PROPAGATION OF LOCALISED FLEXURAL VIBRATIONS ALONG PLATE EDGES DESCRIBED BY A POWER LAW

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1. INTRODUCTION

Localised flexural vibrations propagating along sharp edges of elastic wedge-like structures are characterised by low propagation velocities (generally much lower than that of Rayleigh waves), and their elastic energy is concentrated in the area of about one wavelength from the edge. Such localised vibrations, also known as wedge acoustic waves, have been investigated in a number of papers (see, e.g. [1-14]) with regard to their possible applications to acoustic non-destructive testing of special engineering constructions and for better understanding vibrations of propellers, turbine blades and some civil engineering constructions. They may be important also for the explanation of many as yet poorly understood phenomena in related fields of structural dynamics, physics, environmental acoustics and may result in many useful practical applications. In particular, it is expected that these waves may play an important role in the dynamics of wedge-shaped off-shore structures (such as piers, dams, wave-breakers, etc.), and in the formation of vibration patterns and resonance frequencies of propellers, turbine blades, disks, cutting tools and airfoils. They may be responsible for specific mechanisms of helicopter noise, wind turbine noise and ship propeller noise. Promising mechanical engineering applications of wedge elastic waves may include measurements of cutting edge sharpness, environmentally friendly water pumps and domestic ventilators utilising wave-generated flows. Another possible application earlier suggested by one of the present authors [10] may be the use of wedge waves for in-water propulsion of ships and submarines, the main principle of which being similar to that used in nature by fish of the ray family.

Initially these localised flexural waves have been investigated for wedges in contact with vacuum [1- 6]. Later on, the existence of localised flexural elastic waves on the edges of wedge-like immersed structures has been predicted [7]. This was followed by the experimental investigations of wedge waves in immersed structures which considered samples made of different materials and having different values of wedge apex angle [8,9]. Recently, finite element calculations have been carried out [10] for several types of elastic wedges with the of apex angle varying in the range from 20 to 90 degrees. Also, the analytical theory based on geometrical-acoustics approach has been developed for the same range of wedge apex angle [11]. In the paper [12] dealing with finite element calculations of the velocities and amplitudes of wedge waves, among other results, calculations have been carried out of the velocities of waves propagating along the edge of a cylindrical wedge-like structure bounded by a circular cylinder and a conical cavity. In the paper [13] different cylindrical and conical wedge-like structures have been investigated using geometrical acoustics approach.

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In the present paper, we report some new analytical results in the theory of localised flexural vibrations propagating along edges of free and immersed structures of "non-linear" shape (see Fig.1). The results are described with the emphasis on methodological aspects of using geometrical-acoustics approach for developing the theory. Some of these results have been recently delivered in the review paper [14]. Using the geometrical acoustics technique, the velocities of localised wedge modes are calculated for edges with a cross section described by a power-law relationship between the local thickness d and the distance from the edge x: $d = \varepsilon x^m$, where m is a positive rational number. It is shown that deviations of a wedge shape from the linear geometry ($m = 1$ and $\varepsilon = \Theta$, where Θ is the wedge apex angle) result in frequency dispersion of wedge modes. It is also shown that for $m \geq 2$ - in free wedges, and for $m \geq 5/3$ - in immersed wedges the velocities of localised waves tend to zero, unless there is a truncation on the wedge tip. In other words, localised waves do not propagate along free or immersed structures with $m \geq 2$ and $m \geq 5/3$ respectively. This phenomenon can be explained by trapping of flexural wave energy near the curved edges considered which represent acoustic 'black holes' for flexural waves. The discussion is given on possible use of these phenomena for vibro-isolation.

2. GEOMETRICAL-ACOUSTICS APPROACH

2.1 Linear Wedges in Vacuum

The approximate analytical theory of localised elastic waves in solid wedges can be based on the geometrical acoustics approach considering a slender wedge as a plate with a local variable thickness $d(x)$, where x is the distance from the wedge tip measured in the middle plane. In the case of "linear" wedge $d(x) = x\Theta$, where Θ is the wedge apex angle.

