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TEACHING PROOF TO UNDERGRADUATES: SEMANTIC AND SYNTACTIC APPROACHES

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This paper contrast the rationales behind semantic and syntactic approaches to teaching an undergraduate transition-to-proof course, using data from interviews with two mathematicians. It addresses the ICMI theme of teachers' views and beliefs, with particular focus on (1) instructors' expectations in proof-based courses and (2) both example-based and logical structure-based skills that we would like students to develop before arriving at university.

INTRODUCTION

The ICMI discussion document says that mathematical proof consists of “explicit chains of inference following agreed rules of deduction, and is often characterized by the use of formal notation, syntax and rules of manipulation” but is also “perhaps most importantly, a sequence of ideas and insights with the goal of mathematical understanding...”. Weber and Alcock (2004) used a similar distinction to characterize a *syntactic* proof production as one that involves drawing inferences by manipulating symbolic formulae in a logically permissible way, and a *semantic* proof production as one that uses instantiations of mathematical concepts to guide the formal inferences drawn. Small-scale studies have indicated that there might be differences in the degree to which individuals tend to adopt each of these strategies, in real analysis (Pinto & Tall, 2002) and in general “transition to proof” courses (Alcock & Weber, 2008), and at the graduate level in number theory (Alcock & Inglis, in press). This is also consistent with mathematicians' self-reports of distinctions within their own activity (e.g., Burton, 2004; Nardi, 2007; Tall, 1991).

But what does this mean for the teaching of mathematical proof? Is it necessary, even, that we agree, or are there multiple valid approaches? The study reported here (and in Alcock, 2008) addresses these points by contrasting perspectives of two mathematicians teaching an undergraduate transition-to-proof course.

THE INTERVIEW STUDY

The excerpts used here come from a study in which five mathematicians each took part in two interviews about their experience of teaching an undergraduate transition-to-proof course called Introduction to Mathematical Reasoning. They each had several years' experience and taught the course regularly to classes of fewer than 25 students, meaning that they were regularly exposed to students' verbal and written attempts to construct proofs. This paper shows similarities in the way that two of these mathematicians described student difficulties, and contrasts their different responses to these perceived problems.

SIMILAR CONCEPTIONS OF THE ISSUES

Both professors were personally committed to teaching the course, because of its importance in supporting higher level courses and out of personal interest.

P1: When I made the decision that I really wanted to understand how to teach this course, and work on it, well then it became a passion.

P3: It's certainly to me an extremely enjoyable experience because I find it challenging and I'm learning a lot. And I have always had a general interest in strange ways of thinking. [...] Probably if I wasn't a mathematician I'd be an anthropologist.

Both remarked that what a mathematician might consider to be natural logical reasoning could not be taken for granted in the student population.

P1: Most mathematicians would like to live in a world where people are born with [basic logic]. And, you know they just – that it just develops naturally, you know that you don't have to do any...specific effort to get it across. And...well it's just not true, certainly for our students.

P3: To me there is no doubt that there are lots of very basic things that people like you and me absorbed somehow without consciously studying them in a systematic way in a course. [...] We have some ability to think along those lines, that our students either don't have, or if they have it, it hasn't been sort of nurtured or developed in the way it should have been.

Both noted that students' early arguments often could not be critically analyzed.

P1: [I say to the students] "...so there is a place in your argument where you lied to me. And there must be a critical place." [...] So of course at this point they're not writing proofs. They're not even writing arguments that you can apply this kind of analysis to. It's so kind of vague or so diffuse...

P3: Most of the proofs, ideas in the homework...I cannot even tell what's wrong with them. I cannot even pinpoint the step where it goes wrong.

Both discussed the need to teach students about using predicates as building blocks for mathematical statements, with particular attention to the need to control for the scope of different variables.

P1: [T]hey have to give everything a different name, to introduce it you must list all the properties that that thing has – you can't later on sneak in something.

P3: If you want to prove that the sum of two even integers is even, then [...] you take x and you write it as $2a$, and you take y and you write it as $2b$, but you cannot write it as $2a$ again.

