Controlling stochastic oscillations close to a Hopf bifurcation by time-delayed feedback

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We study the effect of a time-delayed feedback upon a Van der Pol oscillator under the influence of white noise in the regime below the Hopf bifurcation where the deterministic system has a stable fixed point. We show that both the coherence and the frequency of the noise-induced oscillations can be controlled by varying the delay time and the strength of the control force. Approximate analytical expressions for the power spectral density and the coherence properties of the stochastic delay differential equation are developed, and are in good agreement with our numerical simulations. Our analytical results elucidate how the correlation time of the controlled stochastic oscillations can be maximized as a function of delay and feedback strength.

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I. INTRODUCTION

During the last decade, control of complex irregular dynamics has evolved as one of the central issues in applied nonlinear science [1, 2]. A particularly simple and efficient scheme based upon delay differential equations has been applied to deterministic chaotic systems [3]. It uses the difference of a system variable $s(t)$ at time t and the same variable at a delayed time, $s(t-\tau)$, to generate a control force which is coupled back to the system. If the control parameters are chosen appropriately, an intrinsically unstable periodic orbit (UPO) embedded in the chaotic attractor can be stabilized (*time–delay autosynchronization*). In particular, the time delay τ should be chosen equal to the period T of the target UPO; then the control force vanishes when the target UPO is reached (*noninvasive control*). This scheme is simple to implement, quite robust, and has been applied successfully in real experiments. An extension to multiple time-delays (extended time–delay autosynchronization) has been proposed by Socolar et al [4], and analytical insight into those schemes has been gained by several theoretical studies [5–8]. Such self–stabilizing feedback control schemes (time–delay autosynchronization) with different couplings of the control force have been applied to various classes of deterministic ordinary and partial differential equations, modelling, e.g., semiconductor oscillators [9–15].

In this paper we study the control of stochastic differential equations by time delayed feedback. In contrast to control of deterministic chaos, the control of noise-induced phenomena is still an open problem. Recently, a number of methods were suggested for the control of stochastic resonance [16, 17] and of self-oscillations in the presence of noise [18]. In [19] an external periodic force was proposed for the control of noise-induced oscillations in a pendulum with a randomly vibrating suspension axis. All methods mentioned above assume the presence of deterministic self-oscillating components in the dynamics, and aim either to enhance the latter component or to exploit the external deterministic oscillating force to manipulate the regularity of motion. In contrast to those investigations, a passive self-adaptive method for the control of oscillations induced merely by noise was proposed only recently [20, 21]. It uses time-delayed feedback control to change the coherence of the oscillations, and tune their timescale. The method was applied to two classes of oscillators, namely, a system near a supercritical Hopf bifurcation (Van der Pol oscillator), and an excitable system (FitzHugh-Nagumo model).

In the present work we study a generic model of a nonlinear oscillator, the Van der Pol oscillator, below the Hopf bifurcation, i.e. in the regime where the deterministic system has a stable fixed point, and does not oscillate autonomously. We present numerical and analytical results on control of noise-induced oscillations. Due to the delay the process described by this equation is essentially non-Markovian, and the powerful methods, like Fokker-Planck equations suitable for Markovian processes, are not applicable here. At present, the theory of stochastic delay differential equations is under development, and the methods created so far mostly cover linear and very specific cases of nonlinear systems [18, 22–28]. Nevertheless, we have been able to obtain some analytical results for the problem considered here.

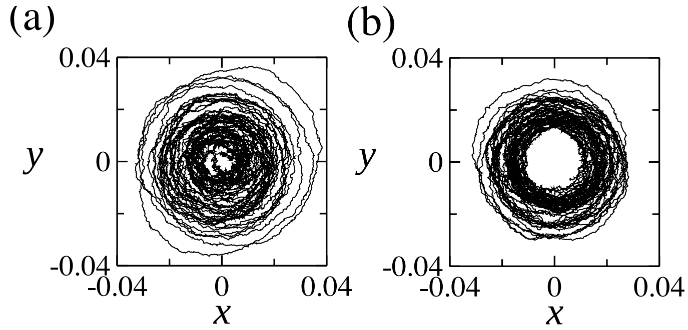

FIG. 1: Numerically simulated phase portraits of noise-induced oscillations of the Van der Pol system at $\omega_0 = 1$, $\varepsilon = -0.01$, $D = 0.003$: (a) without feedback $K = 0$; (b) with feedback $K = 0.2$, $\tau = T_0$. In both cases the system was integrated during 300 time units.

