Review of "Proof in Mathematics Education: Research, Learning and Teaching" (David Reid and Christine Knipping)

Matthew Inglis
Mathematics Education Centre, Loughborough University

In the last few years edited books discussing issues that surround the teaching and learning of mathematical proof have been published at an alarming rate (e.g. Boero 2007; Stylianou et al. 2009; Hanna and De Villiers in press). This is not surprising in view of the growing research literature on proof and proving in educational settings. While most mathematicians and mathematics educators consider proof to be a defining characteristic of mathematics, proof is known to be difficult for students to master, and epistemologically controversial amongst both philosophers and research mathematicians. As a consequence there are complex and unsettled issues surrounding how proof can and should be taught at all levels of education. Current research efforts include investigating students' beliefs about the nature of proof, different approaches to reading and validating proofs, and instructional designs that can facilitate a deeper appreciation of proof in the classroom.

Reid and Knipping (R&K) have made an interesting and unusual contribution to this growing literature. Rather than compile another edited volume bringing together new empirical or theoretical analyses, or write a short chapter reviewing the literature on proof as part of a larger handbook, R&K have attempted to produce a lengthy, coherent synthesis of this entire subsection of the research literature. In a sense their book can be seen as an advanced level textbook, suitable for beginning doctoral or masters level students who may be writing dissertations related to proof.

Compared to other disciplines, such introductions to the literature are surprisingly rare in mathematics education. As R&K point out in their introduction, getting to grips with a large body of literature is a difficult and daunting task (especially a literature that is as fragmented and contradictory as the educational literature on proof). So, the existence of a book-length synthesis of a subsection of the literature is extremely valuable, but what characteristics should it have? Several immediately spring to mind: such books should faithfully present other authors' views, have comprehensive coverage of the literature, and should present debates and controversies in a balanced but critical fashion. In the remainder of this review I discuss R&K's contribution with reference to these three criteria.

With respect to the first criterion, in my view R&K accurately and carefully present the positions of the authors they discuss (although I did not always agree with the arguments they developed based on these summaries, of which more below). With respect to the second, the book's coverage is broad, falling into three main sections. The first concerns itself with a discussion of the nature of proof, considering historical and epistemological perspectives in turn. It concludes with an interesting and helpful discussion of the perspectives adopted by different research traditions. The second section of the book delves deeper into the literature by synthesising empirical findings regarding several "important research foci" (within this category R&K include amongst other things the role of proof, the difference between proof and argumentation, approaches to teaching proof, and schemes for the classification of arguments). The third section concentrates on a detailed discussion of the processes students adopt when proving. Notably, this section is somewhat different to the others as it is primarily based on R&K's own research work, rather than a synthesis of the literature. The book concludes with a discussion of teaching implications, and suggestions for future research. What then, of the second criterion? Other than some minor cavils[1], I think the book covers the literature well. I would be happy to recommend it to an incoming PhD student to use as an introduction to the field.

What of the remaining criterion? Do R&K present their synthesis in a reliable and balanced fashion? In my view they do. Although there were issues on which I disagreed with R&K (discussed below), in general the presentation seemed to me to be a constructive and helpful representation of the current state of the literature.

I did find some exceptions to this rule; places where I felt R&K's treatment was unduly simplistic or offering a partial summary. The clearest came early on in the book when R&K presented their take on various philosophical accounts of mathematics. R&K suggest that there are four basic philosophies: *a priorist*, infallibilist, quasi-empiricist and social-constructivist. The first, they say, is characterised by a belief in platonic objects (real mathematical objects that exist independently of human cognition) that are accurately and completely described by axiomatic structures, coupled with a belief that deductive inferences from those axioms are truth preserving. R&K tell us that since the discovery of non-Euclidean geometries "most mathematicians" have abandoned this view. In contrast, an infallibilist, according to R&K, accepts the truth-preserving nature of deductive inferences, but rejects the suggestion that mathematical objects exist in some platonic sense. Quasi-empiricists, in contrast, agree with the infallibilists about the non-existence of mathematical objects, but reject the standard view of the relationship between theorems and axioms: deductive reasoning does not transmit truth from axioms to theorems, rather, following Lakatos (1976), falsity is transmitted from theorems to axioms. R&K characterise their final position, adopted by social constructivists, as being associated with a rejection of the unique status of the deductive method. As deduction is just a social construction, on this view what counts as a valid argument will vary from community to community and culture to culture.

