (16) with respect to $R^j,\ j \neq i$, and keeping in mind (15) and (1) one obtains the explicit expressions for $\partial_j \lambda^i$ and $\partial_j \mu^i$ in the form

$$\partial_j \lambda^i = (\lambda^i - \lambda^j) B_{ij} \partial_j a, \quad \partial_j \mu^i = (\mu^i - \mu^j) B_{ij} \partial_j a \tag{18}$$

where B_{ij} are rational expressions in $\lambda^i, \lambda^j, \mu^i, \mu^j$ whose coefficients depend on partial derivatives of the density $f(a, b, c)$ up to third order. Explicitly, one has

$$B_{ij} = \frac{N_{ij}}{D_{ij}} = \frac{N_{ij}}{2(f_{aa} + f_{ab}(\mu^i + \mu^j) + f_{ac}(\lambda^i + \lambda^j) + f_{bb}\mu^i\mu^j + f_{bc}(\mu^i\lambda^j + \mu^j\lambda^i) + f_{cc}\lambda^i\lambda^j)};$$

notice that, modulo the dispersion relation (16), the denominator D_{ij} equals $4D\left(\frac{\lambda^i+\lambda^j}{2},\ \frac{\mu^i+\mu^j}{2}\right)$. The numerator N_{ij} is a polynomial expression of the form

$$N_{ij} = f_{aaa} + f_{aab}(\mu^j + 2\mu^i) + f_{aac}(\lambda^j + 2\lambda^i) + f_{abb}\mu^i(\mu^i + 2\mu^j) + f_{acc}\lambda^i(\lambda^i + 2\lambda^j) +$$

$$2f_{abc}(\lambda^i\mu^j + \lambda^j\mu^i + \lambda^i\mu^i) + f_{bbb}(\mu^i)^2\mu^j + f_{ccc}(\lambda^i)^2\lambda^j +$$

$$f_{bbc}\mu^i(\lambda^j\mu^i + 2\lambda^i\mu^j) + f_{bcc}\lambda^i(\lambda^i\mu^j + 2\lambda^j\mu^i).$$

The equation (17) takes the form

$$\partial_i \partial_j a = -(B_{ij} + B_{ji}) \partial_i a \partial_j a. \tag{19}$$

The compatibility conditions $\partial_k \partial_j \lambda^i = \partial_j \partial_k \lambda^i$, $\partial_k \partial_j \mu^i = \partial_j \partial_k \mu^i$ and $\partial_k \partial_j \partial_i a = \partial_j \partial_k \partial_i a$ are equivalent to the equations

$$\partial_k B_{ij} = (B_{ij} B_{kj} + B_{ij} B_{ik} - B_{kj} B_{ik}) \partial_k a, \tag{20}$$

which must be satisfied identically by virtue of (15), (16), (18). The details of this calculation (which is computationally intense) can be summarized as follows.

In order to obtain equations with 'smallest possible' coefficients at the fourth order derivatives of $f(a, b, c)$ we rewrite (20) as

$$\partial_k N_{ij} = N_{ij} \left(\frac{1}{D_{ij}} \partial_k D_{ij} + B_{kj} \partial_k a + B_{ik} \partial_k a \right) - D_{ij} B_{kj} B_{ik} \partial_k a. \tag{21}$$

The fourth order derivatives of $f(a, b, c)$ are present only in the l.h.s. term $\partial_k N_{ij}$. Further reduction of the complexity of the expression in the r.h.s. is achieved by representing $1/D_{ij}$ in the form

$$\frac{1}{D_{ij}} = U_{ij} = [2(\lambda^i f_{bc} + f_{ab})(\lambda^j f_{bc} + f_{ab}) - f_{bb}(\lambda^i \lambda^j f_{cc} + (\lambda^i + \lambda^j) f_{ac} + f_{aa})$$

$$+ f_{bb}(\lambda^j f_{bc} + f_{ab}) \mu^i + f_{bb}(\lambda^i f_{bc} + f_{ab}) \mu^j + f_{bb}^2 \mu^i \mu^j] / ((\lambda^i - \lambda^j)^2 H)$$