II. SIMULATION OF THE NOISY VAN DER POL SYSTEM WITH DELAY

The following dynamic system describes a Van der Pol oscillator with time delayed feedback in the presence of noise:

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= (\varepsilon - x^2)y - \omega_0^2 x + K(y(t-\tau) - y(t)) + D\xi(t) \end{aligned} \tag{1}$$

Here ω_0 denotes the natural oscillation frequency without feedback, K represents the delayed feedback strength, τ is the time delay, $\xi(t)$ is Gaussian white noise with zero mean and intensity defined by the parameter D.

$$\begin{aligned} \langle \xi(t) \rangle &= 0 \\ \langle \xi(t)\xi(t') \rangle &= \delta(t - t') \end{aligned} \tag{2}$$

The bifurcation parameter ε governs the dynamics of the system. At $K = 0$, $D = 0$ and $\varepsilon < 0$, the system has a stable fixed point $(0, 0)$ and does not exhibit self-sustained oscillations. At $\varepsilon = 0$ a supercritical Hopf bifurcation occurs, after which at $\varepsilon > 0$ a stable limit cycle exists in the phase space of the system. In the following we fix $\omega_0 = 1$ and $\varepsilon = -0.01$ at a value slightly below the Hopf bifurcation. In this case the only attractor in the system is a stable focus. However, inclusion of noise ($D > 0$) into the system can evoke motion in the phase space that in many respects resembles noisy oscillations with basic period [29] $T_0 \approx 2\pi/\omega_0$ (see Fig. 1(a)). To quantify the regularity or coherence of these oscillations we introduce the correlation time t_{cor} as [30]

$$t_{cor} = \frac{1}{\sigma^2} \int_0^\infty |\Psi(s)| ds, \tag{3}$$

where

$$\Psi(s) = \langle (y(t) - \langle y \rangle)(y(t+s) - \langle y \rangle) \rangle \tag{4}$$

is the autocorrelation function of $y(t)$, and $\sigma^2 = \Psi(0)$ is its variance. In Fig. 2(a) the dependence of t_{cor} on the noise intensity is shown by a dashed line. The regularity of the oscillations decreases with increasing noise intensity D. Fig. 2(b) shows the dependence of the basic period T_0 of the noise-induced oscillations (determined by the highest peak in the Fourier power spectrum) upon the noise intensity D. T_0 hardly changes with variation of D, which is quite different, e.g., from the FitzHugh-Nagumo system [20, 21].

To check if the delayed feedback can enhance the regularity of noise-induced motion, we set $K = 0.2$, and fix $\tau = 6.1728$ at the value of the basic period T_0 of the stochastic oscillations without feedback. For negative ε perturbations due to noise $\xi(t)$ result in damped oscillations. The phase coherence of these oscillations is characterized by how many rotations (on average) a phase trajectory can make before collapsing into the fixed point at the origin and is measured by the correlation time introduced above. Once a phase trajectory comes close to the origin the coherence is destroyed. The coherence is determined by the effective dissipation in the system: weak dissipation results in a slow approach of phase trajectories to the fixed point and thus to many rotations, while strong dissipation leads to fewer rotations. This is illustrated in Fig.1 where we have plotted phase trajectories of Eqs.(1) integrated for

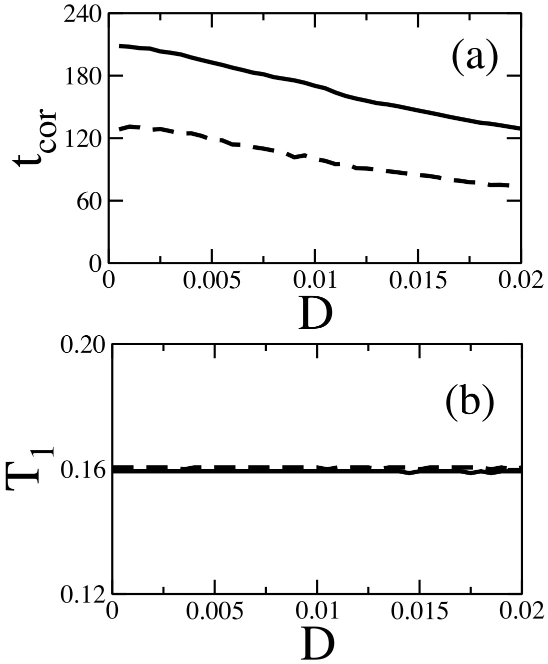

FIG. 2: (a) Correlation time t_{cor} *vs* noise intensity D: without delayed feedback, $K = 0$ (dashed line), and with delayed feedback with $K = 0.2$ at $\tau = T_0$ (solid line). (b) Basic period *vs* noise intensity D for $K = 0$ (T_0: dashed line) and for $K = 0.2$, $\tau = T_0$ (T_1: solid line). Numerical simulation.

the same noise intensity with and without delayed feedback control. For $K = 0$ the phase trajectory comes close to the origin many times during integration (Fig. 1(a)). For the same length of time the phase trajectory of the system with delayed feedback never crossed the origin (Fig. 1(b)) which indicates significant enhancement of coherence of noise-induced oscillations. It is indeed confirmed by the correlation time t_{cor} vs D for $K = 0.2$ which is shown in Fig. 2(a) by a solid line. A remarkable increase of coherence is observed for all values of noise intensity. Thus, we can conclude that delayed feedback can make noise-induced oscillations more ordered if the delay τ matches the basic period of the oscillations.

More generally, an increase of coherence of noise-induced oscillations is observed if τ is close to T_0 or to its integer multiples. If, however, τ is chosen close to $T_0(2n-1)/2$ with integer n, the coherence is decreased substantially by the feedback. The correlation time t_{cor} estimated numerically oscillates as a function of τ, as shown in Fig. 3(a). The local maxima increase approximately linearly with τ.

