

# Solitary wave solution for a non-integrable, variable coefficient nonlinear Schrödinger equation

R. Grimshaw\*

*Department of Mathematical Sciences,  
Loughborough University, Loughborough, LE11 3TU, UK*

K. W. Chow<sup>†</sup> and C. K. Poon

*Department of Mechanical Engineering,  
University of Hong Kong, Pokfulam Road, Hong Kong*

K. Nakkeeran<sup>‡</sup>

*Department of Engineering, Fraser Noble Building,  
King's college, University of Aberdeen, Aberdeen AB24 3UE, UK*

(Dated: January 26, 2006)

## Abstract

A non-integrable, variable coefficient nonlinear Schrödinger equation which governs the nonlinear pulse propagation in an inhomogeneous medium is considered. The same equation is also applicable to optical pulse propagation in averaged, dispersion-managed optical fiber systems, or fiber systems with phase modulation and pulse compression. Multi-scale asymptotic techniques are employed to establish the leading order approximation of a solitary wave. A direct numerical simulation shows excellent agreement with the asymptotic solution. The interactions of two pulses are also studied.

PACS numbers: 42.81.Dp, 42.65.Tg, 05.45.Yv

Keywords: variable coefficient, nonlinear Schrödinger equation (NLSE), soliton solutions, optical fibers

---

\*Electronic address: R.H.J.Grimshaw@lboro.ac.uk

†Electronic address: kwchow@hkusua.hku.hk

‡Electronic address: k.nakkeeran@abdn.ac.uk

Although localized modes in integrable, conservative systems have been studied intensively, the dynamics of non-conservative systems, where gain or dissipation is allowed, is usually less well understood. As an example, consider the variable coefficient nonlinear Schrödinger equation (NLSE) relevant in nonlinear optics:

$$iA_z + A_{tt} + 2|A|^2A + \kappa^2 t^2 A + i\Gamma A = 0, \quad (1)$$

where  $A$  is the envelope of the axial electric field,  $\kappa$  is related to the group-velocity dispersion and other pulse parameters (see [1] for more details) and  $\Gamma$  is the loss/gain coefficient. Equation (1) finds application in optical fiber systems with phase modulation or pulse compression. Without the loss/gain term ( $\Gamma = 0$ ), Eq. (1) has been studied in different settings [2–4], e.g., nonlinear compression of chirped solitary waves [3] and quasi-soliton propagation in DM optical fiber [2]. With the loss/gain term ( $\Gamma \neq 0$ ), Eq. (1) has been reported earlier in the literature [5–8] from the integrability point of view, where by choosing a special parameter, one soliton solution has been obtained by a Bäcklund transformation. It is also the governing equation for an averaged, dispersion-managed (DM) optical fibre system [1, 9–13]. This averaging procedure investigates the energy enhancement of DM solitons analytically [10] and also assists the management of large dispersion variations in a DM fiber system [11]. Equation (1) is completely integrable when  $\kappa = \Gamma$  [14], and we have recently reported the exact bright and dark soliton solutions in that case [1].

Here in this brief report, we consider the variable coefficient NLSE (1) in the non-integrable case of  $\kappa \neq \Gamma$ , which is physically more significant compared to the completely integrable form reported earlier in the literature. We derive a leading order approximation for the solitary wave for the case of arbitrary damping or gain, or in other words, the non-integrable regime of Eq. (1).

To derive the soliton solution of Eq. (1) using modulation theory we first let

$$A = B(t, z) \exp\left(\frac{i\kappa t^2}{2}\right). \quad (2)$$

Equation (1) can then be rewritten as

$$iB_z + B_{tt} + 2i\kappa t B_t + i(\Gamma + \kappa)B + 2|B|^2B = 0. \quad (3)$$

On assuming that

$$B = B(\xi, z), \quad \text{and} \quad \xi = \xi(t, z), \quad (4)$$

we get

$$i(B_z + B_\xi \xi_z) + B_{\xi\xi} \xi_t^2 + B_\xi \xi_{tt} + 2i\kappa t B_\xi \xi_t + i(\Gamma + \kappa)B + 2|B|^2 B = 0. \quad (5)$$

