THE INFLUENCE OF THE BASIC ELECTRONIC CALCULATOR ON THE
TEACHING AND LEARNING OF MATHEMATICS IN THE 11 - 16 AGE RANGE

by

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ABSTRACT

The electronic calculator is now invariably the device used by people in employment and everyday life to deal with complicated and tedious calculations. The aim of this dissertation is to examine the effect it may have on the secondary school mathematics curriculum and, especially, to examine its potential as a powerful teaching aid which can be used to help pupils to acquire understanding of mathematical concepts.

Chapter 1 investigates the contribution the basic calculator makes as a calculating aid which should cause the teacher to reassess the place of the standard pencil and paper algorithms in the curriculum. Some of the fears associated with this innovation are also discussed. The final section emphasises the importance of knowing the idiosyncrasies of different calculators.

Chapter 2 suggests, in some detail, ways in which the teacher may use the calculator to enhance the understanding of certain topics such as fractions and place value. Applications of the calculator to everyday life problems, such as compound interest, are also included as well as the possibility of more interesting and enjoyable topics being introduced into the syllabus. New methods, such as iterative procedures, are discussed and the potential of the calculator as an aid to investigations is ascerted.

Chapter 3 looks at the beneficial influence of the calculator on the mathematics curriculum generally and the possible effect on the mathematical content in particular with further suggestions following on from Chapter 2. Some contentious issues are considered and it is emphasised that more must be done to encourage the _effective_ use of the calculator and not allow it to be overshadowed by its more 'glamorous' counterpart - the microcomputer.

ACKNOWLEDGMENTS

I declare that this dissertation is entirely my own work.

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Chapter 1 : <u>THE CALCULATOR AS A CALCULATING AID</u>

1) Historical Background
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It is generally accepted that the man who designed and produced the first calculating machine was Blaise Pascal (1623-62), a Frenchman whose father was a government official. The father always seemed to be weighed down with tedious calculations, so Blaise at the age of 19 years produced his first machine in 1642.

"by means of which you alone may, without any effort, perform all the operations of arithmetic and may be relieved of the work which has often fatigued your spirit ...."

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HISTORICAL BACKGROUND

Most people are aware that there is a great debate going on about
mathematical education in primary and secondary schools, but most
people are not aware that it has been going for over 100 years.

In recent years there has been considerable discussion of the
nature of mathematical understanding. There is general agreement
that understanding in mathematics implies an ability to recognise
and to make use of a mathematical concept in a variety of settings,
including some which are not immediately familiar. The idea that
"the ability to solve problems is at the heart of mathematics"
(Cockcroft, 1982, 231) is certainly not a new one. Polya (1961)
put this well many years ago: "Mathematical know-how is the
ability to solve problems - not merely routine problems but
problems requiring some degree of independence, judgement, origin-
ality and creativity".

The debate has centred mainly on the emphasis that should be put
on the computational techniques involved in arithmetic and on
the ability to understand and solve the problem and how this
understanding may be achieved.

The following comment was made by an H.M.I. in 1895: "The
accuracy of the work is all that can be desired, and in many
cases marvellous: at the same time the oral test shows that the
children are working in the dark .... (This) shows itself in
the inability .... to solve very simple problems".

The situation seems to have changed little over the years. The
first report from the Assessment of Performance Unit based on
a 1978 survey, concludes: "Most 11 year olds can do maths
involving fundamental concepts and skills and do simple applica-
tions. There is a fairly sharp decline as understanding is

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probed more deeply into more complex settings and unfamiliar contexts."

One factor which has contributed to the difficulty in problem-solving is that the standards by which the results of mathematical education are judged too often only concern the childrens' facility with tools - multiplication, adding fractions and so on. The pressure on the teacher is predominantly in these terms. Far less external concern is expressed about childrens' problem-solving. The Cockcroft Committee received very many submissions which showed concern for the 'basics' usually defined in terms of purely arithmetic skills, with stress on the operations of addition, subtraction, multiplication and division treated in isolation from application to real situations. The skills which are basic are presumably needed as a basis either for the mathematics required in employment or in adult life or for further study. Although many of the requirements may be considered to be 'element-ary' in terms of their position within the hierarchy of learning mathematics and the stage of schooling at which they are first introduced, it does not follow that they are necessarily either simple or straight forward for most pupils to learn, and, more importantly, to apply.

The most significant event in recent times in mathematical education has been the introduction of so called 'modern' mathematics. A direct outcome of a conference held in South-amption in 1961 was the setting up of the School Mathematics Project. A Director of S.M.P. wrote in his report for 1962-63 that "a major aim of the syllabus is to make school mathematics more exciting and more enjoyable, and to impart a knowledge of the nature of mathematics and its uses in the modern world".

The course was designed for those pupils whose mathematical attainment was in the top quarter of their age group and the

classroom materials designed for the course took for granted that the teachers using them would possess sufficient mathematical insight and experience to enable them to work in the ways intended. However, as the movement spread the materials were used by lower attaining pupils and by teachers who did not have the necessary mathematical background and training to appreciate the intentions underlying the new course. The materials were often presented as a collection of disconnected topics whose relevance to the mathematics course as a whole did not become apparent to the pupils. Many teachers who had previously concentrated all their efforts on the basic skills now went to the other extreme and neglected them. This led to claims and counter-claims by opponents and proponents of the movement and at one stage a teacher claimed to be either 'modern' or 'traditional'. Public concern for standards grew to such an extent, that pressure was created in some quarters for a 'back to the basics' movement which culminated in the reknowned Ruskin College speech by James Callaghan in 1976.

So pity the poor teacher who has been tugged this way and that way in a relatively short period of time. It is no wonder that the 1979 H.M.I. report on secondary education referred to the problem in these terms: "Many schools are not finding it easy to strike an appropriate balance between the varied demands which are being made on them at the moment ..... The dilemma .... is accentuated by demands for 'greater numeracy' because the notion of numeracy is so often ill-defined. In a great many cases the schools are responding by concentrating narrowly on computational skills, devoid of context or application, in a way which easily becomes counter productive. The notion of numeracy should certainly include more than accurate computation .... it should include .... the ability to apply knowledge in fresh circumstances".

The importance of developing the skills of computation for all members of society has never been doubted. The problem has been

that so much time must be devoted to mastering these skills for
many pupils, that the reason for developing them is too often
neglected - that is, their use which enables an individual to cope
with the practical demands of his everyday life. There have been
times when the educational system has been in danger of producing
a nation of clerks. Too many children and adults who have been
quite incapable, for a variety of reasons, of mastering these
underestimated skills have become disillusioned at best and at
worst left with "feelings of fear, helplessness, anxiety and
even guilt" (Cockcroft, 20)

What joy if somehow these skills could be mastered more quickly
or their importance nullified to some extent! More avenues would
then be opened to the really rewarding, enjoyable and even
exciting aspects of mathematics.

About 30 years ago the electronic calculator was invented. In
the mid 1970's their price started to fall so rapidly that education-
alists recognised the very great implications for the teaching
of mathematics in schools.

ATTITUDES TOWARDS THE USE OF CALCULATORS IN SCHOOL

These implications are far reaching and include the ways in which
calculators can be used to assist and improve the teaching of
mathematics in the classroom and the extent to which they should
change the content of what is taught and the relative stress
which is placed on different topics within the mathematics syllabus.

However, the real controversy surrounding the introduction of
calculators is about the use of them by children who have not
yet mastered the traditional pencil-and-paper methods of computat-
ion. It is feared that children who use them too early will not
acquire fluency in computation nor confident recall of basic
number facts. These fears are understandable and may be ill-
ustrated by views expressed in submissions to the Cockcroft
Report (375) For instance:
"Exercise of the basic skills should not depend upon use of
calculators: these should be limited to higher education".
"Mathematics must be a compulsory subject, taught to a reasonable
standard using one's loaf and not a calculator".

These views are not just shared by some teachers and parents
but also by employers. The Bath and Nottingham studies, report-
ing to the Cockcroft Committee, believes that there is an ambivalent
attitude to the use of calculators in industry and commerce at
the present time. In many types of employment which require a
considerable amount of calculation and analysis of data, they
are regarded as desirable aids to speed and accuracy. However,
their use is still viewed with suspicion by some shop floor
managers and supervisors (especially those who supervise engin-
eering and other technical apprentices and craftsmen of various
kinds) who themselves were trained to use slide rules or logarithm
tables. However, the majority of young employees who were seen
to be using calculators at work had not been trained in their
use either at school or on the job. In consequence, calculators

were frequently not being used in the most effective way. On one fact, however, there is general agreement - calculators are here to stay. They are becoming increasingly available to young and old, in all walks of life, and the proper use of them must be taught at some stage. Most would also agree that this is best accomplished at school. However, questions still surround the stage at which they should be introduced in the school life of the student; their use in examinations; which ability groups would benefit most from them; and so on.

Some would argue that low-attaining pupils should not use them at all but should concentrate on mastering the basics, while others support the view that it is these pupils who would benefit most from the use of a calculator.
"Calculators have revolutionised computation and barely numerate students can overcome their weaknesses with these". (Cockcroft 375)

Proponents stress that the availability of a calculator in no way reduces the need for mathematical understanding on the part of the person who is using it. For example, knowing how to multiply is a different skill from knowing when to multiply. A calculator is useless until the student knows how to solve the problem.

There remains too, some uncertainty about introducing calculators in the primary classrooms. Views vary from complete freedom to replace pencil and paper calculations to a complete ban on them. Their is little evidence about the eventual balance to be obtained at the primary stage between calculations carried out mentally, on paper, or with a calculator. However, one submission to the Cockcroft Committee from a parent may well prove significant: "Following professional advice of mathematical colleagues we kept calculators away from our children until their late teens. But the youngest at age 6 got hold of a calculator to help out with his 'tables' and found it such fun that he has been much

more mathematically inclined since. So perhaps it would be wise to introduce simple calculators at an early age".

In the secondary school, support for the use of the calculator is much more prevalent.

In their 1981 questionnaire to Heads of Department, the Mathematics Advisers of Hertfordshire asked for details of policy concerning the use of the calculator. Most replies from schools indicated that a policy did exist with only a few making reference to a 'flexible policy' or 'at the teacher's discretion'. Most schools gave a stage at which its use was first allowed which indicated the popular attitude was for mastering basic computational skills with pencil and paper first and that other uses of the calculator did not rate highly. The information is:-

Allowed from 1 st Year onwards 6%
   "   "   2   "   "   8%
   "   "   3   "   "   12%
   "   "   4   "   "   45%
   "   "   5   "   "   8%

However, the Advisers have reason to believe that this information is already out of date. Since the publishing of the Cockcroft Report there is evidence that its strong advocacy of the use of calculators has already encourage their greater use in the classroom.

Two significant events in 1983 provide evidence that the move towards accepting the calculator in the classroom is gaining momentum. In January 1983, the G.C.E. and C.S.E. Boards Joint Council for 16+ National Criteria published its recommendations for mathematics. The written submissions from all walks of life overwhelmingly demanded that electronic calculators should be used

in at least part of the assessment. The list of assessment
objectives includes 'the use of the electronic calculator'
and the content list includes 'the efficient use of an electronic
calculator' for all children. Thus if the criteria are approved
by the Education Secretary, it will be assumed that all children
from 11 to 16 will use them throughout secondary school and
will take them into public examinations in their final year.

An equally significant event will be the introduction of the first
calculator-based course. In September 1983 the new S.M.P.
11 - 16 course will be available to all schools. At its inception
5 years ago, the team of teachers who planned the course assumed
that calculators would be available to all pupils, and devised
the materials accordingly. The course has been highly acclaimed
by schools involved in the pilot scheme. Much of the material
necessitates the use of a calculator and takes advantage of the
power of the calculator to promote understanding as well as allow
realistic problems to be introduced. Other material requires
work to be done without its use. To ensure mental skills are
not neglectedthe team is devising oral tests for all pupils when
they leave school.

NUMERACY AND THE STANDARD WRITTEN ALGORITHMS

Numeracy

The concept of 'numeracy' and the word itself were introduced in
the Crowther Report published in 1959. The definition is intended
to imply a quite sophisticated level of mathematical understanding,
but over the last few years the meaning seems to have changed and
it is now associated with the ability to perform basic arithmetic
operations. However, Cockcroft emphasises the wider aspect and
would wish the word 'numerate' to imply the possession of two
attributes. The first of these is an 'at-homeness' with numbers
and an ability to cope with the practical mathematical demands of
everyday life. The second is an ability to have some appreciation
and understanding of information which is presented in mathematical
terms, for instance graphs, charts or tables or by reference to
percentage increase or decrease.

"Our concern is that those who set out to make their
pupils 'numerate' should pay attention to the wider aspects of
numeracy and not be content merely to develop the skills of
computation". (39)

Perhaps the most significant definition of numeracy is the one
given by Girling (1977)

"Basic numeracy is the ability to use a 4-function
electronic calculator sensibly".

The implications of this definition are more far-reaching than is
at first apparent. In brief, Girling believes this implies:
1) the need to be able to check that the user and/or
calculator have not made a mistake
2) the need to understand the relative size of numbers
3) the need to be able to perform mental calculations
for speed, convenience and for use in the commercial and industrial
world.

The extract below comes from 'Practical Arithmetic' a
book published in 1884.

Example 4.—Divide 396·293 by 41·967, so as to have six
decimal places in the quotient.

41,9,6,7,9,6,7)396293989(9·442771
877711703
18582236
16787187
1795049
1678719
116380
83936
32394
29878
3016
2988
78
42
36

The divisor and dividend are
extended by repeating the
figures of the respective recurring deci-
mals. The operation is then carried on in
the same way as in the contracted process
in Example 3. It is necessary to observe,
that when the divisor does not contain so
many figures as those required in the
quotient, the division must be carried on
entire, till the figures required in the
quotient be one less than the figures of
the divisor, before we begin to cut off
figures from the divisor. In this example,
for instance, there is one place of whole
numbers (this is obvious from dividing
396, the integral part of the dividend, by
41, the integral part of the divisor), and
six places of decimals, so that the divisor
must contain at least seven figures : to secure greater accuracy, a figure
more is taken in the annexed example.

REASON OF THE RULE.—The removal of the decimal points in the divisor and
dividend to the same number of places to the right, is just multiplying the
divisor and dividend by the same number; and this operation does not affect
the value of the quotient. The divisor being now a whole number, the number
of decimal places in the quotient must be equal to the number of decimal places
in the dividend, since the dividend is the product of the divisor and quotient.

Exercises.

1. Divide 4·98152 by ·073, and 7·3, . Ans. 68·24 and ·6824.
2. ″ 546·145811575 by 6·8531, and ·0068531,
Ans. 79·69325, and 79693·25.
3. ″ 78·5 by ·01953125, & 195·3125, Ans. 4019·2, and ·40192.
4. ″ 16·7235 by 98·7629, ... Ans. ·1693297.
5. ″ 71·237 by ·069184, ... Ans. 1029·6857.
6. ″ 9·0182 by ·271884, ... Ans. 33·157.
7. ″ 17·391 by 26·5, so as to have six decimal places in
the quotient, ... Ans. ·654905.
8. ″ 41·3374 by ·93, so as to have five decimal places in
the quotient, ... Ans. 44·53667.

To DIVIDE by 10, 100, or 1 WITH ANY OTHER NUMBER OF
NOTHINGS ANNEXED, it is only necessary to remove the decimal
point as many places to the *left*, as there are nothings in the
divisor ; thus—124·5 divided by 100, becomes 1·245. Also,

3·765 ÷ 10 = ·3765, for 3·765 ÷ 10 = $\frac{3765}{1000 \times 10} = \frac{3765}{10000} = ·3765$.

