ON THE MEROMORPHIC SOLUTIONS TO AN EQUATION OF HAYMAN

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ABSTRACT. The behaviour of meromorphic solutions to differential equations has been the subject of much study. Research has concentrated on the value distribution of meromorphic solutions and their rates of growth. The purpose of the present paper is to show that a thorough search will yield a list of all meromorphic solutions to a multi-parameter ordinary differential equation introduced by Hayman. This equation does not appear to be integrable for generic choices of the parameters so we do not find all solutions — only those that are meromorphic. This is achieved by combining Wiman-Valiron theory and local series analysis. Hayman conjectured that all entire solutions of this equation are of finite order. All meromorphic solutions of this equation are shown to be either polynomials or entire functions of order one.

1. INTRODUCTION

Much research has been undertaken concerning the behaviour of meromorphic solutions to differential equations (see [19] and references therein). In this paper we will consider the problem posed by Hayman [11], p. 93 of showing that all meromorphic solutions to the ordinary differential equation (ODE)

$$ff'' - f'^2 = k_0 + k_1 f + k_2 f' + k_3 f'' \tag{1.1}$$

where the k_j are constants, are of finite order. We will solve this problem by finding all meromorphic solutions. The key mathematical methods that we use are Wiman-Valiron theory, local series analysis, and reduction of order. It should be stressed that we do not find the general

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solution of equation (1.1) explicitly, which may well be impossible — we only find the meromorphic solutions.

In general, finding explicit solutions to nonlinear differential equations in terms of finite combinations of known functions is difficult, if not impossible. However, it was observed in the late nineteenth and early twentieth centuries that ODEs whose general solutions are meromorphic appear to be integrable in that they can be solved explicitly or they are the compatibility conditions of certain types of linear problems (see, e.g. [1], chapter 7). In the 1880s Kowalevskaya [17, 18] considered the equations of motion for a spinning top, which is a sixth-order system depending on parameters describing the mass, centre of mass, and moments of inertia of the top. For special choices of these parameters the equations of motion had been solved by Euler and Lagrange. Kowalevskaya observed that these solutions were meromorphic when extended to the complex plane. She determined all choices of the parameters for which the general solution was meromorphic. She found one new case, which she then solved explicitly in terms of ratios of hyper-elliptic functions [5]. No further cases in which these equations can be solved explicitly have been discovered in the intervening 112 years.

From the many examples known in the literature it appears that many, perhaps all, ODEs whose general solutions are meromorphic can be solved explicitly or are the compatibility condition for a related spectral problem. Furthermore, the condition that the general solution is meromorphic can be replaced by the condition that the ODE possesses the Painlevé property (that all solutions are single-valued about all movable singularities) [1]. The Painlevé property will be discussed in section 5.

The philosophy underlying Kowalevskaya's work is that we should be able to find the general solution of an ODE if its general solution is meromorphic. Here we extend this idea to the problem of finding all (particular) meromorphic solutions of an ODE, regardless of whether the general solution is meromorphic. Hence meromorphicity can be used to uncover explicit particular solutions of non-integrable equations.

We begin by discussing the significance of equation (1.1) in complex function theory. Finite order functions have nice properties and so they have been the subject of intense study (see [10] and the reference therein). The major result concerning the order of growth of meromorphic solutions of first-order ODEs is the following theorem due to Gol'dberg [6]. The major result for higher-order ODEs that admit finite order solutions is due to Hayman [11] which will be discussed below.

For the standard notation and terminology of Nevanlinna theory, see [10, 19].

Theorem 1.1. (Gol'dberg) *All meromorphic solutions of the first-order ODE*

$$\Omega(z,\, f,\, f') = 0, \tag{1.2}$$

where Ω is polynomial in all its arguments, are of finite order.

A generalization of Gol'dberg's result to second-order algebraic equations has been conjectured by Bank [4]. Let f be any meromorphic solution of the ODE

$$\Omega\big(z,\, f,\, f',\, f''\big) = 0, \tag{1.3}$$

where Ω is polynomial in all of its arguments. In terms of the Nevanlinna Characteristic $T(r, f)$ (see, e.g., [10] or [19]), Bank [4] conjectured that

$$T(r,\, f) < K_2 \exp(K_1 r^c), \qquad 0 \le r < +\infty, \tag{1.4}$$

where K_1, K_2 and c are positive constants. In [11], Hayman described a generalization of this conjecture to n^{th}-order ODEs, known as the *classical conjecture. If $f(z)$ is a meromorphic solution of*

$$\Omega\big(z,\, f,\, f',\cdots,f^{(n)}\big) = 0, \tag{1.5}$$

where Ω is polynomial in $z,\, f',\, \cdots, f^{(n)}$, then we have

$$T(r,\, f) < a \exp_{n-1}(br^c), \qquad 0 \le r < +\infty \tag{1.6}$$

where a, b and c are constants and $\exp_\ell$ is defined by

$$\exp_0(x) = x, \qquad \exp_1(x) = e^x, \qquad \exp_\ell = \exp\big\{\exp_{\ell-1}(x)\big\}.$$

Clearly the Bank conjecture (1.4) is a special case of the Classical Conjecture when $n = 2$. Hayman credited the conjecture to S. Bank and L. Rubel.