The velocities c of the localised modes propagating along the wedge tip (in y-direction) can be calculated from the following Bohr - Sommerfeld type equation [4-6]:

$$\int_0^{x_t} \left[k^2(x) - \beta^2\right]^{1/2} dx = \pi n . \tag{1}$$

Here $k(x)$ is a local wavenumber of a quasi-plane plate flexural wave (as a function of the distance x from the wedge tip), $\beta = \omega/c$ is yet unknown wavenumber of a localised wedge mode, $n = 1,2,3, \ldots$ is the mode number, and x_t is the so called ray turning point being determined from the equation $k^2(x) - \beta^2 = 0$.

In the case of linear wedge in vacuum

$$k(x) = 12^{1/4}\, k_p^{1/2} (\Theta x)^{-1/2}, \tag{2}$$

where $k_p = \omega/c_p$ is the wavenumber of a symmetrical plate wave, $c_p = 2c_t\,(1-c_t^2/c_l^2)^{1/2}$ is its phase velocity, and c_l and c_t are longitudinal and shear velocities in wedge material. Hence $x_t = 2\sqrt{3}k_p/\Theta\beta^2$, and the solution of eqn (1) yields the extremely simple analytical expression for wedge wave velocities [4-6]:

$$c = c_p n \Theta/\sqrt{3}. \tag{3}$$

The expression (3) agrees well with the other theoretical calculations [1-3] and with the experimental results [1]. Note that, although the geometrical acoustics approach is not valid for the lowest order wedge mode ($n = 1$) [5], in practice it provides quite accurate results for wedge wave velocities in this case as well.

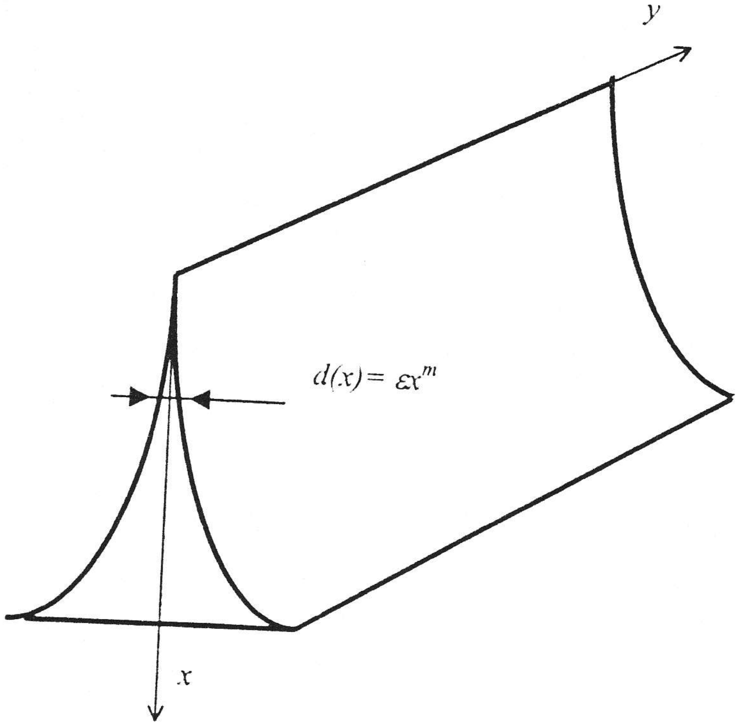

Fig. 1. Elastic wedge-like structure of non-linear shape

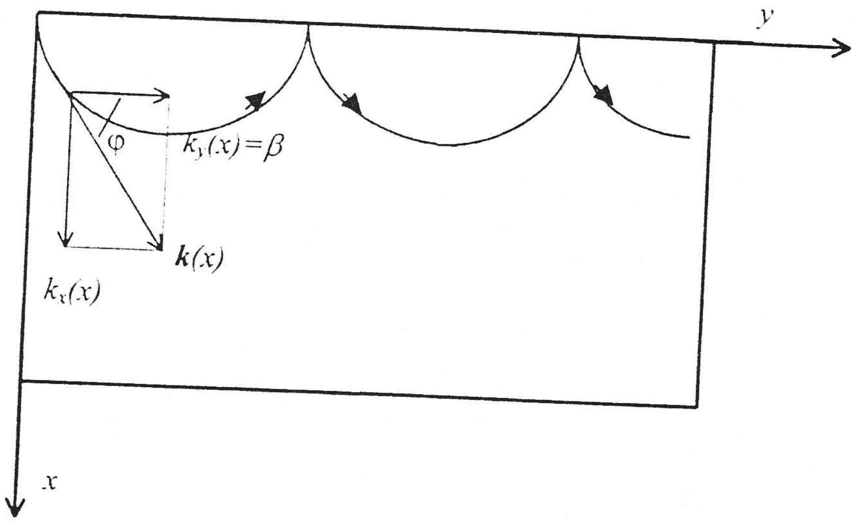

Fig. 2. Localised wedge modes as successively reflecting flexural waves.

