

LARGE-AMPLITUDE SOLITARY WAVES WITH VORTEX CORES IN STRATIFIED AND ROTATING FLOWS

R.H.J. Grimshaw*
and O.G. Derzho†

Abstract

Most theoretical studies of solitary waves are for the weakly nonlinear regime, where models such as the Korteweg-de Vries equation are commonly obtained. However, observations of solitary waves often show that these waves can have large amplitudes, to the extent that they may contain vortex cores, that is, regions of recirculating flow. In this work, we report on theoretical asymptotic models, which describe explicitly the structure of solitary waves with recirculation zones, for certain special but important upstream flow configurations.

The key feature which enables this construction is that, both for stratified shear flows and for axisymmetric swirling flows, the steady state vorticity equation is almost linear when the upstream flow is almost uniform. That is, for stratified shear flows the upstream flow and the upstream stratification are almost constant, while for rotating flows the upstream axial flow and angular velocity are almost constant. This feature enables the asymptotic construction of solitary waves described by a steady-state generalised Korteweg-de Vries equation in an outer zone, matched to an inner zone containing a recirculation zone. These recirculation zones exist for wave amplitudes just greater than a certain critical amplitude for which there is incipient flow reversal. The recirculation zones have a universal structure such that their width increases without limit as the wave amplitude increases from the critical amplitude to a certain maximum amplitude, but their existence can be sensitive to the actual upstream flow configuration. Applications are made to observations and numerical simulations of large amplitude internal solitary waves, and to the phenomenon of vortex breakdown.

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1 Introduction

Solitary waves in stratified flows confined to a channel of finite depth have been the subject of many studies, beginning with the pioneering work of Dubreil-Jacotin [1] and Long [2]. For this special but important class of flows, they showed that the fully nonlinear, steady, two-dimensional, inviscid and incompressible equations can be reduced to a single nonlinear elliptic equation for the streamfunction. It is usually derived on the assumption that all streamlines originate upstream, thus in particular excluding regions of closed streamlines, and can then be used to establish the existence of solitary waves of permanent form. An important special case arises when the stratification is nearly uniform and the upstream flow is also nearly uniform. In this limit the equation is almost linear for finite-amplitude waves, and this circumstance has been exploited by Benney and Ko [3] and Derzho [4] amongst others.

However, these theories are usually limited to amplitudes less than a certain critical amplitude for which there is incipient flow reversal. Recently Derzho and Grimshaw [5] (see also Aigner et al [6]) showed that asymptotic solutions could be constructed for solitary waves with amplitudes just greater than this critical amplitude, provided that the

*Department of Mathematical Sciences, Loughborough University, Loughborough LE11 3TU, UK, email: R.H.J. Grimshaw@lboro.ac.uk

†Department of Mathematics and Statistics, Monash University, Boz 28M, Vic. 3800, Australia

the waves incorporated a small vortex core containing recirculating fluid. Their results were generally in agreement with several experimental and numerical investigations.

An analogous situation exists for the axisymmetric, steady, rotating flow of an inviscid, incompressible fluid contained in a circular tube. Here again a single nonlinear elliptic equation can be derived for the streamfunction, namely the Bragg-Hawthorne equation [7], based on the assumption that all streamlines originate upstream. This equation has been the basis for several studies of solitary waves and the phenomena of vortex breakdown (see, for instance, Benjamin [8, 9], Leibovich and Kribus [10] and Keller [11]). However, recently Derzho and Grimshaw [12] pointed out that generally these theories were limited to amplitudes less than a certain critical amplitude for which there is incipient flow reversal. They then used an asymptotic construction to find solitary waves with a small recirculating zone on the tube axis. This result was found to be in good agreement with experimental and numerical investigations.

In this short note we review and summarise these recent results. In the next section we describe the case for stratified flow, and then follow with the analogous results for rotating flow.