(which holds identically modulo the dispersion relation (16)), and the subsequent substitution $B_{st} = N_{st}/D_{st} = N_{st} U_{st}$. The denominators of the r.h.s. terms in (21) cancel out as explained in the program file 2-Lagr3dim.frm (see [11]), producing a polynomial in λ^i, λ^j, λ^k, μ^i, μ^j, μ^k with coefficients depending on the derivatives of the density $f(a, b, c)$. This was the most essential technical part of the calculation: the starting expression for the r.h.s. of (21) has more than 500.000 terms with different denominators; after properly organized cancellations we get a polynomial expression with less than 2000 terms. Using (16) and assuming $f_{bb} \neq 0$ (this can always be achieved by a linear change of independent variables x, y, t), we simplify this polynomial by substituting the powers of $(\mu^i)^s$, $(\mu^j)^s$, $(\mu^k)^s$, $s \geq 2$, arriving at a polynomial of degree one in each of μ^i, μ^j, μ^k and degree four in λ's. Equating similar coefficients of these polynomials in both sides of (21) we arrive at a set of 45 equations for the derivatives of the Lagrangian density $f(a, b, c)$ (linear in the fourth derivatives). Solving it, we get closed form expressions for all fourth order derivatives of $f(a, b, c)$ in terms of its second and third order derivatives. A straigtforward computation shows that the compatibility conditions are satisfied identically. Although the arising expressions are very long indeed, they can be rewritten in a compact form (9). This finishes the proof of Theorem 1.

Further particulars and the programs in FORM [33] and Maple[1] [34] for the computations described above are given in [11]. The file 1-README explains the overall structure of this program package.

Remark. The integrability conditions (9) are invariant under the obvious equivalence transformations generated by

(e1) linear transformations of a, b, c corresponding to linear changes of the independent variables x, y, t;

(e2) translations in a, b, c corresponding to the transformation $u \to u + \alpha x + \beta y + \gamma t$;

(e3) transformations of the form $f \to \mu f + \nu a + \eta b + \tau c + \rho$ which do not effect the Euler-Lagrange equations.

These transformations generate a 17-parameter symmetry group of the system (9). Notice that (e1) and (e2) generate the group of affine transformations. These symmetries allow one to considerably simplify the classification results. For instance, given any quadratic polynomial $Q(a, b, c)$ and a linear function $l(a, b, c)$, both not necessarily homogeneous, the density

$$f(a, b, c) = Q(a, b, c)/l(a, b, c)$$

[1] Maple(TM) is a trademark of Waterloo Maple Inc.

is a solution of (9). Up to the equivalence (e1) – (e3) this expression can be transformed to either of the non-equivalent (over reals) canonical forms

$$\frac{b^2+c^2+1}{a}, \qquad \frac{bc+1}{a}, \qquad \frac{b^2+c}{a}.$$

The classification below is carried out up to this natural equivalence.

3 Integrable polynomial Lagrangians

We will start with a useful remark on homogeneous solutions of the equation (9). Recall that a function f is said to be homogeneous of degree k if it satisfies the Euler identity $af_a + bf_b + cf_c = kf$. This implies, in particular, that f_a, f_b, f_c are homogeneous of degree $k-1$, that is,

$$G\begin{pmatrix} a \\ b \\ c \end{pmatrix} = (k-1)\begin{pmatrix} f_a \\ f_b \\ f_c \end{pmatrix}; \tag{22}$$

here the 3×3 matrix G is the Hessian matrix of f.

Lemma *Let f be a homogeneous solution of the equation (9) with non-zero Hessian $H = det\ G$. Then the degree of homogeneity k can take only one of the three values $0, 2$ or 3.*

Proof:

The equation (9) is an identity in da, db, dc. Let us replace da, db, dc by a, b, c, respectively. Under this identification one has: $df \to kf,\ d^2f \to (k-1)kf,\ d^3f \to (k-2)(k-1)kf,\ d^4f \to (k-3)(k-2)(k-1)kf$. Moreover,

$$\frac{dH}{H} = \frac{H_a da + H_b db + H_c dc}{H} \to \frac{H_a a + H_b b + H_c c}{H} = 3(k-2),$$

since H is homogeneous of degree $3(k-2)$. Finally, the matrix one-form dM reduces to