Similarly, 3·765 ÷ 1000 = ·003765, &c.

Exercises.—Divide 7·2754 by 10, 100, 10000, 1000000.

The Standard Written Algorithms

If we accept that the calculator will change but by no means
eliminate the need for reasonable mental and written skills of
calculation, an important question arises. What is to be regarded
as appropriate levels of skills in the calculator age?

Go into any classroom in this country and it is a fair bet that
the children have been taught the same standard written algortithms
for the four rules of number. With a few minor variations they will
probably look like this:

$$
\begin{array}{cccc}
\begin{array}{r} 68 \\ +29 \\ \hline 97 \end{array}
&
\begin{array}{r} {}^{5}\!\!\not{6}7 \\ -49 \\ \hline 18 \end{array}
&
\begin{array}{r} 64 \\ \underline{\times 8} \\ 512 \end{array}
&
\begin{array}{r} 13 \\ 5\overline{)65} \end{array}
\end{array}
$$

And yet is this the way people really calculate given a free
choice? It may well be an illuminating exercise for the
experienced, traditional teacher to ask his pupils how they would
tackle similar calculations mentally given a free choice. It
may well be an interesting exercise for the same teacher to examine
how he approached the calculations himself. A fascinating variety
of methods would probably emerge.

There are, of course, important reasons why the written algorithms
have been developed and persevered with over many years. They
are _written_, so the calculation is permanent and correctible.
They are _standardised_, that is, it is possible to arrange that
everyone does the same thing. They are _general_, in the sense
that they will work for any numbers, large or small, whole or
decimal. This is possible because the methods require breaking
a number up into hundreds, tens and units and dealing with these
as digits separately. However, this process does not correspond
to the ways in which people tend to think about numbers, and the

methods are usually carried out automatically without a real need
to understand what is going on. Rote learning is, thus, usually
associated with the teaching of these methods. However, they have
remained attractive because they are so general and efficient and
because they are much easier to manage and mark than mental
techniques. Add to this the fact that they are traditional and we
see why they remain so sacrosanct in the eyes of the general public.

Do real alternatives to these methods exist?

In 1973, D.A. Jones, investigated the methods used by each of 80
11 year olds to calculate

$$67 + 38, \ 83 - 26, \ 17 \times 6, \text{ and } 116 \div 4^2$$

The questions were written in this form, and the children were free
to use written or mental methods. Over half of the 320 calculations
were successfully completed by non-standard methods despite the
heavy teaching of standard algorithms. This suggests, at the least,
that the standard methods are not suitable for mental work. At
the same time, quite often they are not understood by children.
Mathematics teachers will, ho doubt, recognise these errors:

```
   44          45          43       2)218
  +28         - 37         x5         1 9
   81          12        2015
    ʌ
```

Further, the standard algorithms are misused. How often do we
see calculations such as these?

```
  1003         36        10)7320
 - 994        x100          732
   009        3600
              000
               00
             3600
```

At least the mental algorithms require understanding, and although
they are not designed for recording they can be if desired. They
are also flexible and can be adapted to suit the numbers involved.
They work with complete numbers rather than separated tens and units
digits e.g.

$$4 \times 45 = 2 \times 90 = 180;$$
$$6 \times 28 = 6 \times 30 - 12 = 168$$

Unfortunately, they are limited in the sense that they cannot be
applied to the most difficult calculations, which is why they
have not been developed further. However, with a greater emphasis
on understanding predominating these days, it would be sensible
to develop these mental techniques at the expense of the traditional
written algorithms for the easier calculations, providing an altern-
ative method can be found for the more difficult calculations.

With the advent of the electronic calculator, Stuart Plunkett
(1979), believes a real alternative now exists, which may remove
the necessity of spending an inordinate amount of time teaching
the standard algorithms. He illustrates his proposals by exam-
ining the wide range of calculations we can do with numbers.

Plunkett, for convenience, divides the range into five rough bands

Red	Orange	Yellow	Green	Blue
6+9	145+100	148+39	693+387	4974+6872+4567+7928
13-8	85-20	85-27	682-376	
5x7	5x30	18x3	841x8	891x678
45-5	60-3	72-4	693-7	8391-57

They are in order of increasing difficulty and yet probably in
order of decreasing frequency of use.
The Red band contains number bonds up to 10+10 and 10x10 and
their inverses, and these facts should be available for instant

recall.

The Orange band can be done by one-step mental methods given thorough knowledge of the Red band.

The Yellow band calculations are also appropriate for mental methods and the average person in the street should be able to do these in his head.

The Green band could be done mentally but it would probably be more appropriate to reach for pencil and paper.

In a practical situation it would be absurd to use a mental process for the Blue band, but if a calculator were available, it would be equally absurd to use a written method.

So Plunkett proposes that there is now a place for mental algorithms, the use of the calculator and non-standard written methods. He believes that the standard written algorithms are out of date because the large amount of time that is at present wasted on attempts to teach and learn them so often leads to frustration, unhappiness and a deteriorating attitude towards mathematics. The Cockcroft Committee will vouch for this.

Three stages are proposed that children may go through, progressing according to ability.

Stage 1. The acquisition of mental techniques for calculations in the red, orange, and yellow bands.

Stage 2. The use of calculators for green and blue calculations.

Stage 3. The development of some casual written methods.

Following on from this stage, an understanding of larger numbers should be made easier, and so children will be more likely to make sensible use of the calculator for more difficult problems. It will be less likely that they will press the wrong button, but their understanding of orange band calculations will enable them to check the machine's answer for reasonableness. In other words, the calculator will make sense.

Stage 3 would be necessary when calculators are not available in an everyday life situation, especially money problems. Non-standard written methods for dealing with these can be adopted by children and teachers from mental algorithms used for more manageable numbers.

The advent of the calculator has, therefore, provided us with a great opportunity. We do not need to teach methods of dealing with calculations of great complexity for the average citizen at least. We can now foster methods more suitable to the minds and purposes of the users. People who need complex calculations for their jobs will find a calculator a much more valuable aid than pencil and paper.

However, we are expected to be able to calculate, and the standard algorithms are the traditional ones. It would be unthinkable for many people that other methods and a calculator should be allowed to replace them. But the argument that most everyday calculations are mental ones and much time is wasted teaching the standard written ones is strong. As Plunkett concludes:

"The calculator should be regarded as a sensible tool for difficult calculations, the ideal complement to mental arithmetic".

GETTING TO KNOW YOUR CALCULATOR

It is inevitable that pupils will possess different makes and
models of calculators, with the consequent difficulties of co-
ordinating their usage in the classroom. Of immediate concern
are the problems which will arise when younger secondary pupils,
particularly, bring to their mathematics lessons calculators
which operate in the reverse Polish system or those which automat-
ically display numbers in exponential form. Clearly the range
of characteristics of individual makes of calculator is greater
than the two referred to above, but these are particularly critical
to the conduct of lessons in which the use of a calculator plays
a fundamental part. Fortunately, more and more schools, often
with encouragement from Mathematics Advisers, are able to make
available a class set of a single model (or at least a half class
set). Free calculators have been received by all Suffolk schools
with pupils in the age range 7-16, "sufficient to launch them into
a new era", while the advisers of Northampton provide a similar
service after the teachers have attended an in-service course
run by them on calculators. Further, schools are also advising
parents as to the type of calculator used by the school and
encouraging them to buy the recommended model. The problems of
differences between calculators will gradually recede and this
chapter will be more concerned with fundamental differences
between basic calculators. (Green and Lewis, 1978, provide a
very full account of all the differences the teacher may encounter
and also look at the inside of the calculator).

Alan Graham, Centre for Mathematics Education, Open University,
identifies three important aspects in which basic calculators
differ:

       1. logic - arithmetic or algebraic

       2. methods of correcting keying-in errors

       3. operation of the constant function - K or automatic

As an introduction a class, who have been requested to bring in their own calculators where available, may be given a set of calculator sequences to predict the answer and then to find the actual calculator answer:

	GUESS	ANSWER
(a) 2 [x] 3 [+] 4 [=]		
(b) 2 [+] 3 [x] 4 [=]		
(c) 4 [+] [-] 3 [=]		
(d) 3 [-] [-] 2 [=]		
(e) 2 [+] 3 [ON/C] 4 [=]		
(f) 3 [+] 2 [=] [=]		

this may be C or CE on your calculator

On the Texas TI 30 the answers were as follows; (a) 10 (b) 14 (c) 1 (d) 1 (e) 6

It is important that the inconsistencies that are likely to crop up be dealt with before any further work is done. A few simple tips will be of help to the pupils.

1. Type of Logic - Algebraic or Arithmetic

This is perhaps the most basic difference between calculators. At the cheap end of the calculator market, most machines are programmed to carry out the operations <u>arithmetically</u> - i.e. in the order in which they are fed into the machine. Thus, for sequence (b)

2 [+] 3 [x] 4 [=]

a calculator with arithmetic logic will perform the addition of
2 and 3 before multiplying the result by 4. A machine with
algebraic logic, however, obeys the conventions of precedence in
algebra, where multiplication and division must be carried out
before addition and subtraction. The pupils may discover ways of
coaxing an 'algebraic' calculator to calculate 'arithmetically'

(1) 2 $\boxed{+}$ 3 $\boxed{=}$ $\boxed{\text{x}}$ 4 $\boxed{=}$

(2) $\boxed{(}$ 2 $\boxed{+}$ 3 $\boxed{)}$ $\boxed{\text{x}}$ 4 $\boxed{=}$

2. Key stroke errors - and how to correct them

A later chapter emphasises the importance of estimating an answer
to a problem so that a ridiculous answer cannot be accepted.
However, quite often we make mistakes in pressing the wrong
button part of the way through a problem and realise the error
at the time. Do we start again?

Sequences (c) and (d) show what course to take if we press the
wrong <u>operation</u> by mistake. On many machines, including the
TI 30, all but the last of a sequence of operations will be
ignored. So if you accidently press $\boxed{-}$ instead of $\boxed{+}$, this
will be corrected by pressing $\boxed{+}$ next time. This does not work
for some calculators. Pressing 6 $\boxed{+}$ $\boxed{-}$ may confuse some machines
into displaying 12, while 6 $\boxed{-}$ $\boxed{+}$ may give O

Sequence (e) should confirm how the calculator copes with pressing
the wrong <u>number</u>. What happens with this sequence:

2 $\boxed{+}$ 3 $\boxed{\text{ON/C}}$ $\boxed{\text{ON/C}}$ 4 $\boxed{=}$?

3. The constant key - $\boxed{\text{K}}$

A constant facility enables the user to set the calculator up
to perform the same 'sum' repeatedly at the touch of the $\boxed{=}$ key.
If the machine has a key marked $\boxed{\text{K}}$, then this can be done directly.

For example, the TI 30 requires 5 [+] [K] to add on 5 to any
number chosen followed by [=] . Thus:

    4 [=]   gives 9
    10 [=]   gives 15
    and so on.

If the machine does not have a K key, it is likely that the
constant facility comes on <u>automatically</u> after a suitable cal-
culation is keyed in.

    For example, if the following sum is keyed in:
    3 [+] 4 [=]
    Now press
    1 [=]
    10 [=]
    100 [=]

With many calculators the addition sum ([+] 4) is carried over
as a constant addition and so 4 will be added to 1, 10, 100,
giving 5, 14, 104. Other models take the 3 [+] part of the original
sum as its constant function. A few models bring in their constant
function when an operation is pressed twice in succession and
kills it when any key other than a number or [=] is pressed.

After practice with an exercise of 'Guess and Press', for example,
2 [x] [K] 1 [=] ; [=] ; [=] ; .... , further practice may be
given in the form of a game:

<u>Guess the Number</u>

Player A chooses any number between 1 and 100
(say 40) and, unseen by B presses 40 [÷] [K] 0
(zero wipes 40 from display)

Player B has then to guess which number
(denominator) A has chosen by trying different
numbers (numerators) and pressing [=]

B's aim is to guess A's number as quickly as
possible, B then chooses a number for A to
guess.

```
┌──────────────────────────────────────────────────┐
│                                                  │
│   Sample Play:   B's attempts:                   │
│                                                  │
│   B Presses        Display          Comment      │
│                                                  │
│   1.      5 [ = ]   0.125    5 is too small      │
│                                                  │
│   2.     24 [ = ]   0.6      24 is too small     │
│                                                  │
│   3.     50 [ = ]   1.25     50 is too big       │
│                                                  │
│   4.     40 [ = ]   1·       40 is the denominator│
│                                                  │
│   B's score is 4                                 │
│                                                  │
└──────────────────────────────────────────────────┘
```

The game can be adapted if a calculator's constant is set up in
a different way.

These three important features, i.d. logic, keying-in errors,
constant facility should be checked systematically by the pupils
as described previously, before the calculator can be used effect-
ively.

Accuracy and Overflow

Another problem which arises in the use of different calculators
is why, for example, $\frac{1}{3} \times 3$ appears on some as 0.999 999 9, on
others as 1.000 000 2 and on others simply as 1; or why $(\sqrt{2})^2$
may turn out to be 1.999 999 8. The answer is connected entirely
with the conventions which have been built into the machine.
Some 'round off', i.e. give the number corrected to the number of
figures on display, some truncate, i.e. cut the end off the
number, some round up, some hold more figures than they display,
some work in scientific (exponent) form, even when they display
in index-free form. There are those machines which convert to
exponent form as soon as numbers become to large or too small
for display and others merely display 'E' when this happens

successfully eradicating all trace of your calculations so far!

The subject can become confusing but can also be an interesting source of investigation. One useful quick test enabling the teacher to diagnose the accuracy to which any particular machine works, when dividing, is to perform:

$$(7 \div 9) - 0.777\ 777\ 7$$

Some calculators are designed particularly for use in scientific problems and can work in exponential notation. Such a calculator will usually display an answer, of say, 38 000 000 as 3.8 07, meaning 3.8×10^7, or will display 0.000 000 76 as 7.6 -07. meaning 7.6×10^{-7}. However, calculators which do not work in this way, do not have enough room to display a very large or very small number, and usually have special symbols to indicate overflow or underflow. If this is the case, matters may be improved by using standard form. For example:

$$5\ 180 \times 69\ 5000 = 5.18 \times 10^3 \times 6.95 \times 10^4$$
$$= (5.18 \times 6.95) \times 10^7$$
$$= 36.001 \times 10^8 \text{ or } 360\ 010\ 000$$

More able pupils may be asked to investigate the problem

$$\frac{0.00086}{.1162} \times 438$$

by comparing and discussing answers to each of the following methods:

(1) $(0.00086 \div 1162) \times 438$;

(2) $(0.00086 \times 438) \div 1162$;

(3) $(438 \div 1162) \times 0.00086$;

(4) $\dfrac{8.6 \times 10^{-4} \times 4.38 \times 10^2}{1.162 \times 10^3}$

Recommended Calculators

A sensible question would seem to be "why can't calculators be

standardised?" It is likely that all L.E.A.'s have set up
working parties to discuss and recommend essential and desirable
features of a calculator and then try to find one on the market
that best fits these features. The Hertfordshire Calculator
Working Party meets regularly to revise its recommendations
according to availability of new models. Liquid Crystal Display
is high on its list, together with an automatic 'power off'
device, a positive 'feel' or 'click' when a key is depressed,
as well as strength and a reasonable size.