Choosing  $\xi_z + 2\kappa t \xi_t = 0$ , and  $\xi_{tt} = 0$  gives  $\xi = t \exp(-2\kappa z)$  and Eq. (5) becomes

$$iB_z + \exp(-4\kappa z) B_{\xi\xi} + i(\Gamma + \kappa)B + 2|B|^2 B = 0. \quad (6)$$

Now we use the slowly-varying envelope approximation for the optical soliton with respect to  $z$  (i.e.  $\kappa \ll 1$ ), and so rewrite the variable  $B$  as

$$B = R \left( \xi - \int^z V dz, z \right) \exp \left( ik\xi - i \int^z \sigma dz \right) + B_1, \quad (7)$$

where the leading order is given by

$$\sigma R - iV R_\xi + (R_{\xi\xi} + 2ikR_\xi - k^2 R) \exp(-4\kappa z) + 2R^3 = 0. \quad (8)$$

Considering the real and imaginary parts of this equation, and noting that we may assume that  $R$  is real-valued, gives

$$V = 2k \exp(-4\kappa z), \quad \text{and} \quad R_{\xi\xi} + [\sigma \exp(4\kappa z) - k^2] R + 2 \exp(4\kappa z) R^3 = 0. \quad (9)$$

Now looking for the soliton solution in the form:

$$R = a \operatorname{sech} \gamma \xi, \quad (10)$$

gives

$$\sigma \exp(4\kappa z) = k^2 - \gamma^2, \quad \text{and} \quad \gamma^2 = a^2 \exp(4\kappa z). \quad (11)$$

In general,  $a, \gamma, k, \sigma$  and  $V$  depend on the slowly varying variable  $\kappa z$ . To determine their behaviour we must look into the next order. We put

$$B_1 = R_1 \exp\left(ik\xi - i \int^t \sigma dt\right), \quad (12)$$

and then get

$$\exp(-4\kappa z)R_{1\xi\xi} + [\sigma - k^2 \exp(-4\kappa z)]R_1 + 2R^2(2R_1 + R_1^*) + iR_z + i(\Gamma + \kappa)R = 0. \quad (13)$$

where the asterisk ( $f^*$ ) represents the complex conjugate of the variable  $f$ . On decomposing  $R_1$  as  $P + iQ$ , we get:

$$\begin{aligned} \exp(-4\kappa z)[P_{\xi\xi} - \gamma^2 P] + 6R^2 P &= 0, \\ \exp(-4\kappa z)[Q_{\xi\xi} - \gamma^2 Q] + 2QR^2 + R_z + (\Gamma + \kappa)R &= 0. \end{aligned} \quad (14)$$

The equation for  $P$  is homogeneous, and the only bounded solution is  $P = R_\xi$ . The equation for  $Q$  is inhomogeneous, and the criterion for a bounded solution is that the inhomogeneous term be orthogonal to the bounded solutions of the homogeneous adjoint operator. Here the homogeneous equation is self-adjoint, and the only bounded solution is  $Q = R$ . Thus we get

$$\int_{-\infty}^{\infty} R[R_z + (\Gamma + \kappa)R]d\zeta = 0 \quad (15)$$

which leads to

$$\int_{-\infty}^{\infty} \frac{R^2}{2}d\zeta = (\text{const.}) \exp[-2(\Gamma + \kappa)z] \quad \text{and so} \quad \frac{a^2}{\gamma} = (\text{const.}) \exp[-2(\Gamma + \kappa)z]. \quad (16)$$

Thus we get

$$\gamma = \gamma_0 \exp[-2(\Gamma - \kappa)z] \quad \text{and} \quad a = a_0 \exp(-2\Gamma z). \quad (17)$$

Also, with  $k$  as an arbitrary constant, we get

$$\begin{aligned} \xi - \int^z V dz &= t \exp(-2\kappa z) + \frac{k}{2\kappa} \exp(-4\kappa z) + \xi_0, \\ \int^z \sigma dz &= -\frac{k^2}{4\kappa} \exp(-4\kappa z) - \int \gamma^2 \exp(-4\kappa z) dz. \end{aligned} \quad (18)$$

