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Managing Contradictory Evidence

C.J. Hinde, R.S. Patching and S.A. McCoy

Abstract— The paper draws on the theory of mass assignment to refine the underlying semantics of intuitionistic fuzzy sets. Inconsistency can arise from several sources and it is dealt with in different ways. All the representations of inconsistency and contradiction in this paper arise from considering restricting and positive evidence lattices. In particular this paper formally addresses the operators, intersection and conjunction in detail. Because union and disjunction are required to compute the values for intersection and conjunction these are also covered as part of the analysis.

Keywords: Contradiction, Inconsistency, Intuitionistic Fuzzy Sets, Mass Assignment

I. Introduction

THE major thrust of this work is to explore the management of contradictory evidence within the paradigm of intuitionistic fuzzy sets (1; 2; 3) with the representation used in mass assignment (4) to refine the semantics. Earlier research on inconsistency by Hinde (5) and (6) developed a representation for the contradiction which could arise due to positive and negative evidence coming from the same source. Later work by Hinde (6) explored the relationship between inconsistent intuitionistic fuzzy sets and mass assignment. Following and extending the work by (7) it developed a prototype relationship between the two representations extending both. In Hinde (5) the operators ∩ and ∪ were defined such that they are t-norms and s-norms respectively within the context of intuitionistic fuzzy sets. The following work by Hinde (6), which linked intuitionistic fuzzy sets to mass assignment touched on the operators giving an example of the ∩ operation. This paper carries more detail, in particular, differential distributions of contradiction can arise between sets as they are combined. These are combined into one consistent combined distribution of contradiction.

A. Overview

Intuitionistic fuzzy set theory goes back to the seminal work of Atanassov (1), and has had considerable attention paid to it since that first paper (2; 3; 7; 8). Intuitionistic fuzzy sets are a generalisation of the classic fuzzy sets using a measure of membership and a measure of non-membership.

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Following (2) the definition of an intuitionistic L-fuzzy set (ILFS) A^* over a universe of discourse $\mathbb{U}$ has the form:

$$A^* = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in \mathbb{U}\}$$

subject to

$$\mu_A(x) + \nu_A(x) \leq 1 \mid x \in \mathbb{U} \qquad (1)$$

Following the notation of Atanassov, Hinde (5) defined inconsistent intuitionistic fuzzy sets, IIFS.

$$A^{\iota*} = \{\langle x, \mu_A(x), \nu_A(x), \iota_A(x)\rangle \mid x \in \mathbb{U}\} \qquad (2)$$

subject to

$$\mu_A(x) + \nu_A(x) + \iota_A(x) \leq 1 \mid x \in \mathbb{U} \qquad (3)$$

Inconsistent Intuitionistic Fuzzy Sets are able to represent a lack of knowledge that arises from a variety of circumstances. Hinde (5) explored some of these.

Subsequently this was mapped across to a mass assignment representation (9) where an element of mass, defined in (9) as a scalar, was extended to be a triple. The triple vector of mass allowed the representation of a membership value, a non-membership value and a value for the contradiction (6) brought about by evidence, obtained from the same source, for membership and non-membership.

B. Evidence

The evidence contained in the sets, and how it is obtained is relevant so the notation for necessary and possible evidence is now introduced. ⊤ is verum a symbol denoting true and only true. ⊥ is falsum a symbol denoting false and only false. Together they make the set of Boolean values {⊤, ⊥} associated with possibility elimination. We choose to use ⊤ rather than t for this because {⊤} will be derived from {⊤, ⊥} by eliminating ⊥. In earlier works, Patching (10) and Hinde (11), possibilities have been eliminated by moving up the evidence lattice, Figure 1, an increase in knowledge. This gives us evidence towards non-membership but only indirectly gives us evidence towards membership in that, if all other possibilities have been eliminated what remains must be the truth; but this is not necessarily the case if the evidence is inconsistent. Another evidence lattice due to Belnap (12) shown in Figure 2 builds membership evidence from no evidence towards evidence for truth or falsity. The tokens ⊤ and ⊥ are inappropriate here as {⊤} will have been derived by eliminating ⊥ leaving "true and only true" or inconsistent. Building evidence from {} requires the use of different symbols as the set {t} means there has been evidence for true but leaves the possibility that evidence for f might occur. If evidence for f occurs we get {t, f} meaning
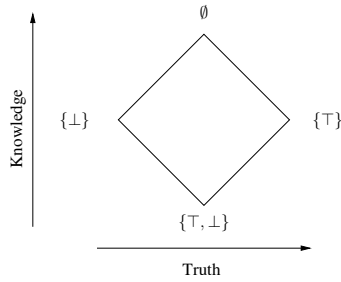
Fig. 1. A representation of a restricting evidence lattice using the notation adopted in this paper showing the increase in evidence (knowledge), and restriction of possibilities from $\{\top, \bot\}$ through to $\{\}$.
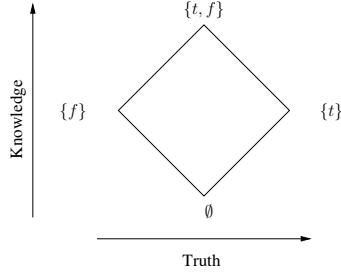

Fig. 2. A representation of Belnap's $\mathcal{FOUR}$ cumulative evidence lattice. The knowledge goes from no evidence represented by $\emptyset$ through to $\{t, f\}$ representing inconsistency.

inconsistent because evidence for both true and false has occurred.

Contradiction occurs when the evidence in the lattice 1 contradicts the evidence in 2. Where there is contradiction it is duplicated in membership values and non-membership values but it still may not violate constraint 1 or even 3. The contradiction here is the same as that discussed in (13), but has arisen without necessarily violating constraint 1. The contradiction described above differs from the inconsistency addressed in Patching (10) as it arises from the interaction of the restriction lattice, Figure 1 and the positive evidence lattice, Figure 2 whereas Patching's arises solely from the restriction lattice.