Algebraic logic, is regarded as essential and for lower
seconday classes the following factors, together with those
above, are thought to be particularly important:

Keyboard: Facility to include the following:

(a) Arithmetic Keys $\boxed{+}$ $\boxed{-}$ $\boxed{x}$ $\boxed{\div}$ $\boxed{=}$

Sign change key $\boxed{+/-}$

Reciprocal key $\boxed{1/x}$

Square Root key $\boxed{\sqrt{x}}$

Power key $\boxed{x^y}$

(b) Clear and entry $\boxed{C}$ $\boxed{CE}$ (may be combined as
a single key)

(c) Memory $\boxed{M+}$ $\boxed{M-}$ $\boxed{MR}$ $\boxed{MC}$

(d) Keys for percentage, squares and 'pi' were felt
to be useful but less important.

Display: No more than eight digits should appear in
the display.

For sixth form (or equivalent) use, a scientific calculator
should incorporate the following additional facilities:

<u>Keyboard</u>

Arithmetic keys   nth root $\boxed{x^{\frac{1}{y}}}$

                factorial $\boxed{x!}$

Trigonometric functions $\boxed{\sin}$ $\boxed{\cos}$ $\boxed{\tan}$

                with both degrees and radians. Inverse
and trigonometric functions should be
available by means of the same keys
in conjunction with the inverse
INV key.

Logarithmic functions $\boxed{e^{x}}$ $\boxed{10^{x}}$ $\boxed{\log^{x}}$ $\boxed{\ln^{x}}$

Brackets; facility for at least three levels of brackets

Statistics $\boxed{\Sigma x}$ $\boxed{\Sigma x^{2}}$
          mean $\boxed{\bar{x}}$
        standard deviation $\boxed{SD}$

<u>Display</u>

(a)   A high degree of accuracy is required. The
calculator should work with at least three digits
more than are displayed.

(b)   When the answer to a calculation exceeds normal
display capacity, the machine should automatically
switch to scientific notation.

Chapter 2 : <u>THE CALCULATOR AS A TEACHING AID</u>

1) Estimation, Errors and Accuracy

2) Fractions

3) A Classroom Investigation into Recurring Decimals

4) Place Value and Decimals

5) The Ordering of Operations

6) Growth and Decay

7) Sequence, Series and Limits

8) The Fibonacci Sequence

9) Iterative Methods

ESTIMATION, ERRORS AND ACCURACY

Whenever a computation is performed, whether mentally, pencil
and paper or with a calculator, due consideration must be given
to whether the answer is a 'reasonable' one. A 'reasonable'
answer may be considered from two important aspects. One,
we should first estimate the answer and consider whether the
estimate is sensible in terms of size, and two, after being
satisfied that it is a sensible estimate, is the answer given
to reasonable degree of accuracy in the context of the question?

Estimation of answers

Estimation is already part of today's curriculum but with the
availability of calculators, the skill of estimation is even
more important. However, we no longer need the precision in
our estimates which was often demanded for checking hand
calculation, and, in fact, to check a calculator result, one
is primarily concerned with order of magnitude and hence the
only real need is the ability to do single digit arithmetic
and work with powers of 10. For example, to estimate
358 x 7 294 we may think of this as 300 x 7 000, that is we
need only consider the leading digit rather than round. Thus,
our estimate is 3 x 100 x 7 x 1 000 = 2 100 000. More
ambitious students may prefer a better estimate and so rounding
off may be introduced together with some number relationship
notions and some common-sense.

There are no fixed rules for estimating answers and different
procedures can result in somewhat different answers. The main
point to keep in mind is that we want to get an estimate that
will tell us whether or not we have made any major error in
the use of the calculator. Some more difficult examples may
help to illustrate the points made.

Example 1)

$$0.024\ 3 \times 11.92 + 14.07 \times 0.003\ 9$$
$$\simeq 0.02 \quad \times 12 \quad + 14 \quad \times 0.004$$
$$= 0.24 + 0.056 \simeq 0.2 + 0.1 = 0.3$$

Notice that we have alternated a round off down with a round off
up and our estimate compares quite closely to the calculator
answer of 3.445 29

Example 2)

$$14.87 \times 9.25 \times 0.002\ 7 - 0.144\ 3$$
$$\simeq 14 \quad \times 10 \quad \times 0.003 \quad - 0.14$$
$$= 140 \times 0.003 - 0.14 = 0.42 - 0.14 = 0.28$$

Here we have not obeyed the accepted rule for rounding off
when giving an answer to the required degree of accuracy. In
fact, we round up 9.25 because multiplying by 10 is easy and
rounded down 14.87 and 0.002 7 to compensate. The estimated
answer is close enough to the calculator answer of 2.270 782 5
to indicate that no major error has been made by the user.

In estimating the answer to a multiplication or an addition
we try to alternate up and down roundings since questions may
arise where obeying the usual rules would result in an estimate
which is really too big or too small. In estimating the answer
to a division or subtraction problem, however, both of the
numbers involved should be rounded up or both rounded down in
order to balance the approximation.

Example 3)

$$(655.45 \div 3.22) - (27.655 \times 5.87)$$
$$- (600 \div 3) \qquad - (27 \times 6)$$
$$= \quad 200 - 162$$
$$= \quad 38$$

Calculator answer = 41.221 05

Other forms of practice would include those such as given by
S.M.P. Book 4 (1968):

'Say which answers you consider are reasonable estimates, and
in the case of the others, put down what you think would be
better'

1) 12.3 x 2.9 = 36 approx.   2) 0.105 x 0.1 = 0.1 approx.
3) 9.6 x 26.2 = 250 approx. 4) 1 023 x 19 = 2 000 approx.
5) $(20.2)^2$ = 400 approx.   6) $\sqrt{170}$ = 13 approx.
7) $\sqrt[3]{0.08}$ = 0.2 approx.   8) $\dfrac{16 \times 1.1}{3.4}$ = 5 approx.

9) $\sqrt{(3.1^2 + 6.9^2)}$ = 10 approx.
10) 62.9 x 0.9 x 0.49 = 3 approx.

Cockcroft (257) makes it clear that "the ability to estimate is
important not only in many kinds of employment but in the
ordinary activities of adult life". Industry and commerce rely
extensively on the ability to estimate but although these skills
develop on the job, employers often complain that young entrants
to industry and commerce lack a 'feel' for both number and
measurement. Cockcroft has observed that although estimation is
included in most schemes of work, it is not practiced in many
classrooms. It is felt that teachers do not appreciate how much
is implied by the words 'ability to estimate' and how long it
takes to develop this ability. The conceptual difficulty of
realising whether an answer is reasonable in the context of a
question, however, must have been experienced by most mathematics
teachers. For instance, we can take those children who have been
quite happy to accept their answer for the area of a desk top
even though that answer is close the the area of a football
pitch or a postage stamp.

There are, therefore, other aspects of estimation besides that
required for computational work on a calculator. Children need

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to estimate lengths, areas, capacities, and weights. They should be able to estimate the amount of wallpaper needed for a room, the amount of petrol needed for a journey and so on.

Limits of Accuracy and Errors

When we have made our estimation of a given computation we then perform the computation on the calculator. We now have to satisfy ourselves that the answer shown on the calculator, say 7.382 074 8, is close enough to our estimation to feel confident that we have not made any major error in pushing the calculator buttons. Do we then write down 7.382 074 8 as our answer and feel pleased with ourselves? Many children are perfectly happy to do this or just forget a few figures on the end and write down, say 7.38. Just as bad, perhaps, is that many textbooks tell them to round off the answer to 3 significant figures or 2 decimal places without explanation or without challenging the pupils to round off to a sensible answer in the context of the question.

Much of our calculation depends on measurement and all measurements are approximate. We must, therefore, think very carefully about the data being fed into the calculator and relate our answer to these data. For example, with an eight-digit display calculator and an ordinary ruler, is there much point in saying that the answer to a particular problem based on measurement with the ruler is 7.382 074 8?

The topic of limits of accuracy and errors takes even greater importance now that we are getting our answers from the calculator. The concept of the reasonableness of answers in terms of physical reality and a feel for errors may be introduced before the child reaches secondary school and then the mathematics of errors may be taught at various levels.

In any measurement, we speak of the GREATEST POSSIBLE ERROR
which is always one-half of the unit of measure used in making
the measurement, and the child should appreciate that when two
measurements are compared for their PRECISION, the more precise
of the two measurements is the one for which the greatest
possible error is smaller. It is important that measurements
be stated in a way which shows clearly how precise they are. One
way of doing this is to make certain agreements about what is
meant when we write the number for a measure in decimal form.
For example, when we write 4.52 cm. for a certain measurement,
we understand that the measurement has been made correct to the
nearest 0.01 cm. and that the greatest possible error is
$\frac{1}{2}$(0.01 cm.) or 0.005 cm. This is equivalent to writing
(4.52 $\pm$ .005 cm.) for the measurement. By this agreement each
of the digits in 4.52 is SIGNIFICANT.

Non-zero digits in a measure are always significant, but
zeros may or may not be. The child should appreciate and under-
stand the rules for zeros in a measure. He should appreciate,
for example, that 4.80 cm. is a more precise measurement than
4.8 cm. and that the zero here is significant, while the zeros
in 0.003 4 are not. Further, that if the distance between two
objects is given as 470 cm, we usually assume that the zero is
not significant and that the measurement is precise to the
nearest 10 cm. (470 $\pm$ 10 cm.) If we indicate that the zero
is significant (for example, as 47_0_ cm.) then the greatest
possible error is 0.5cm. Similarly, if the measurement of
93 000 000 miles is precise to the nearest 100 000 miles we
may indicate this as 93 _0_00 000 but then tie this in with a
more familiar way of expressing very large or very small numbers,
that is, by scientific notation. For example, 9.3 x 10^7 miles
has two significant digits, while 9.30 x 10^7 miles has three.

What about computations with numbers involving measurements?

If we are given two measurements of, say, 2.63 cm. and 6.8 cm.,
is it significant to give their sum as 9.43 cm.? The child
should appreciate that, by considering the sum of the smallest
possible measure and the sum of the largest possible measure,
the sum of the two measures is somewhere between 9.375 and 9.485
and that it is only reasonable to give an answer with the same
precision as the least precise measurement. Thus, we round
each measure to the same precision as the least precise measure
involved and then perform the addition (or subtraction).

Similarly, a rule frequently used when multiplying numbers
obtained from measurements or from approximations of any kind
('keep as many significant digits in the product as there are
in the factor with the fewest significant digits') may be
illustrated by considering the difference between the smallest
possible area of a rectangle and the largest possible area.
For example, the approximate area of a rectangle whose dimensions
are 72 cm. by 42 cm. may be written as 3024 ± 57 cm.^{2}, and we
see that the computed area cannot be more accurate than two
significant digits and may not even be that accurate.

The mathematical notion of RELATIVE AND ABSOLUTE errors
should probably be introduced after pupils have more facility
with fractions. Pupils should have an idea of the importance of
the concepts and how to actually calculate them numerically.
That is, the pupil should be able to tell you that dropping
the 8 in 0.700 8 is a small relative error, while dropping the
8 in 0.007 8 is a large relative error. The absolute error for
both, however, is the same.

The discussion about data will, no doubt, lead on to the
implications of interpreting calculator answers when the initial
data is known to be EXACT. An example is money. An amount such
as £3.27 is exact and may be written with as many zeros after
it as you wish i.e. £3.270 000 ....   Suppose we found the

simple interest on this amount for one year at $1\frac{3}{4}\%$ per annum.
The interest would be 0.057 225 0, that is 5.722 5 pence.
This is an exact answer and in the circumstances is quite correct.
How you pay it is another matter! However, since this interest
is payable yearly, the decimal parts of a penny can make a
difference over a number of years.

However, the situation is potentially confusing to the child.
Science teachers tend to use theoretical models in their
everyday problems. A physics teacher asking a child to find
the density of a 3 kg. mass, having volume 4 m.^{3}, probably
anticipates the answer 0.75 kg.m.$^{-3}$, not 0.8 kg.m.$^{-3}$. Here he
is more interested in checking understanding and/or knowledge
of the definition of density, than awareness of accuracy.
Mathematics teachers (and examiners) are often thoroughly
careless about making it clear whether a problem, apparently
involving measurements, is intended to be regarded as a
description of a real model situation or a model. In 'A circle
of 1.5 cm. ....', is 1.5 a number x such that $1.45 \leqslant x < 1.55$
or is it an exact pure number?

Nigel Webb (1980) suggests simplified versions of some rules
of accuracy already discussed. They are intended as rules of
thumb for everyday use, but he emphasises the need for the
teacher to discuss problems of accuracy and the anomalies
arising. In brief, he suggests that:

1) Where specific units of measurement are involved,
the number of significant figures given in the measurement
will be assumed to indicate the accuracy of the measurement.
In this case the result of a calculation should be given to the
same number of significant figures as the least accurate item
of data. However, care must be taken in formulating questions
otherwise the following may result:

The two sides of a rectangle are of length 3 cm. and 4 cm. respectively.

$$\text{The Area} = 3 \times 4 \text{ cm}^2$$
$$= 12 \text{ cm}^2$$
$$= 10 \text{ cm}^2 \text{ (to 1 sig. fig.)}$$

2) Where specific units of measurement are not involved, numbers given as data for calculation will be assumed to be exact, pure numbers unless otherwise described. Then the result of a calculation should be given to the maximum accuracy of the calculator.

However, John Hersee (1981) argues that measurements are implicit wherever any real situation is being modelled. "We want to develop and encourage a critical sense in young people, an attitude which assesses the reasonableness of any result achieved in the light of the original situation from which it arose. In the classroom, this critical attitude will be developed only through constant attention; it must be part and parcel of any 'modelling' of any attempt to apply mathematics to a problem".

Hersee adds that it is for this reason that the S.M.P. take the view that arbitrary, or apparently arbitrary rules should be avoided. Each situation should be assessed as it arises, unless a rule does exist. However, simplifying rules may be used for the time being, if a new topic is being introduced and the pupil needs to concentrate on the new idea and not be distracted by worries about accuracy. It is also probable that some simplifying implication of exactness may be required in the artificial environment of examining.

All this is not new, calculators have helped to draw it to our attention afresh.

FRACTIONS

The mathematics curriculum has always placed great emphasis
on the skills of computations involving fractions, and the
experienced mathematics teacher will tell you that it is one
of the most difficult and frustrating topics to teach. Most
adults will remember with trepidation the seemingly endless
exercises involving computations such as $(2\frac{3}{4} + 4\frac{5}{7}) \div (1\frac{1}{2} \times 2\frac{2}{3})$,
but how many could remember how to do them now? And, indeed,
why should they? How often are they used in everyday life or
even industry?

The Bath and Nottingham study (Cockcroft, 75-6) reports that
although fractions are still widely used within engineering
and some other craft work these are almost always fractions
whose denominators are included in the sequence 2, 4, 8, ... 64.

'The need to perform operations such as $\frac{2}{5} + \frac{3}{7}$ does not normally
arise, and the manipulation of fractions of the kind which
is commonly practiced in the classroom is hardly ever carried
out. In the rare instances in which it is necessary to multiply
or divide, it is usual to convert each to a decimal before
performing the operation, if necessary with the help of a
calculator". Further, "the notation of fractions appears in
some clerical and retail jobs, for instance $4\frac{3}{7}$ to represent
4 weeks and 3 days or $2\frac{5}{12}$ to represent 2 dozens and 5 singles.
However, school-type manipulation is rarely found and then
only in very simple cases; for instance, the calculation
required to find the charge for 3 days based on a weekly
rate, is division by 7, followed by multiplication by 3".

