ON INTUITIONISTIC FUZZY NEGATIONS AND INTUITIONISTIC FUZZY EXTENDED MODAL OPERATORS. Part 2.

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Abstract—**Some relations between intuitionistic fuzzy negations and intuitionistic fuzzy extended modal operations $F_{\alpha,\beta}$ and $G_{\alpha,\beta}$ are studied.**

I. On some previous results

The concept of the Intuitionistic Fuzzy Set (IFS, see [1]) was introduced in 1983 as an extension of Zadeh's fuzzy set. All operations, defined over fuzzy sets were transformed for the IFS case. One of them - operartion "negation" now there is 27 different forms (see [2]). In [1] the relations between the "classical" negation and the two standard modal operators "necessity" and "possibility" are given. Here, we shall study the relations between the intuitionistic fuzzy negations and the intuitionistic fuzzy extended modal operations $F_{\alpha,\beta}$ and $G_{\alpha,\beta}$.

In some definitions we shall use functions sg and $\overline{\text{sg}}$:

$$\text{sg}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases},$$

$$\overline{\text{sg}}(x) = \begin{cases} 0 & \text{if } x > 0 \\ 1 & \text{if } x \leq 0 \end{cases}$$

For any two IFSs A and B the following relations are valid:

$$A \subset B \text{ iff } (\forall x \in E)(\mu_A(x) \leq \mu_B(x)\nu_A(x) \geq \nu_B(x)),$$

$$A \supset B \text{ iff } B \subset A,$$

$$A = B \text{ iff } (\forall x \in E)(\mu_A(x) = \mu_B(x) \& \nu_A(x) = \nu_B(x)).$$

Let A be a fixed IFS. In [1] definitions of standard modal operators are given:

$$\Box A = \{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle | x \in E\},$$

$$\Diamond A = \{\langle x, 1 - \nu_A(x), \nu_A(x)\rangle | x \in E\}.$$

The first extended modal operator is

$$D_\alpha(A) = \{\langle x, \mu_A(x) + \alpha.\pi_A(x),$$
$$\nu_A(x) + (1-\alpha).\pi_A(x)\rangle | x \in E\},$$

where $\alpha \in [0, 1]$. It is extended to

$$F_{\alpha,\beta}(A) = \{\langle x, \mu_A(x) + \alpha.\pi_A(x), \nu_A(x) + \beta.\pi_A(x)\rangle | x \in E\},$$

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where $\alpha, \beta \in [0, 1]$ and $\alpha + \beta \leq 1$. Another non-standard modal operator is

$$G_{\alpha,\beta}(A) = \{\langle x, \alpha.\mu_A(x), \beta.\nu_A(x)\rangle | x \in E\},$$

where $\alpha, \beta \in [0, 1]$.

Obviously,

$$\Box A = D_0(A) = F_{0,1}(A),$$

$$\Diamond A = D_1(A) = F_{1,0}(A),$$

$$D_\alpha(A) = F_{\alpha,1-\alpha}(A).$$

In [2], [3], [4], [5], [6] the following 27 different negations are described.

$$\neg_1 A = \{\langle \nu_A(x), \mu_A(x)\rangle | x \in E\},$$

$$\neg_2 A = \{\langle \overline{\text{sg}}(\mu_A(x)), \text{sg}(\mu_A(x))\rangle | x \in E\},$$

$$\neg_3 A = \{\langle \nu_A(x), \mu_A(x).\nu_A(x) + \mu_A(x)^2\rangle | x \in E\},$$

$$\neg_4 A = \{\langle \nu_A(x), 1 - \nu_A(x)\rangle | x \in E\},$$

$$\neg_5 A = \{\langle \overline{\text{sg}}(1 - \nu_A(x)), \text{sg}(1 - \nu_A(x))\rangle\rangle | x \in E\},$$

$$\neg_6 A = \{\langle \overline{\text{sg}}(1 - \nu_A(x)), \text{sg}(\mu_A(x))\rangle\rangle | x \in E\},$$

$$\neg_7 A = \{\langle \overline{\text{sg}}(1 - \nu_A(x)), \mu_A(x)\rangle\rangle | x \in E\},$$

$$\neg_8 A = \{\langle 1 - \mu_A(x), \mu_A(x)\rangle | x \in E\},$$

$$\neg_9 A = \{\langle \overline{\text{sg}}(\mu_A(x)), \mu_A(x)\rangle | x \in E\},$$

$$\neg_{10} A = \{\langle \overline{\text{sg}}(1 - \nu_A(x)), 1 - \nu_A(x)\rangle\rangle | x \in E\},$$

$$\neg_{11} A = \{\langle \text{sg}(\nu_A(x)), \overline{\text{sg}}(\nu_A(x))\rangle\rangle | x \in E\},$$

$$\neg_{12} A = \{\langle \nu_A(x).(\mu_A(x) + \nu_A(x)),$$
$$\mu_A(x).(\mu_A(x) + \nu_A(x)^2)\rangle | x \in E\},$$