The degree of indeterminacy arises from two sources now, rather than have indeterminacy arising from a lack of evidence we now have indeterminacy arising from a superfluity of evidence, but which differs from the superfluity of evidence addressed in the work on mass assignment, (10; 11).

The evidence needs to be separated into 4 elements.

1) Evidence only for membership.
2) Evidence only for non-membership.
3) Evidence for both membership and non-membership.
4) Lack of any evidence.

II. Relationship to Mass assignment

An initial exploration of the relationship to mass assignment is given in (6). The main contribution which shall be used here is the representation of mass as a triple:

$$m_A(X) = \langle \mu_A(X), \nu_A(X), \iota_A(X) \rangle$$

This allows several possible denotations of inconsistency, as enumerated below:

1) $m_A(\{\}) = (\delta, 0, 0)$
   - inconsistency arising from the membership curve.
2) $m_A(\mathbb{U}) = (0, \delta, 0)$
   - inconsistency arising from the non-membership curve. This states that all members of the support set have evidence for non-membership.
3) $m_A(X) = (0, 0, \delta)$
   - inconsistency arising from contradictory evidence about the subset X.

We need a set of selection functions to extract sub masses so the equations can be sensibly expressed, these are defined in Equation 4.

$$
\begin{aligned}
\underset{A}{m}(X) &= \langle \mu_A(X), \nu_A(X), \iota_A(X) \rangle \quad (4) \\
\mu(\underset{A}{m}(X)) &= \mu_A(X) \\
\nu(\underset{A}{m}(X)) &= \nu_A(X) \\
\iota(\underset{A}{m}(X)) &= \iota_A(X)
\end{aligned}
$$

The notation here has used the quantities μ, ν and ι as projection operators to extract the values μ, ν and ι from the mass triples. Although this is overloading we believe this is better as it is clear what they are extracting.

III. Votes for and against

Fuzzy sets can be built up using a voting model, (4) and a membership curve established. A non-membership curve may be established similarly. So there are votes for membership, votes for non-membership and potentially there may be votes that are contradictory. These are counted in the triple mass assignment. Once all votes have been cast we should be in a position to calculate the inconsistency, the ignorance and the contradictions. However, without knowing whether a vote has been cast twice or not at all it would not be possible to determine whether there is ignorance or contradiction. Figure 3 shows the membership and non-membership curve for the assignment shown in Equation 5. Inconsistency and contradiction are both zero in this example assignment.

$$
\begin{aligned}
\underset{A}{m}(\{\}) &= \langle 0.0, 0.7, 0.0 \rangle \quad (5) \\
\underset{A}{m}(\{a\}) &= \langle 0.2, 0.0, 0.0 \rangle \qquad \underset{A}{m}(\{c\}) = \langle 0.0, 0.2, 0.0 \rangle \\
\underset{A}{m}(\{a, b\}) &= \langle 0.2, 0.0, 0.0 \rangle \qquad \underset{A}{m}(\{b, c\}) = \langle 0.0, 0.1, 0.0 \rangle \\
\underset{A}{m}(\{a, b, c\}) &= \langle 0.6, 0.0, 0.0 \rangle
\end{aligned}
$$

This needs explanation. There have been votes for the sets $\{a\}$, $\{a, b\}$ and $\{a, b, c\}$. These alone would give rise to Equation 6. All the votes against would be represented in Equation 7.

$$\underset{A}{\mathrm{m}}(\{a\}) = \langle 0.2, 0.0, 0.0 \rangle \qquad\qquad\qquad (6)$$

$$\underset{A}{\mathrm{m}}(\{a,b\}) = \langle 0.2, 0.0, 0.0 \rangle \qquad \underset{A}{\mathrm{m}}(\{a,b,c\}) = \langle 0.6, 0.0, 0.0 \rangle$$

$$\underset{A}{\mathrm{m}}(\{\}) = \langle 0.0, 0.7, 0.0 \rangle \qquad\qquad\qquad (7)$$

$$\underset{A}{\mathrm{m}}(\{c\}) = \langle 0.0, 0.2, 0.0 \rangle \qquad \underset{A}{\mathrm{m}}(\{b,c\}) = \langle 0.0, 0.1, 0.0 \rangle$$
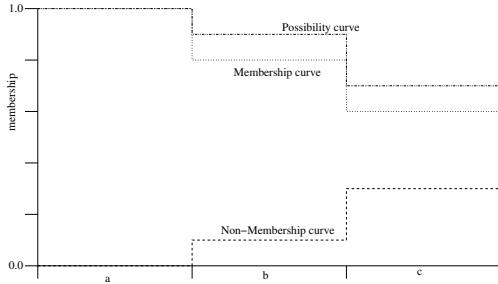

Fig. 3. Showing the three curves, the non-membership curve, the possibility curve arising from the non-membership curve and the membership curve; all arising from the mass assignment in Equation 5.