$$\begin{pmatrix} 0 & (k-1)f_a & (k-1)f_b & (k-1)f_c \\ (k-1)f_a & (k-2)f_{aa} & (k-2)f_{ab} & (k-2)f_{ac} \\ (k-1)f_b & (k-2)f_{ab} & (k-2)f_{bb} & (k-2)f_{bc} \\ (k-1)f_c & (k-2)f_{ac} & (k-2)f_{bc} & (k-2)f_{cc} \end{pmatrix};$$

its determinant equals $(k-1)^2(k-2)^2 det\ M$. Thus, for homogeneous f, the equation (9) implies

$$(k-3)(k-2)(k-1)kf = 3(k-2)^2(k-1)kf + \frac{3(k-1)^2(k-2)^2}{H} det\ M. \tag{23}$$

We also point out the identity

$$\frac{det\ M}{H} = -\begin{pmatrix} f_a & f_b & f_c \end{pmatrix} G^{-1}\begin{pmatrix} f_a \\ f_b \\ f_c \end{pmatrix}.$$

Taking into account (22), this implies

$$(k-1)\frac{det\ M}{H} = -\begin{pmatrix} f_a & f_b & f_c \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = -kf.$$

Thus, both terms on the right hand side of (23) cancel, leaving the identity $(k-3)(k-2)(k-1)k = 0$. It remains to point out that Lagrangians of homogeneity $k = 1$ are automatically degenerate. Q.E.D.

This result is particularly useful for the classification of polynomial solutions. Indeed, let us represent a polynomial f of degree k in the form

$$f = Q_k + Q_{k-1} + ...$$

where Q_k is a *homogeneous* polynomial of degree k in the variables a, b, c. Writing the equation (9) in the homogeneous form

$$Hd^4 f = d^3 f dH + 3det(dM),$$

we readily see that the leading term Q_k must be a solution itself. This leads to two possibilities: either $k = 3$ (see Sect. 3.1) or, if $k \geq 4$, the Hessian of Q_k must vanish identically. In the last case the classical result [15] (see also [26], p. 234) implies the existence of a linear change of variables a, b, c after which Q_k becomes a binary form, that is, a homogeneous function of two variables only (say, a and b). This partial 'separation of variables' considerably simplifies all calculations (see Sect. 3.2 for a complete analysis of the case $k = 4$).

3.1 Classification of cubic Lagrangians

The integrability conditions (9) provide a straightforward classification of integrable cubic Lagrangians with the densities

$$f(a, b, c) = C(a, b, c) + \alpha a^2 + \beta b^2 + \gamma c^2 + \mu ab + \nu ac + \eta bc$$

where C is a homogeneous cubic form in a, b, c. Using equivalence transformations of the type (e1) one can bring C to a canonical form, thus simplifying the analysis. Direct calculations using (9) reveal that the cubic C must necessarily be totally reducible, that is, a product of three linear forms (this condition is independent of the quadratic part). Thus, we have four cases to consider, depending on the mutual position of the corresponding three lines.

Case 1: Three lines in a general position. Without any loss of generality we can assume $C = abc$. The substitution of f into the integrability conditions (9) readily implies $\alpha = \beta = \gamma = 0$. Since the remaining constants μ, ν, η can be eliminated by the equivalence transformations (e2), we have just one integrable Lagrangian density in this class,

$$f = u_x u_y u_t,$$

with the corresponding Euler-Lagrange equation

$$u_t u_{xy} + u_y u_{xt} + u_x u_{yt} = 0. \tag{24}$$

Notice that this Lagrangian density is equivalent to

$$f = u_x^3 + u_y^3 + u_t^3 - 3u_x u_y u_t;$$

the corresponding (complex) linear change of variables x, y, t can be reconstructed from the factorization

$$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a + \epsilon b + \bar{\epsilon} c)(a + \bar{\epsilon} b + \epsilon c), \quad \epsilon = -\frac{1}{2} + i\frac{\sqrt{3}}{2};$$

in this case two of the three lines are complex conjugate. One can show that the equation (24) possesses the Lax pair

$$S_t = q(S_x/u_x)\ u_t, \quad S_y = r(S_x/u_x)\ u_y, \tag{25}$$

where the functions $q(s)$ and $r(s)$ ($s \equiv S_x/u_x$) satisfy a pair of ODEs