2 Stratified Flows

2.1 Formulation

Let us first consider the steady, two-dimensional flow of an inviscid, incompressible density-stratified fluid. In standard notation the equations of motion are,

$$(1) \quad \begin{aligned} u_x + w_z &= 0, \\ u\rho_x + w\rho_z &= 0, \\ \rho(uu_x + ww_z) + p_x &= 0, \\ \rho(uw_x + ww_z) + p_z + g\rho &= 0. \end{aligned}$$

Here the coordinates are (x, z) with x directed horizontally and z vertically upwards, (u, w) are the corresponding velocity components, ρ is the density and p is the pressure. The streamfunction ψ is defined in the usual way,

$$(2) \quad u = -\psi_z, \quad w = \psi_x.$$

From the second equation in (1) it follows that the density is constant on streamlines, so that $\rho = \rho(\psi)$. Here the functional form of $\rho(\psi)$ is yet to be determined. For those streamlines which originate upstream, it follows that it is determined from the upstream density profile, $\rho_0(z)$ where the streamfunction is given by $\psi_0(z)$. Finally, eliminating the pressure from the last two equations in (1) yields the vorticity equation,

$$(3) \quad \psi_{xx} + \psi_{zz} + \frac{1}{\rho} \frac{d\rho}{d\psi} [gz + \frac{1}{2}(\psi_x^2 + \psi_z^2)] = G(\psi).$$

As for $\rho(\psi)$ the functional form of $G(\psi)$ is found from the upstream boundary conditions for those streamlines which originate upstream, but otherwise is as yet undetermined. The boundary conditions are that

$$(4) \quad \psi \rightarrow \psi_0(z) \quad \text{and} \quad \rho \rightarrow \rho_0(z), \quad \text{as } x \rightarrow \infty,$$

while on the channel walls,

$$(5) \quad \psi_x = 0, \quad \text{on } z = 0, h.$$

Equation 3, together with the boundary conditions 4 and 5, is a nonlinear elliptic boundary value problem. Our aim here is to construct, asymptotically for long waves, a family of finite-amplitude solitary wave solutions including those containing vortex cores, that is a region of recirculating fluid. The account given here is necessarily brief, and more details can be found in [5] and [13].

2.2 Asymptotic Analysis

The basis of our approach is the observation that for nearly uniform, weak stratification and nearly uniform flow, then equation (3) is nearly linear without any restriction in wave amplitude. Hence we assume that

$$(6) \quad N^2(z) = -\frac{g}{\rho_0} \frac{d\rho_0}{dz} = N_0^2 + \sigma\Omega(z), \quad \text{and} \quad \psi_0(z) = cz + \kappa\Psi(z).$$

Here N_0 is a constant value characterizing the weak stratification, and c is the wave speed to be determined, while the functions $\Omega(z)$ and $\Psi(z)$ characterise the small but significant departures from the uniform state, so that σ and κ are two small parameters, both of $O(\epsilon^2)$ where the small parameter ϵ characterises the ratio of the vertical scale h to the long horizontal length scale. Thus we introduce the new horizontal variable $X = \epsilon x/h$, and seek solutions for which ψ is a function of X and z . An asymptotic solution is sought as a power series in ϵ^2 , whose leading term is given by

$$(7) \quad \psi^{(0)} = c^{(0)}z + A(X)\sin(\pi z/h), \quad \text{and} \quad c^{(0)} = N_0 h/\pi.$$

At the next order in the asymptotic expansion an inhomogeneous linear equation is obtained whose compatibility condition leads to the required equation for the wave amplitude $A(X)$,

$$(8) \quad A_{XX} - \Delta A + M(A) = 0.$$

Here Δ is proportional to the speed correction term $c^{(1)}$, while $M(A)$ is a nonlinear term related in quite a complicated way to the functions $\Omega(z)$ and $\Psi(z)$ in 6. Equation (8) has the form of a generalised KdV equation. When, all streamlines originate upstream, solutions of (8) such that $A \rightarrow 0$ as $|X| \rightarrow \infty$ describe the required solitary waves.

Next we observe that a sufficient condition for all streamlines to originate upstream is that $\psi_z > 0$. Here we are concerned with the situation when this condition is violated at some point on the wave profile. Hence we suppose that $\psi_z = 0$ when $|X| = X_0$ for some critical amplitude A_* , so that equation (8) holds for $|X| > X_0$ and $|A| < A_*$. To leading order $A_* = N_0 h^2/\pi^2$ when the breakdown occurs at the upper channel wall.