Steinmetz proved the classical conjecture for any second-order polynomial equation which is homogeneous in its dependent variable and its derivatives. Furthermore, he showed how the solution of such an equation can be expressed in terms of entire functions of finite order.

Theorem 1.2. (Steinmetz) *Suppose that in (1.3), Ω is homogeneous in f, f', f''. Then all meromorphic solutions of (1.3) take the form*

$$f(z) = \frac{g_1(z)}{g_2(z)} \exp\{g_3(z)\}, \tag{1.7}$$

where $g_j(z)$, $j = 1, 2, 3$, are entire functions of finite order. In particular f satisfies (1.4).

In particular, the function $f(z) = e^{e^z}$ satisfies (1.4) and the differential equation

$$ff'' - (f')^2 - ff' = 0 \tag{1.8}$$

and is of infinite order.

Bank proved in [4] that if a meromorphic solution f of (1.3) satisfies $N(r, a_j, f) = O(e^{r^c})$ for a_j, $j = 1, 2$ belongs to the extended complex plane $\hat{\mathbf{C}}$ where c is some positive constant, then f satisfies (1.4). This result improved upon Bank's own result [3] where a weaker assumption that $N(r, a_j, f) = O(r^c)$ for a_j, $j = 1, 2$ is assumed. In fact, Gol'dberg [7] proved a stronger result for a special subclass of (1.9). Hayman [11] generalized this result to higher-order algebraic ODEs of the form (1.5). Let Ω take the form

$$\Omega = \sum_{\lambda\in\Lambda} d_\lambda(z) f^{i_0}(f')^{i_1} \cdots (f^{(n)})^{i_n}, \tag{1.9}$$

and where $\Lambda = \{(i_0, i_1, \cdots, i_n) \in \mathbf{N^n} : n_i \in \mathbf{N}\}$ is a finite set and d_λ are polynomials in z.

Hayman formulated the following theorem in terms of the *degree* $|\lambda| = i_0 + i_1 + \cdots + i_n$ and the *weight* $\|\lambda\| = i_0 + 2i_1 + \cdots + (n+1)i_n$ of the terms in equation (1.5).

Theorem 1.3. (Hayman) *Let $f(z)$ be an entire solution of the equation (1.5) where Omega is given by (1.9). Let Γ be the subset of Λ in (1.5) such that it contains those terms in (1.9) with the highest weights among those with the highest degree. Let the highest degree among all the polynomials $d_\lambda(z)$ be d and suppose further that*

$$\sum_{\lambda\in\Gamma} d_\lambda(z) \neq 0. \tag{1.10}$$

Then $f(z)$ has finite order of growth $\max\{2d, d+1\}$ *at most.*

Hayman [11] has suggested the problem of showing that all entire solutions to equation (1.1) where the k_j are either constants or rational functions of the independent variable z, are of finite order. As explained in [11], this is in some sense the simplest differential equation that is neither covered by the results of Steinmetz (since (1.1) is not homogeneous) nor Hayman (since (1.10) is violated) and yet appears to have only finite-order solutions.

In this paper we will consider the case in which the k_j are constants. Not only will we show that Hayman's conjecture is correct, namely that all entire solutions to (1.1) have finite order, we will also show by explicit construction that all meromorphic solutions are either polynomials or entire functions of order one.

Note that the transformation $f = w + a_2$ takes equation (1.1) to

$$w\frac{d^2w}{dz^2} - \left(\frac{dw}{dz}\right)^2 = \alpha w + \beta\frac{dw}{dz} + \gamma, \tag{1.11}$$

where $\alpha = k_1$, $\beta = k_2$, and $\gamma = k_0 + k_1k_3$. For some purposes, which will be apparent later, it will be convenient to write equation (1.11) as

$$(w'' - \alpha)w = (w' - a_+)(w' - a_-), \tag{1.12}$$

where

$$a_\pm = \frac{-\beta \pm \sqrt{\beta^2 - 4\gamma}}{2}.$$

We will see that equation (1.11) always contains some particular meromorphic solutions, however its general solution is meromorphic if and only if either $\alpha = \gamma = 0$ or $\beta = 0$. In these cases it is straightforward (see section 5) to prove the following.