P3: [T]here is a distinction between statements and other utterances that are not statements, and what you prove is statements. [...] You may prove various properties of the derivative, but you don't prove the derivative.

Both discussed the problem that at least the “weaker” students seemed to expect mathematics to be about manipulating symbols in order to “get something”.

P1: One problem is that they do not think at all about the statements that they’ve proved. In other words, they are looking to do everything in a purely syntactic way. [...] They give a proof which is [...] complete nonsense because using a proof like this you could prove absolutely anything.

P3: I would say that 99% of the questions students ask are formulated in procedural terms. Like, “Can I do this?”, “Can I do that?” rather than “Is this true?”, “Is this right?” And they have to do with symbol manipulation.

However, the two expressed contrasting beliefs about how to respond.

PROFESSOR 1: A SEMANTIC RESPONSE

Professor 1 stressed his concern about the need to emphasize meaning in the students’ thinking, contrasting this with Professor 3’s approach.

P1: When we do a proof, [...] although we know in principle that it can be done syntactically, we can bring it down to some axiomatic system...that, in fact, you think very semantically. [...] You’re thinking about the objects themselves, and you are confident with your ability to do...to do legal reasoning on this thing.

P1: [The students] come with a pretty syntactic view of mathematics. They see a mathematics problem and they know certain rules for manipulating. [Professor 3] wants them to really understand the syntax of what they’re doing, and to work on writing it. My problem is that it’s more of the same. In other words, it encourages students to think of mathematics as a syntactic enterprise, rather than trying to assign meaning to what they’re doing.

He explained that he liked to “start with interesting problems where the answer is not obvious” and that he liked a book based on discrete mathematics because

P1: It’s a domain in which you can actually draw nice pictures, and the pictures have a direct correspondence with the proofs you’re writing. So you can do examples. [...] You can scribble out numbers to test out the proof ideas.

This emphasis on examples continued as he talked about the way he used them in understanding definitions and in checking arguments.

P1: If I see a definition, I immediately instantiate it. You know, try some examples. [...] You describe a new definition, you say, “Let f be a function, let x be a real number, we say that...” and then some relationship between f and x holds if *blah blah blah*. [...] [T]hey have to realize that this definition only makes sense in the context of, I have to have a function in mind and I have to have a number in mind.

P1: This particular part of the argument is more actually much more general than the specific situation you’re dealing with. And therefore I can do a check on this part of the argument by thinking about that more general situation and doing examples within that more general situation.

He lamented students’ apparent lack of similar activity.

P1: [If] they don't understand the definition, they'll just skip that sentence and go on. They will come in for help on a particular problem, and five or ten minutes into the discussion I'll realize that they never bothered to process this particular definition. They have no idea what this means.

P1: I also tell them things like, before you try to prove a statement, let's say it's "For all...something," then you should, look, illustrate. [...] Pick some examples, just so you can see how...how this theorem works. So maybe you don't see exactly how it's going to help you do what you have to do, but trust me, it will. You have to just get into the habit of understanding it. And they just refuse to do this.

He discussed his construction of tasks that focused on building these skills.

P1: I have one where I define what a partition of a set means [...]: a collection of subsets such that the empty set is not one of the subsets, for every element of the underlying set there is a subset that contains it, for any two sets in the partition the intersection is empty. Which is pretty abstract, when you look at something like that. And then I just ask, okay, construct three examples of a partition on the set $\{1,2,3,4,5\}$. And then, okay, construct an example of a collection of sets on $\{1,2,3,4,5\}$ which satisfies the first two properties but not the third, the first and the third properties but not the second, the second and third properties but not the first.

He noted, however, that experience had also led him to introduce an increasing number of "guide rails" to head off common syntactic errors. He expressed regret about this and stressed his view that work on syntax should be minimized.

P1: It's a little bit of a disappointment [that] I can't build so heavily on these interesting problems. [...] They have to learn enough syntax, syntactic things about proofs...um...so that they can stay within the lines, somehow. And I'm trying to find the minimum point that I can get that to, because it's so boring. And because it's – and also because that's their habit of thinking, it's syntactic. And if they're going to succeed in this course then they have to get away from that.