For $|X| < X_0$ we construct an asymptotic solution which contains a vortex core attached to the upper channel wall, within the region defined by $h - \eta(x) < z < h$. For the flow within the vortex core, equation (3) again holds, but the density $\rho(\psi)$ and the function $G(\psi)$ cannot be determined from the upstream conditions. Instead, we assume that the density is a constant, equal to its value on the vortex core boundary which is a streamline; this is to ensure that there is static stability. Further it can be shown that within the vortex core the vorticity $G(\psi)$ is sufficiently small that the flow is stagnant to leading order in the asymptotic analysis.

The vortex core and the inner region ($|X| < X_0$ and $0 < z < h - \eta$) are characterised by a different length scale from the outer region ($|X| > X_0$), and here we put $\xi = \beta x/h$. Further we assume that the height of the vortex core is $O(\delta)$ so that $\eta = \delta f(\xi)$. In the inner region the asymptotic solution is again given by (7) to leading order, but now we set

$$(9) \quad A = A_* + \mu B.$$

Here β , δ and μ are small parameters. An optimal balance shows that the required scaling is given by

$$(10) \quad \beta = \epsilon^{1/3}, \quad \delta = \epsilon^{2/3}, \quad \text{and} \quad \mu = \epsilon^{4/3}.$$

A compatibility condition applied to the next order equation in the asymptotic expansion, together with the use of matching conditions that the streamfunction and the velocity field are continuous across the vortex core boundary, yields the desired equation for B ,

$$(11) \quad B_{\xi\xi} - \Delta A_* + M(A_*) = \frac{2}{3}\nu B^{3/2}$$

Here ν is a supercriticality parameter, which is of order unity, and given by $\nu = 2^{1/2}\pi^2/N_0^{1/2}h$. The interface with the vortex core is given by $f = (2B/N_0)^{1/2}$. This equation is to be solved in the region $|\xi| < \xi_0$ subject to the matching conditions that $B = 0$ at $|\xi| = \xi_0$ (so that $A = A_*$ there) and that $A_X = \mu^{1/2}B_\xi$ also at $|\xi| = \xi_0$. Of course the solution in the outer zone, $|X| > X_0$, must also be found from (8) simultaneously. Both equations can be integrated once, and in particular (11) yields

$$(12) \quad B_\xi^2 = R(A_*)(B_0 - B) - \frac{8\nu}{15}(B_0^{5/2} - B^{5/2}),$$

where $R(A_*) = 2M(A_*) - \frac{4}{A_*} \int_0^{A_*} M(A)dA$

Here we have used the matching conditions at $|\xi| = \xi_0$ and the normalisation condition that $B = B_0$ at the wave maximum at $|\xi| = 0$, where $B_0 = N_0 h^2 / \pi$. In particular it follows that solutions only exist when $4\nu/3 < R(A_*)$, implying a maximum possible amplitude above A_* for these waves with a vortex core. Further, as this maximum amplitude is approached, the vortex core width increases indefinitely. The structure of equation (11) indicates that the inner zone and the vortex core have a universal structure, and depend on the upstream conditions only through the single parameter $R(A_*)$. However, whether or not these solitary waves can exist at all depends in general on the details of the nonlinear function $M(A)$ in (8).

3 Rotating flows

3.1 Formulation

Let us next consider the steady, axisymmetric flow of an inviscid, incompressible fluid, confined to a circular tube. In standard notation the equations of motion are,

$$(13) \quad \begin{aligned} u_x + \frac{1}{r}(rv)_r &= 0, \\ uu_x + vu_r + p_x &= 0, \\ uw_x + vv_r + p_r - \frac{w^2}{r} &= 0, \\ uw_x + vw_r + \frac{vw}{r} &= 0. \end{aligned}$$

Here the coordinates are (x, r, θ) with x in the axial direction, r in the radial direction, and θ in the azimuthal direction, (u, v, w) are the corresponding velocity components, and p is the pressure. The streamfunction ψ is defined in the usual way,

$$(14) \quad ru = -\psi_r, \quad rv = \psi_x.$$

The last equation in (13) shows that the circulation is constant along a streamline, so that $rw = C(\psi)$. Here the functional form of $C(\psi)$ is yet to be determined. For those streamlines which originate upstream, it is determined from the upstream angular velocity profile, $\Omega_0(r)$ where $w = r\Omega$. Finally, eliminating the pressure from the middle two equations in (13) yields the vorticity equation,

$$(15) \quad \psi_{xx} + \psi_{rr} - \frac{\psi_r}{r} + C(\psi)C'(\psi) = G(\psi).$$