Proposition 1.4. *If $\alpha = \gamma = 0$ then the general solution to equation (1.11) is given by*

$$w(z) = \frac{\beta}{c_1} + c_2 e^{c_1 z}, \tag{1.13}$$

$$w(z) = -\beta z + c_1, \tag{1.14}$$

$$w(z) = 0, \tag{1.15}$$

where c_1 and c_2 are arbitrary constants.

Proposition 1.5. *If $\beta = 0$ then the general solution to equation (1.11) is given by*

$$w(z) = c_1 \exp\left(\pm i \frac{\alpha}{\sqrt{\gamma}} z\right) + \frac{\gamma}{\alpha}, \qquad \text{if } \alpha \neq 0, \tag{1.16}$$

$$w(z) = c_1 \pm i\sqrt{\gamma} z, \tag{1.17}$$

$$w(z) = \frac{1}{c_1^2}\left[\alpha + \sqrt{\alpha + \gamma c_1^2}\, \cosh\left(c_1 z + c_2\right)\right], \tag{1.18}$$

$$w(z) = -\frac{\alpha}{2} z^2 + c_2 \alpha z - \frac{\gamma + c_2^2 \alpha^2}{2\alpha}, \qquad \text{if } \alpha \neq 0, \tag{1.19}$$

where c_1 and c_2 are arbitrary constants.

The central result of this paper is the following.

Theorem 1.6. *If α and γ are not both zero and if $\beta \neq 0$ then the only meromorphic solutions to equation (1.11) are*

$$w(z) = c_1 \exp\left(\frac{\alpha z}{a_\mp}\right) - \frac{\gamma}{\alpha}, \tag{1.20}$$

if $\alpha \neq 0$ and

$$w(z) = c_1 + a_\pm z, \tag{1.21}$$

if $\alpha = 0$, where c_1 is an arbitrary constant. If $\alpha = \gamma = 0$ or $\beta = 0$ then the general solution of equation (1.11) is meromorphic and given by Proposition 1.4 and Proposition 1.5 respectively.

The general solution of equation (1.11) depends on two parameters (c_1 and c_2 in Propositions 1.4 and 1.5). The solutions described by (1.20) and (1.21) each represent two one-parameter (c_1) families of special solutions to equation (1.11). The two families are parametrized by the choice of a_+ and a_- (there is only one family if $a_+ = a_-$). In the generic case, all solutions other than those given in Theorem 1.6 are branched.

The order of the transcendental meromorphic solutions to equation (1.11) comes as an immediate corollary to Theorem 1.6.

Corollary 1.7. *All transcendental meromorphic solutions of equation (1.11), are entire and of order one.*

In section 2 we use asymptotic estimates from Wiman-Valiron theory to show that the only non-vanishing entire solutions to equation (1.11) are of the form $c_2\mathrm{e}^{c_1 z}$, where c_1 and c_2 are constants. Cauchy's existence and uniqueness theorem (see, e.g., [13], page 284) guarantees that the initial value problem $w(z_0) = w_0$ and $w'(z_0) = w_p$ for equation (1.11) has a unique analytic solution in a neighbourhood of $z = z_0$ provided that w_0 and w_p are finite and $w_0 \neq 0$. Hence checking the existence of local series expansions will only provide information regarding expansions about either the zeros or the poles of w. A straightforward leading-order analysis (see section 3) shows that no solution of equation (1.12) can possess a pole of any order. This implies that all meromorphic solutions are entire.

In section 3 we use local series analysis about a zero of w to show that either the only entire solutions of equation (1.11) are those given in equations (1.20) and (1.21) or at least one of the parameters β and γ must be zero. In section 4 we complete the classification of entire solutions by finding all entire solutions in the cases $\beta = 0$ and $\gamma = 0$. Here we use the fact that equation (1.11) is autonomous (i.e. it does not contain the independent variable z explicitly) to reduce it to a first-order equation for $y := w'(z)$ as a function of $x := w(z)$. This equation is of Abel type which we solve by transforming it to a separable equation. This leads to a first-order equation for w as a function of z.

Although we do not construct the general solution (which is branched) of equation (1.11) in the generic case (i.e. $\beta \neq 0$ and α, γ not both zero), we are nonetheless able to find all entire (and therefore all meromorphic) solutions.