The mass assignment of the membership curve is given by Equation 8.

$$\underset{A}{\mathrm{m}}(\{a,b,c\}) = 0.6 \qquad\qquad\qquad (8)$$

$$\underset{A}{\mathrm{m}}(\{a,b\}) = 0.2 \qquad \underset{A}{\mathrm{m}}(\{a\}) = 0.2$$

The mass assignment of the possibility curve derived from the non-membership curve is derived from taking the complement of the mass supports, giving Equation 10.

$$\underset{A}{\mathrm{m}}(\overline{X}) = \nu(\underset{A}{\mathrm{m}}(X)) \qquad\qquad\qquad (9)$$

$$\underset{A}{\mathrm{m}}(\{a,b,c\}) = 0.7 \qquad\qquad\qquad (10)$$

$$\underset{A}{\mathrm{m}}(\{a,b\}) = 0.2 \qquad \underset{A}{\mathrm{m}}(\{a\}) = 0.1$$

It is now relatively straightforward to calculate two types of ignorance. There is the ignorance that exists as a result of no votes being cast, where the mass is all held in the set of support for both the membership curve and the possibility curve derived from the non-membership curve. There is the value of π that is the difference between the membership curve and the complement of the non-membership curve. Calculating values of μ, ν and π for the 3 elements results in:

$$\mu(a) = 1.0, \quad \neg\nu(a) = 1.0, \quad \pi(a) = 0.0$$
$$\mu(b) = 0.8, \quad \neg\nu(b) = 0.9, \quad \pi(b) = 0.1$$
$$\mu(c) = 0.6, \quad \neg\nu(c) = 0.7, \quad \pi(c) = 0.1$$

These values are constrained by Equations 11. The contradictory evidence associated with X belongs to the membership curve of that set and so might imply negative evidence
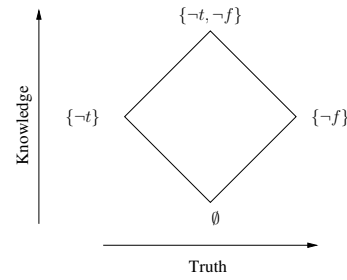

Fig. 4. A representation of Belnap's $\mathcal{FOUR}$ cumulative evidence lattice applied to the Boolean values $\{\neg t, \neg f\}$..

for $\overline{X}$; the contradictory evidence also belongs to the non-membership curve of X and again so might imply evidence for membership of the set $\overline{X}$. However, to simplify and clarify we explicitly denote the complementary sets in our equations. The evidence is for and against the one set X.

$$\sum \mu(\mathrm{m}(X)) + \frac{\sum \iota(\mathrm{m}(X))}{2} = 1 \qquad (11)$$
$$\sum \nu(\mathrm{m}(X)) + \frac{\sum \iota(\mathrm{m}(X))}{2} = 1$$

IV. Inconsistent and contradictory evidence

One sort of inconsistency has already been dealt with, that which arises from assignment of mass to the empty set {}. Typically the non-membership curve, when expressed as a mass assignment, has mass assigned to the empty set if the possibility curve is to have mass assigned to ignorance. So the non-membership curve starts off with all the mass assigned to {} and moves it upwards, as in Figure 4. As this is complemented it serves to produce a possibility curve. Inconsistency is mass assigned to $\mathbb{U}$ in the non-membership curve and corresponds to mass assigned to {} in the membership curve. Contradiction is mass assigned to both the membership curve and the non-membership curve. Cubillo (13) measures self contradiction in Intuitionistic Fuzzy Sets but does not consider that contradiction may arise before the set is complete. So the applicable constraint is now as shown in Equation 12. In Cubillo's terms a nonzero value of $\iota_A(x)$, contradiction, would imply a zero value of $\pi_A(x)$, hesitation; here we allow all four values to be non-zero.

$$\mu_A(x) + \nu_A(x) + \iota_A(x) + \pi_A(x) = 1 \mid x \in \mathbb{U} \qquad (12)$$

A. Inconsistency examples

We now illustrate some mass assignment examples following our definition of inconsistency given above and also in (5) with the corresponding belief and possibility curves that arise.

1) Inconsistent due to contradiction between types of evidence: The masses here have a degree of inconsistency resulting from a contradiction between the membership and the non-membership curve arising from votes for membership and votes for non membership from the same event; the example illustrated is taken from Equation 13.

$$\operatorname*{m}_{A}(\{\}) = \langle 0.1, 0.6, 0.0 \rangle \qquad \operatorname*{m}_{A}(\{a\}) = \langle 0.2, 0.0, 0.1 \rangle \text{ (13)}$$
$$\operatorname*{m}_{A}(\{a,b\}) = \langle 0.1, 0.0, 0.0 \rangle \qquad \operatorname*{m}_{A}(\{c\}) = \langle 0.0, 0.1, 0.0 \rangle$$
$$\operatorname*{m}_{A}(\{b,c\}) = \langle 0.0, 0.2, 0.0 \rangle \qquad \operatorname*{m}_{A}(\{a,b,c\}) = \langle 0.5, 0.0, 0.0 \rangle$$

The mass assignment for the membership curve is given in Equation 14, which sums to 0.9.

$$\operatorname*{m}_{A}(\{\}) = 0.1 \qquad \operatorname*{m}_{A}(\{a\}) = 0.2 \qquad (14)$$
$$\operatorname*{m}_{A}(\{a,b\}) = 0.1 \qquad \operatorname*{m}_{A}(\{a,b,c\}) = 0.5$$

It is tempting to make the sum of the mass elements total 1.0 by either adding a mass of 0.1 to {} or $\{a,b,c\}$; but that does not represent where the votes have been assigned correctly. The non-membership mass assignment in this case could be as given in Equation 15.

$$\operatorname*{m}_{A}(\{\}) = 0.6 \qquad (15)$$
$$\operatorname*{m}_{A}(\{a\}) = 0.1 \qquad \operatorname*{m}_{A}(\{a,b\}) = 0.2$$

Let the balance be due to contradictory votes for membership and non-membership concerning the set $\{a\}$. So there is contradictory evidence about the membership and non-membership curve for $\{a\}$. Furthermore there is ignorance about the remaining votes for membership and non-membership curves as there is mass assigned to the set $\{a,b,c\}$ or $\mathbb{U}$ corresponding to uncast votes. There is inconsistency in that some votes have eliminated all options and so there is also mass assigned to {}. There are also votes cast for the non-membership curve that all possible elements have had positive evidence for their falsity, see Figure 4. All votes are accounted for, some have not been cast, some are inconsistent and some have been cast twice.