In Fig 3(b) the variance $\langle F^2 \rangle$ of the control force $F(t) = y(t-\tau) - y(t)$ estimated numerically is given as a function of τ by a solid line. Note that the minima of $\langle F^2 \rangle$ correspond to the maxima of t_{cor} at $\tau = T_0 n$. That means that less force is required to control more regular behavior. However, as we expect, in contrast to the deterministic case, this force never vanishes.

A second important feature of the delayed feedback control is that it allows one to manipulate the timescale of the noise-induced oscillations. The control parameter that is expected to affect the system timescales is the time delay τ. To assess the effect of changing τ, we numerically calculate the Fourier power spectral density $S(\omega)$ of the stochastic oscillations, that we will further refer to as spectrum for brevity, for a range of τ. For $K = 0.2$ and $D = 0.003$ the spectra are shown in Fig. 4(a). Without feedback ($\tau = 0$), the spectrum has only one pronounced peak with period T_1 equal to T_0. However, as τ increases from zero more peaks appear in the spectrum. For a better illustration of the spectral properties, we extract all spectral peaks, order them with respect to decreasing heights, denote their periods as $T_i,\ i = 1, 2, \ldots$ and plot T_i *vs* τ in Fig. 3(d). The basic period T_1 is shown by white circles, while the other periods $T_i,\ i = 2, 3, \ldots$ are denoted by grey circles. The following features can be observed: (i) the periods T_i of all peaks change with τ, (ii) the evolution of $T_1(\tau)$ is almost piecewise linear; (iii) the heights and the widths of spectral peaks vary as τ changes. The first two observations mean that the delayed feedback entrains the timescales of noise-induced oscillations: the change in τ results in an almost proportional change in T_i. The third observation implies that by varying the delay one can control the regularity of oscillations. Fig. 5 shows the spectral peak periods T_i in dependence on τ for two different strengths of the control force K. While the general behavior remains the same, the modulation of the timescales with τ becomes stronger with increasing K.

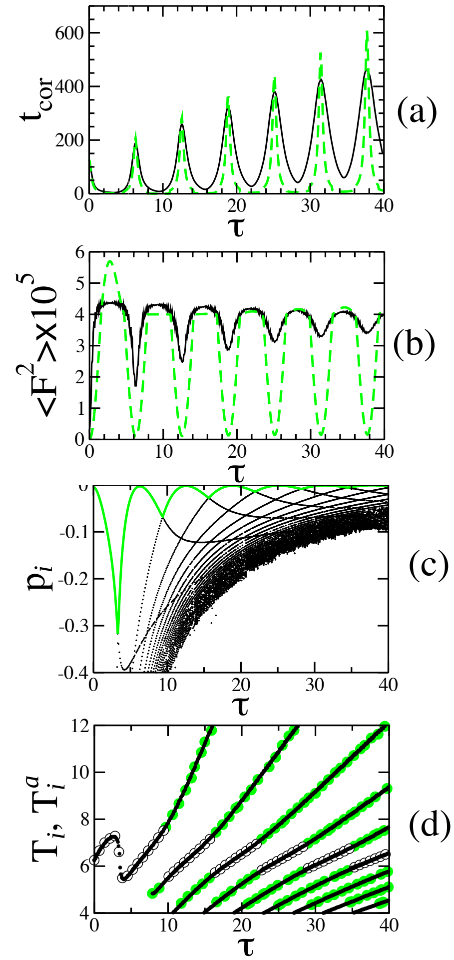

FIG. 3: Dependence of controlled oscillations in the Van der Pol system on the time delay τ. (a) Correlation time t_{cor}: solid line - numerical simulation, dashed line - analytic estimation by Eq. (28). (b) Variance $\langle F^2 \rangle$ of control force F: solid line - numerical simulation, dashed line - analytic estimation by Eqs. (28) and (31). (c) real parts $p_i,\ i = 1, 2, \ldots$ of eigenvalues of the stable fixed point, grey line marks the largest p_1. (d) Circles: peak periods T_i from numerically simulated spectra. White circles - basic period T_1, grey circles - other periods $T_i,\ i = 2, 3, \ldots$. Dots: peak periods $T_i^a,\ i = 1, 2, \ldots$ from the analytically estimated spectrum given by Eq. (14). Parameters: $D = 0.003$ and $K = 0.2$.

Next we study the dependence of noise-induced oscillations upon the control strength K. Fig. 6 shows the correlation time and the periods $T_{1,2}$ of the two highest spectral peaks as functions of K. We set $D = 0.003$ and consider two values of τ which correspond approximately to a maximum and a minimum of coherence at fixed $K = 0.2$, respectively: $\tau = 6.1728 \approx T_0$ (Fig. 6(a),(c)), and $\tau = 3.3$ (Fig. 6(b),(d)). The correlation time increases approximately linearly with K for optimum $\tau = 6.1728 \approx T_0$, and K not too large (Fig. 6(a), while it sharply drops with K and then remains at a low level with only very slight increase if τ is chosen close to the minimum coherence (Fig. 6(b)). This means that the coherence of the oscillations can in fact be considerably enhanced by a stronger control force, e.g. by more than a factor of 6 for $K = 2$, if the delay time is carefully adjusted to its optimum value. For τ approximately equal to the basic period T_0 of uncontrolled oscillations, the increase of K does not change the position of the basic spectral peak with period $T_1 = T_0$ (white cirles in Fig. 6(c)). Note that at $K \approx 0.25$ another, lower peak appears whose period T_2 (grey circles in Fig. 6(c)) monotonically increases with K. If, however, $\tau = 3.3$ (Fig. 6(d)), T_1 remains close to T_0 only for a range of $K \lessapprox 0.21$. As K is increased further, the basic peak splits into two, with periods T_1 and T_2. T_1 grows monotonically with K, while T_2, remaining smaller, tends to coincide with τ for large K.