2. **Zero-free solutions**

In this section we will consider non-vanishing entire solutions w to equation (1.11). In this case there exists an entire function g such that the solution w has the form

$$w(z) = \mathrm{e}^{g(z)}. \tag{2.1}$$

We will show that g is necessarily a linear function of z. Specifically, we will prove the following.

Proposition 2.1. *The only zero-free entire solutions of equation (1.11) are given by*

$$w(z) = \begin{cases} c_2 \mathrm{e}^{c_1 z}, & \text{if } \alpha = \beta = \gamma = 0, \\ c_1 \mathrm{e}^{-\alpha z/\beta}, & \text{if } \beta \neq 0, \gamma = 0, \\ -\gamma/\alpha, & \text{if } \alpha \neq 0, \end{cases} \tag{2.2}$$

where c_1 and c_2 are arbitrary non-zero constants.

We note that each of the three solutions given by (2.2) above is a special case of the solutions in the list in Theorem 1.6. Our argument relies on the classical result given below in Lemma 2.3, which states that if g is transcendental then near its maximum on a large circle, $|z| = r$, there is a simple asymptotic relationship between g and its derivatives. We will use this relationship together with the fact that g satisfies a particular third-order polynomial ODE (equation (2.7)) to constrain the parameters α, β, and γ in equation (1.11). Subject to these constraints, we are able to solve equation (1.11) exactly.

Substituting equation (2.1) into equation (1.11) and rearranging gives

$$\mathrm{e}^{2g} g'' = (\alpha + \beta g')\mathrm{e}^{g} + \gamma. \tag{2.3}$$

Differentiating equation (2.3) with respect to z and dividing by e^g gives

$$\mathrm{e}^{g} \left(g''' + 2g'g''\right) = \alpha g' + \beta \left[g'' + (g')^2\right]. \tag{2.4}$$

We wish to divide equation (2.4) by $g''' + 2g'g''$ which we can only do provided this expression does not vanish identically. If g is entire and

$$g''' + 2g'g'' = 0 \tag{2.5}$$

then g is linear in z. It follows from equation (2.1) that

$$w(z) = A\mathrm{e}^{Bz}, \tag{2.6}$$

where A and B are arbitrary constants. Substituting equation (2.6) into equation (1.11) yields $(\alpha + \beta B)A\mathrm{e}^{Bz} + \gamma = 0$ for all z. Solving this equation for A and B and using equation (2.6) shows that the only solutions of equation (1.11) arising from equation (2.5) are those given by (2.2).

Consider the case in which equation (2.5) is not satisfied identically. Solving equation (2.4) for e^g as a function of g', g'' and g''' and using this to eliminate the e^g and e^{2g} terms in equation (2.3) shows that g

satisfies the third-order ODE

$$\begin{aligned} g'' \left\{\alpha g' + \beta[g'' + (g')^2]\right\}^2 &= \gamma(g''' + 2g'g'')^2 \\ &+(\alpha + \beta g')(g''' + 2g'g'') \left\{\alpha g' + \beta[g'' + (g')^2]\right\}. \end{aligned} \tag{2.7}$$

We note that g cannot be a non-linear polynomial since, if it were, then the left side of equation (2.4) would grow exponentially while the right side would be a polynomial.

For the case in which g is transcendental entire, we will use Lemma 2.3 below to compare g and its derivatives in equation (2.7). Before introducing the lemma, however, we define the central index of an entire function.

Definition 2.2. Let

$$g(z) = \sum_{n=0}^{\infty} a_n z^n$$

be entire. The *central index* $\nu(r, f)$ is the greatest non-negative integer m such that

$$|a_m| r^m = \max_{n \geq 0} |a_n| r^n.$$

Note that if g is a polynomial of degree N then $\nu(r, g) = N$ for sufficiently large r. If g is transcendental then $\nu(r, g)$ is increasing, piecewise constant, right-continuous, and tends to $+\infty$ as $r \to +\infty$.

In terms of the central index we have the following (see, for example, [14], pp. 33–35, pp. 197–199; [9]; [19]; pp. 50–52,).