After some algebra this yields

$$A = a_0 \operatorname{sech} \left\{ \gamma_0 \left[ t \exp(-2\Gamma z) + \frac{k}{2\kappa} \exp(-2\kappa z - 2\Gamma z) \right] + \eta^{(0)} \right\} \\ \times \exp \left[ -2\Gamma z + \frac{i\kappa t^2}{2} + ikt \exp(-2\kappa z) + \frac{ik^2}{4\kappa} \exp(-4\kappa z) - \frac{i\gamma_0^2}{4\Gamma} \exp(-4\Gamma z) \right], \quad (19)$$

where  $\eta^{(0)}$  is a phase constant arising from the choice of origin. For the case of  $\kappa = \Gamma$ , this reduces to the recently reported 1-soliton solution by the Hirota bilinear method [1].

To test the validity of this asymptotic approximation, we develop a numerical code for the NLSE (1) using the Hopscotch scheme [15]. Figure 1 (a) shows the propagation of a solitary wave under fairly weak damping, while Fig. 1 (b) demonstrates the excellent agreement between the computational results and the asymptotic approximation (19).

Besides the context of DM fiber system, families of evolution equations related to (1) are relevant to optical systems with pulse compression or phase modulation, and hence stabilities of localized modes are critical issues. To supplement the analytical progress in this brief communication, we initiate some preliminary numerical simulations on interactions of pulses.

In the ordinary NLSE, two identical pulses will interact if they are sufficiently close initially [16]. This interaction consists typically of periodic merger and separation. In the present case, the scenario of merger and separation can persist for a few cycles if the damping (or gain) is sufficiently weak, see Fig. 2. The two pulses will soon diverge from each other and never interact again. For very strong damping or gain, this energy loss or input will of course completely erase even the initial merger and separation cycles.

We have thus obtained a leading order approximation for a solitary wave propagating in a dispersion-managed optical fiber medium. For families of variable coefficient NLSE (1) documented in the literature, it appears that analytical treatment has only been given for the case of  $\Gamma = \kappa$ , which is the integrable case. We have thus removed this constraint and demonstrated that, at least asymptotically, localized pulse can exist for arbitrary values of  $\Gamma$  and  $\kappa \ll 1$ . For the special case of  $\Gamma = \kappa$ , our results reproduce the known 1-soliton solution.

From the perspective of fiber optics,  $\kappa$  is related to the group-velocity dispersion of the fiber and other pulse parameters such as the chirp, and  $\Gamma$  is the loss/gain coefficient of the

transmission system. The integrability condition of  $\kappa = \Gamma$  will dictate a constraint on the fiber and pulse parameters of the optical transmission system. The new expression (19) with  $\kappa$  and  $\Gamma$  independent of each other will thus permit more flexibility in the design of the optical fiber communication system.

## ACKNOWLEDGMENTS

KWC and KN wish to thank The Royal Society for their support in the form of an International Joint Project Grant. Partial financial support has been provided to KWC by the Research Grants Council through contracts HKU 7184/04E and 7123/05E.