For the purpose of this paper, it is convenient to consider changing the integration over x in eqn (1) to the integration over the angle φ between the vector $\mathbf{k}(x)$ and its horizontal projection $k_y(x) = \beta$ [15]. Using the obvious relationships (see Fig.2)

$$k(x)\cos(\varphi) = \beta, \tag{4}$$

$$k_x(x) = \beta\tan(\varphi) \tag{5}$$

and expressing x from eqn (4) as function of the angle φ, with β being a parameter: $x = f(\varphi,\beta)$, one can rewrite eqn (1) in the form

$$\beta\int_{\pi/2}^{0}\tan(\varphi)f'(\varphi,\beta)d\varphi = \pi n, \tag{6}$$

where $f'(\varphi,\beta) = \partial f(\varphi,\beta)/\partial\varphi$. This form of the velocity equation is useful if the dependence of $k(x)$ is such that there exists an explicit analytical relationship $x = f(\varphi,\beta)$. This is obviously the case for the above mentioned wedge of linear geometry. It is easy to check that for such a linear wedge eqn (6) combined with eqns (4) and (5) gives the same result (3) that follows from eqn (1). As we will see below, for wedges of non-linear cross-section, the use of eqn (6) brings more advantages, essentially simplifying the derivation.

2.2 Immersed Linear Wedges

To apply the geometrical acoustics approach for calculating the velocities of localised modes in a wedge immersed in liquid one has to make use of the expression for a plate wave local wavenumber $k(x)$ which takes into account the effect of liquid loading [7,11]. The starting point to derive $k(x)$ for this case is the well known dispersion equation for the lowest order flexural mode in an immersed plate.

For shortness, we dwell in this paper only on the case $\rho_f/\rho_s \approx 1$ typical for light solid materials in water and limit our analysis by a subsonic regime of wave propagation ($k > \omega/c_f$), where ρ_s and ρ_f are respectively the mass densities of solid and liquid, and c_f is the velocity of sound in liquid. For the sake of simplicity, we impose even a more severe restriction on wave velocities considering very slow propagating plate flexural modes ($k >> \omega/c_f$) and using the approximation of incompressible liquid. Then, for $kd << 1$ typical for thin plates, the wavenumber $k(x)$ in the case of linear wedge $d = d(x) = x\Theta$ has the form

$$k(x) = \left[\sqrt{6}\,\frac{c_l}{c_t}\,\frac{1}{\sqrt{c_l^2 - c_t^2}}\,\frac{\sqrt{\rho_f}}{\sqrt{\rho_s}}\,\frac{\omega}{(x\theta)^{3/2}}\right]^{2/5}. \tag{7}$$

Substituting eqn (7) into (1) or (6) and performing some simple transformations, one can derive the following analytical expression for wedge wave velocities c [11]:

$$c = c_t A^{-5/2} D^{-3/2} (\pi n)^{3/2} \Theta^{3/2}, \tag{8}$$

where $A = 6^{1/5}(\rho_f/\rho_s)^{1/5}(1 - c_t^2/c_l^2)^{-1/5} = 6^{1/5}(\rho_f/\rho_s)^{1/5}\,[2(1-\sigma)]^{1/5}$ is a nondimensional parameter which depends on the relation between the mass densities ρ_f/ρ_s and on the Poisson ratio σ, and $D = \int_0^1 (x^{-6/5} - 1)^{1/2} dx = 2.102$. Comparison of eqns (3) and (8) shows that the effect of liquid loading results in significant decrease of wedge wave velocities in comparison with their values in vacuum, especially for small angles Θ.