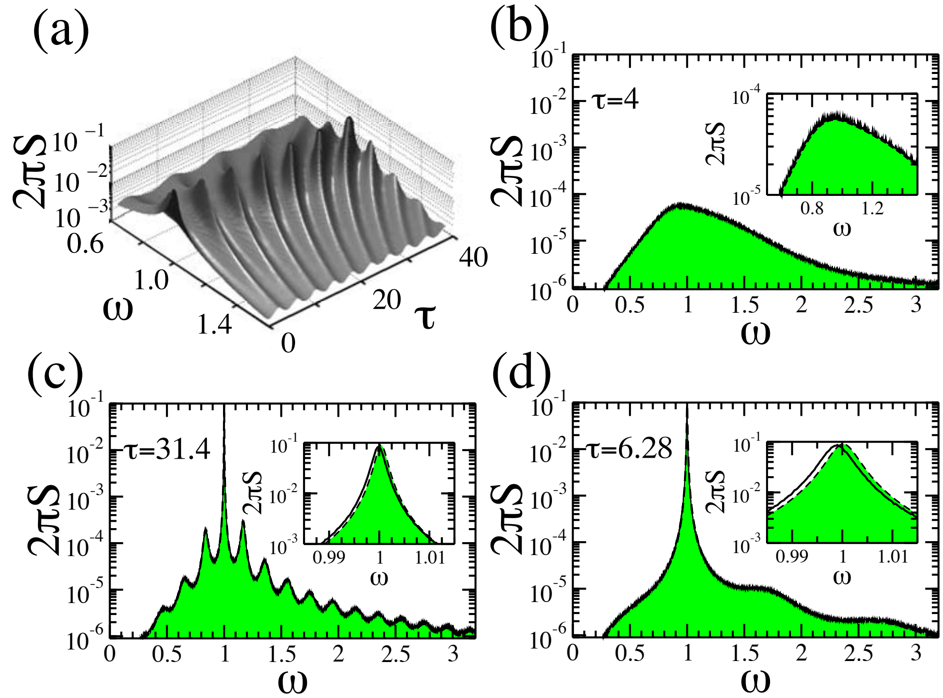

FIG. 4: Power spectral density $S(\omega)$ of noise-induced oscillations in the Van der Pol system in the presence of delayed feedback for a range of τ at $D = 0.003$, $K = 0.2$. (a) Numerically simulated $S(\omega)$ in dependence upon τ. (b), (c), (d) $S(\omega)$ for three different values of τ as given in the figures. Solid line - numerically simulated spectra, shaded - spectra estimated analytically by Eq. (14). The insets give details of the main spectral peak.

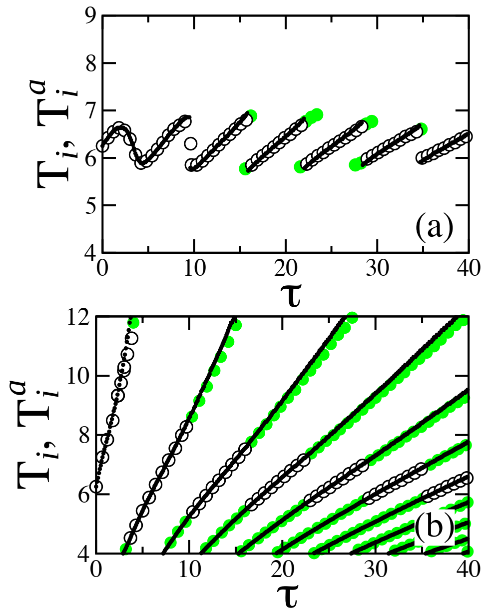

FIG. 5: Evolution of spectral peak periods T_i of noise-induced oscillations as a function of τ at $D = 0.003$ for different values of K: (a) $K = 0.1$, (b) $K = 0.5$. Symbols as in Fig. 3(d).