- 
- [1] C. C. Mak, K. W. Chow and K. Nakkeeran, *J. Phys. Soc. Japan*, **74**, 1449 (2005).
  - [2] S. Kumar and A. Hasegawa, *Opt. Lett.*, **22**, 372 (1997).
  - [3] J. D. Moores, *Opt. Lett.*, **21**, 555 (1996).
  - [4] M. L. Quiroga-Teixeiro, D. Anderson, P. A. Anderson, A. Berntson and M. Lisak *Opt. Soc. Am.*, **13B**, 687 (1996).
  - [5] M. N. Vinoj, V. C. Kuriakose and K. Porsezian, *Chaos, Solitons and Fractals*, **21**, 2569 (2001).
  - [6] K. Nakkeeran, *J. Phys. A: Mathematical and General*, **34**, 5111 (2001).
  - [7] K. Nakkeeran, *Nonlinearity*, **15**, 1747 (2002).
  - [8] R. Ganapathy, V. C. Kuriakose and K. Porsezian, *Opt. Commun.*, **194**, 299 (2001).
  - [9] A. Hasegawa and Y. Kodama, *Opt. Lett.*, **16**, 1385 (1991).
  - [10] T. S. Yang and W. L. Kath, *Opt. Lett.*, **22**, 985 (1997).
  - [11] M. J. Ablowitz and G. Biondini, *Opt. Lett.*, **23**, 1668 (1998).
  - [12] A. Hasegawa, Y. Kodama and A. Maruta, *Opt. Fiber Tech.*, **3**, 197 (1997).
  - [13] Zhiyong Xu, Lu Li, Zhonghao Li, Guosheng Zhou and K. Nakkeeran, *Phys. Rev. E*, **68**, 046605 (2003).
  - [14] P. A. Clarkson, *Proc. Royal Soc. Edinburgh*, **109A**, 109 (1988).
  - [15] C. S. Tsang and K. W. Chow, *Mathematics and Computers in Simulations*, **66**, 551 (2004).
  - [16] C. Desem and P. L. Chu, in *Optical Solitons: Theory and Applications*, Cambridge University Press, Editor: J. R. Taylor (1992).

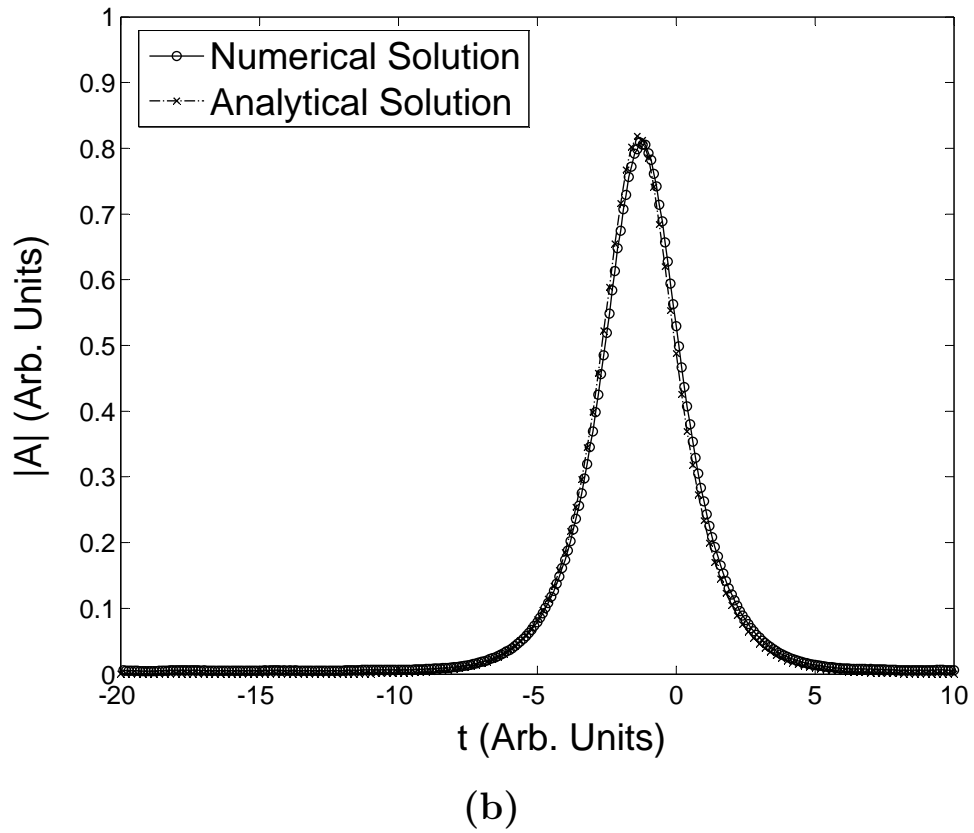
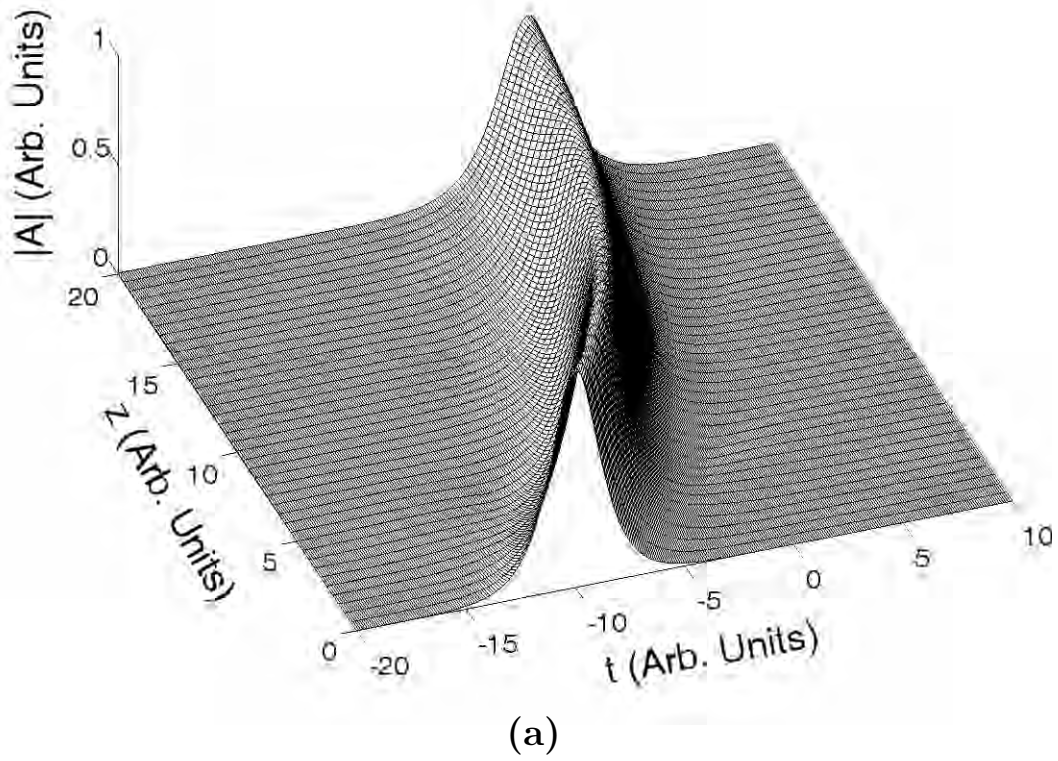