3. WAVES IN WEDGES DESCRIBED BY A POWER LAW

In what follows we generalise the above theory by introducing a power law relationship between the local thickness d and the distance from the tip x: $d = \varepsilon x^m$, where m is any positive rational number. Then, for free and immersed wedges, eqns (2) and (7) should be replaced by the following expressions respectively:

$$k(x) = 12^{1/4} k_p^{1/2} (\varepsilon x^m)^{-1/2}, \qquad (9)$$

$$k(x) = \left[\sqrt{6} \frac{c_l}{c_t} \frac{1}{\sqrt{c_l^2 - c_t^2}} \frac{\sqrt{\rho_f}}{\sqrt{\rho_s}} \frac{\omega}{(\varepsilon x^m)^{3/2}} \right]^{2/5}. \qquad (10)$$

Substituting eqns (9) and (10) into eqn (6) and using (4), (5), one can obtain the general relationships for wedge wave velocities of localised elastic modes propagating in non-linear free and immersed wedges respectively:

$$c = \frac{\omega^{(m-1)/(m-2)} P^{1/(m-2)} F_m^{m/(m-2)}}{(2\pi m \varepsilon^{1/m})^{m/(m-2)}}, \qquad (11)$$

$$c = \frac{\omega^{3(m-1)/(3m-5)} B^{2/(3m-5)} G_m^{3m/(3m-5)}}{(2\pi m \varepsilon^{1/m})^{3m/(3m-5)}}. \qquad (12)$$

Here

$$P = \frac{12^{1/2}}{c_p} = \sqrt{3} \frac{c_l}{c_t} \frac{1}{\sqrt{c_l^2 - c_t^2}}, \qquad (13)$$

$$B = \frac{24^{1/2}}{c_p} \frac{\sqrt{\rho_f}}{\sqrt{\rho_s}} = \sqrt{6} \frac{c_l}{c_t} \frac{1}{\sqrt{c_l^2 - c_t^2}} \frac{\sqrt{\rho_f}}{\sqrt{\rho_s}}, \qquad (14)$$

and

$$F_m = (4/m) \int_0^{\pi/2} \sin^2 \varphi \cos^{(2-2m)/m} \varphi d\varphi, \qquad (15)$$

$$G_m = (10/3m) \int_0^{\pi/2} \sin^2 \varphi \cos^{(5-6m)/3m} \varphi d\varphi. \qquad (16)$$

It is clearly seen from eqns (11) and (12) that deviation of a wedge shape from linear geometry ($m = 1$ and $\varepsilon = \Theta$) results in dispersion of wedge wave velocities. For linear wedges, as expected, the velocity c is independent of ω and reduces to the earlier derived expressions (3) and (8).

It is interesting to notice that for $m \geq 2$ (in the case of free wedges) and for $m \geq 5/3$ (in the case of immersed wedges) the integrals in (15) and (16) become divergent. The analysis shows that the corresponding wave velocities determined by eqns (11) and (12) tend to zero for $m \to 2$ and $m \to 5/3$. This is in agreement with the earlier investigation of wedge modes in a quadratic wedge ($m = 2$) in contact with vacuum [16]. In the latter case the velocities of all wedge modes are equal to zero, unless there is a truncation of the wedge tip. For $m > 2$ and $m > 5/3$ not only the integrals in (15) and (16), but also the expressions (11) and (12) diverge, indicating that they are not valid for these cases. Nevertheless, it is quite reasonable to assume that for $m > 2$ and $m > 5/3$ the wedge wave velocities are zero as well. The reason for this will be clear from the analysis in the next section.

4. ELASTIC WEDGES AS ACOUSTIC BLACK HOLES

Let us compare the above mentioned conclusions for localised wedge modes with the results for flexural wave propagation in normal direction towards the wedge tip. In the case of quadratic wedge in vacuum ($m = 2$) such analysis has been first performed by Mironov [17] who noticed that in this case the incident wave requires indefinite time to reach the tip and thus never reflects back. He has suggested to use such wedges as absorbers of vibration energy which are equivalent to astronomical "black holes". As we will see below, this property relates not only to quadratic wedges in vacuum, but to a wider class of wedges in contact with both vacuum and liquid if their profiles satisfy the conditions $m \geq 2$ (for free wedges) and $m \geq 5/3$ (for immersed wedges), i.e. the above mentioned conditions associated with zero velocities of wedge modes in free and immersed wedges.