It would seem therefore, that the calculator has enabled us
to shift the emphasis from fraction notation and manipulation
to decimal notation and manipulation in order to describe
the world around us when whole numbers are not adequate.

In any case, the advisability of introducing fraction manipulation before secondary school age has been reassessed, and even new research has concluded that for many the abstraction of fraction manipulation is entirely inappropriate for many children who leave school still at Piaget's concrete stage.

The Concept in Secondary Mathematics and Science project (1979) concludes that many children do not feel confident with the use of fractions and try whenever possible to apply the rules of whole numbers to operations on fractions. Most children avoided the use of the addition and subtraction algorithms when another method was available. When the addition rule was used the younger children were more successful than the older ones because the rules were fresh in their memories. Other common difficulties found, which any mathematics teacher must be only too familiar with, include;

finding equivalent fractions such as $\frac{2}{7} = \frac{10}{*}$;
comparing the sizes of fractions which do not have the same denominator;
adding fractions using 'add tops add bottoms';
multiplying the whole numbers and fractions separately
e.g. $3\frac{3}{4} \times 2\frac{2}{3} = 6\frac{6}{12}$;
their own concept that 'multiplying only makes it bigger';
application of fractions to problems.

What skills should be taught then to secondary school children? Cockcroft's list (455-8) suggests that all pupils should be able to:

"Perform calculations involving the word 'of' such as $\frac{1}{3}$ of £4.50.
Be able to add and subtract fractions with denominators 2, 4, 8, or 16 in the context of measurement.

Know the decimal equivalents of $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$, $\frac{1}{10}$, $\frac{1}{100}$ and also that $\frac{1}{3}$ is about 0.33.

Be able to convert fractions to decimals with the help of a calculator".

However, the higher attaining pupils should not be limited to these concepts but should be introduced to the whole range of fractions especially those who would be studying higher mathematics. Some of the ideas that follow therefore, are mainly aimed at this ability group. While the emphasis for the average and below average ability group is on understanding the concepts behind easier fractions. Hopefully, some of the examples are interesting and worthwhile in their own right.

After the idea of EQUIVALENCE has been introduced, it may be verified by using the calculator. Once pupils are convinced that a principle is true, and a calculator is a powerful tool to this end, they are more inclined to try to understand and learn the principle. Thus,

$$\frac{15}{8} = 1.875 \quad \text{and}$$

$$\frac{15 \times 7}{8 \times 7} = \frac{105}{56} = 1.875$$

and the pupils are convinced that the two fractions are the same size. Conversely, they may be asked to check $\frac{2}{5}$, $\frac{10}{25}$, $\frac{14}{35}$ etc. and to find a rule for themselves.

Equivalence is fundamental to comparing sizes of fractions with unequal denominators and addition of fractions. The following questions may be set and verified with the calculator:

'Using the four numbers 2, 5, 7, 9

(a) Write as many fractions as you can (you may use
a number more than once)

(b) Which fraction is the smallest? The largest?

(c) Which fractions are less than 1? Greater than 1?
Equal to 1?

(d) Which fractions are equivalent?

(e) Write down the fractions in order of size.'

After ADDITION of fractions has been introduced with the aid
of equivalence, practice with more interesting problems may
be achieved by posing problems of the following type:

(1) Arrange the numbers 5, 6, 7, and 8 into two fractions
that will give

(a) the largest sum

(b) the smallest sum (Do not use any number
more than once)

(2) Consider the following sequence of numbers
$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$ etc.

(a) Add the first two numbers in the sequence

(b) Add the first three numbers in the sequence

(c) Add the first four numbers in the sequence
Do you begin to see a pattern? .

(d) Predict the sum of the first ten numbers in
the sequence.

Dubisch/Hood (1979) provide many similar exercises.

By considering the area of a rectangle, the rule for MULTIPLICATION
of fractions can be developed, that is,

$$\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d} \; ;$$

again, we can use a calculator to check this rule. For example,

to check that

$$\frac{7}{6} \times \frac{5}{3} = \frac{7 \times 5}{6 \times 3}, \text{ check on the calculator}$$

that $\quad \frac{7}{6} = 1.1666667 \text{ and } \frac{5}{3} = 1.6666667$

so $\quad \frac{7}{6} \times \frac{5}{3} = 1.666667 \times 1.6666667 = 1.9444445$

on the other hand,

$$\frac{7}{6} \times \frac{5}{3} = \frac{7 \times 5}{6 \times 3} = \frac{35}{18} = 1.9444444$$

The small difference in the answer may provoke some discussion, another desirable feature of calculator usage.

After the procedure for MULTIPLICATION OF MIXED NUMBERS has been explained and verified on the calculator, the more-able pupil may be able to discover a method of multiplying without recourse to changing to improper fractions first. For example,

$$2\frac{2}{3} \times 1\frac{4}{5} = \frac{8}{3} \times \frac{9}{5} = 4\frac{4}{5} \quad \text{or } 4.8 \text{ (on the calculator)}$$

alternatively,

$$2\frac{2}{3} \times 1\frac{4}{5} = \left(2 + \frac{2}{3}\right) \times \left(1 + \frac{4}{5}\right)$$

$$= (2 \times 1) + \left(2 \times \frac{4}{5}\right) + \left(\frac{2}{3} + 1\right) + \left(\frac{2}{3} \times \frac{4}{5}\right)$$

$$= 2 + \frac{8}{5} + \frac{2}{3} + \frac{8}{15}$$

$$= 4.7999999 \text{ (on the calculator)}$$

Again, the small difference in answers should provoke the question: Is 4.8 the same as 4.7999999?

The usual rule for division of fractions, $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$

may also be easily checked on the calculator. For example,

$$\frac{28}{93} \div \frac{17}{27} = \frac{28}{93} \times \frac{27}{17}$$

$$= 0.47817836$$

To check,

$$\frac{28}{93} \div \frac{17}{27} = .30107526 \div .62962962 = .47817836$$

It may be shown also that

$$\frac{a}{b} \div \frac{c}{d} = \frac{a \div c}{b \div d}$$

with the above question, although it does not usually tell us what the answer is as a fraction.

Similarly, it may be shown, for example, that

$$3\frac{1}{3} \div 2\frac{1}{5} \text{ is equal to } (3 \div 2\frac{1}{5}) + (\frac{1}{3} \div 2\frac{1}{5})$$

rather than

(a) $(3 \div 2) + (\frac{1}{3} \div \frac{1}{5})$ or

(b) $(3\frac{1}{3} \div 2) + (3\frac{1}{3} \div \frac{1}{5})$ or

(c) $(3\frac{1}{3} \div 2) + \frac{1}{5}$

In the C.S.M.S. tests 35% of pupils opted for (a) and only 5.1% of 15 year olds chose the correct answer. Although it

was recognised that if they had been asked to do the computation, the children may have used an algorithm which gave a correct answer, but the 35% certainly did not check the correspondence between this version and the rule.

To test the ingenuity of the pupils, they may be asked to find the value of $\frac{5}{7} + \frac{8}{9}$ on the calculator without using the memory or writing down an intermediate value. Of course, a simple calculator cannot cope with the task as it stands but if it is rearranged as

$$\left\{ \left(\frac{5}{7}\right) \times 9 + 8 \right\} \div 9$$

it can be dealt with, usually with the key sequence

$$5 \boxed{\div} 7 \boxed{\times} 9 \boxed{+} 8 \boxed{\div} 9 \boxed{=}$$

Can the method be extended to a sequence of fractions, involving either addition or subtraction i.e.

$$\frac{a}{b} \pm \frac{c}{d} \pm \frac{e}{f} \pm \ \dots \ \text{etc?}$$

Careful thought will show that it can. For example,

$$\frac{5}{7} + \frac{8}{9} + \frac{2}{3} \quad \text{may be rearranged as}$$

$$\frac{\left\{ \left(\frac{5}{7}\right) \times 9 + 8 \right\}}{9} + \frac{2}{3}$$

$$= \left[\left\{ \frac{\left(\frac{5}{7}\right) \times 9 + 8}{9} \right\} \times 3 + 2 \right] \div 3$$

Possible with key sequence

$$5 \boxed{\div} 7 \boxed{\times} 9 \boxed{+} 8 \boxed{-} 9 \boxed{\times} 3 \boxed{+} 2 \boxed{\div} 3 \boxed{=}$$

A more interesting exercise involves manipulation with CONTINUED
FRACTIONS. The child gains practice with several skills of
fractions and is also required to predict a pattern from the
results. The calculator becomes useful for converting the
fractions to decimals so that the child may concentrate on
the manipulation and predictions themselves.
An example of a continued fraction is

$$1 + \cfrac{2}{1 + \cfrac{2}{1 + \cfrac{2}{1 + \dots}}}$$

The exercise, illustrated by Green (1981), is to calculate
each stage as follows:

(1) 1 $= 1.0000$

(2) $1 + \dfrac{2}{1} = 3$ $= 3.0000$

(3) $1 + \cfrac{2}{1 + \cfrac{2}{1}} = 1 + \dfrac{2}{3}$ $\simeq 1.667$

(4) $1 + \cfrac{2}{1 + \cfrac{2}{1 + \cfrac{2}{1}}} = 1 + \cfrac{2}{1 + \cfrac{2}{3}} = 1 + \cfrac{2}{\dfrac{5}{3}} = 1 + \dfrac{6}{5} = 2.2000$

(5) $1 + \cfrac{2}{1 + \cfrac{2}{1 + \cfrac{2}{1 + \cfrac{2}{1}}}} = 1 + \dfrac{10}{11} \simeq 1.909$

(6) $1 + \dfrac{22}{21}$ $\simeq$ 2.048

(7) $1 + \dfrac{42}{43}$ $\simeq$ 1.977

(8) $1 + \dfrac{86}{85}$ $\simeq$ 2.012

The children may then be asked to PREDICT the next few results by spotting a pattern in the fractional part:

$$\dfrac{\text{Double previous denominator} + 2}{\text{Double previous numerator} - 1} \quad \text{THEN} \quad \dfrac{\text{Double} - 2}{\text{Double} + 1}$$

A check may be desirable.

Some may also spot that the decimal answers are getting closer and closer to 2 and a reasonable conclusion is that :

$$1 + \cfrac{2}{1 + \cfrac{2}{1 + ...}} = 2 \quad \text{eventually if enough stages are taken.}$$

Further investigations for rules or patterns may be carried out such as :

(1) $2 + \cfrac{3}{2 + \cfrac{3}{2 + \cfrac{3}{2 + ...}}}$

(2) $3 + \cfrac{4}{3 + \cfrac{4}{3 + \cfrac{4}{3 + ...}}}$

(3) $1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \dots}}}$ (4) $2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \dots}}}$

It is surprising how much interest is generated by this type
of investigation, and the next section will illustrate how
the pupils' enthusiasm and energies may be nurtured by an
experienced and sympathetic teacher.

A CLASSROOM INVESTIGATION INTO RECURRING DECIMALS

Most mathematics teachers would agree that some of the most successful lessons spring up from unexpected questions (and sometimes unexpected interest) during routine lessons. It can be most rewarding if the teacher allows a particular line of enquiry to continue and even branch out as long as interest is maintained. This was the particular experience of David Wiseman, Head of Mathematics at a Hertfordshire secondary school, during a routine fractions lesson with an average ability third year form. He gives a full account of how the topic developed and the questions and reactions of his pupils in a series of articles for Hertsmaths (1981-2). The initial work can be done without a calculator, but a calculator allows the lesson to proceed more fluently and quickly. This is particularly important with a low ability class who are more likely to make computational mistakes and so miss the main point of the exercise. Boredom and frustration are likely to follow unless a calculator is used at an early stage.

In short, it all began when he was discussing the decimal equivalence of $\frac{1}{2}$, $\frac{1}{4}$, $\frac{3}{4}$, and subsequently $\frac{1}{8}$, $\frac{3}{8}$, $\frac{5}{8}$, and $\frac{7}{8}$ with 3rd year set 4.

They worked out $\frac{1}{8}$ and $\frac{3}{8}$ by "dividing the bottom into the top", but one boy, of his own accord, decided to try $\frac{1}{3}$ and with a mixture of worry and excitement exclaimed "it keeps going on". This all too rare opportunity in the class for child-initiated discussion led to other children trying other fractions and resulted in:

$$\frac{1}{3} = 0.3333 \qquad\qquad \frac{1}{6} = 0.1666$$

$$\frac{2}{3} = 0.6666 \qquad\qquad \frac{5}{6} = 0.8333$$

The word 'RECURRENCE' was then introduced, and although they surprisingly refused the aid that, for example,

$$\frac{2}{3} = 2 \times \frac{1}{3} = 2 \times 0.3333 = 0.6666$$

all members of the class found decimal equivalents of each member of the 'ninths family':

$$\frac{1}{9} = 9 \overline{\smash{\big)}\ 1.0^10^10^10}\ \ldots \quad \begin{array}{c} 0.1 \cdot 1\ 1\ 1\ \ldots \end{array}$$

$$\frac{2}{9} = 9 \overline{\smash{\big)}\ 2.0^20^20^20}\ \ldots \quad \begin{array}{c} 0.2\ 2\ 2\ 2\ \ldots \end{array}$$

$$\frac{3}{9} = 9 \overline{\smash{\big)}\ 3.0^30^30^30}\ \ldots \quad \begin{array}{c} 0.3\ 3\ 3\ 3\ \ldots \end{array}$$

A reinforcement of fractional equivalence was experienced when a few realised that $\frac{3}{9} = \frac{1}{3}$.

Some children noticed the pattern in the remainders, and as this particular class had already done some work on "Clock" or "Modular" Arithmetic, the explanation for the increasing size of the remainder was given by reference to the module NINE clock :
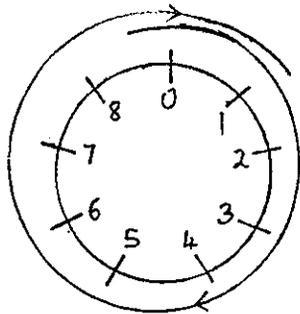
$10 \equiv 1 \pmod 9$

$20 \equiv 2 \pmod 9$

$30 \equiv 3 \pmod 9$
etc.

A homework assignment on the elevenths family resulted in excited discussion during the following lesson:

$$\frac{1}{11} = 1 \div 11 = 11 \overline{)1.0\ 0^10\ 0^10}^{\,0.0\ 9\ 0\ 9\ 0}$$

$$\frac{2}{11} = 2 \div 11 = 11 \overline{)2.0^90^20^90^20}^{\,0.1\ 8\ 1\ 8\ 1}$$

$$\frac{3}{11} = 3 \div 11 = 11 \overline{)3.0^80^30^80^30}^{\,0.2\ 7\ 2\ 7\ 2}$$

$$\frac{4}{11} = 4 \div 11 = 11 \overline{)4.0^70^40^70^40}^{\,0.3\ 6\ 3\ 6\ 3} \qquad \text{etc.}$$

Again patterns were observed in the remainders and relationships
between digits in the quotients and remainders. These observations
may be explained through knowledge of Modular Arithmetic.

It was during the following term that David Wiseman was asked
the question "When can we do some more of those divides that
go on, Sir?"

