Reflection and transmission of Rayleigh waves in a wedge

V. V. Krylov and V. G. Mozhaev

Physics Faculty, M. V. Lomonosov State University, Moscow

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An approximate theory of the reflection and transmission of Rayleigh waves in an elastic wedge is developed on the basis of the representation of the wedge by a system of two coupled surface-wave waveguides. The theory is in good agreement with the existing experimental data and permits the prediction of a number of new results, which can be tested experimentally.

The interaction of Rayleigh waves with the edge of an elastic wedge is investigated in such scientific and engineering fields as seismology, ultrasonic surface flaw detection, and fracture mechanics.[1-4] It has recently stimulated a certain interest among specialists in acoustoelectronics in connection with the promising outlook for the application of the edges of wedges (corners) as effective frequency-independent surface-wave reflectors,[5] for the wideband conversion of bulk to surface waves and vice versa,[6,7] and for the suppression of spurious signals in surface-acoustic-wave (SAW) delay lines and filters.[8] Despite the considerable time devoted to the study of this problem and the large number of publications pertaining to it (see, e.g., Knopoff's survey[9]), it is still far from solution in the theoretical plan, even for the isotropic case. This is because of fundamental difficulties in the formulation of exact analytical solutions of the dynamical equations of the theory of elasticity, subject to the condition of the absence of normal stresses on the surface of the wedge-shaped region.[10] The existing computer-generated numerical solutions[11,12] refer only to the simplest case of a right-angle wedge. The current approximate theories,[9,13-16] on the other hand, are in poor agreement with the experimental data, particularly in the range of small wedge angles θ. Specifically, none of these theories affords a satisfactory description of the experimentally observed[1-3] multiple oscillations of the Rayleigh-wave reflection and transmission coefficients as a function of the angle θ. Also lacking is an explanation of why almost total reflection or transmission of Rayleigh waves is possible for small angles θ.

In the present article we propose a straightforward approximate theory of the reflection and transmission of Rayleigh waves in a wedge, where the indicated oscillations are described. In contrast with the majority of the existing approaches, which are valid for $\theta \sim \pi$, the proposed theory proceeds from the opposite limiting case $\theta \sim 0$. The wedge in this case can be regarded as a combination of two coupled waveguides, because each of its faces represents a guiding structure for surface waves. A Rayleigh wave incident on the edge of the wedge is then represented by the sum of a symmetric and an antisymmetric mode of the given "coupled waveguide system." For sufficiently sharp wedges these modes can be approximated by the lowest symmetric (longitudinal) and the lowest antisymmetric (flexural) Lamb (plate) modes in a plane-parallel plate of thickness h corresponding to the local thickness of the wedge. We note that the actual idea of representing the modes of a sharp wedge by plane-parallel plate modes is quite well known.[17,18] However, the idea has not been used before in application to the analysis of Rayleigh-wave reflection and transmission in the wedge.

We now undertake the direct solution of the stated problem (Fig. 1). We first consider the simpler case of normal incidence of a Rayleigh wave on the edge of the wedge (the angle of incidence $\alpha = 0$). The displacements u_I in the incident wave on the surface of the wedge are represented by the sum of a symmetric wave u_s and an antisymmetric wave u_a:

$$u_I = u_s + u_a, \tag{1}$$

$$u_s = (u_0/2)\exp\left[-i\int_{x_0}^{x} k_s(x',\theta)\,dx'\right], \tag{2}$$

$$u_a = (u_0/2)\exp\left[-i\int_{x_0}^{x} k_a(x',\theta)\,dx'\right]. \tag{3}$$

Here u_0 is the amplitude of the incident Rayleigh wave [the factor $\exp(-i\omega t)$ is omitted]; $k_s(x, \theta)$ and $k_a(x, \theta)$ are the and antisymmetric wave numbers, which depend on the coordinate x measured from the edge of the edge as a result of the variation of its effective local thickness $h(x, \theta) = 2x\tan(\theta/2)$; x_0 is a certain initial coordinate, which is sufficiently far from the edge and satisfies the condition $h(x_0, \theta) > \lambda_R$, where λ_R is the Rayleigh wavelength at a given frequency ω. Excitation of the Rayleigh wave incident on the edge of the wedge takes place at the point x_0, and the phases of the symmetric and antisymmetric modes are measured relative to this reference point. Either the longitudinal or the transverse component of the displacement vectors in the Rayleigh and Lamb waves can be taken as the displacements u. The equality of the amplitudes of the symmetric and antisymmetric waves in Eqs. (1)-(3) follows from the fact that the excitation of the incident Rayleigh wave takes place only on one face of the wedge, so that the symmetric and antisymmetric waves must exactly cancel one another on the opposite face at $x \sim x_0$. We note that the representation (1) is standard in the theory

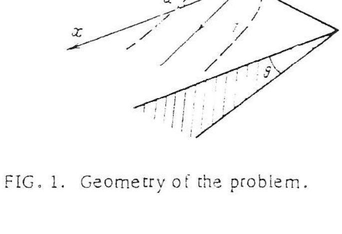

FIG. 1. Geometry of the problem.

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of coupled waveguides (see, e.g., Louisell's book[19]). Equations (2) and (3), on the other hand, are essentially a first approximation of the well-known WKB method. The variation of the wave amplitudes u_s and u_a with the distance traveled is disregarded, because their growth as the edge is approached is subsequently compensated in the process of reverse propagation of the reflected symmetric and antisymmetric waves to the reception point.

At sufficiently large x in the interval $x < x_0$, for which $h(x, \theta) > \lambda_R$, the equality $k_s = k_a = k_R$ holds with a high degree of accuracy, where k_R is the Rayleigh wave number, and both modes involved in Eq.(1) propagate in phase. As the waves approach the edge of the wedge, i.e., for $h(x, \theta) < \lambda_R$, the quantities k_s and k_a differ appreciably from one another, creating a phase difference between the modes. After reflection of the symmetric and antisymmetric modes from the edge of the wedge, which can be regarded as reflection from the free end of a thin plate in the given approximation, the phase separation process continues during propagation of the waves in the reverse direction. Clearly, the θ-dependent phase difference between the reflected symmetric (u_s') and antisymmetric (u_a') waves at the reception point is in fact the cause of the oscillations of the Rayleigh-wave reflection and transmission coefficients as the wedge angle θ is varied. Of course, this is true under the assumption that the moduli of the reflection coefficients of both the symmetric and antisymmetric wave are individually close to unity, i.e., the conversion of energy from the investigated lowest Lamb modes into higher modes, corresponding to the scattering of surface into bulk waves at the edge of the wedge, is practically absent. This assumption is fully justified in the case of sharp wedges, because it holds true for waves in thin plane-parallel plates.

The expressions for the reflected (u_R) and transmitted (u_T) Rayleigh waves measured at distances x_0 from the edge of the wedge are written as follows in light of the foregoing considerations:

$$u_R = u_s' + u_a', \tag{4}$$

$$u_T = u_s' - u_a', \tag{5}$$

$$u_s' = (u_0/2)\exp(-i\varphi_s - i\Phi_s), \tag{6}$$

$$u_a' = (u_0/2)\exp(-i\varphi_a - i\Phi_a), \tag{7}$$

where $\varphi_{s,a} = 2\int_{x_0}^{0} k_{s,a}(x,\theta)\,dx$ are the phase leads to the symmetric and antisymmetric waves in transmission from the point x_0 to the edge of the wedge and back again, and Φ_s, Φ_a are the phase shifts of these waves in direct reflection from the edge. The main sign in Eq. (5) is associated with the phase relations between the oscillations in the waves u_s and u_a. If the oscillations in these wave have a phase difference $\Delta\varphi$ on one of the faces of the wedge, the phase difference on the other face will be equal to $\Delta\varphi + \pi$ by virtue of the antisymmetry of the wave u_a, and this is equivalent to the placement of a minus sign in front of u_a' in Eq. (5). From the relations (4)-(7) we readily obtain expressions for the coefficients of reflection $R \equiv u_R/u_I$ and transmission $T \equiv u_T/u_I$ of Rayleigh waves in the wedge:

$$R = \sin\Psi_- \exp(i\Psi_+ - i\pi/2), \qquad T = \cos\Psi_- \exp(i\Psi_+), \tag{8}$$