As for $C(\psi)$ the functional form of $G(\psi)$ is found from the upstream boundary conditions for those streamlines which originate upstream, but otherwise is as yet undetermined. The boundary conditions are that

$$(16) \quad \psi \rightarrow \psi_0(z) \quad \text{and} \quad \Omega \rightarrow \Omega_0(z), \quad \text{as } x \rightarrow \infty,$$

while at the tube centre, and on the tube wall,

$$(17) \quad \psi_x = 0, \quad \text{on } r = 0, a.$$

Equation (15), together with the boundary conditions (16) and (17), is a nonlinear elliptic boundary value problem. The analogy with the stratified flow formulation in the previous section is clear. As for that case, our aim is to construct, asymptotically for long waves, a family of finite-amplitude solitary wave solutions, including those containing a zone of recirculating fluid. The account given here is necessarily brief, and more details can be found in [12].

3.2 Asymptotic Analysis

As for the stratified flow problem of the previous section, the basis of our approach is the identification of the upstream conditions which lead to a nearly linear equation without any restriction in wave amplitude. Thus here we observe that this is achieved when the upstream axial flow and the upstream angular velocity are both nearly uniform. Hence we assume that

$$(18) \quad \Omega_0(z) = \Omega_1 + \sigma\Omega(z), \quad \text{and} \quad \psi_0(z) = \frac{1}{2}cr^2 + \kappa\Psi(z).$$

Here Ω_1 is a constant value characterizing the nearly uniform angular velocity, and c is the wave speed to be determined, while the functions $\Omega(z)$ and $\Psi(z)$ characterise the small but significant departures from the uniform state,

so that σ and κ are two small parameters, both of $O(\epsilon^2)$ where the small parameter ϵ characterises the ratio of the radial scale a to the long axial length scale. Thus we introduce the new horizontal variable $X = \epsilon x/a$, and seek solutions for which ψ is a function of X and r . An asymptotic solution is sought as a power series in ϵ^2 , whose leading term is given by

$$(19) \quad \psi^{(0)} = \frac{1}{2}c^{(0)}r^2 + A(X)rJ_1(\lambda^{(0)}r/a), \quad \text{and} \quad J_1(\lambda^{(0)}) = 0.$$

Here $\lambda = |2\Omega_1/c|$ and J_1 is the Bessel function of order 1. At the next order in the asymptotic expansion an inhomogeneous linear equation is obtained whose compatibility condition leads to the required equation for the wave amplitude $A(X)$, which has precisely the same form and meaning as (8). As there, when all streamlines originate upstream, solutions of (8) such that $A \rightarrow 0$ as $|X| \rightarrow \infty$ describe solitary waves.

Next we observe that a sufficient condition for all streamlines to originate upstream is that $\psi_r > 0$. Here we are concerned with the situation when this condition is violated at some point on the wave profile. Hence we suppose that $\psi_r = 0$ when $|X| = X_0$ for some critical amplitude A_* , so that equation (8) holds for $|X| > X_0$ and $|A| < A_*$. To leading order $A_* = 2ca/\lambda^{(0)}$ when the breakdown occurs on the tube axis.

The analysis now closely parallels that for the stratified flow case. Thus, for $|X| < X_0$ we construct an asymptotic solution which contains a recirculation zone located now on the tube axis, within the region defined by $0 < r < \eta(x)$. For the flow within the recirculation zone, equation (15) again holds, but the circulation $C(\psi)$ and the function $G(\psi)$ cannot be determined from the upstream conditions. Instead, we assume that the circulation is zero, or at least asymptotically small; this is to ensure that the flow satisfies the Rayleigh stability criterion, namely that $(u - c)C(\psi)C'(\psi) > 0$. Further it can be shown that within the recirculation zone the vorticity $G(\psi)$ is sufficiently small that the flow is stagnant to leading order in the asymptotic analysis.