Lemma 2.3. *Let g be a transcendental entire function, and $\nu = \nu(r,\ g)$ be its central index. Let $0 < \delta < 1/4$ and z be such that $|z| = r$ and*

$$|g(z)| > M(r,\ g)\nu(r,\ g)^{-\frac{1}{4}+\delta} \tag{2.8}$$

holds. Then there exists a set $F \subset \mathbf{R}$ of finite logarithmic measure, i.e., $\int_F dt/t < +\infty$ such that

$$g^{(m)}(z) = \left(\frac{\nu(r,\ g)}{z}\right)^m (1 + o(1))g(z) \tag{2.9}$$

holds for all $m \geq 0$ and $r \notin F$. If g has finite order σ

$$\sigma = \limsup_{r\to+\infty} \frac{\log\log M(r,\ g)}{\log r} = \limsup_{r\to+\infty} \frac{\log \nu(r,\ g)}{\log r}. \tag{2.10}$$

We now return to our analysis of transcendental entire solutions of equation (2.7). Choose z on $|z| = r \notin F$ such that (2.8) holds and assume that g is transcendental. Using the asymptotic relation (2.9) in (2.7) gives, to leading order, a polynomial equation in ν/z and $g(z)$. The terms $\beta^2(g')^4g''$ and $2\beta^2(g')^4g''$ on the left and right sides of equation (2.7) respectively, are the only terms which generate the factor $(\nu/z)^6\,(1+o(1))g(z)^5$ on application of (2.9). All other terms have degrees strictly less than both the exponents 6 and 5 appearing in the leading order terms above. Therefore the only way that equation (2.7) can hold for a transcendental entire function g is when $\beta = 0$. If $\beta = 0$ then equation (2.7) becomes,

$$4\gamma g'g''g''' + (4\gamma + \alpha^2)g'^2g'' + \alpha^2 g'g''' + \gamma g'''^2 = 0. \tag{2.11}$$

The leading term in (2.11) is given by the term $4\gamma g'g'' = g'''(\nu/z)^6\big(1 + o(1)\big)g^3$. Thus $\gamma = 0$. Similarly we deduce that $\alpha = 0$ and this corresponds to the case when $\alpha = \beta = \gamma = 0$ in the solution (2.2) and so g is linear — a contradiction.

3. Local Series Expansions

In this section we will consider local series expansions of solutions to equation (1.11). We will show that all meromorphic solutions are entire. We will also show that if w is an entire solution to equation (1.11) that vanishes at a point $z = z_0$ then either w is given by the solutions (1.20–1.21) or at least one of the parameters β, γ in equation (1.11) must vanish. In the last case, we will show in section 4 how to obtain all entire solutions that have a zero using the method of reduction of order.

Note that Cauchy's existence and uniqueness theorem (see, e.g., [13, 12]) guarantees the existence of a unique locally analytic solution of equation (1.11) with the initial conditions $w(z_0) = w_0$ and $w'(z_0) = w_p$ provided w_0 and w_p are finite and $w_0 \neq 0$. We will investigate the case where $w(z_0)$ is zero or infinity.

Let w be a meromorphic solution of equation (1.11) that either vanishes or has a pole at some point z_0 in the finite complex plane. Then w has a Laurent expansion which converges in a punctured disc centred at $z = z_0$,

$$w(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^{p+n}, \tag{3.1}$$

where $a_0 \neq 0$ and $p \neq 0$ is an integer. We substitute the expansion (3.1) into equation (1.11) and keep only the leading-order behaviour of each of the terms in the equation. This yields

$$
\begin{aligned}
& [a_0^2 p(p-1)(z-z_0)^{2p-2}+\cdots] - [a_0^2 p^2 (z-z_0)^{2p-2}+\cdots] \\
(3.2) \quad = \;& \alpha[a_0(z-z_0)^p+\cdots] + \beta[a_0 p(z-z_0)^{p-1}+\cdots] + \gamma.
\end{aligned}
$$

The lowest power of $z-z_0$ on the left of equation (3.2) is $2p-2$. The lowest power of $z-z_0$ on the right is either $p-1$ or 0 (from the constant term γ). We see that there is only one possible balance of these powers, namely $p=1$. When $p=1$, we see on equating constant terms in equation (3.2) that $a_0 = a_\pm$. The following two propositions follow immediately.

Proposition 3.1. *Any solution, w, of equation (1.11) does not possess a pole of any order. In particular, any meromorphic solution of equation (1.11) is entire.*

Proposition 3.2. *Let w be any solution of equation (1.11) analytic in a neighbourhood of the point $z=z_0$ and let $w(z_0)=0$. Then $w'(z_0)=a_\pm$.*

Having obtained the leading-order behaviour of any meromorphic solution to equation (1.11) that vanishes at $z=z_0$, we will now derive a recurrence relation for the a_n in the expansion (3.1) with $p=1$ and $a_0=a_\pm$. Equation (1.11) becomes

$$
\begin{aligned}
(3.3) \quad & \sum_{n=0}^{\infty}\left[\sum_{m=0}^{n}(n-m+1)(n-2m-1)a_m a_{n-m}\right](z-z_0)^n \\
& = [\beta a_0+\gamma] + \sum_{n=1}^{\infty}\left[\alpha a_{n-1}+\beta(n+1)a_n\right](z-z_0)^n.
\end{aligned}
$$