Equation 13 has mass assigned to contradiction associated with the set $\{a\}$, but if contradictory votes have been cast then perhaps there should be mass assigned to the contradictory part of the complementary set $\{b,c\}$? However, what this states is that there have been contradictory votes cast about the set $\{a\}$, it only indirectly makes a statement about $\{b,c\}$.

Equation 13 has most aspects that we are interested in, the relevant curves are shown in Figure 5. We can extract all the relevant data from this to produce the following:
The conventional fuzzy memberships:
$\{0.9 \mid a + 0.6 \mid b + 0.5 \mid c\}$
The intuitionistic fuzzy memberships:
$\{\langle 0.9, 0.0 \rangle \mid a + \langle 0.6, 0.2 \rangle \mid b + \langle 0.5, 0.3 \rangle \mid c\}$
The inconsistent intuitionistic fuzzy memberships:
$\{\langle 0.9, 0.0, 0.1 \rangle \mid a + \langle 0.6, 0.2, 0.0 \rangle \mid b + \langle 0.5, 0.3, 0.0 \rangle \mid c\}$

This gives a unifying semantics to mass assignment and to intuitionistic fuzzy sets. It also introduces the ability to manage contradictory evidence.
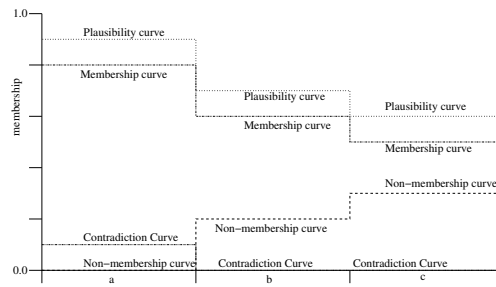

Fig. 5. Showing the four curves, the membership curve, the non-membership curve, the corresponding possibility curve arising from it and the contradiction curve; all arising from the mass assignment in Equation 13.

That the mass assignment triple is a true generalisation is easily shown by the following mass assignment triple in Equation 16. All the curves in Figure 5, and the corresponding assignments above are unchanged but the entries for {}, $\{b\}$ and $\{c\}$ have changed.

$$\operatorname*{m}_{A}(\{\}) = \langle 0.1, 0.4, 0.0 \rangle \qquad \operatorname*{m}_{A}(\{a\}) = \langle 0.2, 0.0, 0.1 \rangle \text{ (16)}$$
$$\operatorname*{m}_{A}(\{a,b\}) = \langle 0.1, 0.0, 0.0 \rangle \qquad \operatorname*{m}_{A}(\{c\}) = \langle 0.0, 0.3, 0.0 \rangle$$
$$\operatorname*{m}_{A}(\{b\}) = \langle 0.0, 0.2, 0.0 \rangle \qquad \operatorname*{m}_{A}(\{a,b,c\}) = \langle 0.5, 0.0, 0.0 \rangle$$

B. Mass assignment operations

The operations on the new mass measure follow the operations defined in (5), the tabular form as defined in (9) is applied to the quantities $\mu_A(x)$ and $\mu_A(x) + \iota_A(x)$ together with the corresponding quantities derived from $\nu_A(x)$. So the contradiction is propagated through the reasoning system accordingly. The mass triples cannot be derived in a single table but membership calculations require a table to perform the operations without contradiction, and another table to perform the operations including contradiction; similarly for non-membership. After collecting the resultant masses together the contradiction is calculated by subtracting the mass results without contradiction from those with contradiction and taking the maximum contradiction values for each mass block. This follows the operations defined in (5), which describes the necessary technology to perform the usual set theoretic operations. This paper moves the formalism forward.

Intersection requires the definition of the intersection values to obtain the membership values, but also requires Union to be defined to obtain the non-membership values. Given that the mass assignment triple has projection operators we need a further set of projection operators to extract complete single valued mass assignments over the universe of discourse. Again we overload the operators.

$$
\begin{aligned}
\mathrm{M}(A) &= \{\langle \mu_A(X), \nu_A(X), \iota_A(X)\rangle \mid X \in A, \quad (17)\\
&\quad \mu_A(X) > 0 \vee \nu_A(X) > 0 \vee \iota_A(X) > 0\}\\
\mu(\mathrm{M}(A)) &= \{\mu_A(X) \mid X \in A, \mu_A(X) > 0\}\\
\nu(\mathrm{M}(A)) &= \{\nu_A(X) \mid X \in A, \nu_A(X) > 0\}\\
\iota(\mathrm{M}(A)) &= \{\iota_A(X) \mid X \in A, \iota_A(X) > 0\}\\
\mu\iota(\mathrm{M}(A)) &= \{\mu_A(X) + \iota_A(X) \mid X \in A,\\
&\quad \mu_A(X) + \iota_A(X) > 0\}\\
\nu\iota(\mathrm{M}(A)) &= \{\nu_A(X) + \iota_A(X) \mid X \in A,\\
&\quad \nu_A(X) + \iota_A(X) > 0\}
\end{aligned}
$$