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where

$$\Psi_\pm = \int_0^{x_0} [k_a(x,\theta) \pm k_s(x,\theta)]\,dx + (\pi + \Phi_a \pm \Phi_s)/2.$$

The phase shifts Φ_s and Φ_a are left undetermined in Eqs. (8). It is known from the theory of vibrations of thin plates[20] that $\Phi_s \equiv \Phi_s(0) = 0$ and $\Phi_a \equiv \Phi_a(0) = \pi/2$ for plates of a constant thickness. In order to extend the domain of validity of the given theory of Rayleigh-wave reflection and transmission to the case of not too sharp wedge angles, it is necessary to take the dependence of Φ_s and Φ_a on θ into account. The simplest way to do this is by linear interpretation of the values of Φ_s and Φ_a at $\theta = 0$ and at $\theta = \pi$. The case $\theta = \pi$ clearly corresponds to the transition from a wedge to a half-space. In this case Φ_s and Φ_a are readily determined by analyzing the opposing motion of two in-phase and two in-phase and two antiphase Rayleigh waves, respectively, on the surface of a half-space, whence it follows that $\Phi_s \equiv \Phi_s(\pi) = 0$ and $\Phi_a \equiv \Phi_a(\pi) = \pi$. The functions $\Phi_s(\theta)$ and $\Phi_s(\theta)$ then acquire the forms $\Phi_s(\theta) \equiv 0$ $\Phi_a(\theta) = (\pi + \theta)/2$ in the given approximation. Using these expressions and replacing the integration with respect to x in Eq. (8) by integration with respect to h, we can write the moduli of the reflection and transmission coefficients as $x_0 \to \infty$, which are the quantities actually determined in experiment, in the form

$$\begin{aligned} |R| &= |\sin[\delta/\mathrm{tg}(\theta/2) - (\pi-\theta)/4]|, \\ |T| &= |\cos[\delta/\mathrm{tg}(\theta/2) - (\pi-\theta)/4]|, \end{aligned} \tag{9}$$

where $\delta = (1/2)\int_0^{\infty} [k_a(h) - k_s(h)]\,dh$ is a dimensionless constant that depends on the elastic properties of the wedge material.

The subsequent analysis entails the calculation of for which it is necessary to know the complete functions $k_s(h)$ and $k_a(h)$. Inasmuch as analytical equations do not exist for $k_s(h)$ and $k_a(h)$, it is convenient in the calculations to use the approximations of the corresponding dispersion relations obtained by numerical methods.[21] We introduce the approximation in such a way that the functions $k_a(h)$ and $k_s(h)$ will tend to the functions for flexural and longitudinal plate waves, respectively as $h \to 0$ and to the dispersion relation for a Rayleigh wave, as $h \to \infty$. Proceeding from this condition, we write the approximating functions in the form

$$\begin{aligned} \xi_s &= \Omega\left[\frac{c_t}{c_p} - \frac{c_t}{c_R}\right][1 + (A\Omega)^4]^{-1} + \Omega\frac{c_t}{c_R}, \\ \xi_a &= \Omega^{1/2}\left(\frac{12}{\pi^2}\right)^{1/4}\left(\frac{c_t}{c_p}\right)^{1/2}[1 + (B\Omega)^4]^{-1} + \Omega\frac{c_t}{c_R}. \end{aligned} \tag{10}$$

Here $\xi_{s,a} = k_{s,a}h/\pi$, $\Omega = \omega h/\pi c_t$, $c_p = 2c_t(1 - c_t^2/c_l^2)^{1/2}$ is the so-called "plate" velocity, to which the velocity of the symmetric mode tends as $h \to 0$, c_l and c_t are the velocities of longitudinal and shear bulk waves, and A and B are dimensionless approximation constants. For Duralumin ($\nu = 0.35$), which has been used in experiments,[1,2] $A \simeq 0.94$, $B \simeq 1.95$, and the calculation of δ in accordance with Eqs. (10) gives $\delta \simeq 2.75$.