Indeed, let us consider propagation of a plane flexural wave in normal direction towards the tip of a wedge described by a power law and calculate the integrated wave phase Φ resulting from the wave propagation from a certain point x to the wedge tip ($x = 0$):

$$\Phi = \int_0^x k(x)dx \,. \tag{17}$$

Substituting the corresponding expressions for $k(x)$ in the cases of free and immersed wedges (eqns (9) and (10) respectively) into eqn (17), one can prove that the integral in eqn (17) diverges for $m \geq 2$ and $m \geq 5/3$ for free and immersed wedges respectively. This means that the phase Φ becomes infinite under these circumstances and the wave never reaches the edge. This gives a very simple interpretation to the wedge wave velocities being equal to zero for such wedge profiles. Indeed, wedge waves in geometrical acoustics interpretation are plate flexural wave propagating from the ray turning point to the wedge tip and reflecting back to the turning point. When the wave propagating along the ray trajectory approaches the edge, it falls on the wedge tip almost at the right angle. Then, if the power-law profile of free or immersed wedges is characterised by $m \geq 2$ and $m \geq 5/3$ respectively, the wave is being trapped near the edge in the same way, as it was described in the case above. In the light of this, it does not propagate along the edge, and its velocity is equal to zero.

In practice, the use of non-linear wedges as vibration energy absorbers is limited by technological difficulties with manufacturing of perfect non-linear profiles. Real wedges always have truncated edges. And this strongly affects their performance as vibration absorbers. For ideal non-linear wedges, it follows from eqn (17) that even an infinitely small material damping characterised by imaginary part of $k(x)$ is sufficient for all the wave energy to be absorbed. However, for truncated wedges the lower integration limit in eqn (17) must be changed from 0 to a certain value x_0 describing the truncation. In the case of normal incidence of a quasi-plane flexural wave this results in drastic reduction of wedge absorbing properties that reveals in the increase of the associated reflection coefficient from zero to $R_0 = \exp(-2\int_{x_0}^{x} \mathrm{Im}\, k(x)dx)$ [17]. According to the calculations made in [17] for a quadratic wedge in vacuum having realistic values of truncation x_0 and material quality factor Q, the corresponding value of R_0 was about 0.7. This means that relatively little absorption actually took place.

The use of localised wedge modes can significantly improve the situation. Since, according to the geometrical acoustics approach [4-6], wedge waves can be interpreted as quasi-plane flexural waves propagating along a curvilinear trajectory and experiencing multiple reflections from the free edge and ray turning points (see Fig. 2), the resulting wave reflection coefficient R taking into account the integrated wave attenuation can be approximated as

$$R = (R_0)^N , \tag{19}$$

where N is the number of edge reflections on the distance from the source to the point of observation. The realistic values of N can be up to 10. Therefore, the use of localised vibration modes in the example described above may result in the reflection coefficient R being as low as $0.7^{10} = 0.028$. The absorbing properties of non-linear wedges can be even more enhanced by covering their surfaces with highly absorbing materials. This problem, however, requires a special consideration that goes beyond the scope of this paper.

5. CONCLUSIONS

Using the geometrical acoustics technique, the analytical expressions for phase velocities of localised wedge modes have been derived for edges with a cross section described by a power-law relationship between the local thickness d and the distance from the edge x: $d = \varepsilon x^m$, where m is a positive rational number.

It was shown that deviations of a wedge shape from the linear geometry ($m = 1$ and $\varepsilon = \Theta$, where Θ is the wedge apex angle) result in frequency dispersion of wedge modes.

For profiles with $m \geq 2$ - in the case of free wedges, and with $m \geq 5/3$ - in the case of immersed wedges the velocities of localised waves tend to zero, unless there is a truncation on the wedge tip. This implies that localised waves do not propagate along free or immersed structures with $m \geq 2$ and $m \geq 5/3$ respectively. This phenomenon can be explained by trapping of flexural wave energy near the curved edges considered which represent acoustic 'black holes' for flexural waves.

The discussion was given on possible use of wedge waves propagating along edges of free and immersed non-linear wedges with $m \geq 2$ and $m \geq 5/3$ respectively for isolation of flexural vibrations.

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