$$q' = \frac{q - r}{r - s}, \quad r' = \frac{r - q}{q - s}.$$

This system can be integrated in elliptic functions as follows. Introducing $f(s) = q(s) - s$ and $g(s) = r(s) - s$ one first rewrites the system in a translation-invariant form

$$f' = f/g - 2, \quad g' = g/f - 2$$

or, equivalently,

$$(fg)' = -(f+g), \quad (f+g)' = \frac{(f+g)^2}{fg} - 6.$$

Setting $fg = u$, $f + g = -u'$ one obtains a second order ODE for u, namely, $uu'' + (u')^2 - 6u = 0$, whose integral is given in parametric form $u = \wp(p),\ s = \zeta(p)$ (that is, $u(s)$ is obtained by excluding p from these two equations). Here $\wp(p)$ is the Weierstrass $\wp$-function, $(\wp'(p))^2 = 4\wp^3(p) + \lambda^2$ (notice that $g_2 = 0,\ g_3 = -\lambda^2$), and $\zeta(p)$ is the corresponding zeta-function: $\zeta'(p) = -\wp(p)$. Thus, in parametric form, $fg = \wp(p),\ f + g = \wp'(p)/\wp(p)$. The Lax pair (25) can be rewritten in parametric form as follows: adding and multiplying equations (25) we have

$$\frac{S_t}{u_t} + \frac{S_y}{u_y} = q + r = 2s + f + g = 2\zeta(p) + \frac{\wp'(p)}{\wp(p)},$$

$$\frac{S_t}{u_t}\frac{S_y}{u_y} = qr = s^2 + s(f+g) + fg = \zeta^2(p) + \zeta(p)\frac{\wp'(p)}{\wp(p)} + \wp(p).$$

Solving these equations for $\frac{S_t}{u_t}$ and $\frac{S_y}{u_y}$ and keeping in mind that $\frac{S_x}{u_x} = s = \zeta(p)$, we arrive at parametric equations

$$\frac{S_x}{u_x} = \zeta(p), \quad \frac{S_y}{u_y} = \zeta(p) + \frac{\wp'(p) + \lambda}{2\wp(p)}, \quad \frac{S_t}{u_t} = \zeta(p) + \frac{\wp'(p) - \lambda}{2\wp(p)}. \tag{26}$$

Notice that these equations imply the algebraic identity

$$\left(\frac{S_x}{u_x} - \frac{S_y}{u_y}\right)^2 \left(\frac{S_x}{u_x} - \frac{S_t}{u_t}\right)^2 \left(\frac{S_y}{u_y} - \frac{S_t}{u_t}\right)^2 = \lambda^2.$$

Case 2: Three lines through a common point. Without any loss of generality we can assume $C = ab(a+b)$. The substitution of the corresponding f into the integrability conditions (9) implies $\gamma = \nu = \eta = 0$ so that there is no c-dependence. Thus, this case gives no non-trivial examples.

Case 3: One double line. Here $C = a^2b$, and the substitution into the integrability conditions implies $\gamma = \nu = 0$. Translations in b and a eliminate α and μ. Furthemore, the linear transformation $\beta b + \eta c \to c$ reduces f to the canonical form $f = bc + a^2b$. The corresponding Lagrangian density

$$f = u_x^2 u_y + u_y u_t$$

generates the equation

$$u_{yt} + u_y u_{xx} + 2u_x u_{xy} = 0 \tag{27}$$

which, up to a rescaling, is a particular form of the so-called dispersionless r-Dym equation [1], [20],

$$u_{yt} = \frac{3-r}{2-r}\left(\frac{1}{2-r}u_y u_{xx} - \frac{1}{1-r}u_x u_{xy}\right).$$