$$\neg_{13} A = \{\langle 1 - \overline{\text{sg}}(1 - \mu_A(x)), \overline{\text{sg}}(1 - \mu_A(x))\rangle\rangle | x \in E\},$$

$$\neg_{14} A = \{\langle \text{sg}(\nu_A(x)), \overline{\text{sg}}(1 - \mu_A(x))\rangle\rangle | x \in E\},$$

$$\neg_{15} A = \{\langle \overline{\text{sg}}(1 - \nu_A(x)), \overline{\text{sg}}(1 - \mu_A(x))\rangle\rangle | x \in E\},$$

$$\neg_{16} A = \{\langle \overline{\text{sg}}(\mu_A(x)), \overline{\text{sg}}(1 - \mu_A(x))\rangle\rangle | x \in E\},$$

$$\neg_{17} A = \{\langle \overline{\text{sg}}(1 - \nu_A(x)), \overline{\text{sg}}(\nu_A(x))\rangle\rangle | x \in E\},$$

$$\neg_{18} A = \{\langle x, \nu_A(x).\text{sg}(\mu_A(x)), \mu_A(x).\text{sg}(\nu_A(x))\rangle | x \in E\},$$

$$\neg_{19} A = \{\langle x, \nu_A(x).\text{sg}(\mu_A(x)), 0\rangle | x \in E\},$$

$$\neg_{20} A = \{\langle x, \nu_A(x), 0\rangle | x \in E\},$$

$$\neg_{21} A = \{\langle x, \nu_A(x), \mu_A(x).\nu_A(x) + \mu_A(x)^n\rangle | x \in E\},$$

where real number $n \in [2, \infty)$,

$$\neg_{22} A = \{\langle x, \nu_A(x),$$

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$$\mu_A(x).\nu_A(x) + \overline{\text{sg}}(1 - \mu_A(x))\rangle | x \in E\},$$

$$\neg_{23}A = \{\langle x, (1 - \mu_A(x)).\text{sg}(\mu_A(x)),$$

$$\mu_A(x).\text{sg}(1 - \nu_A(x))\rangle | x \in E\},$$

$$\neg_{24}A = \{\langle x, (1 - \mu_A(x)).\text{sg}(\mu_A(x)), 0\rangle | x \in E\},$$

$$\neg_{25}A = \{\langle x, 1 - \nu_A(x), 0\rangle | x \in E\},$$

$$\neg^{\varepsilon}A = \{\langle x, \min(1, \nu_A(x) + \varepsilon), \max(0, \mu_A(x) - \varepsilon)\rangle | x \in E\},$$

where $\varepsilon \in [0, 1]$,

$$\neg^{\varepsilon,\eta}A = \{\langle x, \min(1, \nu_A(x) + \varepsilon), \max(0, \mu_A(x) - \eta)\rangle | x \in E\},$$

where $0 \leq \varepsilon \leq \eta \leq 1$.

II. Main results

Now, following and extending the idea from [7], [8] we shall prove following assertions.

Theorem 1: For every IFS A and for every $\alpha, \beta \in [0, 1]$ so that $\alpha + \beta \leq 1$, the following properties are valid:
(1) $\neg_1 F_{\alpha,\beta}(A) = F_{\beta,\alpha}(\neg_1 A)$,
(2) $\neg_2 F_{\alpha,\beta}(A) \subset F_{\alpha,\beta}(\neg_2 A)$,
(3) $\neg_4 F_{\alpha,\beta}(A) \supset F_{\alpha,\beta}(\neg_4 A)$,
(4) $\neg_5 F_{\alpha,\beta}(A) \supset F_{\alpha,\beta}(\neg_5 A)$,
(5) $\neg_8 F_{\alpha,\beta}(A) \subset F_{\alpha,\beta}(\neg_8 A)$.
(6) $\neg_{11} F_{\alpha,\beta}(A) \supset F_{\alpha,\beta}(\neg_{11} A)$.

Proof: Let $\alpha, \beta \in [0, 1]$ be given so that $\alpha + \beta \leq 1$, and let A be an IFS. Then we obtain directly that:

$$\neg_1 F_{\alpha,\beta}(A)$$
$$= \neg_1\{\langle x, \mu_A(x) + \alpha.\pi_A(x), \nu_A(x) + \beta.\pi_A(x)\rangle | x \in E\}$$
$$= \{\langle x, \nu_A(x) + \beta.\pi_A(x), \mu_A(x) + \alpha.\pi_A(x)\rangle | x \in E\}$$
$$= F_{\beta,\alpha}(\{\langle x, \nu_A(x), \mu_A(x)\rangle | x \in E\})$$
$$= F_{\beta,\alpha}(\neg_1 A).$$

Therefore equality (1) is valid.

The rest of the assertions can be proved by another manner. Let us prove, for example (5).