In order to correctly project the masses of the two sets A and B we need to form 4 mass assignment operations. The $\cap$ and $\cup$ operations are the usual mass assignment operations performed on the respective sets of single mass assignments extracted from the sets of triple mass assignments. Let the function SUP construct the supremum of two sets of mass assignments. The supremum of two mass assignments is the most restricted assignment that can be restricted to both mass assignments. Calculation of this is non-trivial but has been coded and is the subject of another paper. We then get:

$$
\begin{aligned}
\mu(\mathrm{M}(A \cap B)) &= \mu(\mathrm{M}(A)) \cap \mu(\mathrm{M}(B)) \quad (18)\\
\nu(\mathrm{M}(A \cap B)) &= \nu(\mathrm{M}(A)) \cup \nu(\mathrm{M}(B))\\
\mu\iota(\mathrm{M}(A \cap B)) &= \mu\iota(\mathrm{M}(A)) \cap \mu\iota(\mathrm{M}(B))\\
\nu\iota(\mathrm{M}(A \cap B)) &= \nu\iota(\mathrm{M}(A)) \cup \nu\iota(\mathrm{M}(B))\\
\mu\iota'(\mathrm{M}(A \cap B)) &= \{\mu\iota(\underset{A\cap B}{\mathrm{m}}(X)) - \mu(\underset{A\cap B}{\mathrm{m}}(X))\\
&\quad \mid X \in A \cap B\}\\
\nu\iota'(\mathrm{M}(A \cap B)) &= \nu\iota(\underset{A\cap B}{\mathrm{m}}(X)) - \nu(\underset{A\cap B}{\mathrm{m}}(X))\\
&\quad \mid X \in A \cap B\}\\
\iota(\mathrm{M}(A \cap B)) &= SUP(\mu\iota'(\mathrm{M}(A \cap B)), \nu\iota'(\mathrm{M}(A \cap B)))
\end{aligned}
$$

We can then form the mass assignment triples of the combination from these assignments. The set defined by $\mu(\mathrm{M}(A \cap B))$ contains the membership values, $\nu(\mathrm{M}(A \cap B))$ the non-membership values and $\iota(\mathrm{M}(A \cap B))$ the contradictory evidence. Note that the contradictory evidence cannot be calculated from the sets, but is obtained from the individual mass assignments. By substituting $\cup$ for $\cap$ and vice versa in Equations 18 the mass assignments for Union can be formed.

Although the operations of intersection and union have the characteristic functions conjunction and disjunction the operations have different details. Following the notation above we can generalise the Boolean operations as follows using Equation 19. This is identical to the equation shown in Equation 18, however the implementation of $\wedge$ is different to that of $\cap$ between single valued mass assignments. Set elements interact differently to truth values.

$$
\mu(\mathrm{M}(A \wedge B)) = \mu(\mathrm{M}(A)) \wedge \mu(\mathrm{M}(B)) \quad (19)
$$

These two sets of operations implement intersection, union, conjunction and disjunction including contradiction

between the restriction lattice and the cumulative evidence lattice. Note that calculation of the non-membership values in the case that the operation is a t-norm follows Atanassov (2) and uses the corresponding s-norm. This paper follows suit.

V. Examples

The examples section will illustrate the operations of intersection and conjunction.

A. Intersection

Starting with the mass assignment in Equation 13 we intersect with Equation 20.

$$
\begin{aligned}
\underset{B}{\mathrm{m}}(\{\}) &= \langle 0.1, 0.6, 0.0\rangle \quad (20)\\
\underset{B}{\mathrm{m}}(\{a\}) &= \langle 0.2, 0.0, 0.1\rangle\\
\underset{B}{\mathrm{m}}(\{a, b\}) &= \langle 0.1, 0.0, 0.0\rangle\\
\underset{B}{\mathrm{m}}(\{c\}) &= \langle 0.1, 0.1, 0.0\rangle\\
\underset{B}{\mathrm{m}}(\{b, c\}) &= \langle 0.0, 0.2, 0.0\rangle\\
\underset{B}{\mathrm{m}}(\{a, b, c\}) &= \langle 0.4, 0.0, 0.0\rangle
\end{aligned}
$$

The membership values for Equations 13 and 20

Equation 13	Equation 20
$\mu\underset{A}{\mathrm{m}}(\{\}) = 0.1$	$\mu\underset{B}{\mathrm{m}}(\{\}) = 0.1$
$\mu\underset{A}{\mathrm{m}}(\{a\}) = 0.2$	$\mu\underset{B}{\mathrm{m}}(\{a\}) = 0.2$
$\mu\underset{A}{\mathrm{m}}(\{a, b\}) = 0.1$	$\mu\underset{B}{\mathrm{m}}(\{a, b\}) = 0.1$
	$\mu\underset{B}{\mathrm{m}}(\{c\}) = 0.1$
$\mu\underset{A}{\mathrm{m}}(\{a, b, c\}) = 0.5$	$\mu\underset{B}{\mathrm{m}}(\{a, b, c\}) = 0.4$

The intersection of the membership mass assignments is shown in Figure 6.

	B				
	{}:0.1	{a}:0.2	{a,b}:0.1	{c}:0.1	{a,b,c}:0.4
{}:0.1	{}:0.01	{}:0.02	{}:0.01	{}:0.01	{}:0.04
{a}:0.2	{}:0.02	{a}:0.04	{a}:0.02	{}:0.02	{a}:0.08
{a,b}:0.1	{}:0.01	{a}:0.02	{a,b}:0.01	{}:0.01	{a,b}:0.04
{a,b,c}:0.5	{}:0.05	{a}:0.1	{a,b}:0.05	{c}:0.05	{a,b,c}:0.2

Fig. 6. Showing the tableau for the intersection of the membership values of the two mass assignments in Equations 13 and 20, which do not include the contradictory evidence.