The recirculation zone and the inner region ($|X| < X_0$ and $\eta < r < a$) are characterised by a different length scale from the outer region ($|X| > X_0$), and here we put $\xi = \beta x/a$. Further we assume that the height of the vortex core is $O(\delta)$ so that $\eta = \delta f(\xi)$. In the inner region the asymptotic solution is again given by (19) to leading order, but now we set $A = A_* + \mu B$, just as in (9). Here β , δ and μ are again small parameters. An optimal balance shows that the required scaling is now given by

$$(20) \quad \beta = \epsilon^{1/2}, \quad \delta = \epsilon^{1/2}, \quad \text{and} \quad \mu = \epsilon.$$

Interestingly, this scaling differs from that in (10) for the stratified flow case. As before, a compatibility condition applied to the next order equation in the asymptotic expansion, together with the use of matching conditions that the streamfunction and the velocity field are continuous across the vortex core boundary, yields the desired equation for B ,

$$(21) \quad B_{\xi\xi} - \Delta A_* + M(A_*) = \nu B^2$$

Note that while this is similar in form to the analogous equation (11) in the stratified flow case, significantly the power of B on the right-hand side is now 2 in place of 3/2. Here again ν is a supercriticality parameter, which is of order unity, and given now by $\nu = 1/IA_*$, where $I = J_0(\lambda^{(0)})^2/2$. The interface with the recirculation zone is given by $f = 2a(B/A_*)^{1/2}/\lambda^{(0)}$. As in the stratified flow case, this equation is to be solved in the region $|\xi| < \xi_0$ subject to the same matching conditions. In place of (12) we now get

$$(22) \quad B_{\xi}^2 = R(A_*)(B_0 - B) - \frac{2\nu}{3}(B_0^3 - B^3),$$

and $R(A_*)$ is defined exactly as in (12). Analogously to the stratified flow case, it follows that solutions only exist when $2\nu < R(A_*)$, implying a maximum possible amplitude above A_* for these waves with a recirculation zone. Further, as this maximum amplitude is approached, the width of the recirculation zone increases indefinitely. The structure of equation (21) indicates that the inner zone and the recirculation zone have a universal structure, and depend on the upstream conditions only through the single parameter $R(A_*)$. However, whether or not these solitary waves can exist at all depends in general on the details of the nonlinear function $M(A)$ in (8). Further details and the implications for the phenomenon of vortex breakdown are discussed in [12].

References

- [1] M. L. Dubreil-Jacotin, *Sur la détermination rigoureuse des ondes permanentes périodiques d'amplitude finie*, J. Math. Pure Appl., 13 (1937), 217.
- [2] R.R. Long, *Some aspects of the flow of stratified fluids I. A theoretical investigation*, Tellus, 5 (1952), 42.
- [3] D.J. Benney and D.R.S. Ko, *The propagation of long, large amplitude internal waves*, Stud. Appl. Math., 66 (1978), 1.
- [4] O.G. Derzho, *The propagation of solitary waves in a stratified flow with a free boundary*, Proc. Int. Symp. on Heat and Mass Transfer, Bailowieza, Poland, 113 (1992), 113.
- [5] O.G. Derzho and R. Grimshaw, *Solitary waves with a vortex core in a shallow layer of stratified fluid*, Phys. Fluids, 9 (1997), 3378.
- [6] A. Aigner, D. Broutman and R. Grimshaw, *Numerical simulations of internal solitary waves with vortex cores*, Fluid Dyn. Res., 25 (1999), 315.
- [7] S.L. Bragg and W.R. Hawthorne, *Some exact solutions of the flow through annular cascade actuator discs*, J. Aero. Sci., 17 (1950), 243.
- [8] T.B. Benjamin, *Theory of vortex breakdown*, J. Fluid Mech., 14 (1967), 593.
- [9] T.B. Benjamin, *Some developments in the theory of vortex breakdown*, J. Fluid Mech., 28, (1967), 593.
- [10] S. Leibovich and A. Kribus, *Large amplitude wavetrains and solitray waves in vortices*, J. Fluid Mech., 216 (1990), 459.
- [11] J.J. Keller, *On the interpretation of vortex breakdown*, Phys. Fluids, 7 (1995), 1695.
- [12] O.G. Derzho and R. Grimshaw, *Solitary waves with recirculation zones in axisymmetric rotating flows*, J. Fluid Mech., (submitted).
- [13] S.R. Clarke and R. Grimshaw, *The effect of weak shear on finite-amplitude internal solitary waves*, J. Fluid Mech., 395 (1999), 125.