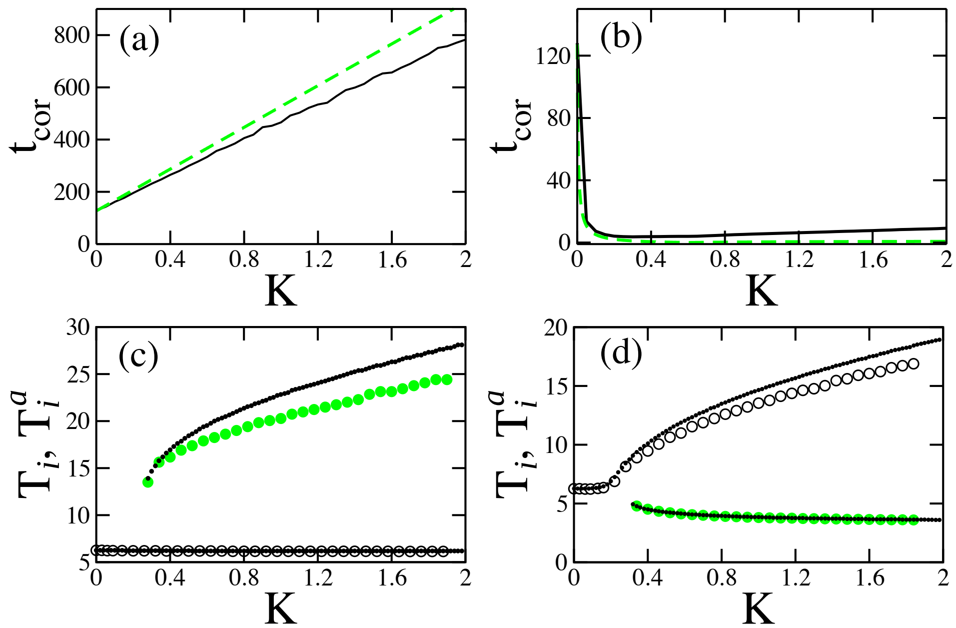

FIG. 6: Dependence of controlled oscillations in the Van der Pol system on the feedback strength K. (a), (c) $\tau = 6.1728$, (b), (d) $\tau = 3.3$. (a), (b) Correlation time t_{cor}: solid line - numerical simulation at $D = 0.003$, dashed line - analytic estimation by Eq. (24) or (27) with $\tau = 2\pi$ or $\tau = \pi$, respectively. (c), (d) Circles: peak periods $T_{1,2}$ from numerically simulated spectra at $D = 0.003$. White circles - basic periods T_1, grey circles - T_2. Dots: peak periods $T^a_{1,2}$ from the analytically estimated spectrum given by Eq. (14).

III. ANALYTICAL RESULTS ON TIME-DELAYED FEEDBACK CONTROL

Since the oscillations occur in the vicinity of the fixed point $(0, 0)$ of the system (1), we shall try to gain some analytical insight by considering the linearized Van der Pol oscillator, including delayed feedback control and noise:

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= \varepsilon y - \omega_0^2 x + K(y(t-\tau) - y(t)) + D\xi(t) \end{aligned} \tag{5}$$

which can be written as a single stochastic delay differential equation (SDE) of second order:

$$\ddot{x} - \varepsilon \dot{x} + \omega_0^2 x - K(\dot{x}(t-\tau) - \dot{x}) = D\xi(t). \tag{6}$$

In the absence of noise, $D = 0$, Eq. (6) can be solved by an exponential ansatz, yielding the characteristic equation for the complex eigenvalues $\lambda = p + iq$ of the fixed point $(0, 0)$:

$$\lambda^2 - \lambda\varepsilon + \omega_0^2 - K\lambda(e^{-\lambda\tau} - 1) = 0 \tag{7}$$

which can be separated into real and imaginary parts:

$$\alpha(p, q) \equiv p^2 - q^2 \;+\; \omega_0^2 - p\varepsilon + K(p - pe^{-p\tau}\cos q\tau - qe^{-p\tau}\sin q\tau) = 0 \tag{8}$$

$$\beta(p, q) \equiv 2pq \;-\; q\varepsilon + K(q + pe^{-p\tau}\sin q\tau - qe^{-p\tau}\cos q\tau) = 0. \tag{9}$$

At $K = 0$, the fixed point is a stable focus for

$$\varepsilon < 0, \quad |\varepsilon| < 2\omega_0. \tag{10}$$

i.e. it possesses a pair of complex conjugate eigenvalues with real part $p = \varepsilon/2 < 0$ and imaginary parts $q = \pm\sqrt{\omega_0^2 - (\varepsilon/2)^2}$.

The delayed feedback $K > 0$ influences the local properties of the fixed point, but does not change its stability for $\varepsilon < 0$. Note that for negative ε the condition for a Hopf bifurcation, $p = 0$, $q \neq 0$, cannot be satisfied in (9) for any K since $\cos q\tau \leq 1$ always holds. This means that a limit cycle is not induced by delayed feedback. However, the delayed feedback introduces infinitely many complex eigenmodes, which are given by a countable set of eigenvalues whose real parts p_i appear from minus infinity at $\tau = 0$ and always remain negative. These are shown in Fig. 3(c) as a function of τ. The real part of each eigenvalue depends non-monotonically upon τ, and hence the different branches cross over. This leads to an oscillatory behavior of the largest real part (highlighted by a grey line in Fig. 3(c)), which will be denoted by p_1. From comparison of Fig. 3(c) with Fig. 3(a) it is evident that the coherence of the noise-induced oscillations is largest (maxima of the correlation time t_{cor}) when the fixed point is deterministically least stable (i.e. p_1 is closest to zero).