FIG. 1: (a) The propagation of a solitary wave under a fairly weak damping,  $\kappa = 0.05$ ,  $\Gamma = 0.005$ . (b) The comparison between the computational and the analytical approximation at  $z = 20$ .

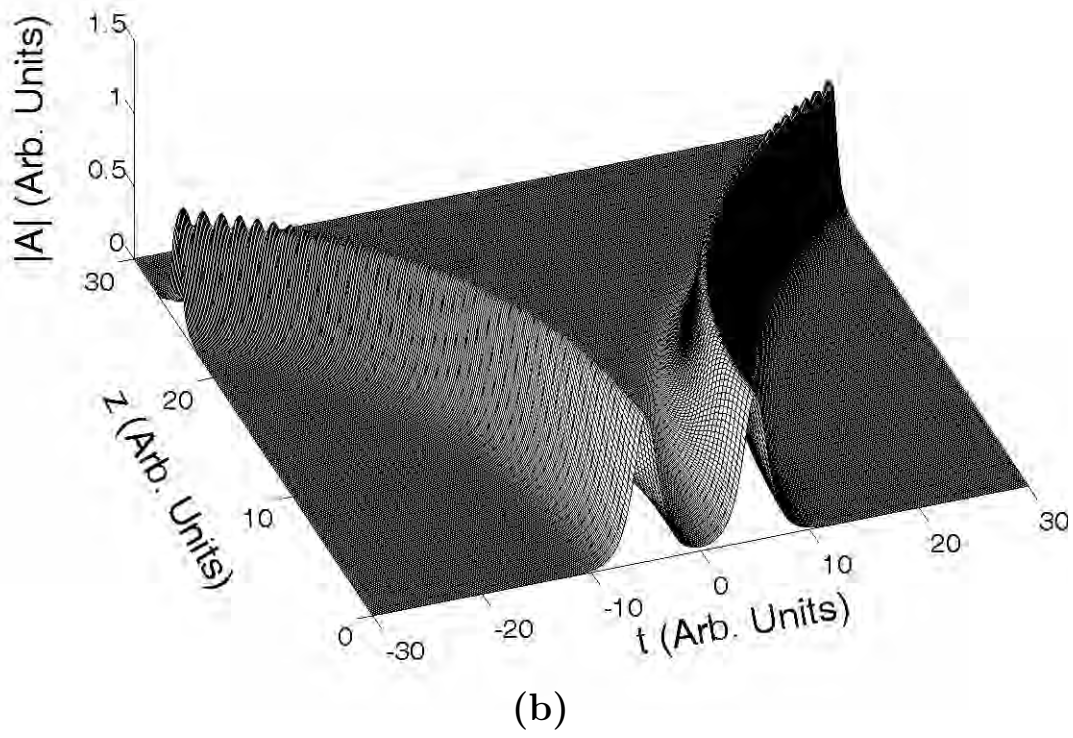
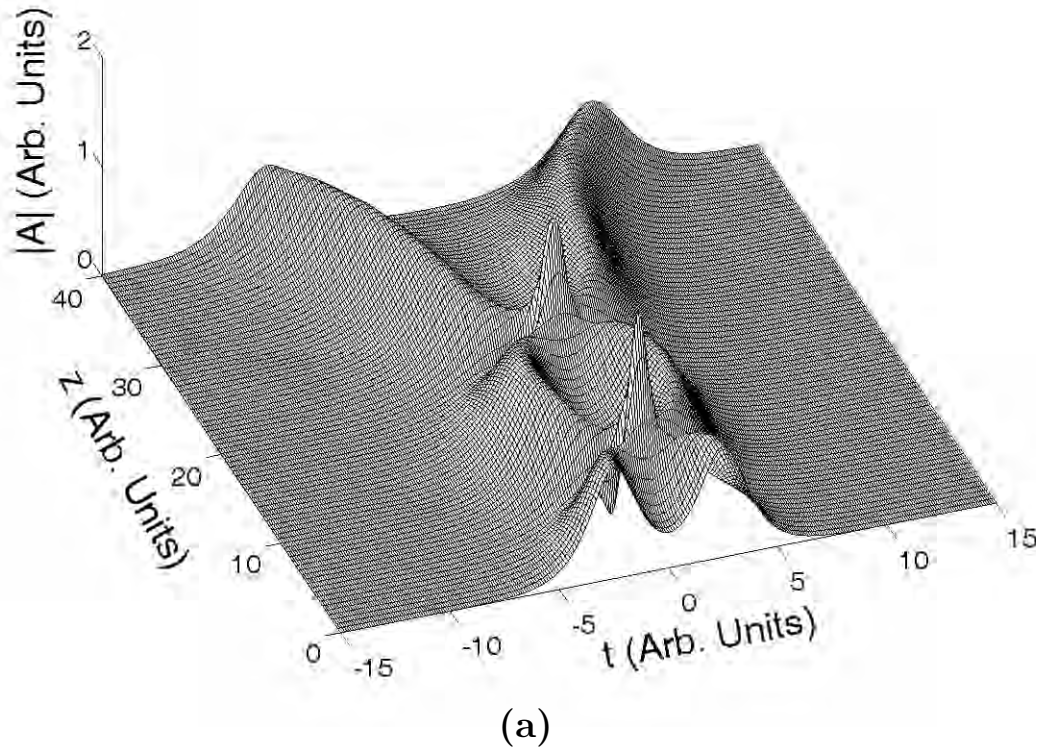


FIG. 2: (a) The interactions of two pulses relatively close initially,  $\kappa = 0.05$ ,  $\Gamma = 0.005$ . (b) The interactions of two pulses relatively far apart initially.