Indeed, for $r = 4/3$ this equation possesses the Lagrangian

$$\int (u_y u_t - \frac{15}{4}u_x^2 u_y)\, dx\, dy.$$

The equation (27) possesses the Lax pair

$$S_t = -2u_x S_x + \frac{2}{5} S_x^{5/2}, \quad S_y = 2u_y S_x^{-1/2}.$$

Case 4: One triple line. Here $C = a^3/3$, and the substitution into the integrability conditions implies a single constraint $4\beta\gamma - \eta^2 = 0$. Eliminating α by a translation of a, one arrives at the expression $f = a^3/3 + (\sqrt{\beta}b + \sqrt{\gamma}c)^2 + a(\mu b + \nu c)$. The linear change of variables $\sqrt{\beta}b + \sqrt{\gamma}c \to b$, $\mu b + \nu c \to -c$ reduces f to the canonical form $f = a^3/3 + b^2 - ac$ which corresponds to the dKP density

$$f = u_x^3/3 + u_y^2 - u_x u_t.$$

Geometrically, the cases 3 and 4 can be viewed as degenerations of the case 1. It would be interesting to perform these degenerations explicitly on the level of the corresponding PDEs and Lax pairs.

3.2 Classification of fourth order Lagrangians

According to the Lemma, the leading quartic part of the Lagrangian density f must necessarily be degenerate and, thus, can be written as a form in two variables (say, a and b) after an appropriate linear transformation. Since any homogeneous binary quartic can be reduced to one of the five non-equivalent forms $a^4 + \mu a^2 b^2 + b^4$, $a^2 b(a + b)$, $a^2 b^2$, $a^3 b$ or a^4 (depending on the mutual location of its four roots), we have five cases to consider. A direct substitution into (9) implies that the first four cases lead to inconsistency (whatever cubic and quadratic terms are), while the last case leads, up to the equivalence transformations (e1)-(e3), to a unique solution $f = a^4 + 2a^2 c - ab - c^2$. The corresponding Lagrangian density

$$f = u_x^4 + 2u_x^2 u_t - u_x u_y - u_t^2$$

generates the equation

$$u_{xy} = -u_{tt} + 2u_t u_{xx} + 4u_x u_{xt} + 6u_x^2 u_{xx}.$$

Up to a rescaling, this is a particular case of the so-called r-th dispersionless modified KP equation [1], [20],

$$u_{xy} = \frac{3-r}{(2-r)^2} u_{tt} - \frac{(3-r)(1-r)}{2-r} u_t u_{xx} - \frac{(3-r)r}{2-r} u_x u_{xt} - \frac{(3-r)(1-r)}{2} u_x^2 u_{xx},$$

corresponding to the parameter value $r = 2/3$ (for which the equation becomes manifestly Lagrangian).

A similar analysis reveals that the integrability conditions (9) possess no polynomial solutions of degree 5. We conjecture that the four examples listed above exhaust the list of polynomial integrable Lagrangians which are essentially three-dimensional.

4 Pseudopotentials: proof of Theorem 2

In this section we prove that any integrable non-degenerate Euler-Lagrange equation (2) possesses a dispersionless Lax pair. We look for a pseudopotential S governed by the equations

$$S_t = F(S_x, u_x, u_y, u_t), \quad S_y = G(S_x, u_x, u_y, u_t). \tag{28}$$

Calculating the consistency condition $S_{ty} = S_{yt}$ and using the equation (2) we arrive at five relations among F and G,

$$
\begin{aligned}
&F_\xi G_a - G_\xi F_a = \tfrac{f_{aa}}{f_{bb}} F_b, \\
&F_\xi G_b - G_\xi F_b = 2\tfrac{f_{ab}}{f_{bb}} F_b - F_a, \\
&F_\xi G_c - G_\xi F_c = 2\tfrac{f_{ac}}{f_{bb}} F_b + G_a, \\
&G_c + \tfrac{f_{cc}}{f_{bb}} F_b = 0, \quad G_b + 2\tfrac{f_{bc}}{f_{bb}} F_b - F_c = 0.
\end{aligned}
\tag{29}
$$

Here the auxiliary variable ξ denotes S_x and $a = u_x$, $b = u_y$, $c = u_t$. We also assume $f_{bb} \neq 0$: for a non-trivial Lagrangian, this can always be achieved by an admissible linear change of x, y, t.