The mass assignment resulting from the intersection of the membership functions is:

$$\mu \underset{A \cap B}{\mathrm{m}}(\{\}) = 0.2$$
$$\mu \underset{A \cap B}{\mathrm{m}}(\{a\}) = 0.26$$
$$\mu \underset{A \cap B}{\mathrm{m}}(\{a,b\}) = 0.1$$
$$\mu \underset{A \cap B}{\mathrm{m}}(\{c\}) = 0.05$$
$$\mu \underset{A \cap B}{\mathrm{m}}(\{a,b,c\}) = 0.2$$

The membership functions including contradictory evidence for Equations 13 and 20:

Equation 13	Equation 20
$\mu\iota \underset{A}{\mathrm{m}}(\{\}) = 0.1$	$\mu\iota \underset{B}{\mathrm{m}}(\{\}) = 0.1$
$\mu\iota \underset{A}{\mathrm{m}}(\{a\}) = 0.3$	$\mu\iota \underset{B}{\mathrm{m}}(\{a\}) = 0.3$
$\mu\iota \underset{A}{\mathrm{m}}(\{a,b\}) = 0.1$	$\mu\iota \underset{B}{\mathrm{m}}(\{a,b\}) = 0.1$
	$\mu\iota \underset{B}{\mathrm{m}}(\{c\}) = 0.1$
$\mu\iota \underset{A}{\mathrm{m}}(\{a,b,c\}) = 0.5$	$\mu\iota \underset{B}{\mathrm{m}}(\{a,b,c\}) = 0.4$

B

	{}:0.1	{a}:0.3	{a,b}:0.1	{c}:0.1	{a,b,c}:0.4
{}:0.1	{}:0.01	{}:0.03	{}:0.01	{}:0.01	{}:0.04
{a}:0.3	{}:0.03	{a}:0.09	{a}:0.03	{}:0.03	{a}:0.12
{a,b}:0.1	{}:0.01	{a}:0.03	{a,b}:0.01	{}:0.01	{a,b}:0.04
{a,b,c}:0.5	{}:0.05	{a}:0.15	{a,b}:0.05	{c}:0.05	{a,b,c}:0.2

A

Fig. 7. Showing the tableau for the intersection of the membership values of the two mass assignments in Equations 13 and 20 including contradictions.

The mass assignment resulting from the intersection of the membership functions including contradictions is:

$$\mu \underset{A \cap B}{\mathrm{m}}(\{\}) = 0.23$$
$$\mu \underset{A \cap B}{\mathrm{m}}(\{a\}) = 0.42$$
$$\mu \underset{A \cap B}{\mathrm{m}}(\{a,b\}) = 0.1$$
$$\mu \underset{A \cap B}{\mathrm{m}}(\{c\}) = 0.05$$
$$\mu \underset{A \cap B}{\mathrm{m}}(\{a,b,c\}) = 0.2$$

Giving interim contradiction values as in Equation 21

$$\iota \underset{A \cap B}{\mathrm{m}}(\{\}) = 0.03 \qquad \iota \underset{A \cap B}{\mathrm{m}}(\{a\}) = 0.16 \qquad (21)$$

The non-membership values for Equations 13 and 20

Equation 13	Equation 20
$\nu \underset{A}{\mathrm{m}}(\{\}) = 0.6$	$\nu \underset{B}{\mathrm{m}}(\{\}) = 0.6$
$\nu \underset{A}{\mathrm{m}}(\{c\}) = 0.1$	$\nu \underset{B}{\mathrm{m}}(\{c\}) = 0.1$
$\nu \underset{A}{\mathrm{m}}(\{b,c\}) = 0.2$	$\nu \underset{B}{\mathrm{m}}(\{b,c\}) = 0.2$

If the operation for the membership functions is intersection then the operation for the non-membership functions will be union. The mass assignment resulting from the union of the non-membership functions is:

$$\nu \underset{A \cap B}{\mathrm{m}}(\{\}) = 0.36$$
$$\nu \underset{A \cap B}{\mathrm{m}}(\{c\}) = 0.13$$
$$\nu \underset{A \cap B}{\mathrm{m}}(\{b,c\}) = 0.32$$

The non-membership functions including contradictory evidence for Equation 13:

Equation 13	Equation 20
$\nu\iota \underset{A}{\mathrm{m}}(\{\}) = 0.6$	$\nu\iota \underset{B}{\mathrm{m}}(\{\}) = 0.6$
$\nu\iota \underset{A}{\mathrm{m}}(\{a\}) = 0.1$	$\nu\iota \underset{B}{\mathrm{m}}(\{a\}) = 0.1$
$\nu\iota \underset{A}{\mathrm{m}}(\{c\}) = 0.1$	$\nu\iota \underset{B}{\mathrm{m}}(\{c\}) = 0.1$
$\nu\iota \underset{A}{\mathrm{m}}(\{b,c\}) = 0.2$	$\nu\iota \underset{B}{\mathrm{m}}(\{b,c\}) = 0.2$

The mass assignment resulting from the union of the non-membership functions including inconsistency is:

$$\nu\iota \underset{A \cap B}{\mathrm{m}}(\{\}) = 0.36 \qquad \nu\iota \underset{A \cap B}{\mathrm{m}}(\{a\}) = 0.13$$
$$\nu\iota \underset{A \cap B}{\mathrm{m}}(\{c\}) = 0.13 \qquad \nu\iota \underset{A \cap B}{\mathrm{m}}(\{a,c\}) = 0.02$$
$$\nu\iota \underset{A \cap B}{\mathrm{m}}(\{b,c\}) = 0.32 \qquad \nu\iota \underset{A \cap B}{\mathrm{m}}(\{a,b,c\}) = 0.04$$

Giving second interim contradiction values as in Equation 22

$$\iota \underset{A \cap B}{\mathrm{m}}(\{a\}) = 0.13 \qquad\qquad (22)$$
$$\iota \underset{A \cap B}{\mathrm{m}}(\{a,c\}) = 0.02 \qquad \iota \underset{A \cap B}{\mathrm{m}}(\{a,b,c\}) = 0.04$$

This is not the same as the contradictory values given by the membership analysis, however they are consistent with one another. We have the two assignments as shown in Equation 21 due to considerations of membership values and Equation 22 due to considerations of non-membership values.