PROFESSOR 3: A SYNTACTIC RESPONSE

Professor 3, in contrast, did not think it reasonable to set challenging problems.

P3: I hear a lot of talk about the fact that it's important to give them challenging problems with interesting mathematics. [...] I can understand [this]. Maybe on this level they have these problems because this stuff is boring, but more challenging mathematics might be interesting. But I have doubts, because even when I put moderately challenging problems, what I always find is that the main obstacle is not that they don't know what to do, the main obstacle is that they will do anything, and don't seem to think it needs justification.

He confirmed that he might expect students to check statements using examples.

P3: They definitely don't have the habit of approaching statements critically, asking themselves "Is this really true? Let me see. Let me try some examples and I will check".

However, this was in response to a specific question and was his only such comment. He spoke more of his concern over the lack of precision in students' use of notation, and their lack of experience of thinking in terms of structures.

P3: The difficulty with understanding the rigidity of mathematical... grammar [...] extends to the point that, for example, students are unaware that mathematical letters, symbolism, is case sensitive. You cannot introduce a natural number little n , and then refer to it as capital N in the next sentence.

P3: The sentence is of a certain form. [...] And the main obstacle is that [...] most students don't seem to have had any courses in English grammar, in anything else that might teach them how to think in terms of structures, logical structures.

He emphasized his response to this in terms of insisting upon standard phrasing.

P3: So for instance I don't accept it when a student says, "a number x is even if x is $2k$, where k is an integer". I don't accept that because there is supposed to be an existential quantifier there.

P3: ...never use the word "any" because in mathematics it tends to mean too many things. "Any" can sometimes mean "some", it sometimes means "all". Now the good student would know how to use the word "any" correctly. But a not-very-good student is likely to use it in a way that is ambiguous, so better never to use it.

He discussed his overall aim that learning to use mathematical syntax in this precise way would prepare students for proving more complex statements.

P3: And I insist very much on minimizing the number of rules because I have become convinced that this is the antidote to the general...amorphousness of their work. [...] There is a need to put a lot of rigidity and they are...maybe it doesn't hurt to go to the other extreme for a while. Once, in one course.

P3: It seems to me that when one makes a thing very minimal in this way, this has a very positive side. [...] When there is only one thing you can do, then it's much easier to know what you can – what you should do. [...] If you have a sentence that is, for example, a universal sentence, so "For all x , *blah blah blah*," then there is only one way to prove that: "Let x be arbitrary, we prove that *blah blah blah*."

He acknowledged that this syntactic approach did not capture "real proof", but expressed his view that going beyond this was unrealistic.

P3: Doing proofs, mathematical proofs, is not a matter of analyzing logical structures and statements, and applying the logical rules of inference. That, if you want, that may be the frame, the...the thing that supports the proof. But the real proof and the real mathematics is not that. And that should also be conveyed in the course. But of course it's too much. That's wishful thinking.

DISCUSSION

These two professors conceptualized the problems of teaching proof in similar ways, but diverged in their beliefs about how to address them. Both had well thought-out and reasonable positions, and the purpose here is not to judge either

as better or worse but to emphasize the fact that teaching at the undergraduate level, particularly at the transition-to-proof stage, might vary considerably in its emphasis. This is pertinent to stated aims of the ICMI study, because the notional smooth development from early forms of argument into formal proof was not experienced by these professors; both found the majority of their students to be unprepared for the reasoning expected in their courses. Indeed, teachers and professors do not all communicate with each other and agree about the ethos of mathematics classes. Students will therefore encounter different approaches, which raises the question of what skills they need to develop in order to respond flexibly to different emphases in later classes. The distinction used here may be useful in characterizing some of these skills: here Professor 1's semantic approach needed students to generate examples, and Professor 3's syntactic approach needed practice in thinking in terms of logical structures. These skills could be incorporated into earlier mathematics lessons in a variety of ways, which I hope might be discussed at the conference.

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