Remark. One has to distinguish between 'true' and 'fake' pseudopotentials, the latter satisfying equations of the form

$$
S_t = p(a, S_x)c + q(a, S_x), \quad S_y = p(a, S_x)b + r(a, S_x), \tag{30}
$$

where the functions $p(a, \xi)$, $q(a, \xi)$, $r(a, \xi)$ solve the system

$$
p_a + pp_\xi = 0, \quad q_a + pq_\xi = 0, \quad r_a + pr_\xi = 0.
$$

Equations (30) are automatically consistent by virtue of the relations $a_y = b_x$, $a_t = c_x$, $b_t = c_y$ and, therefore, generate no non-trivial PDEs. Such pseudopotentials can be ruled out, for instance, by the requirement $F_b \neq 0$. We assume this hereafter. For a 'true' dispersionless Lax pair (28) the compatibility conditions must be *equivalent* to the equation (2). One can readily verify that the compatibility conditions for the Lax pair (28), (29) with $F_b \neq 0$ imply (2).

Equations (29) imply the expressions for the first derivatives of G in terms of the first derivatives of F,

$$
\begin{aligned}
&G_c = -\tfrac{f_{cc}}{f_{bb}} F_b, \\
&G_b = F_c - 2\tfrac{f_{bc}}{f_{bb}} F_b, \\
&G_a = -F_\xi F_b \tfrac{f_{cc}}{f_{bb}} - F_\xi F_c \left(\tfrac{F_c}{F_b} - 2\tfrac{f_{bc}}{f_{bb}}\right) + 2\tfrac{f_{ab}}{f_{bb}} F_c - \tfrac{F_a F_c}{F_b} - 2\tfrac{f_{ac}}{f_{bb}} F_b, \\
&G_\xi = F_\xi \left(\tfrac{F_c}{F_b} - 2\tfrac{f_{bc}}{f_{bb}}\right) - 2\tfrac{f_{ab}}{f_{bb}} + \tfrac{F_a}{F_b},
\end{aligned}
\tag{31}
$$

along with the following extra relation involving the first derivatives of F only:

$$
\begin{aligned}
&f_{cc} F_\xi^2 F_b^2 + F_\xi^2 F_c (f_{bb} F_c - 2 f_{bc} F_b) + 2 f_{ac} F_\xi F_b^2 + \\
&2 F_\xi F_a (f_{bb} F_c - f_{bc} F_b) + f_{bb} F_a^2 + f_{aa} F_b^2 - 2 f_{ab} F_b (F_\xi F_c + F_a) = 0.
\end{aligned}
\tag{32}
$$

To close this system one proceeds as follows. Differentiating the relation (32) by a, b, c, ξ and calculating the compatibility conditions of the equations (31) (that is, $G_{cb} = G_{bc}$, $G_{ca} = G_{ac}$, $G_{c\xi} = G_{\xi c}$, etc, six conditions altogether), one arrives at a linear system of ten equations for the second partial derivatives of F. Solving this system (we skip the resulting expressions for the second order derivatives of F due to their complexity) and calculating the resulting compatibility conditions $F_{aab} = F_{aba}$, etc, one arrives at exactly the same integrability conditions (9) as obtained in Sect 2. from the requirement of the existence of hydrodynamic reductions. This proves Theorem 2.

The necessary details of this computation are explained in the file 1-README, see [11]. The main problem was to simplify the expressions for second order derivatives of F: the original expressions found using Maple were extremely complicated and would put the calculation of compatibility conditions ($F_{aab} = F_{aba}$, etc.) beyond our reach: for instance, the number of terms in the expression for F_{aa} was equal to 18328, the corresponding number for F_{ac} was 9045, etc. Thus, the hypothetical expression for F_{aab} would have approximately 10^9 terms. After taking into consideration the identity (32) we have simplified F_{aa} to an expression with 367 terms only; respectively, F_{ac} had 173 terms. Verifying the necessary compatibility conditions and their equivalence to (9) thus became feasible.

Remark. The functions F and G which determine the Lax pair (28) depend on five arbitrary constants. Indeed, the values of G, F, F_a, F_b, F_c can be prescribed arbitrarily at any initial point. On the other hand, equations for pseudopotentials remain form-invariant under the transformations

$$S \to \alpha S + \beta x + \gamma y + \delta t + \mu u$$

which allow one to eliminate this arbitrariness in the generic situation. Hence, for a given non-degenerate equation (2), the dispersionless Lax pair is essentially unique.