After a rapid revision of the previous lessons, a discussion
arose as to a suitable notation to save both time and space.
Several ideas were put forward by the pupils, but finally
they agreed to adopt the familiar "dot" above the recurring
number, or numbers, symbol:

$$\text{e.g.} \quad 0.1666 \ldots = 0.1\dot{6} \quad \text{and} \quad 0.272727 \ldots = 0.\dot{2}\dot{7}$$

This invitation to the children to devise their own shorthand
helps them to recognize the importance of a clear unambiguous
but concise notation. One pupil coined the phrase "dotty
number" and another pupil asked whether it was possible to
get a "dotty number", in which more than two digits recurred.
It was decided to investigate the family of sevenths. Although
it takes longer, it is again more fruitful to find each recurring
decimal by division (and leave the calculator for checking) as

there is an additional pattern in the remainders which is
also worthy of analysis.

$$\frac{1}{7} = 7 \overline{)1.0^3 0^2 0^6 0^4 0^5 0^1 0} \quad \overset{0.1\ 4\ 2\ 8\ 5\ 7\ 1\ \dots}{} = 0.\dot{1}4285\dot{7}$$

$$\frac{2}{7} = 7 \overline{)2.0^6 0^4 0^5 0^1 0^3 0^2 0} \quad \overset{0.2\ 8\ 5\ 7\ 1\ 4\ 2\ \dots}{} = 0.\dot{2}8571\dot{4}$$

$$\frac{3}{7} = 7 \overline{)3.0^2 0^6 0^4 0^5 0^1 0^3 0} \quad \overset{0.4\ 2\ 8\ 5\ 8\ 1\ 4\ \dots}{} = 0.\dot{4}2857\dot{1}$$

$$\frac{4}{7} = 7 \overline{)4.0^5 0^1 0^3 0^2 0^6 0^4 0} \quad \overset{0.5\ 7\ 1\ 4\ 2\ 8\ 5\ \dots}{} = 0.\dot{5}7142\dot{8}$$

$$\frac{5}{7} = 7 \overline{)5.0^1 0^3 0^2 0^6 0^4 0^5 0} \quad \overset{0.7\ 1\ 4\ 2\ 8\ 5\ 7\ \dots}{} = 0.\dot{7}1428\dot{5}$$

$$\frac{6}{7} = 7 \overline{)6.0^4 0^5 0^1 0^3 0^2 0^6 0} \quad \overset{0.8\ 5\ 7\ 1\ 4\ 2\ 8\ \dots}{} = 0.\dot{8}5714\dot{2}$$

Nearly all pupils discovered that each fraction recurred in
a cycle of six, and that each decimal equivalent used the same
digit; 1,4,2,8,5,7 and in that order. Only a few discovered
a similar recurring decimal in the remainders; 4,5,1,3,2,6.
An explanation may be given to the pupils in terms of "modular"
arithmetic.

For example, the cycle of six can be explained in terms of
the set of possible remainders 1,2,3,4,5,6 when we divide
unity by seven. At each stage of the division one of these
elements is the remainder. Once all the members of this
set have appeared as the remainder, then one of them appears
again.

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MODULO SEVEN	REMAINDERS

$10 \equiv 3 \pmod 7$

$30 \equiv 2 \pmod 7$

$20 \equiv 6 \pmod 7$

$60 \equiv 4 \pmod 7$

$40 \equiv 5 \pmod 7$

$50 \equiv 1 \pmod 7$
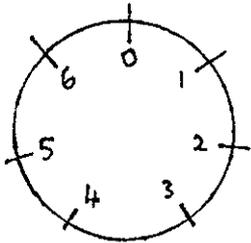
$10 \equiv 3 \pmod 7$ etc.

COUNT ROUND 10

Further, this suggests that if the remainders are re-arranged
they will form an arithmetic sequence in modulo seven:
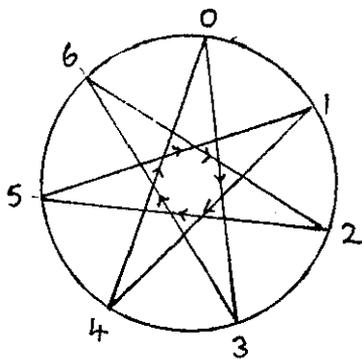
3, 6, 2, 5, 1, 4

+3 +3 +3 +3 +3

What number completes the cyclic
sequence? Why?

Does the recurring cycle 1, 4, 2, 8, 5, 7 .... form an arithmetic
sequence? If so, in which modulo?
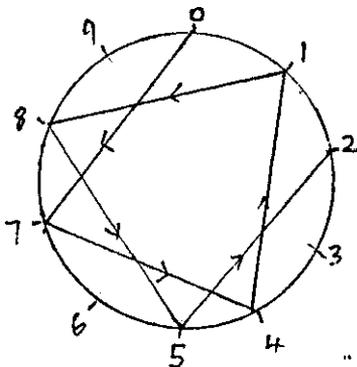
7, 4, 1, 8, 5, 2

+7 +7 +7 +7 +7

OR

$1 \times 7 \equiv 7 \pmod{10}$

$2 \times 7 \equiv 4 \pmod{10}$

$3 \times 7 \equiv 1 \pmod{10}$

$4 \times 7 \equiv 8 \pmod{10}$

$5 \times 7 \equiv 5 \pmod{10}$

$6 \times 7 \equiv 2 \pmod{10}$

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How does this sequence continue? Why do only the first six
members of the sequence occur in the recurring cycle?

Finally, one observant pupil noticed that with $\frac{1}{7}$ there is a
connection between the first four digits:

i.e. 0.1428 ....
28 = 2 x 14

Is there any interest in this?

```
.14
   28
      56
       112
          224
ADD        448  etc.
────────────────────
.1428571428 ........
```

And so the decimal equivalent of $\frac{1}{7}$ is obtained another way.

"What about the twelths?" asked Nigel.

Wiseman was able to show the class, through a question and answer exchange, that each member of the twelths was in fact related to families already considered.

$$\frac{1}{12} = \frac{1}{2 \times 6} = \frac{1}{2} \text{ of } \frac{1}{6} = 2\overline{)\underset{0.1666}{0.0833 \ldots}} = 0.08\dot{3}$$

$$\frac{2}{12} = \frac{1}{6} = 0.1666 \ldots = 0.1\dot{6}$$

$$\frac{3}{12} = \frac{1}{4} = 0.25$$

$$\frac{4}{12} = \frac{1}{3} = 0.3333 \ldots = 0.\dot{3}$$

$$\frac{5}{12} = 5 \times \frac{1}{12} = 5 \times 0.0833 \ldots = 0.41\dot{6}, \text{ etc.}$$

It was established that the thirteenths was the next family unrelated to any previous as were the thirds, fifths, sevenths, elevenths. So a valuable reinforcement of the concept of PRIME NUMBERS developed.

The class then began to find the decimal equivalent of the thirteenths family, and it is at this stage that a calculator would prove invaluable. However, the divisions still had to be done by long hand because the remainders were required, but with the assistance of the calculator.

$$13\overline{)\underset{1.0\ 0^90^{12}0^30^40^10^{10}0}{0.0\ 7\ 6\ 9\ 2\ 3\ 0}}$$

Notice the cycle of six in the decimal equivalent:

$$\left\{0, 7, 6, 9, 2, 3\right\}$$

and, as you would expect, a matching cycle of six in the remainders:

$$\{9,\ 12,\ 3,\ 4,\ 1,\ 10\}$$

$$\frac{2}{13} = 13\overline{)2.0^70^50^{11}0^60^80^20} \quad = 0.1\dot{5}384\dot{6}$$

with the quotient line: $0.1\ 5\ 3\ \ 8\ 4\ 6\ 1\ \dots$

with cycle $\{1,\ 5,\ 3,\ 8,\ 4,\ 6\}$

Discussion followed about other possible cycles for

$$\frac{3}{13} = 13\overline{)3.0^40^10^{10}0^90^{12}0^30} \quad = 0.\dot{2}3076\dot{9}$$

with the quotient line: $0.2\ 3\ 0\ \ 7\ 6\ 9\ 2\ \dots$

Which has the same cycle and remainder cycle as $\frac{1}{13}$

What about $\frac{4}{13}$? Would this have the same cycle as $\frac{2}{13}$?

After only 3 steps:

$$14\overline{)4.0^10^{10}0\ 0}$$

with the quotient line: $0.3\ 0\ 7$

an excited voice rang out "It's part of the first cycle. It's like $\frac{1}{13}$ ...

The class were then able to write in the remaining figures with further division from:
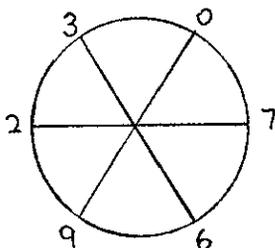
$$\{0,\ 7,\ 6,\ 9,\ 2,\ 3\}$$
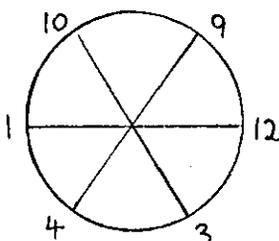
so $\frac{4}{13} = 0.\dot{3}0769\dot{2}$

"I wonder whether any other members of the thirteenth family have a decimal equivalence belonging to the second cycle, 1, 5, 3, 8, 4, 6?"

The investigation continued and pupils' answers pooled to discover:

$$\frac{1}{13} \quad \frac{3}{13} \quad \frac{4}{13} \quad \frac{9}{13} \quad \frac{10}{13} \quad \text{and} \quad \frac{12}{13} \quad \text{had cycle pattern}$$

with remainder cycle

while

$$\frac{2}{13} \quad \frac{5}{13} \quad \frac{6}{13} \quad \frac{7}{13} \quad \frac{8}{13} \quad \text{and} \quad \frac{11}{13} \quad \text{had cycle pattern}$$
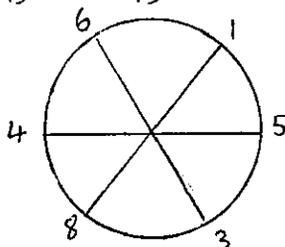
with remainder cycle
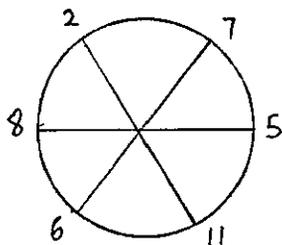
Other questions which may be investigated:

Why are there two and only two cycles?

Can you detect any symmetries in the 'bicycle' pattern?

Explanations can be given again in terms of modular arithmetic.

Further, it is interesting to note that the decimal equivalence of $\frac{1}{13}$ can be obtained in a similar kind of way to $\frac{1}{7}$:

$$
\begin{array}{lll}
7 & = & .07 \\
7 \times 9 & = & 63 \\
7 \times 9^2 & = & 567 \\
7 \times 9^3 & = & 5103 \\
7 \times 9^4 & = & 45927 \\
7 \times 9^5 & = & \underline{413343} \\
& & .0769230 \ \ldots
\end{array}
$$

It was at this stage that the classroom investigation was curtailed. However, there is usually one enthusiastic pupil who has become gripped with the work and he may be encouraged to continue with the 'seventeenths' and 'nineteenths' families on his own.

$$\frac{1}{17} = 0.0\overset{\bullet}{5}88235 \ 2941176\overset{\bullet}{4}70 \ \ldots$$

but as there is no special merit in continuing a study of remainders by long division, the work may proceed much more quickly with the calculator doing all the computation.

The result raises interesting questions such as:

Why do you get a sixteen digit cycle?

Would $\frac{2}{17}$ reproduce the same cycle, or would a second cycle appear on the scene as is the case of the thirteenths family?

For $\frac{2}{17}$ the Texas T.I. 30 calculator displays

$\boxed{0.1176471}$,

and predictiong and checking for $\frac{2}{17}$ and $\frac{3}{17}$ would continue until the sixteen members of the cycle are confirmed.

It is then easy to write down the decimal equivalents of the other members of the family of seventeenths without doing any more working:

$$\frac{4}{17} = .\dot{2}35294227647058\dot{8}$$

$$\frac{5}{17} = .\dot{2}94117647058823\dot{5}$$

. . . . .

. . . . .

$$\frac{16}{17} = .\dot{9}41176470588235\dot{2}$$

These results could be summarised in a table and this is illustrated with the family of nineteenths working on the assumption that there is only one cycle and this has 18 digits :

$\dfrac{1}{19}$ = | .0526315 |

$\dfrac{2}{19}$ = | .1052631 | 052631 belongs to   .052631 ...
                                the cycle. The
                                initial "1" may be
                                final digit of
                                cycle

$\dfrac{3}{19}$ = | .1578947 | .. 15 .. could   .052631578947 ... 1
                                belong to the
                                cycle

$\dfrac{4}{19}$ = | .2105263 | Since the cycle   .052631578947 ... 21
                                begins 05263
                                it must end
                                with 21

To find these four digits $\dfrac{18}{19}$ which is approximately equal to
0.9 was tried next.

$\dfrac{18}{19}$ = | .9473684 | 947 must be $\dfrac{1}{19}$ =
                                 followed by 3684.
                                 Since there are   .052631578947368421
                                 18 digits in the
                                 cycle it suggests ..

To check $\dfrac{16}{19}$ = | .8421052 | and the result was confirmed.

Other members of the family could now be written out without
difficulty.

Finally, in a similar way the calculator, rather than long
division, can be profitably used to find the two cycles of
the family of thirteenths.

The topic of recurring decimals has long fascinated teachers
and pupils alike and many interesting accounts of their exper-
iences may be found in articles such as Beldon (1975, pp 38),
Hewitt (1982, pp 48), Pallister (1978, pp 53), Ounsted (1978,
pp 54).

The calculator, therefore, has enabled this type of investigation
to become accessible to many more pupils than otherwise would
have been possible. In return, it gives useful calculator
practice and provides one of the many different opportunities
for children to appreciate the power and limitations of this
computational device; reinforcing, for example, the fact that
many of its answers are approximate rather than exact.

To conclude with a quote from F.R. Watson (1979)

      " ..... most pupils can be lead to an interest in
the properties of numbers, provided the associated computational
effort involved is not excessive. An important aspect of
mathematics, and one the pupils find attractive, is the
search for pattern and regularity".

PLACE VALUE AND DECIMALS

The aims of this area of the C.S.M.S work was to find whether
children could meaningfully use the base-ten place-value
notation for both whole numbers and decimals, in the sense
of both understanding how it worked and applying it to appropriate
situations. The study concentrated on the area of decimals since
most children in the 11 - 15 year age group were expected to
have a reasonably sound basic knowledge of whole numbers.
There was a major problem in trying to differentiate between
the 'relational' understanding of properties of number, and a
mere 'instrumental' ability to carry out a technique.

An important conclusion arrived at by the team was that the
learning of whole numbers and decimals is not just a matter
of recalling some place-names and a few rules of computation.
Indeed the children who did rely blindly on the rules more
often misapplied them than not.

Instead it involves internalising a whole chain of relation-
ships and connections, some with place-structure itself (e.g. 0.9
is equivalent to 0.90); some linking to other concepts like
those of fractions (e.g. the notion of one hundreth and its
relationship to one tenth); some visual correspondences and
some connecting to applications in the 'real' world.

In the interviews, the weakest group of children displayed only
a tenuous grasp of place value when asked questions involving
adding one to $\boxed{6|3|9|9}$ (with answers such as $\boxed{0|6|3|1|0|0}$
and $\boxed{0|6|4|9|9}$) and to adding ten to 3597. Writing and
verbalising large numbers also caused problems with some children.
and although the calculator's visual display can help with
these concepts, these children still need other visual models
to bring out the relationships in a more concrete way.

More problems were found with the concept of decimals and some children had difficulty in understanding that the figures after the point indicated the part of the number which was less than one unit. This came out with answers to the questions

Ring the BIGGER of the two numbers:

0.75 or 0.8.