R&K suggest that these four positions constitute a single dimension onto which researchers' beliefs can be mapped. But it is extremely hard to see how many respectable modern philosophical positions can be coherently categorised using these labels. As a concrete example, consider Brown's (2008) rather attractive version of platonism. Brown accepts that mathematical objects exist independently of humans in some platonic world (like R&K's *a priorists*), but he rejects the assumption that they are perfectly described by axioms (unlike R&K's *a priorists*). Instead he argues that axioms are hypotheses (like R&K's infallibilists) that attempt to describe the platonic world, but he does not believe that deductive reasoning obtains mathematical truths (unlike R&K's infallibilists). Instead Brown suggests that the goal of deductive reasoning is to test axioms (i.e. hypotheses about platonic objects) by deriving consequences from them and assessing them against one's intuitions of the platonic world (a method essentially identical to that favoured by R&K's quasi-empiricists). Indeed, Brown would go further, and suggest that Lakatos may himself have been a platonist. Finally, Brown, like R&K's social-constructivists, agrees that deductive reasoning is socially constructed, and holds out hope that one day better methods of revealing truths about mathematical objects may be found (in particular, methods that don't fall foul of the incompleteness theorems). If it is possible to adopt a coherent philosophical position by taking some aspects of each of R&K's four positions and rejecting others, it seems hard to justify placing them on a single dimension.[2]

A second point of disagreement was with R&K's argument, presented throughout the book, that the multiplicity of research perspectives which exist in the literature should be considered a strength, because "a diversity of perspectives offers opportunities to make sense of phenomena that might be seen in a limited way from a single perspective" (p. xiv). Unfortunately the authors failed to convince me of this argument, and indeed I found some of their (very competently arranged) summaries of the literature to be rather depressing. Take, for example, their review of argument classification schemes: R&K point out that what they would call an "empirical-perceptual" argument would be classified by Bell as "extrapolation", by the preformalists as "experimental", by Balacheff as "naive empiricism" and by van Dormolen as an argument of the "first level". Similarly they explain that, depending on which perspective is adopted, the same argument might be classified as "symbolic", "a complete deductive explanation", "formal scientific", "mathematical proof", or of the "third level". Are these numerous labels for the same (or at least extremely similar) phenomena really a sign of the strength of the literature? Or are they a sign that as a discipline we have failed to build a cumulative body of knowledge?

While I was reading these sections I was reminded of comments made by two influential researchers that have stuck with me. First, John Mason's lament that mathematics education consists of "a plethora of distinctions, sometimes several labels for at best subtly distinct distinctions, and sometimes the same label is used for different distinctions" (2009, 11). Second, Tommy Dreyfus's suggestion that mathematics educators "tend to invent theories, or at least theoretical ideas, at a pace faster than we produce data to possibly refute our theories" (2006, 78). R&K's summaries of the various argumentation classification schemes provide ample evidence for those who might doubt the wisdom of Mason and Dreyfus's comments. So, unlike R&K, I do not see this proliferation of near identical classification systems as a strength, but rather as an indication of researchers' unwillingness or inability to build on each others' ideas.

Nevertheless, across the book the points where I disagreed with R&K's arguments were comfortably outnumbered by the points where I found myself appreciating the thrust and sophistication of their discussion. But although I did appreciate the efforts of the authors in producing their book, I felt that they have been rather let down by their publishers. An example, the irony of which some may enjoy, is the consistent misspelling of "rigorous" throughout the text (as "rigourous"). If publishers aspire to be more than mere printers, then their copy-editing procedures really should be capable of spotting errors like this.

In summary, R&K's book is an extremely valuable addition to the growing mathematics education literature on proof. There is no doubt that it will be of considerable use to researchers who wish to familiarise themselves with the research literature. But the book's influence could be wider still: for a group of researchers that concentrate on pedagogy, mathematics educators devote surprisingly little effort towards developing pedagogical materials for the teaching of research-level mathematics education. Hopefully, R&K's book will provide the impetus for other researchers to write similar syntheses of further areas of the mathematics education literature.

Notes

1. I was disappointed to see little or no discussion of the small but growing literature on the reading of (rather than the construction of) mathematical proofs. Reading proofs seems to be a critical (but under-researched) aspect of how students engage with higher-level mathematics. It also seemed strange to omit discussion of influential papers by Weber (2001, 2008) and Stylianides (2007).

2. A further problem for R&K's characterisation comes from Balaguer's surprisingly compelling argument that platonism and "fictionalism" (similar to R&K's description of social-constructivism) are in fact the same philosophy (Balaguer 2008).

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