5 Differential-geometric aspects of the integrability conditions

Since the integrability conditions (9) are preserved by the transformations (e1)-(e3), they should be expressible in terms of the corresponding invariants. Among the simplest of these invariants is the Hessian metric d^2f. We point out that *flat* Hessian metrics have been discussed recently in a series of publications [4, 24, 30] in the context of equations of associativity of two-dimensional topological field theory. Let us recall that, for a Hessian metric d^2f, Christoffel's symbols and the curvature tensor take particularly simple forms,

$$\Gamma^k_{ij} = \frac{1}{2} f^{kl} f_{lij}, \quad R_{ijkl} = \frac{1}{4} f^{mp}(f_{imk}f_{pjl} - f_{iml}f_{pjk});$$

here $f(x^i, ..., x^n)$ is a function of n variables, $f_{ij},\ f_{ijk}$ denote partial derivatives of f, and the matrix f^{ij} is the inverse of f_{ij}. Notice that the Riemann tensor does not contain fourth order derivatives of f. The Ricci tensor R_{ij} and the scalar curvature R are defined as $R_{ij} = f^{mp}R_{pimj},\ R = f^{ij}R_{ji}$. The main object of conformal geometry is the Schouten tensor

$$P_{ij} = \frac{1}{n-2} R_{ij} - \frac{1}{2(n-1)(n-2)} R f_{ij}.$$

In three dimensions ($n = 3$) the condition of conformal flatness is expressible in the form $\nabla_k P_{ij} = \nabla_j P_{ik}$ which is equivalent to the vanishing of the so-called Cotton tensor. Our main observation is the following

Proposition. *For any density $f(a, b, c)$ satisfying the integrability conditions (9) the Hessian metric d^2f is conformally flat.*

The proof is a straightforward computer calculation. For instance, the density $f = abc$ generates the Hessian metric $abc\left(\frac{da}{a}\frac{db}{b} + \frac{da}{a}\frac{dc}{c} + \frac{db}{b}\frac{dc}{c}\right)$ which is manifestly conformally flat.

We also present the equivalent tensorial formulation of the integrability conditions (9):

$$f_{ijkl} = Sym\left(f_{ijk}f^{pq}f_{pql} - \frac{3}{2} f_{pqi}f_{rsj}f_{mk}f_{nl}\epsilon^{prm}\epsilon^{qsn}\right). \tag{33}$$

Here ϵ^{ijk} is the totally antisymmetric tensor dual to the volume form of the Hessian metric d^2f (so that $\epsilon^{123} = 1/\sqrt{H},\ \epsilon^{213} = -1/\sqrt{H}$, etc), and Sym denotes the total symmetrization:

$$Sym\ T_{ijkl} = \frac{1}{4!} \sum_{\sigma \in S_4} T_{\sigma(i)\sigma(j)\sigma(k)\sigma(l)}.$$

The formula (33) can be rewritten in a completely invariant differential-geometric way as follows. First of all, the Hessian metric f_{ij} should be replaced by a metric g_{ij} and a flat affine connection ∇ such that

$$\nabla_k g_{ij} = \nabla_j g_{ik}, \tag{34}$$

indeed, the metric g_{ij} assumes the Hessian form in the flat coordinates of ∇. Let us define the 3-tensor $g_{ijk} = \nabla_k g_{ij}$ (which is automatically totally symmetric) and the antisymmetric tensor ϵ^{ijk} such that $\epsilon^{123} = 1/\sqrt{det\ g_{ij}}$, $\epsilon^{213} = -1/\sqrt{\det\ g_{ij}}$, etc. Then (33) can be rewritten as

$$\nabla_k \nabla_l\ g_{ij} = Sym \left(g_{ijk} g^{pq} g_{pql} - \frac{3}{2} g_{pqi} g_{rsj} g_{mk} g_{nl} \epsilon^{prm} \epsilon^{qsn} \right). \tag{35}$$

Thus, the differential-geometric object underlying the classification of integrable Lagrangians is a pair consisting of a metric g_{ij} and a flat affine connection ∇ which satisfy the relations (34), (35).

Acknowledgements

We thank Dr. Maxim Pavlov for drawing our attention to the problem of integrability of multi-dimensional Lagrangians and numerous helpful discussions. We also thank the Royal Society for their financial support of SPT to Loughborough making this collaboration possible.

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