Why is it bigger?

Ring the BIGGER number:

4.06 or 4.5

Ring the number NEAREST IN SIZE to

0.18 ⇥ 0.1 / 10 / 0.2 / 20 / .01 / 2

The interviews revealed some weaknesses in comparing the sizes of 4.9 and 4.90, adding one-tenth to 2.9 (typical answers 2.19, 2.10), and multiplying 5.13 by 10. In the latter case, the rule for whole numbers was usually known (add a nought) but wrongly applied to decimals resulting in answers such as 5.130, 50.130.

There seemed to be a marked reluctance to admit that the answer to the division of one whole number by another might be expressible as a number containing decimals, or even one containing fractions. This difficulty surfaced in the responses to questions such as:

"Divide by twenty the number 24", and dividing a smaller number by a larger one caused even more problems:

"Divide by twenty the number 16"

Over 50% of 12 year olds responded "there is no answer" to the latter question, although this reduced to 23% for 15 year olds. This may arise because you cannot share fairly 16 sweets among 20 people.

It was also clear that the idea that "multiplication always makes it bigger, division always makes it smaller" was very much entrenched. The question below illustrated this:

Ring the one which gives the BIGGER answer

(a) 8 x 4 or 8 ÷ 4

(b) 8 x 0.4 or 8 ÷ 0.4

(c) 0.8 x 0.4 or 0.8 ÷ 0.4

A reasonable response was given to the question:

"Write down any number between 0.41 and 0.42"
(37% correct from 12 year olds to 71% for 15 year olds),
but the poorest response of all items resulted from the question:

"How many different numbers could you write down which lie between 0.41 and 0.42?

It is important that all mathematics teachers should be familiar with the misconceptions and mistakes that children make which have been highlighted here. The advent of the calculator will obviously make children more familiar with decimal represent-ation, but without careful structuring of the work it may just be used to produce meaningless answers which can be copied down faithfully to eight decimal places. The calculator may be used effectively by allowing the pupil to use it to generate output, with the purpose that the output will demonstrate a concept or relationship, or with the actual generation of the output serving to help reinforce a concept which has been taught previously.

Several of the problems encountered in the C.S.M.S. tests might be overcome by using the calculator for concept-reinforcement. At a very basic level after the number system has been introduced by using a concrete method such as Dienes base 10 blocks or even just squared paper divided, the relation-ships between adjacent columns could be reinforced by successive

division by 10. For example,

$$1000 \div 10 = 100$$
$$100 \div 10 = 10$$
$$10 \div 10 = 1$$
$$1 \div 10 = .1$$
$$.1 \div 10 = .10 \quad \text{etc.}$$

or by multiplying by 10

The number 273.4894 may be written in expanded notation as

$$200 + 70 + 3 + 0.4 + 0.08 + 0.009 + 0.0004$$

and then actually performed on the calculator.

Other difficulties which arose may be approached in a similar
way or by carefully guiding the pupil to a desired result using
the technique of concept-demonstration:

What happens if you multiply 8 by 3, then 8 by 1,
8 by .2, 8 by .06 etc?

Then concept-reinforcement may be used to check other numbers
multiplied by a number less than 1. This concept may be illus-
trated by WHIRLPOOLS.
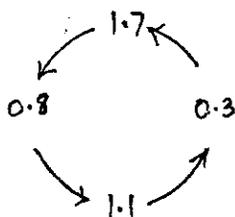
Go round the whirlpool on your
calculator. Multiply the numbers
together:

1.7 x 0.8 x 1.1 x 0.3 x 1.7 x 0.8 x
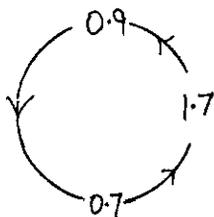and so on.

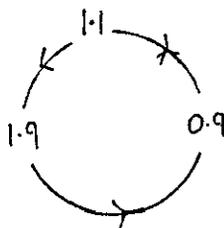What happens to the answers?
Sometimes the answer increases and sometimes decreases, but
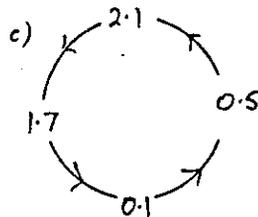the answer shrinks eventually.

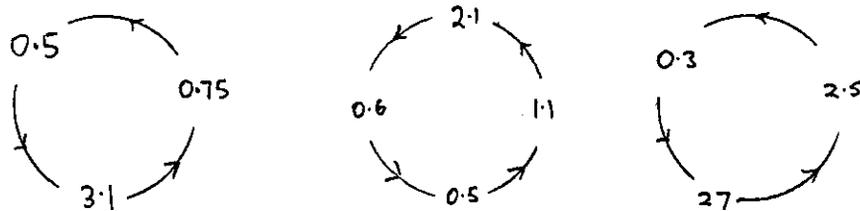Follow the whirlpools:

a)

b)

c)

For each, say whether the answers grow or shrink eventually.

First estimate whether the answers will grow or shrink and then
check using your calculator

0·5   0·75   3·1

2·1   0·6   1·1   0·5

0·3   2·5   27

Similarly for dividing by a number less than one; dividing a
number by a larger number; and developing rules for multiplying
and dividing by powers of 10.

The calculator may also be of some help when comparing the
size of numbers involving decimals, for example, 4.75 and 4.8,
by subtracting the chosen smaller from the chosen larger.
A plus sign shows a correct answer and a negative sign an
incorrect answer. If the answer is incorrect, interest may
be stimulated in the pupil to discover why.

At a higher level, the traditional approach to decimal multi-
plication is to show how the multiplication is done by a
number of examples in which the decimal fraction is first
converted to the standard fraction representation with den-
ominators of ten, hundred etc. The examples are used to justify
the rule regarding the placement of the decimal point. Thus,
the usual pedagogical sequence is first to justify the new
rule and then actually to implement it in practice with decimals.
Some pupils overlook the justification stage and tend to
concentrate on memorizing the rule often with disasterous
effects. An alternative approach could be to first ask the
pupils to look for a pattern when the calculator is used to do
some calculations (discover a rule), then justify the result
with a few, well-chosen examples, and then practice. In other
words, using the calculator for concept-demonstration -
Johnson (1978), particularly illustrates this approach.

The emphasis is on finding the pattern and then asking why it works.

Use your calculator to find the products:

84 x 0.2

0.9 x 0.7

4.5 x 0.8

4.3 x 2.7

0.04 x 0.35

1.33 x 4.75   etc.

None of the items should have a zero in the trailing digit in one of the factors, or a 5 as a trailing digit when the other factor has an even number as the trailing digit.
The class may then arrive at, or be guided to, a generalisation about the placement of the decimal point, further examples given as a check and then the justification given. For example:

Now try the following with your calculator:

3.54 x 0.45

146 x 0.35

4.70 x 0.60

Does your rule still work? What's wrong?
This is used to help reinforce the idea that the rule still holds as the calculator suppresses the trailing zeros in the result. Also this reinforces the idea that two-tenths is twenty-hundreths (further reinforced by converting $\frac{2}{10}$, $\frac{20}{100}$, $\frac{200}{1000}$, etc. on the calculator).

Some calculator games have the potential for a very real contribution to the learning of mathematics, especially for concept-reinforcement in the field of decimals. One such game is Wipeout.

The game involves entering a given number, with all digits different, and then asking the pupils to use subtraction to remove the specified digit (i.e. replace it with a zero)

without changing any other digit in the number. For example,
if the game is used to reinforce concepts of place-value, one
might start with something like 749.65128 and ask pupils to
remove the 2 and then the 5 and so on. This reinforces place-
values, since to remove the 2 requires the subtraction of .0002.
It is important that the game is complemented by other forms
of place-value practices as it could become a rote activity.

Other calculator games will be demonstrated and discussed in
a later chapter.

The C.S.M.S. Project team conclude that pencil and paper
computation techniques with complicated whole numbers or
decimals are no longer essential, but anyone who understands
enough about decimals to be able to use a calculator sensibly
should be able, given time, to work out most of the 'rules'
from first principles anyway, as a number of children managed
to do during the interviews.

"It is certainly to be hoped that the presence of
calculators will shift the emphasis from routine techniques,
which did not seem to be performed very reliably, to the
understanding of the principles, especially since this latter
aspect seems to have been neglected at secondary school level
in this particular area" (Hart, 1981 pp. 65)

THE ORDERING OF OPERATIONS

The suggestion has been made by Booth (1981) and by Hart (1981)
that part of the difficulty which some people experience in
mathematics is due to their use of intuitive "child-methods"
which, while being adequate for the "easy" questions, do not
generalize to the "hard" ones. One of the consequences of
the intuitive approach is that you do not have to be so
rigorous in what you write down, since this is always to be
interpreted in terms of 'common-sense', or in terms of the
context of the question. Consequently, as Brown (1981) points
out, in answer to the question:

> A bar of chocolate can be broken into 12 squares
> There are 3 squares in a row
> How do you work out how many rows there are?
>
> | 12 + 3 | 3 x 4 | 12 x 3 | 3 - 12 |
> |--------|-------|--------|--------|
> | 6 + 6 | 12 - 3 | 12 - 3 | 3 - 12 |

for children operating in this way, there is no ambiguity
between 12 ÷ 3 and 3 ÷ 12 since the context has determined
the meaning of the expression. Either way means "the number
of 3's in 12". (However, entering both expressions in the
written order on a calculator will, of course, produce differ-
ent answers.)

Further, Kiernan (1979) remarks that such an approach also
eliminates the need for such rules as "order of operations"
and "use of brackets", since the obvious thing to do with
3 x 2 + 4, for example, is to work it out the way it comes,
unless, of course, the context requires that the addition be
done first, in which case the expression means 3 x (2 + 4)
and not (3 x 2) + 4. However, where the child is concerned,
there is no need to actually use brackets to indicate which

interpretation is intended for the meaning is defined by the
question. If the child can handle many problems without
worrying about conventions for ordering operations or brackets,
one might ask how important a knowledge of these conventions
is. Since algebraic operations such as $x + y$ cannot be
'performed' in the sense of being replaced by a single value,
any need to record a sequence of operations immediately requires
consideration of these conventions. If the child has not seen
the need for them in arithmetic problems, it is perhaps unlikely
that he will concern himself with them in generalised arithmetic.

The S.E.S.M. Project (1982) uncovered this possibility when
investigating the mistakes that children make in beginning
algebra or 'generalised arithmetic'. Forty-eight second,
third and fourth form children from middle mathematics streams
were interviewed and were posed questions of the type:

---

(1) What can you write for the area of this rectangle?
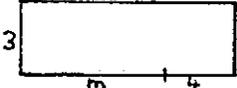
(2) Multiply $k + 2$ by 3

(3) Which of the following can you write for $e + 2$
     multiplied by 3? Tick everyone you think is
     correct.

$e + 6$	$3 \times (e + 2)$
$3 \times e2$	$3(e + 2)$
$3e + 6$	$e + 2 \times 3$
none correct	

Over one third of the children interviewed could explain how
questions 1 and 2 were solved, but only one realised that it
was necessary to use brackets in recording the answer. The
remaining children recorded their answers as:

$$m + 4 \times 3 \text{ or } 3 \times m + 4 \text{ and}$$
$$3 \times k + 2 \text{ or } k + 2 \times 3,$$

even though they stated that each problem required the addition to be performed first. In question 3, only one child selected both bracket expressions and nothing else. Some children answered "you can put the brackets in if you want, its just the same", while others excluded them altogether because they did not know what they meant.

Consequently, if we do in fact require children to have an understanding of these conventions (and for those children who are going to be taught algebra such conventions may be essential), then it would seem that we must first address ourselves to the problem of creating an awareness of the need for such conventions. The calculator can prove a useful aid to understanding this need and provides a useful source for practicing the conventions.

The class may be asked to work out 5 + 4 x 3 mentally and then on the calculator. This is one occasion when different types of logic, algebraic and arithmetic, on different calculators would prove an asset. It soon becomes obvious to the class that some rule must be formulated to avoid ambiguity.

The other rules may be built up in a similar way culminating in the overall M.D.A.S. convention (sometimes recalled as 'My Dear Aunt Sally') Brackets may then be introduced to emphasise the M.D.A.S. rule, but the problem of using the calculator for expressions such as (173 x 42) - (1638 ÷ 234) + (427 x 14) must be discussed.

Some basic calculators now offer brackets (or parenthesis) either singly or multiple nested, which act as a form of memory, storing operands and pending operations. This means that expressions are evaluated properly before being combined with other expressions. However, if these keys are not

available then the memory may be used. For example,

(3 x 4) + (5 x 6) would require:

3 | x | 4 | = | STO | 5 | x | 6 | + | RCL | =

If the memory was not available then the result of each bracket
would have to be written down first before the final answer
could be achieved. It may even be desirable to use this latter
method first even if the calculator does have a memory for
the emphasis is on use of brackets and ordering of operations.
Children at this stage may not be familiar with the use of the
memory.

When the M.D.A.S. rule needs to be broken, then the use of
brackets becomes essential (the BOMDAS convention may now be
introduced). Further calculations should be tried, firstly
by writing down an intermediate answer from the brackets, and
then the final answer calculated, and secondly by writing
down the final answer only. For example:

$$(279 - 142) \times 15$$
$$28 \times (192 + 45)$$
$$(372 + 639) \div 55$$
$$\frac{43 + 17 \times 8}{51 \times 16 - 13}$$

The check without memory may seem tedious, but at an early
stage, it may be necessary for the confidence of the child.

A third kind of check involves ESTIMATION, and this process
is important in its own right, since we are very often not
so much concerned with the exact answer as we are with an
approximate answer: the distance is about 50 miles, the
tank will hold about 50 gallons, etc. That is, learning
to estimate is useful in general as well as an aid in detecting
any major errors in computation as already discussed.

All students should be asked to check the validity of the two
basic properties of addition and multiplication: the commutative
and associative properties, by comparing results using the
calculator. At some later stage, they should compare answers
to questions such as:

Compute 352 x (1894 + 693)
and Compute (352 x 1894) + (352 x 693)

Clearly, the first is easier to compute than the second and
this leads to establishing the usefulness of another basic
property: the distributive property of multiplication over
addition. In general, for any numbers a, b, c,

$$a \times (b + c) = (a \times b) + (a \times c)$$

Also by the commutative property of multiplication, we have

$$(b + c) \times a = (b \times a) + (c \times a)$$

By devising suitable numeric examples, other properties which
are used to simplify calculations, can be validated by checking
a given result or even encouraging the student to discover the
propert himself. For example,

$$a - (b - c) = a - b + c$$
$$a - (b + c) = a - b - c$$
$$(a + b) \div c = (a \div c) = (b \div c)$$

Dubisch / Hood provide a rich source of suitable material to
demonstrate these basic principles.

The calculator, then, provides a useful teaching aid to the
topic 'ordering of operations', on the other hand, it becomes
apparent that the topic itself has become even more important
because an understanding of it is essential to the proper use
of the calculator.

GROWTH AND DECAY - AN APPLICATION OF THE CALCULATOR

How money grows.

A study of annual interest rates and compound interest can be
a useful introduction to develop the idea of a growth factor
and to exhibit the power of the calculator in evaluating
exponential growth. Interest rates are a common feature in
the high-street windows of banks and building societies and
are prominent in their press advertisements.

The work can be aimed at third year and above and assumes
some experience of percentages and their decimal equivalents,
simple sequences, indices and formation of simple algebraic
formulae.