Figure 2 shows |R| and |T| as a function of the angle θ, calculated according to Eqs. (9) for $\delta = 2.75$. Also shown are the experimental values of |R| and |T| obtained for Duralumin samples[1,2] (it is important to note that the measurements reported in Refs. 1 and 2 are the

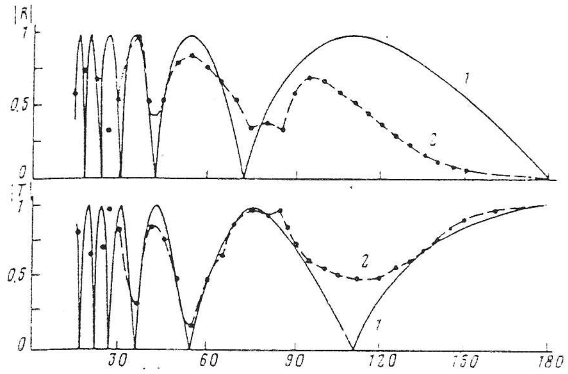

FIG. 2. Moduli of the Rayleigh-wave reflection coefficient $|R|$ and transmission coefficient $|T|$ vs wedge angle θ. 1) Theoretical; 2) experimental.[1,2]

most detailed of any published to date). It is evident from Fig. 2 that the theoretical curves, on the whole, are in fairly good agreement with the experimental data. In particular, they correctly describe the experimentally observed decrease in the oscillation period with a decrease in the wedge angle θ, and the correspondence of the maxima of the reflection coefficient with the minima of the transmission coefficient and vice versa. Good qualitative agreement is also observed in the case of large angles θ, where the given theory is patently invalid, specifically because of its disregard for the appreciable scattering into bulk waves that occurs for such angles. The presence of this kind of scattering is manifested in the fact that the quantity $|R|^2 + |T|^2$ calculated according to the experimental data[1,2] (see Fig. 2) becomes smaller than unity. On the other hand, according to Eqs. (8) and (9), $|R|^2 + |T|^2 = 1$ always. In the range of ultimately small angles θ, the experimental points given in Refs. 1 and 2 are clearly inadequate for the quantitative comparison of the theory with experiment. The measurements described in these papers were performed on individual wedges whose angle θ was incremented in 3.5° steps in the small-angle range. But the periods of the oscillations of the theoretical curves are already commensurate with and even smaller than the measurement step for $\theta < 30°$.

We now discuss the case of oblique incidence of a Rayleigh wave on the edge of a wedge at an angle α (see Fig. 1). Now both the symmetric and the antisymmetric mode experience refraction as the edge is approached, and the nature of the refraction for these modes differs owing to the difference in the laws governing the variation of their velocities near the edge. Clearly, the symmetric mode will be incident on the edge of the wedge at a more oblique angle (curve 1 in Fig. 1), and the antisymmetric mode will arrive practically at a right angle (curve 2 in Fig. 1) due to the tendency of its velocity to zero near the edge. Refraction can be neglected for small angles of incidence α. It then follows from simple geometrical considerations that Eqs. (9) can once again be used if the quantity δ is replaced by $\delta/\cos\alpha$ in them. Consequently, even for a fixed wedge angle, oscillations of $|R|$ and $|T|$ can occur with a variation of the angle of incidence α. This effect has not been described before in the literature.

With an increase in α, the refraction difference in the paths traversed by the symmetric and antisymmetric modes increases, and it must be taken into account (δ depends on α in this case). Obviously, the refraction-induced increase in the acoustic path for the symmetric mode and its decrease for the antisymmetric mode will have the effect of decreasing the phase difference between them and thus decreasing the number of oscillations of $|R|$ and $|T|$ as a function of the angle α.

Consequently, the theory developed in the present study of the reflection and transmission of Rayleigh waves in a wedge agrees quite well with the published experimental data and makes it possible to predict new effects, which can be tested experimentally.

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