Let $\alpha, \beta \in [0, 1]$ be given so that $\alpha + \beta \leq 1$, and let A be an IFS. Then:

$$\neg_8 F_{\alpha,\beta}(A)$$
$$= \neg_8\{\langle x, \mu_A(x) + \alpha.\pi_A(x), \nu_A(x) + \beta.\pi_A(x)\rangle | x \in E\}$$
$$= \{\langle x, 1 - \mu_A(x) - \alpha.\pi_A(x), \mu_A(x) + \alpha.\pi_A(x)\rangle | x \in E\}$$

and

$$F_{\alpha,\beta}(\neg_8 A) = F_{\alpha,\beta}(\{\langle x, 1 - \mu_A(x), \mu_A(x)\rangle | x \in E\})$$
$$= \{\langle x, 1 - \mu_A(x), \mu_A(x)\rangle | x \in E\}.$$

Now, we see easily that

$$1 - \mu_A(x) - (1 - \mu_A(x) - \alpha.\pi_A(x)) = \alpha.\pi_A(x)) \geq 0$$

and

$$\mu_A(x) + \alpha.\pi_A(x) - \mu_A(x) \geq 0.$$

Therefore inclusion (5) is valid.

Theorem 2: For every IFS A and for every $\alpha, \beta \in [0, 1]$ the following properties are valid:
(1) $\neg_1 G_{\alpha,\beta}(A) = G_{\beta,\alpha}(\neg_1 A)$,
(2) $\neg_7 G_{\alpha,\beta}(A) \subset G_{\beta,\alpha}(\neg_7 A)$,
(3) $\neg_{15} G_{\alpha,\beta}(A) \subset G_{\beta,\alpha}(\neg_{15} A)$,
(4) $\neg_{19} G_{\alpha,\beta}(A) \subset G_{\beta,\alpha}(\neg_{19} A)$,
(5) $\neg_{20} G_{\alpha,\beta}(A) = G_{\beta,\alpha}(\neg_{20} A)$,
(6) $\neg_{25} G_{\alpha,\beta}(A) \supset G_{\beta,\alpha}(\neg_{25} A)$.

Theorem 3: For every IFS A and for every $\alpha, \beta \in [0, 1]$ the following properties are valid:
(1) $\neg^{\varepsilon} G_{\alpha,\beta}(A) \supset G_{\beta,\alpha}(\neg^{\varepsilon} A)$, where $0 \leq \varepsilon \leq 1$,
(2) $\neg^{\varepsilon,\eta} G_{\alpha,\beta}(A) \supset G_{\beta,\alpha}(\neg^{\varepsilon,\eta} A)$, where $0 \leq \varepsilon \leq \eta \leq 1$.

Proof: Let $\alpha, \beta \in [0, 1]$ be given so that $\alpha + \beta \leq 1$, let A be an IFS and let ε, η be given so that $0 \leq \varepsilon \leq \eta \leq 1$. Then

$$\neg^{\varepsilon,\eta} G_{\alpha,\beta}(A) = \neg^{\varepsilon,\eta}\{\langle x, \alpha.\mu_A(x), \beta.\nu_A(x)\rangle | x \in E\}$$
$$= \{\langle x, \min(1, \beta.\nu_A(x) + \varepsilon), \max(0, \alpha.\mu_A(x) - \eta)\rangle | x \in E\}$$

and

$$G_{\beta,\alpha}(\neg^{\varepsilon,\eta} A)$$
$$= G_{\beta,\alpha}(\{\langle x, \min(1, \nu_A(x) + \varepsilon), \max(0, \mu_A(x) - \eta)\rangle | x \in E\})$$
$$= \{\langle x, \beta.\min(1, \nu_A(x) + \varepsilon), \alpha.\max(0, \mu_A(x) - \eta)\rangle | x \in E\}.$$

Now, we obtain:

$$\min(1, \beta.\nu_A(x) + \varepsilon) - \beta.\min(1, \nu_A(x) + \varepsilon)$$
$$= \min(1, \beta.\nu_A(x) + \varepsilon) - \min(\beta, \beta.\nu_A(x) + \beta.\varepsilon) \geq 0$$

and

$$\alpha.\max(0, \mu_A(x) - \eta) - \max(0, \alpha.\mu_A(x) - \eta)$$
$$= \max(0, \alpha.\mu_A(x) - \alpha.\eta) - \max(0, \alpha.\mu_A(x) - \eta) \geq 0.$$

Therefore, inclusion (2) is vaild.

There are other, more complex relations, e.g., if $0 \leq \alpha \leq \beta \leq 1$, then for the IFS A the inclusions:

$$\neg_8 G_{\alpha,\beta}(A) \supset G_{\beta,\alpha}(\neg_8 A),$$
$$\neg_9 G_{\alpha,\beta}(A) \supset G_{\beta,\alpha}(\neg_9 A),$$

are valid, but an open problem is to find all similar inclusions.

III. Conclusion

In a next research authors will study the above properties for the case of other extended intuitionistic fuzzy modal operators and for the intuitionistic fuzzy topological operators.

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