After practice with simple interest rates for one year, followed
by calculating compound interest rates one year at a time using
the calculator, the results may be displayed as a sequence of
growing amounts of money:

$$£200 \rightarrow £224 \rightarrow £250.88 \rightarrow £280.99 \rightarrow$$

To search for a pattern of change from one number to the next
two different plans may be tried:

Plan 1 uses subtraction:
224 - 200 = 24
250.88 - 224 = 26.88
280.99 - 250.88 = 30.11

These results give the interest for successive years but do
not suggest how you might continue the sequence after £280.99

Plan 2 uses division (a calculator is a great help here)

$224 \div 200 = 1.12$

$250.88 \div 224 = 1.12$

$280.99 \div 250.88 = 1.12$

The questioning may continue as follows:

What number should you multiply £224 by to get £250.88 ?

Continue the sequence beyond £280.99 and calculate
the number of years it takes to almost double £200 with an
interest rate of 12% per annum.

What interest rate would produce this sequence?

£125 → £135 → £145.80 → ?

Can you explain this sequence:

£75 → £84 → £96.60 → £115.92 → ?

(Change of interest rate)

The amount at the end of any year can be calculated by repeated
multiplication by a constant GROWTH FACTOR.

For an interest rate of 15% the growth factor is 1.15, for
$13\frac{1}{2}$% it is 1.135, for $8\frac{3}{4}$% it is 1.0875 and so on.

For a principal of £200 and a 12% p.a. rate of compound
interest, the pattern of growth can be seen from:

£200 – multiply by $(1.12)^1$ → £224 after 1 year

£200 – multiply by $(1.12)^2$ → £250.88 after 2 years

£200 – multiply by $(1.12)^3$ → £280.99 after 3 years

. . . .

£200 – multiply by $(1.12)^7$ → £442.14 after 7 years

and so on.

Further questions to consider:

Make a formula for calculating the amount after n years

Make a formula which applies to any principal £P and
any growth factor R.

Growth Factors and Decay Factors.

The compound interest formula

   Amount after n years = Principal x (Growth Factor)

introduces the formation and use of growth factors and now the
idea of a decay factor may be explored. If scientific calculators
are being used it is worth paying some attention to the use of
a y^x key (or its equivalent), and where a calculator result
is to be followed by a realistic approximation, it is a good
idea to ask pupils to mark the calculator result in some way
e.g. Ⓒ

A growth factor is always greater than 1.

   e.g. 1.12 (= 1 + 0.12) for a 12% rate of growth
   or appreciation.

A decay factor is always less than 1.

   e.g. 0.88 (= 1 - 0.12) for a 12% rate of decay
   or depreciation.

Further exercises highlight the differences between growth
and decay factors and introduce everyday terminology associated
with them:

  (1) If inflation runs at 8% p.a. for the next
five years, how much would I expect to pay at the end of that
time for a leather coat which costs £50 now?

  (2) What value would a £1 note have, under constant
devaluation of 12% per year after 10 years?

  (3) A car depreciates in value by 30% in its first
year and then by 20% in each year after that. What % of its
value at new will the car have after 5 years?

Regular Savings

Many children will be saving regularly with a building society using the idea of the sum of a geometric progression. If there are n payments of £P each, and interest is added at a rate of R%, then the final amount is:

$$£P\left(\dfrac{\left[1 + \dfrac{R}{100}\right]^n - 1}{\dfrac{R}{100}}\right)$$

The power of the calculator may be used to combat formidable but impressive formulae such as the modification of the above formula when interest is compounded more frequently than annually:

$$\text{Amount} = P\left(k + \frac{(k + 1)}{2} \times \frac{R}{100}\right)\left[\dfrac{\left(1 + \dfrac{R}{100}\right)^{2n/k} - 1}{\dfrac{R}{100}}\right]$$

where k is the number of months before compounding occurs. (Green and Lewis, 1978 pp. 77)

Regular Payments

Many children would have heard of a <u>repayment mortgage</u>, for a house purchase, and, although it would not be of any immediate practical significance, the calculator provides another opportunity for the more-able child to take advantage of the challenge of following its workings.

For example, payment per year for a loan of £5000 at $8\frac{1}{2}\%$ for 25 years may be built up as follows:

Year 1 : Amount owed = 5000

Amount plus interest = 5000 x 1.085

Repayment = A

Year 2   :   Amount owed = 5000 x 1.085 - A

Amount plus interest = (5000 x 1.085 - A) x 1.085

$$= 5000 \times 1.085^2 - A \times 1.085$$

Repayment = A

Year 3   :   Amount owed = $5000 \times 1.085^2 - A \times 1.085 - A$

Amount plus interest = $(5000 \times 1.085^2 - A \times 1.085$

$$- A) \times 1.085$$

$$= 5000 \times 1.085^3 - A \times 1.085^2$$

$$- A \times 1.085$$

Repayment = A

. . . .

Year 10   : $5000 \times 1.085^9 - A \times 1.085^8 - \ldots - A \times 1.085 - A$

Year 20   : $5000 \times 1.085^{19} - A \times 1.085^{18} - \ldots - A \times 1.085 - A$

The end of the 25th year must see the loan completely repaid.
The amount at the start of the 26th year must therefore be
zero:

$$= 5000 \times 1.085^{25} - A \times 1.085^{24} \ldots - A \times 1.085 - A = 0$$

$$= 5000 \times 1.085^{25} = A(1.085^{24} + 1.085^{23} + \ldots + 1.085 + 1)$$

$$= 5000 \times 1.085^{25} = \frac{A(1.085^{25} - 1)}{1.085 - 1}$$

(the sum of a G.P. has already been developed)

$$= 38433.8 = A \times 78.6678$$

$$= A = 488.56$$

The annual payment is £488.56

The following table shows how the loan decreases during the
25 years and the proportions of interest and repayment, the
yearly payment change.

Total £488.56

| | | Interest Payment | |
Year	Outstanding Loan	(8½% of loan)	Loan repayment
1	£5000	£425	£63.56
2	£4936.44	£419.60	£68.96
3	£4867.48	£413.74	£74.82
	. . . .		
23	£1247.80	£106.06	£362.50
24	£865.30	£73.55	£415.01
25	£450.29	£38.27	£450.29

SEQUENCES, SERIES AND LIMITS

The study of simple sequences and series can give pupils
valuable insight into interesting mathematical ideas and
generalisations. This can lead into the use of algebraic
formulae for the terms of a sequence and the sum of a series.
The role of the calculator in the topic is simply to carry
out additions, subtractions, multiplication and division and
so enable the pupil to concentrate on investigating patterns
and relationships more easily and quickly.

The series of odd numbers is considered first, including
investigating

    a)   the number of terms in the series and it sum and

    b)   the last term and the sum of the series.

Further pupil material develops the topic by working with
the sequences of whole numbers and triangle numbers.

Sequence of Odd Numbers

Although the sequence of odd numbers may not be of any practical
value to the vast majority of pupils, interest can be heightened
by presenting it in a physical context which aids the imagination.
This may be achieved by considering stacking tins of beans in
a supermarket, an idea included in "calculators Count", Schools
Council (1983)
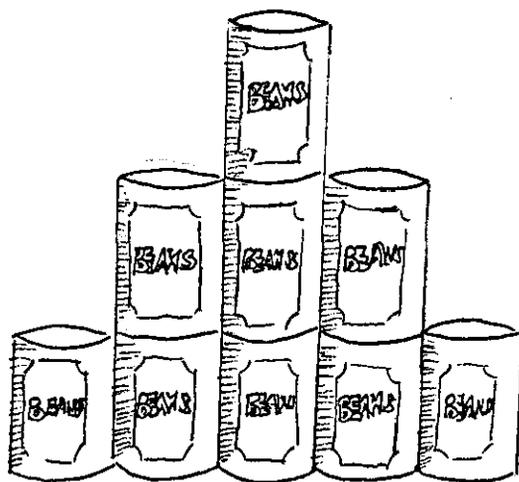
Number of layers in the stack	Number of tins in the bottom layer	Total number of Tins in the stack
1	1	1
2	3	1 + 3 = 4
3	5	1 + 3 + 5 = 9
.		
.		
.		
.		
10		

The build up to the concept may proceed with a similar sequence
of questions and instructions to:

Use a calculator to complete the table.

How can you quickly calculate the numbers of tins
needed to make the stack 15 layers high? The more able pupils
should be encouraged to express the relationship algebraically
i.e. $S_n = n^2$ when n is the number of terms.

Check the relationship using the calculator

All children should find that:

Number of layers $\longrightarrow$ | squared | $\longrightarrow$ number of tins needed

How many layers could be built with a) 169 tins

b) 247 tins?

The square root button may be used or a trial-and-error method.

From the table, what has to be done to each number in
the second column to make the corresponding number in the first
column?

How quickly can you calculate how many tins are needed
if there are to be 27 tins in the bottom layer?

This exercise is designed to develop the general relationship
between the last term in the series and its sum. With many
pupils it is enough to express the relationship in words.

Number of tins in $\longrightarrow$ | ? | $\longrightarrow$ Number of tins
the bottom layer needed altogether

Mor able children should be encouraged to go on to express
it as an algebraic formula

$$e.g. \quad S = \left[\frac{1 + 1}{2}\right]^2 \quad \text{where 1 is the last term in the series.}$$

How many tins would there be altogether if the bottom
layer contained
   a) 33 tins
   b) 397 tins?

Challenge: One of the stacks has some layers missing
at the top. The bottom layer had 89 tins. The top layer has
11 tins. Can you calculate how many tins were in the stack?

Whole Numbers and their Sum

To investigate the relationship between the number of terms in
the series of whole numbers and its sum, the tins may be
stacked in a different way
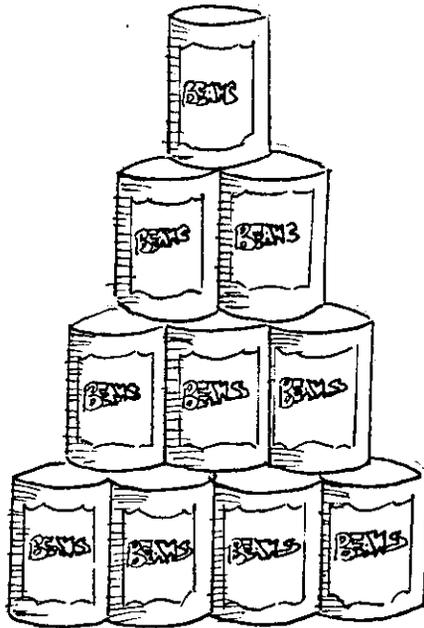
Following a similar build up to odd numbers, most children
should be able to work out a relationship by considering the
table for 12 layers

```
┌────────────┐
│ 12         │────────┐
└────────────┘        ↘   ┌──────────┐      ┌──────────┐
                          │ WORK OUT │ →    │ DIVIDE   │ →  78 tins
┌────────────┐        ↗   │ 12 x 13  │      │ BY 2     │
│ 12 + 1 = 13│────────┘   └──────────┘      └──────────┘
└────────────┘
```
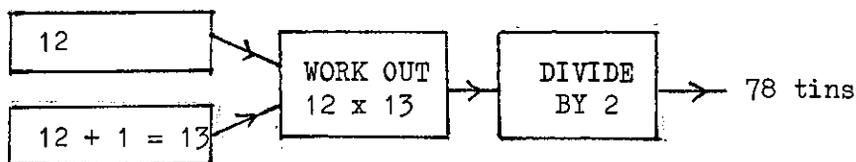
More able children should be able to express this algebraically

$$e.g. \quad S = \frac{n(n + 1)}{2} \quad \text{when n is the number of terms.}$$

It should be noted that each sim is called a TRIANGLE number.

Triangle Numbers and their sum

To introduce the series formed by the sequence of triangle
numbers, and the relationship between the number of terms
and its sum, stacking in the supermarket may be used again.
This time, however, we consider stacking oranges in a pyramid
that has a triangular base with:

      4 oranges in the top layer

      3 oranges in the second layer

      6 oranges in the third layer, and so on
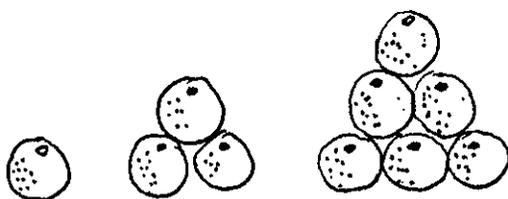
It will be difficult for many pupils to visualize the successive
layers and so table-tennis balls glued together can be used for
each separate layer.

Is there an easy way to calculate the number of oranges needed?

A table of results leads to: (for 12 layers)
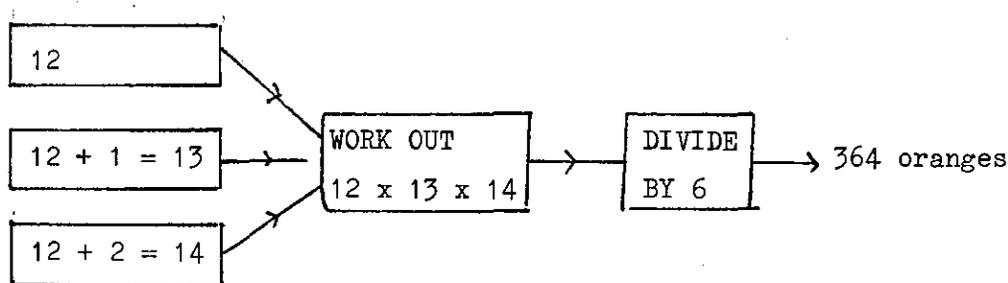
Again the more-able pupils may express the relationship as

$$\text{e.g.} \quad S = \frac{n(n + 1)(n + 2)}{6}$$

After using the calculator to practice the generalisation,
the applications of the relationship in reverse may be tested
and this involves seeking three consecutive whole numbers
whose product is known.

e.g. How many layers could be built with 1540 oranges?
All pupils should be encouraged to experiment with their
calculators initially, using trial and error to find the solution.
For many pupils this may be as far as they can go. However,
it should be possible for more able pupils, with a suitable
background of knowledge, to develop a better strategy using
the integer immediately below the cube root of the known
product as a starting point:

e.g. $n(n + 1)(n + 2) = 9240$ (For 1540 oranges)

$\sqrt[3]{9240} = 20.98$ using the $\left[x^y\right]$ key

so try n = 20.
check 20 x 21 x 22 = ?

"When you multiply any three consecutive numbers together and
then divide the answer by 6, you always finish up with a whole
number".

Can you explain why this MUST happen?

Many pupils will again benefit from experimenting with their calculators before trying to suggest a formal answer.

The following style of argument should be expected from pupils:

At least one of the consecutive numbers must be even and have 2 as a factor.

Exactly one of the consecutive numbers must be in the '3 times' table and have 3 as a factor.

Therefore the product must have 6(= 2 x 3) as a factor.

More searching questions:

A big stack had some layers missing from the top.
The bottom layer had 120 oranges and the top layer had 55 oranges.
How many layers were missing?
How many oranges are in the stack?

The Geometric Series and a Limit

To study some properties of a simple convergent geometric series and to introduce the idea of a mathematical limit, the topic, at its simplest level, can be approached by using decimals alone, in conjunction with a calculator. At a more advanced level fractions should be used alongside decimals.

A jumping frog is used to introduce sequences in which successive terms are generated by a process of halving. For example:

A clockwork frog can leap 32 cm when fully wound up.
Its next leap is half as long (16 cm)
Its next leap is half again (8 cm) and so on.