The imaginary parts of the eigenvalues describe the velocity of rotation of the deterministic phase trajectory around the fixed point that is associated with its ith eigenmode. The corresponding eigenperiods $T_i^e = 2\pi/|q_i|$ coincide remarkably with the spectral peak periods T_i in Fig. 3(d) as shown in [20, 21]. The jumps in the dominant spectral peak periods T_1 as a function of τ (Fig. 3(d)) can now be explained by the crossover of the different branches of p_i in Fig. 3(c): at the point of intersection, another eigenmode becomes the least stable, and the associated eigenperiod T_i^e determines the corresponding spectral peak which becomes the highest one.

For $p \approx 0$ and $|\varepsilon| \ll K$ it follows from eq. (9) that $\cos q\tau \approx 1$ and $q\tau \approx 2\pi n$ with integer n. Then the eigenperiod T_1^e is

$$T_1^e = 2\pi/|q_1| \approx \tau/n. \tag{11}$$

Eq. (11) holds the better, the closer p_1 is to zero. This explains the almost piecewise linear behaviour of the basic periods T_1 in Figs. 3(d) and 5 at large τ, at which the condition $p_1 \approx 0$ holds more accurately (Fig. 3(c)).

Next we shall explore the spectral properties of the linearized Van der Pol oscillator under control and noise. Consider the Fourier transform of (6) which can be expressed as

$$\hat{x}(\omega) = \frac{D\hat{\xi}(\omega)}{\omega_0^2 - \omega^2 + i\omega\varepsilon + i\omega K(e^{i\omega\tau} - 1)} \tag{12}$$

where $\hat{x}(\omega)$ and $\hat{\xi}(\omega)$ denote the Fourier transforms of $x(t)$ and $\xi(t)$, respectively. Using $\hat{y}(\omega) = -i\omega\hat{x}(\omega)$ and $\langle\hat{\xi}(\omega)\hat{\xi}^*(\omega')\rangle = \delta(\omega - \omega')/(2\pi)$ by eq.(2), we can obtain the power spectral density $S(\omega)$ of y, applying the Wiener-Khinchin theorem [31]

$$\langle\hat{y}(\omega)\hat{y}^*(\omega')\rangle = \delta(\omega - \omega')S(\omega) \tag{13}$$

It yields

$$S(\omega) = \frac{D^2}{2\pi}\frac{\omega^2}{(\omega^2 - \omega_0^2 + \omega K sin(\omega\tau))^2 + \omega^2(\varepsilon - K(1 - cos(\omega\tau))^2}. \tag{14}$$

The power spectral density (14) of the linearized oscillator is connected with the linear modes (8), (9) of the free running oscillator by setting $p = 0, q = -\omega$. $S(\omega)$ may be viewed as the transfer function which converts the flat white noise input into a spectrally modulated output by amplifying certain modes of the noise. Eq. (14) has the form

$$S(\omega) = \frac{D^2}{2\pi}\frac{\omega^2}{\alpha(0, -\omega)^2 + \beta(0, -\omega)^2}, \tag{15}$$

A Hopf bifurcation ($\alpha(0, -\omega) = \beta(0, -\omega) = 0$) leading to the birth of a limit cycle would correspond to a pole of the power spectral density at the corresponding frequency of self-oscillation, but as we have shown above, this cannot occur for $\varepsilon < 0$. However, those eigenmodes which are least stable, i.e. which have smallest $|p|$, correspond to maxima of $S(\omega)$ and can thus be excited by noise. This explains why the basic periods in Fig. 3(d) match nicely with $2\pi/q$ of the linear modes $p + iq$ [20, 21].

For weak noise D, the power spectral density $S(\omega)$ of the *linear* oscillator (14) is in excellent agreement with that of the *nonlinear* Van der Pol oscillator obtained from our simulations (Fig. 4). The basic periods T_i^a calculated from the maxima of (14) as a function of τ coincide with T_i in Figs. 3(d), 5.

In order to gain analytical insight into the dependence of the coherence properties upon the control parameters, we will derive simple analytical approximations from (14) in the following. First consider the case $K = 0$. Eq. (14) reduces to a Lorentzian

$$S(\omega) = \frac{D^2}{2\pi}\frac{\omega^2}{(\omega^2 - \omega_0^2)^2 + \omega^2(2\Gamma)^2} \tag{16}$$

with peak frequency ω_0 and approximate half-width $\Gamma = -\varepsilon/2$. This, by inverse Fourier transform according to the Wiener-Khinchin Theorem, for $|\varepsilon| \ll \omega_0$ approximately leads to the normalized autocorrelation function [30]

$$\frac{\Psi(s)}{\sigma^2} = e^{-\frac{|\varepsilon|}{2}s} \cos(\omega_0 s) \tag{17}$$

The correlation time

$$t_{cor} = \int_0^\infty e^{-\frac{|\varepsilon|}{2}s} |\cos(\omega_0 s)| ds \tag{18}$$

can be evaluated approximately for $|\varepsilon| \ll \omega_0$ by substituting the cos term by the filling factor $\frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \cos\phi d\phi = 2/\pi$, which yields

$$t_{cor} = \frac{2}{\pi}\int_0^\infty e^{-\frac{|\varepsilon|}{2}\tau} d\tau = \frac{4}{\pi|\varepsilon|} \tag{19}$$

It describes the behavior for small noise intensity D and is in good agreement with the numerical results from the *nonlinear* simulations of the Van der Pol system in Fig. 2(a), i.e., $t_{cor} \approx 127$ for $\varepsilon = -0.01$, $\omega_0 = 1$.