The constant term in (3.3) vanishes identically since $a_0=a_\pm$ solves $a_0^2+\beta a_0+\gamma=0$. Equating the coefficients of $(z-z_0)^n$ for $n=1,2,\ldots$ gives the recurrence relation

$$
(3.4) \quad (n+1)([n-2]a_0-\beta)a_n = G_n(a_0,a_1,\ldots,a_{n-1}), \qquad n=1,2,\ldots,
$$

where

$$
G_n(a_0,a_1,\ldots,a_{n-1}) := \alpha a_{n-1} - \sum_{m=1}^{n-1}(n-m+1)(n-2m-1)a_m a_{n-m}.
$$

Note that if the coefficient of a_n on the left side of equation (3.4) does not vanish for any positive integer n then we can uniquely determine the power series expansion of w about $z = z_0$ (after choosing either $a_0 = a_+$ or $a_0 = a_-$). We have proved the following.

Proposition 3.3. *Suppose $(n-2)a_0 - \beta \neq 0$ for all positive integers n, where $a_0 = a_+$ or $a_0 = a_-$. Then there is at most one solution w of equation (1.11) satisfying $w(z_0) = 0$ and $w'(z_0) = a_0$.*

For any choice of the parameters α, β, and γ we can in fact produce an explicit solution to equation (1.11) which satisfies

$$w(z_0) = 0 \quad \text{and} \quad w'(z_0) = a_\pm. \tag{3.5}$$

This solution is given by choosing the constant c_1 in the solutions (1.20) and (1.21) listed in Theorem 1.6 such that $w(z_0) = 0$. These solutions will be derived systematically in section 4, for now it is sufficient to note that they are indeed solutions. This gives the solutions

$$w(z) = \frac{\gamma}{\alpha}\left[\exp\left(\frac{\alpha}{a_\mp}(z - z_0)\right) - 1\right] \tag{3.6}$$

if $\alpha \neq 0$ and

$$w(z) = a_\pm(z - z_0) \tag{3.7}$$

if $\alpha = 0$. So the following is a consequence of Propositions 3.2 and 3.3.

Proposition 3.4. *Suppose $(n-2)a_0 - \beta \neq 0$ for all positive integers n. Then (3.6–3.7) are the only solutions of equation (1.11) that are analytic in a neighbourhood of $z = z_0$ satisfying $w(z_0) = 0$.*

Now we consider the case in which the left side of equation (3.4) vanishes for some positive integer n. Recall that solutions of equation (1.11) can have at most two types of zeros as described in proposition 3.2. First we consider the case in which w vanishes at z_+ and z_- and $w'(z_+) = a_+$ and $w'(z_-) = a_-$ ($a_+ \neq a_-$). Since w is not one of the solutions (3.6–3.7), it follows from Proposition 3.4 that the left side of equation (3.4) must vanish at both z_+ and z_- for positive integers $n = N_+$ and $n = N_-$ respectively. It follows that

$$\beta = (N_+ - 2)a_+ = (N_- - 2)a_-.$$

Recall that $a_+ + a_- = -\beta$, so if $\beta \neq 0$ then

$$\frac{1}{N_+} + \frac{1}{N_-} = 1,$$

which is only possible if $N_+ = N_- = 2$. That is, if $\beta = 0$ — a contradiction. So the only case in which a solution could have both types of zeros considered above is when $\beta = 0$. This corresponds to case 3 of Proposition 3.5 below.

The only case remaining is that in which w is entire and has at least one zero and all the zeros of w are the same type (i.e. either $w'(z_0) = a_+$ at all zeros z_0 or $w'(z_0) = a_-$ at all zeros). Without loss of generality we assume $w'(z_0) = a_+$ at all points z_0 such that $w(z_0) = 0$. We will assume for the time being that $\gamma \neq 0$ so that $a_\pm \neq 0$. In this case, the function

$$v := \frac{w' - a_+}{w} \tag{3.8}$$

is entire since the numerator vanishes at the zeros of the denominator and these zeros are simple.

Re-writing equation (3.8) together with its derivative gives

$$w' = vw + a_+, \tag{3.9}$$

$$w'' = (v' + v^2)w + a_+v. \tag{3.10}$$

Equation (1.12) becomes

$$v'w = \alpha - a_-v. \tag{3.11}$$

Note that if v is a constant then the solution of equation (3.9) is equation (1.20–1.21). We will show that if v is a non-constant entire function then $\gamma = 0$. If v is not a constant then solving equation (3.11) for w and substituting it into equation (3.9) gives

$$a_-(v^2v' + vv'' - v'^2) - a_+v'^2 = \alpha(v'' + vv'). \tag{3.12}$$

We wish to show that there are no non-constant entire solutions of equation (3.12).