The most general solution to this is the least upper bound of the two assignments which in this case is given in Equation 23:

$$\iota \underset{A \cap B}{\mathrm{m}}(\{a\}) = 0.13 \qquad\qquad (23)$$
$$\iota \underset{A \cap B}{\mathrm{m}}(\{a,c\}) = 0.02 \qquad \iota \underset{A \cap B}{\mathrm{m}}(\{a,b,c\}) = 0.04$$

The intersection of the mass assignments in Equations 13 and 20 results in the final triple mass assignment in Equation 24.

$$\underset{A \cap B}{\mathrm{m}}(\{\}) = \langle 0.23, 0.36, 0.0 \rangle \qquad (24)$$

$$\underset{A \cap B}{\mathrm{m}}(\{a\}) = \langle 0.42, 0.0, 0.13 \rangle$$

$$\underset{A \cap B}{\mathrm{m}}(\{a,c\}) = \langle 0.0, 0.0, 0.02 \rangle$$

$$\underset{A \cap B}{\mathrm{m}}(\{a,b\}) = \langle 0.1, 0.0, 0.0 \rangle$$

$$\underset{A \cap B}{\mathrm{m}}(\{c\}) = \langle 0.05, 0.13, 0.0 \rangle$$

$$\underset{A \cap B}{\mathrm{m}}(\{b,c\}) = \langle 0.0, 0.32, 0.0 \rangle$$

$$\underset{A \cap B}{\mathrm{m}}(\{a,b,c\}) = \langle 0.2, 0.0, 0.04 \rangle$$

B. Conjunction

The next examples concentrate on conjunction. There are two tuples involved here, a tuple giving the membership, non-membership and contradiction of truth, and similarly for false. We are saying here, as before, that elimination of true does not represent positive evidence for false, and neither does the converse situation.

Let the two truth sets be as in Equations 25 and 26. The symbols $\top$ and $\bot$, together with t and f are substituted for the symbols T and F as the class of evidence used, restrictive or positive, is denoted by the values in the mass tuples.

$$\underset{A}{\mathrm{m}}(\{\}) = \langle 0.1, 0.6, 0.0 \rangle \qquad (25)$$

$$\underset{A}{\mathrm{m}}(\{T\}) = \langle 0.2, 0.1, 0.1 \rangle \qquad \underset{A}{\mathrm{m}}(\{F\}) = \langle 0.1, 0.2, 0.0 \rangle$$

$$\underset{A}{\mathrm{m}}(\{T,F\}) = \langle 0.5, 0.0, 0.0 \rangle$$

$$\underset{B}{\mathrm{m}}(\{\}) = \langle 0.1, 0.5, 0.0 \rangle \qquad (26)$$

$$\underset{B}{\mathrm{m}}(\{T\}) = \langle 0.2, 0.2, 0.0 \rangle \qquad \underset{B}{\mathrm{m}}(\{F\}) = \langle 0.2, 0.1, 0.1 \rangle$$

$$\underset{B}{\mathrm{m}}(\{T,F\}) = \langle 0.4, 0.1, 0.0 \rangle$$

These two assignments exhibit all the kinds of uncertainty that we wish to model. The membership values for Equations 25 and 26

	Equation 25	Equation 26
	$\mu \underset{A}{\mathrm{m}}(\{\}) = 0.1$	$\mu \underset{B}{\mathrm{m}}(\{\}) = 0.1$
	$\mu \underset{A}{\mathrm{m}}(\{T\}) = 0.2$	$\mu \underset{B}{\mathrm{m}}(\{T\}) = 0.2$
	$\mu \underset{A}{\mathrm{m}}(\{F\}) = 0.1$	$\mu \underset{B}{\mathrm{m}}(\{F\}) = 0.2$
	$\mu \underset{A}{\mathrm{m}}(\{T,F\}) = 0.5$	$\mu \underset{B}{\mathrm{m}}(\{T,F\}) = 0.4$

The mass assignment resulting from the conjunction of the membership functions without contradiction is:

$$\mu \underset{A \wedge B}{\mathrm{m}}(\{\}) = 0.05 \qquad \mu \underset{A \wedge B}{\mathrm{m}}(\{T\}) = 0.04$$

$$\mu \underset{A \wedge B}{\mathrm{m}}(\{F\}) = 0.34 \qquad \mu \underset{A \wedge B}{\mathrm{m}}(\{T,F\}) = 0.38$$

The membership functions including contradictory evidence for Equation 25 and 26:

	Equation 25	Equation 26
	$\mu\iota \underset{A}{\mathrm{m}}(\{\}) = 0.1$	$\mu\iota \underset{B}{\mathrm{m}}(\{\}) = 0.1$
	$\mu\iota \underset{A}{\mathrm{m}}(\{T\}) = 0.3$	$\mu\iota \underset{B}{\mathrm{m}}(\{T\}) = 0.2$
	$\mu\iota \underset{A}{\mathrm{m}}(\{F\}) = 0.1$	$\mu\iota \underset{B}{\mathrm{m}}(\{F\}) = 0.3$
	$\mu\iota \underset{A}{\mathrm{m}}(\{T,F\}) = 0.5$	$\mu\iota \underset{B}{\mathrm{m}}(\{T,F\}) = 0.4$

The mass assignment resulting from the conjunction of the membership functions including inconsistency is:

$$\mu\iota \underset{A \wedge B}{\mathrm{m}}(\{\}) = 0.06 \qquad \mu\iota \underset{A \wedge B}{\mathrm{m}}(\{T\}) = 0.06$$

$$\mu\iota \underset{A \wedge B}{\mathrm{m}}(\{F\}) = 0.46 \qquad \mu\iota \underset{A \wedge B}{\mathrm{m}}(\{T,F\}) = 0.42$$

Giving interim contradiction values as in Equation 27:

$$\iota \underset{A \wedge B}{\mathrm{m}}(\{\}) = 0.01 \qquad \iota \underset{A \wedge B}{\mathrm{m}}(\{T\}) = 0.02 \qquad (27)$$

$$\iota \underset{A \wedge B}{\mathrm{m}}(\{F\}) = 0.12 \qquad \iota \underset{A \wedge B}{\mathrm{m}}(\{T,F\}) = 0.04$$