For $K \neq 0$ Eq. (14) describes the power spectral density with multiple peaks. A very rough estimate of the position and width of the dominant spectral peak can be obtained for $|\varepsilon| \ll \omega_0, K \ll 1$ by replacing ω with ω_0 in the resonant terms in the denominator of (14). This means that the spectral width is modulated as

$$2\Gamma = -\varepsilon + K(1 - \cos\omega_0\tau) \tag{20}$$

and the correlation time is modulated as

$$t_{cor} = \frac{4}{\pi(-\varepsilon + K(1 - \cos\omega_0\tau))} \tag{21}$$

This crude approximation correctly describes the periodic oscillations in the correlation time as a function of τ, with minima at $\tau = \frac{\pi}{\omega_0}(2n-1)$ and maxima at $\tau = \frac{2\pi}{\omega_0}n \approx T_0 n$ with integer n as numerically found, see Fig. 3(a), but it does not correctly describe the absolute values of the coherence maxima at $\tau \approx T_0 n$, i.e., it does not yield the enhancement of the correlation time by delayed feedback. In order to quantify the enhancement of coherence we consider the optimum values of delay time, i.e. $\tau = \frac{2\pi}{\omega_0}n$ and expand the power spectral density (14) for $\omega = \omega_0(1+\delta)$, $\delta \ll 1$ around ω_0, assuming a narrow spectral width of the dominant peak.

With $\sin(\omega\tau) \approx \omega_0\tau\delta$ and $\cos(\omega\tau) \approx 1 - \frac{1}{2}(\omega_0\tau\delta)^2$ we obtain

$$S(\omega) = \frac{D^2}{2\pi}\frac{\omega^2}{(\omega^2 - \omega_0^2 + \omega K\omega_0\tau\delta)^2 + \omega^2(\varepsilon - \frac{K}{2}(\omega_0\tau\delta)^2)^2} \tag{22}$$

Expressing $S(\omega)$ in terms of δ up to lowest order yields

$$\begin{aligned} S(\omega) &= \frac{D^2}{2\pi}\frac{\omega_0^2(1+\delta)^2}{\left[\omega_0^2\left((1+\delta)^2 - 1 + (1+\delta)\delta K\tau\right)\right]^2 + \omega_0^2(1+\delta)^2(\varepsilon - \frac{K}{2}(\omega_0\tau\delta)^2)^2} \\ &= \frac{D^2}{2\pi}\frac{\omega_0^2(1+\delta)^2}{(1+\frac{K}{2}\tau)^2\left[\omega_0^2\left((1+2\delta+O(\delta^2) - 1\right)\right]^2 + \omega_0^2(1+\delta)^2\left[\varepsilon + O(\delta^2)\right]^2} \\ &= \frac{D^2}{2\pi(1+\frac{K}{2}\tau)^2}\frac{\omega^2}{\left[\omega^2 - \omega_0^2\right]^2 + \omega^2\left[\frac{\varepsilon}{1+\frac{K}{2}\tau}\right]^2} \end{aligned} \tag{23}$$

This is a Lorentzian with peak frequency ω_0 and half-width $\Gamma = \frac{1}{2}|\varepsilon|/(1+\frac{K}{2}\tau)$. Therefore the correlation time at optimum coherence $\tau = \frac{2\pi}{\omega_0}n$ at low noise is

$$t_{cor} = \frac{4}{\pi|\varepsilon|}(1 + \frac{K}{2}\tau) \tag{24}$$

This constitutes an enhancement of coherence by time-delayed feedback by a factor of $1 + \frac{K}{2}\tau$, which is in excellent agreement with the simulations in Fig. 2(a), yielding, e.g., a factor of 1.62 or $t_{cor} \approx 206$ for $K = 0.2$, $\tau \approx 2\pi$,

$\varepsilon = -0.01$. The dependence of t_{cor} upon K described by Eq. (24) is shown by a dashed line in Fig. 6(a). Note that for large K or τ the lowest order expansion is not adequate, and the approximation breaks down.

For minimum coherence, i.e. $\tau = \frac{\pi}{\omega_0}(2n-1)$, a similar expansion of the power spectral density (14) for $\omega = \omega_0(1+\delta)$, $\delta \ll 1$ can be performed with $\sin(\omega\tau) \approx -\omega_0\tau\delta$ and $\cos(\omega\tau) \approx -1 + \frac{1}{2}(\omega_0\tau\delta)^2$:

$$S(\omega) = \frac{D^2}{2\pi} \frac{\omega^2}{(\omega^2 - \omega_0^2 - \omega K \omega_0 \tau\delta)^2 + \omega^2 \left[\varepsilon - K(2 - \frac{1}{2}(\omega_0\tau\delta)^2)\right]^2} \tag{25}$$