A simple leading-order analysis shows that equation (3.12) has no non-constant polynomial solutions. If v has a transcendental entire solution then applying the Wiman-Valiron formula (2.9) to equation (3.12) shows that the only candidates for leading-order terms are $a_-(\nu/z)v^3$ and $-a_+(\nu/z)^2v^2$ where $\nu \equiv \nu(r,v)$ is the central index of v (see definition 2.2). For large $|z| = r$, the central index $\nu(r,v)$ is negligible compared to the magnitude of v, $M(r,f) = \max_{|z|=r} |v(z)|$. More precisely, Wiman (see, e.g., Hille [12], p. 168) showed that, given an entire function v and a real number $\delta > 0$ then $\nu(r,v) < [\log M(r,v)]^{1/2+\delta}$

outside a set of finite logarithmic measure. It follows that the only leading-order term is $a_-(\nu/z)v^3$ which must vanish. Hence $a_- = 0$ which implies that $\gamma = 0$.

We have proved the following.

Proposition 3.5. *Let w be a solution of equation (1.11) such that there is a point $z_0 \in \mathbf{C}$ such that $w(z_0) = 0$ and w is analytic in a neighbourhood of $z = z_0$. Then either*

(1) $w(z) = \dfrac{\gamma}{\alpha}\left[\exp\left(\dfrac{\alpha}{a_\mp}(z-z_0)\right) - 1\right]$, *(if $\alpha \neq 0$), or*

(2) $w(z) = a_\pm(z - z_0)$, *(if $\alpha = 0$), or*

(3) $\beta = 0$, *or*

(4) $\gamma = 0$.

Cases 1 and 2 of the above proposition correspond to the solutions (1.20) and (1.21) of Theorem 1.6.

4. Reduction to First Order

In order to complete our analysis of equation (1.11), we need to find all entire solutions when either $\beta = 0$ or $\gamma = 0$. First we will solve the case $\beta = 0$ (case 1) exactly. We will then reduce equation (1.11) to a first-order ODE for general parameters, which we will analyse in the case $\gamma = 0$ (case 2).

Case 1: $\beta = 0$.
If α and γ are both zero then any constant will satisfy equation (1.11), otherwise the only constant solution is $w(z) = -\gamma/\alpha$ (provided $\alpha \neq 0$). If w is not a constant then multiplying equation (1.11) by w'/w and integrating gives

$$w_z^2 = c_1^2 w^2 - 2\alpha w - \gamma, \tag{4.1}$$

where c_1 is a constant. Equation (4.1) can be integrated to give the solutions (1.18), for $c_1 \neq 0$, and (1.19), for $c_1 = 0$.

We will consider the case in which $\gamma = 0$. Before considering this case, however, we will show how equation (1.11) can be reduced to a first-order ODE for w as a function of z for any choice of the parameters α, β, and γ.

Since equation (1.11) is autonomous (i.e., it admits the symmetry $z \mapsto z + \epsilon$), it can be reduced to a first order equation for $y := w_z$ as a function of $x := w$ (in an domain in which w is one-to-one). This yields the equation

$$\frac{dy}{dx} = \frac{y^2 + \alpha x + \beta y + \gamma}{xy} \Leftrightarrow \frac{dy}{dx} = \frac{(y - a_+)(y - a_-) + \alpha x}{xy}. \tag{4.2}$$

Equation (4.2) is an Abel equation of the second kind (see, e.g., [16]). We first consider the case in which $\alpha = \gamma = 0$. The general solution of equation (4.2) is then given by

$$y(x) = c_1 x - \beta,$$

where c_1 is an arbitrary constant, which corresponds to the solutions (1.13) and (1.14) of equation (1.11). If α and γ do not both vanish and y is not identically zero, then in terms of the new dependent variable

$$u(x) = \frac{\alpha x + \gamma}{y(x)}, \tag{4.3}$$

equation (4.2) becomes the separable equation

$$x(\alpha x + \gamma)\frac{du}{dx} + (u - a_+)(u - a_-)u = 0.$$

Hence, either

$$u \equiv a_\mp \tag{4.4}$$

or separation of variables gives

$$\frac{du/dx}{u(u - a_+)(u - a_-)} + \frac{1}{x(\alpha x + \gamma)} = 0. \tag{4.5}$$

The solutions (4.4) correspond to

$$y(x) = a_\pm + \frac{\alpha}{a_\mp} x \qquad \Leftrightarrow \qquad w'(z) = a_\pm + \frac{\alpha}{a_\mp} w(z),$$

leading (again) to the solutions (1.20) and (1.21) in Theorem 1.6.