The non-membership values for Equations 25 and 26

	Equation 25	Equation 26
	$\nu \underset{A}{\mathrm{m}}(\{\}) = 0.6$	$\nu \underset{B}{\mathrm{m}}(\{\}) = 0.5$
	$\nu \underset{A}{\mathrm{m}}(\{T\}) = 0.1$	$\nu \underset{B}{\mathrm{m}}(\{T\}) = 0.2$
	$\nu \underset{A}{\mathrm{m}}(\{F\}) = 0.2$	$\nu \underset{B}{\mathrm{m}}(\{F\}) = 0.1$
	$\nu \underset{A}{\mathrm{m}}(\{T,F\}) = 0.0$	$\nu \underset{B}{\mathrm{m}}(\{T,F\}) = 0.1$

If the operation for the membership functions is conjunction then the operation for the non-membership functions will be disjunction. The mass assignment resulting from the disjunction of the non-membership functions is:

$$\nu \underset{A \wedge B}{\mathrm{m}}(\{\}) = 0.46 \qquad \nu \underset{A \wedge B}{\mathrm{m}}(\{T\}) = 0.31$$

$$\nu \underset{A \wedge B}{\mathrm{m}}(\{F\}) = 0.02 \qquad \nu \underset{A \wedge B}{\mathrm{m}}(\{T,F\}) = 0.02$$

The non-membership functions including contradictory evidence for Equations 25 and 26:

	Equation 25	Equation 26
	$\nu\iota \underset{A}{\mathrm{m}}(\{\}) = 0.6$	$\nu\iota \underset{B}{\mathrm{m}}(\{\}) = 0.5$
	$\nu\iota \underset{A}{\mathrm{m}}(\{T\}) = 0.2$	$\nu\iota \underset{B}{\mathrm{m}}(\{T\}) = 0.2$
	$\nu\iota \underset{A}{\mathrm{m}}(\{F\}) = 0.2$	$\nu\iota \underset{B}{\mathrm{m}}(\{F\}) = 0.2$
	$\nu\iota \underset{A}{\mathrm{m}}(\{T,F\}) = 0.0$	$\nu\iota \underset{B}{\mathrm{m}}(\{T,F\}) = 0.1$

The mass assignment resulting from the conjunction of the non-membership functions including inconsistency is:

$$\nu\iota \underset{A \wedge B}{\mathrm{m}}(\{\}) = 0.52 \qquad \nu\iota \underset{A \wedge B}{\mathrm{m}}(\{T\}) = 0.42$$

$$\nu\iota \underset{A \wedge B}{\mathrm{m}}(\{F\}) = 0.04 \qquad \nu\iota \underset{A \wedge B}{\mathrm{m}}(\{T,F\}) = 0.02$$

Giving second interim contradiction values as shown in Equation 28:

$$\iota\,\underset{A \wedge B}{\mathrm{m}}(\{\}) = 0.06 \qquad \iota\,\underset{A \wedge B}{\mathrm{m}}(\{T\}) = 0.11 \qquad (28)$$
$$\iota\,\underset{A \wedge B}{\mathrm{m}}(\{F\}) = 0.02 \qquad \iota\,\underset{A \wedge B}{\mathrm{m}}(\{T,F\}) = 0.0$$

Again this is not the same as the contradictory values given by the membership analysis, however they may be combined to form the supremum mass assignment. We have the two assignments as shown in Equation 27 due to considerations of membership values and Equation 28 due to considerations of non-membership values .

The most general solution to this is the supremum of the two assignments which in this case is given in Equation 29; it is relatively easy to calculate:

$$\iota\,\underset{A \wedge B}{\mathrm{m}}(\{\}) = 0.01 \qquad \iota\,\underset{A \wedge B}{\mathrm{m}}(\{T\}) = 0.02 \qquad (29)$$
$$\iota\,\underset{A \wedge B}{\mathrm{m}}(\{F\}) = 0.12 \qquad \iota\,\underset{A \wedge B}{\mathrm{m}}(\{T,F\}) = 0.04$$

The supremum of the mass assignments in Equations 25 and 26 results in the final triple mass assignment in Equation 30.

$$\begin{aligned}
\underset{A \wedge B}{\mathrm{m}}(\{\}) &= \langle 0.05, 0.46, 0.01 \rangle \qquad (30)\\
\underset{A \wedge B}{\mathrm{m}}(\{T\}) &= \langle 0.04, 0.31, 0.02 \rangle\\
\underset{A \wedge B}{\mathrm{m}}(\{F\}) &= \langle 0.34, 0.02, 0.12 \rangle\\
\underset{A \wedge B}{\mathrm{m}}(\{T,F\}) &= \langle 0.38, 0.02, 0.04 \rangle
\end{aligned}$$

VI. Conclusions

Two essentially different ways of representing different kinds of inconsistency have been introduced to Intuitionistic Fuzzy Sets. The first type of inconsistency involves either evidence contributing to the empty set associated with the membership values or evidence contributing to the universal set associated with the non-membership values. This is analogous to the inconsistency developed by Patching (10) and further analysed by Hinde (11). This was not analysed in any detail in this paper as it has been in other works (10). A second type is where evidence for the membership curve and evidence for the non-membership curve conflict. This is distinct from the first type and arises directly out of the treatment using Intuitionistic Fuzzy Sets. A mass assignment treatment has introduced a new element of mass which is a triple involving mass due to the membership curve, mass due to the non-membership curve and finally an element of mass due to conflicts between the membership and non-membership curve. The operators used to combine mass assignments, (9; 10; 11) are defined and used in the context of the mass assignment triples. Intersection and conjunction

were defined and examples shown. Research and development of semantic unification, semantic separation and other operators is required.

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