Expressing $S(\omega)$ in terms of δ up to lowest order yields

$$S(\omega) = \frac{D^2}{2\pi(1-\frac{K}{2}\tau)^2} \frac{\omega^2}{\left[\omega^2 - \omega_0^2\right]^2 + \omega^2 \left[\frac{\varepsilon - 2K}{1-\frac{K}{2}\tau}\right]^2} \tag{26}$$

This is a Lorentzian with peak frequency ω_0 and half-width $\Gamma = \frac{1}{2}\frac{|\varepsilon|+2K}{|1-\frac{K}{2}\tau|}$. Therefore the correlation time at minimum coherence $\tau = \frac{\pi}{\omega_0}(2n-1)$ at low noise is

$$t_{cor} = \frac{4}{\pi}\frac{|1-\frac{K}{2}\tau|}{|\varepsilon| + 2K} \tag{27}$$

which describes the slow increase of the minima as a function of $\tau > 2/K$, and the sharp drop with initial slope $-8/(\pi|\varepsilon|^2)$ at $K = 0$ and the very slight increase at a low level for $K > 2/\tau$ as a function of K (see dashed line in Fig. 6(b)), in very good agreement with the numerical results.

Combination of Eqs. (24) and (27) with (21) yields an improved formula for continuous values of τ (dashed line in Fig. 3(a)):

$$t_{cor} = \frac{4}{\pi(-\varepsilon + K(1 - \cos\omega_0\tau))}|1 + \frac{K}{2}\tau\,\mathrm{sign}(\cos\omega_0\tau)| \tag{28}$$

The Lorentzian approximation of the power spectral density also allows for a qualitative explanation of the τ-dependence of the variance of the control force $F(t) = y(t-\tau) - y(t)$

$$\langle F^2 \rangle = 2(\langle y^2 \rangle - \langle y(t)y(t-\tau)\rangle) \tag{29}$$
$$= 2(\sigma^2 - \Psi(\tau)) \tag{30}$$
$$\approx 2\sigma^2(1 - e^{-\Gamma\tau}\cos\omega_0\tau) \tag{31}$$

where $\langle y \rangle = 0$ has been used in (30) and $\sigma^2 = \int_{-\infty}^{\infty} S(\omega)d\omega$ depends upon τ and K through the power spectral density (14).

The autocorrelation function Ψ correctly describes the damped oscillations of $\langle F^2 \rangle = 2(\sigma^2 - \Psi(\tau))$ as a function of τ, eq. (30), with minima at $\tau = \frac{2\pi}{\omega_0}n$ at values given by

$$\langle F^2 \rangle = 2\sigma^2(1 - e^{-\Gamma\tau}) \tag{32}$$

with $\Gamma = \frac{1}{2}\frac{|\varepsilon|}{1+\frac{K}{2}\tau}$. They correspond to maxima of t_{cor}.

The maxima of the control force are also predicted at the correct positions $\tau = \frac{\pi}{\omega_0}(2n-1)$, and their values are given by

$$\langle F^2 \rangle = 2\sigma^2(1 + e^{-\Gamma\tau}) \tag{33}$$

with $\Gamma = \frac{1}{2}\frac{|\varepsilon|+2K}{|1-\frac{K}{2}\tau|}$. They correspond to minima of t_{cor}.

Combining these results we obtain an approximate expression for the variance of the control force (31) as a function of τ ($\tau \neq 2/K$) with the envelope decay constants determined by

$$\Gamma(\tau) = \frac{1}{2}\frac{|\varepsilon|}{1+\frac{K}{2}\tau}\Theta(\cos\omega_0\tau) + \frac{1}{2}\frac{|\varepsilon|+2K}{|1-\frac{K}{2}\tau|}\Theta(-\cos\omega_0\tau) \tag{34}$$

with the Heaviside function $\Theta(z)$. A rough analytical approximation with constant σ^2 is shown as a dashed line in Fig. 3(b); it neglects the variation of σ^2 with τ, and furthermore the decay constant of the minima (32) gives too small values, but the qualitative features of the numerical result are reproduced. This approximation can be improved by evaluating $\sigma^2(\tau, K)$ using a Lorentzian for $S(\omega)$, in particular for small K, but note that the Lorentzian approximation in (31) introduces a large error for most values of τ, cf. e.g. Fig. 4(b).

IV. CONCLUSION

We have investigated a self-adaptive method for controlling stochastic oscillations by a delay in the form of the difference between the delayed and the current values of a system variable. As a specific example of a nonlinear oscillator we have considered the Van der Pol system close to but below the Hopf bifurcation, where the only deterministic attractor is a stable fixed point. The coherence and the time scales of the noise-induced oscillations can be controlled by appropriately adjusting the control parameters τ (delay) and K (feedback strength). An analysis of the linear modes of the fixed point and the spectral properties of the linearized differential equation has elucidated the control mechanism. We have complemented our numerical investigations by a theory which allows us to predict analytically the dependence of the correlation time and the variance of the control force upon τ, K, and the system parameters, and thus, e.g., optimize the coherence. The oscillatory behavior of the correlation time (which can be substantially increased or decreased depending upon the delay), and the entrainment of the basic oscillations period as a function of τ is correctly described within this framework.

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