We now consider the case $\gamma = 0$. We assume that $\beta \neq 0$ since the solutions for which β is also zero have been considered in case 1.

Case 2: $\gamma = \alpha = 0$.
If w is not identically zero then we can divide equation (1.11) by w^2 and integrate to find

$$\frac{dw}{dz} = cw - \beta,$$

where c is an arbitrary constant. Solving this linear ODE gives the solutions (1.13–1.14).

Case 3: $\gamma = 0$, $\alpha \neq 0$, $\beta \neq 0$. So $a_+ = 0$ and $a_- = -\beta$.
Using partial fractions to integrate equation (4.5) together with the fact that $u = (\alpha w + \gamma)/w_z$, we obtain

$$\frac{w_z}{w} + \frac{\alpha}{\beta} = c_1 \exp\left(\frac{\beta}{\alpha}\left[\frac{w_z + \beta}{w}\right]\right). \tag{4.6}$$

Recall that we were led to consider the case $\gamma = 0$ in Proposition 3.5 under the assumption that w vanishes at some point $z_0 \in \mathbf{C}$. From equation (4.6) we see that the left side has a pole at $z = z_0$ but according to $w'(z_0) = a_\pm$ the right side either has an essential singularity or a regular point at z_0 respectively. Hence there are no entire solutions that vanish in this case.

5. Discussion

In this paper we have addressed the problem of showing that all meromorphic solutions to equation (1.1) are of finite order by providing a complete list of all such solutions. The advantage of producing such lists for classes of differential equations is that from a large number of examples further observations and conjectures can be generated (e.g., the non-polynomial entire solutions of equation 1.1 are of order one) and also to illustrate the relative scarcity of meromorphic solutions in the solution space of generic differential equations.

For differential equations, meromorphic solutions are the exception rather than the rule – even for rational equations. Indeed, Malmquist's theorem [20] states that the only equation of the form

$$\frac{dw}{dz} = R(z, w),$$

where R is rational in w and z that admits a transcendental meromorphic solution is the Riccati equation,

$$\frac{dw}{dz} = a(z)w^2 + b(z)w + c(z),$$

where a, b, and c are rational functions of z. Although no analogous result is known for the case in which a second-order equation admits a transcendental meromorphic solution, much is known about second-order rational ODEs whose general solutions are meromorphic. In fact, much is known in the case that a second order ODE possesses the Painlevé property, which we will now discuss.

An ODE is said to possess the *Painlevé property* if all solutions are single-valued about all movable singularities. In particular, any equation whose general solution is meromorphic possesses the Painlevé property. Equations possessing the Painlevé property have attracted much interest because of their connection with integrable systems and the so-called soliton equations (see, e.g. [1]).

Painlevé, Gambier, and Fuchs classified all second-order equations of the form

$$w'' = F(w, w'; z), \tag{5.1}$$

that possess the Painlevé property, where F is rational in w and w' and locally analytic in z (see [13, 12] and references therein). The notion of the order of meromorphic solutions appears to play an important role in the generalization of the Painlevé property to difference equations [2].

All the equations found in this work of Painlevé et al can be solved in terms of classically-known functions (e.g. elliptic functions, hypergeometric functions, etc.) except those equations that can be mapped to one of six canonical equations, called the Painlevé equations. The first two Painlevé equations (P_I and P_{II}) are

$$\frac{d^2y}{dz^2} = 6y^2 + z, \tag{5.2}$$

$$\frac{d^2y}{dz^2} = 2y^3 + zy + \alpha, \tag{5.3}$$

where α is an arbitrary complex constant. Each of the Painlevé equations can be written as the compatibility of an associated linear (*isomonodromy*) problem [15]. The Painlevé equations are themselves used to define new transcendental functions.

The general solution of equation (1.11) is meromorphic if and only if either $\beta = 0$ or $\alpha = \gamma = 0$ and is branched in all other cases. Therefore it possesses the Painlevé property only for these choices of the parameters α, β, and γ and we can solve the equation explicitly. In the generic case in which the general solution is branched, we can nonetheless find those special solutions that are meromorphic. This suggests the possibility of cataloguing all meromorphic solutions to particular classes of ODEs. In [8] one-parameter families of solutions to an ODE arising in general relativity are found such that all movable singularities are poles. This method appears to generate all exact solutions of this equation in the literature again suggesting that meromorphicity or the absence of movable branch points can lead to explicit particular solutions even when the equation is not integrable.

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