Copy the table and use it to answer the questions

Number of jumps	Length of jump (cm)	Total length jumped (cm)
1	32	32
2	16	48
2	8	56
.		
12		

If the frog had 12 jumps, how far would it jump
altogether?

At its basic level, this exercise provides an opportunity
for discussing place value and calculator technique e.g. use
of a memory, if present. If fractions are also used, it is
easier to tackle the further question:

If a frog had an unlimited number of jumps, could
it ever jump 64 cm altogether?

In considering the seventh jump onwards, this could lead to the
sequence $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32} \ldots\ldots$ and $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \frac{31}{32}$ as the
corresponding partial sums.

The idea of a mathematical limit can then become most apparent.

Another clockwork frog can leap only 20 cm when fully
wound up.

Its next leap is half as long – and so on.

In only 8 jumps, can this frog cross a rug that is
40 cm wide?

A 'super-de-luxe' frog has an adjustable first jump.

This can be set in centimetres and tenths of a centi-
metre; for example 21.3

1) What is the smallest first jump that must be set
so that this frog can cross a 40 cm rug in 5 leaps?

2) Find the smallest jump that must be set so that
this frog can cross the rug in 7 leaps.

At a basic level, these questions give an opportunity for
systematic decimal search (restricted to one decimal place)
using a calculator. At a more advanced level, pupils might
be asked to predict the answer on theoretical grounds, before
checking with a calculator.

The argument might run as follows .....

First jump is 1 unit long

Second jump is $\frac{1}{2}$ unit long

Third jump is $\frac{1}{4}$ unit long

Fourth jump is $\frac{1}{8}$ unit long

Fifth jump is $\frac{1}{16}$ unit long

In 5 jumps a total of $1\frac{15}{16}$ units

So

$40 \text{ cm} = \frac{31}{16}$ units

$40 \text{ cm} - 31 = \frac{1}{16}$ unit

$(40 \text{ cm} - 31) \times 16 = 1$ unit

Thus, length of first jump should be 20.645 cm (theoretically)
Hence, the 'super-de-luxe' frog should be set at 20.7 cm in
order to cross the rug in 5 leaps
In general, if the frog is allowed n jumps, then

$$40 \text{ cm} = \frac{2^n - 1}{2^n - 1} \text{ units}$$

THE FIBONACCI SEQUENCE

In the year 1202 the Italian mathematician Leonardo Fibonacci
completed his book 'Liber abaci'. In the book was the follow-
ing problem, which has inspired mathematicians up to the present
day.

"A pair of rabbits can breed when they are two months
old. They will produce another pair of rabbits each month
after that. The new pair of rabbits will also be able to breed
after two months and will also produce a new pair of rabbits
each month. Starting from a single pair, how many rabbits will
there be in 12 months?"
The illustration shows part of the solution
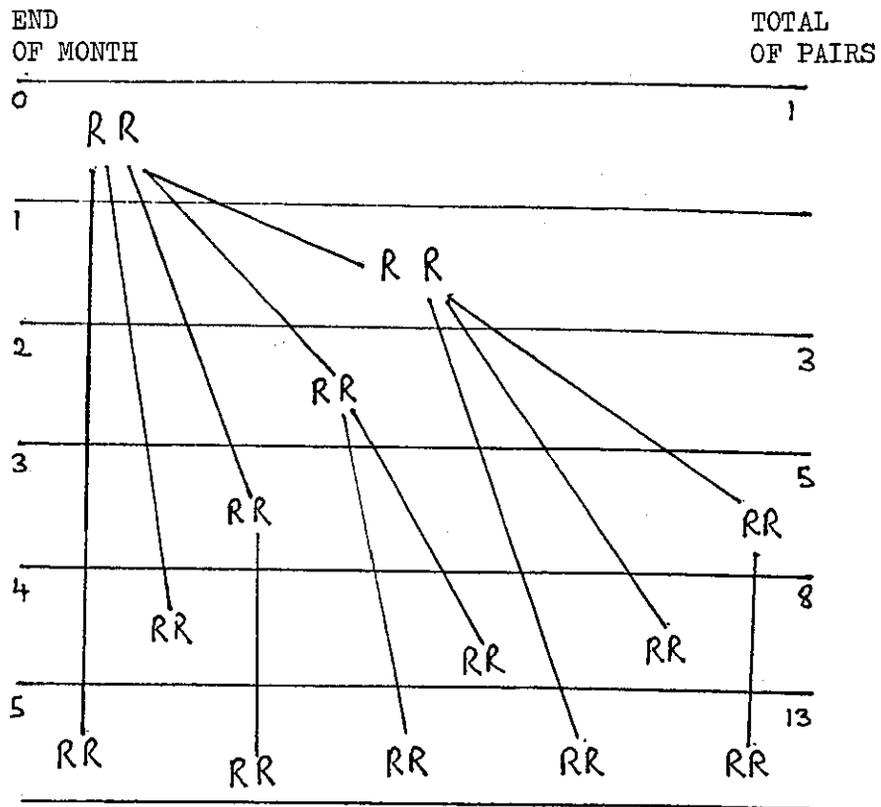
Tree graph for Fibonacci's rabbits

What has interested mathematicians is not the solution itself
but the sequence of numbers which is generated in the finding
of it. Each number in the series can be made by adding the
two previous numbers together: 1, 1, 2, 3, 5, 8, ...
This sequence is known as the Fibonacci Numbers.

Many other examples can be used to demonstrate their generation.
One way is to list all the ways of using 1p coins and 2p coins
to make up 1p, 2p, 3p, etc.

1p	o	1
2p	oo, 0	2
3p	ooo, Oo, oO	3
4p	oooo, Ooo, oOo, ooO, OO	5
5p	ooooo, Oooo, oOoo, ooOo, oooO, OOo, OoO, oOO	8

Other ways include the dichotomous branching of a tree year by
year and a progression of increasing sets of notes which is
musically very tidy and in line with the conventional devel-
opment of western music.

The Fibonacci Numbers provide a wealth of patterns which may
be investigated using the calculator. This material can be
made suitable for a wide ability range.

The most interesting and significant sequence can be produced
by working out the ratios of neighbouring pairs of numbers
in the Fibonacci sequence as follows:

Fibonacci numbers	Ratios
1	
1	$\frac{1}{1} = 1.0000$
2	$\frac{2}{1} = 2.0000$
3	$\frac{3}{2} = 1.5000$
5	$\frac{5}{3} - 1.6667$
8	$\frac{8}{5} = 1.6000$

Continue this pattern writing down the ratios to 4 decimal places. The calculator will enable pupils to conclude that the ratios get closer and closer to 1.6180 .. a number known as the 'GOLDEN RATIO'.

The pupils may then be asked to investigate the sequence produced by using the same rule but with different starting numbers. They should be encouraged to use different starting jumbers amongst themselves too. They should all arrive at the same limit as the numbers get bigger – the Golden Ratio – 1.6180.

A further investigation would change the rule by adding three consecutive terms to produce the next. Will this sequence also have neighbouring numbers whose ratios get closer to 1.6180? A discussion should preceed the calculations. The results of investigating adding 4, 5 or more numbers may be practiced in a table and graphically.

Since the time of the ancient Greeks the Golden Rectangle has played an important part in art and in architecture. A Golden Rectangle can be drawn using the ratio 1:1.618. It is said to be one of the most pleasing of all geometric shapes and the dotted lines in the drawing show how the Golden Rectangle fits the famous Parthenon at Athens.
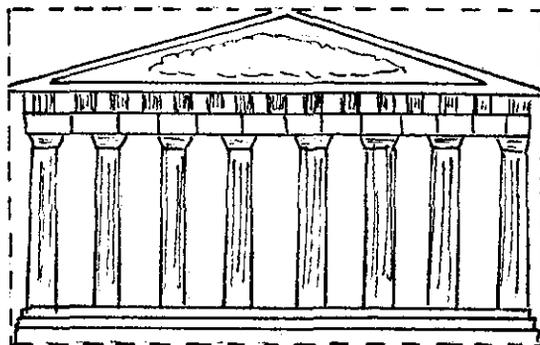
The pupils may produce an approximate one, without measuring,
by adding squares on squared paper several times as follows:
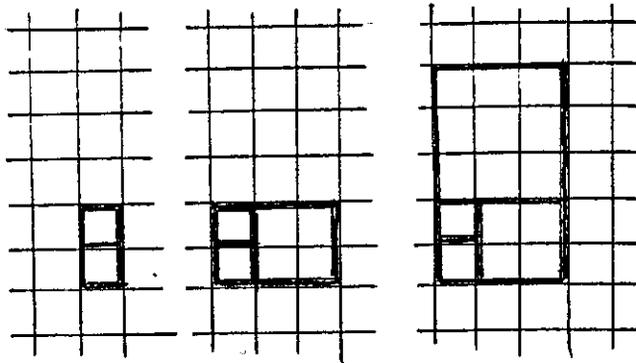
After many stages you will have an almost perfect Golden
Rectangle .

It follows that if you draw a Golden Rectangle and take away
a square you are left with another Golden Rectangle. If you
take away a square from this Golden Rectangle you get ...
another Golden Rectangle. The Golden Rectangle is unique in
this way.

The more able pupil may be shown this in another way:

Given a rectangle of width 1 unit and length x, find x such
that if a square of side 1 unit is cut away, the remaining
rectangle is similar to the original.
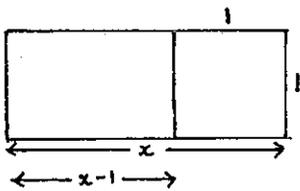
$$\frac{x}{1} = \frac{1}{x-1} = x^2 - x - 1 = 0$$

$$x = \frac{1 \pm \sqrt{5}}{2}$$

$\frac{1 + \sqrt{5}}{2}$ is the Golden Ratio

The continued fraction $\quad 1 + \cfrac{1}{1 + \cfrac{1}{1 + \dots\dots}}$

may also be investigated and be shown to approach the Golden Ratio.

Successive stages give $1, \frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \ldots$

The calculator proves an invaluable aid to checking Binet's formula for the nth term of a Fibonacci Sequence, namely:

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1 + \sqrt{5}}{2}\right)^n - \left(\frac{1 - \sqrt{5}}{2}\right)^n\right)$$

Furthermore as $n \to \infty$ $\left(\frac{1 - \sqrt{5}}{2}\right)^n \to 0$

and so $F_n \to \frac{1}{\sqrt{5}}\left(\frac{1 + \sqrt{5}}{2}\right)^n \simeq 0.447214 \times (1.618034)^n$

Other interesting properties of the Finonacci Sequence may be investigated by studying patterns and extending them.

e.g.   a) $1 + 1 = 3 - 1$ b) $1 + 2 = 3$

                 $1 + 1 + 2 = 5 - 1$ $1 + 2 + 3 = 8$

                 $\ldots\ldots\ldots$ $\ldots\ldots\ldots$

       c) $1^2 + 1^2 = 1 \times 2$

                 $1^2 + 1^2 + 2^2 = 2 \times 3$ d) $1 \times 2 = 1^2 + 1$

                 $\ldots\ldots\ldots$ $1 \times 3 = 2^2 - 1$

                                           $2 \times 5 = 3^2 + 1$

                                           $\ldots\ldots\ldots$

Pupils should be encouraged to explain them.

The Fibonacci Sequence is only one of many sequences and number patterns that may be investigated more thoroughly and enjoyably now a calculator is available to deal quickly with calculations.

ITERATIVE METHODS

One method of finding a reasonably close answer to a problem
which does not have an exact solution is to find an approximate
one, and then use this to obtain a second approximation which
is more accurate than the first. The process may be repeated
so that we obtain a third approximation which is more accurate
than the second. We may go on like this until we obtain the
degree of accuracy that we desire. Unfortunately, some of the
calculations involved are rather difficult, although many of
the methods themselves are fairly simple. Now that calculators
(and computers) have become available to tackle the tedious
series of operations, the methods are a more feasible proposition
for the 11 - 16 year range. The methods are called ITERATIVE
METHODS and the process is ITERATION.

Perhaps the simplest place to start for this range is deter-
mining the square root of a number correct to a given number
of decimal places by the method of decimal search. It is an
example of the way in which the systematic use of trial and
error can help solve a problem and illustrates a particular
advantage which calculators can bring to problem solving.

After some initial work on the concept of a square and a
square root the pupils may be asked to find 13 to a given
number of decimal places.

The number line is a useful visual aid to many pupils.

Step 1
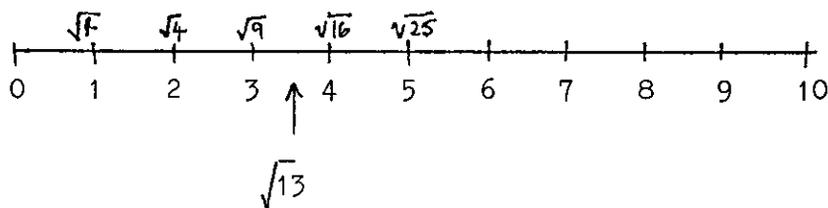
$\sqrt{13}$ is between 3 and 4, but is it nearer to 3 or 4?

Step 2

```
S  |___|___|___|___|___|___|___|___|___|___|  L
  3.0  3.1  3.2  3.3  3.4  3.5  3.6  3.7  3.8  3.9  4.0
```
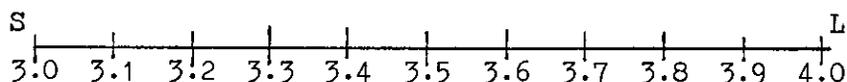
3.0 is too small (S) and 4.0 is too large (L)

Try half way between 3 and 4, that is 3.5

The calculator is used to test if this is too small or too large.

3.5 is too small, so try 3.6, 3.7 until one is too large.

3.7 is too large.

Mark on the diagram therefore, $\sqrt{13}$ is between 3.6 and 3.7.

Step 3

```
S |___|___|___|___|___|___|___|___|___|___|  L
 3.60 3.61 3.62 3.63 3.64 3.65 3.66 3.67 3.68 3.69 3.70
```
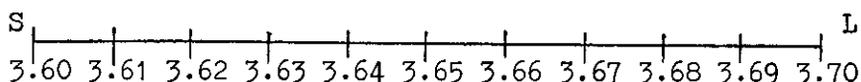
Continue by trying 3.65. Mark all the numbers on the diagram with S or L.

Continue until $\sqrt{13}$ is found correct to two decimal places.

The method also increases an awareness and understanding of place value in decimals, especially in relation to the number line. A similar pattern could be used to find the cube root of a number.

A more interesting method than 'interval halving' for finding the square root of a number is the iterative method known as Hero's Method. However, pupils should not gain the impression that square roots should normally be obtained in this manner! The square root is used here as a vehicle for the introduction of iterative methods and when a square root is needed in a calculation the usual way is to use a calculator with a square root key.

In order to find the square root of 70, the calculator is used to fill in the table

Estimate (E)	70 ÷ E (F)	New Estimate (F + E) ÷ 2
1 st 2 nd 3 rd		

The plausibility of 8 as a first estimate should be discussed and why $F > \sqrt{70}$ if $E < \sqrt{70}$ (and vice-versa).

The iterative formula for producing the sequence of estimates of the square root of some number x, that is $E' = \left[\dfrac{x}{E} + E\right] \div 2$ when E' is the new estimate calculated from the previous estimate E, should then be investigated.

When E' agrees with E to calculator accuracy, we could say that E' and E are both approximately equal to some number N. We could then write the formula as

$$N \simeq \left[\dfrac{x}{N} + N\right] \div 2$$

Rearrange this formula so that the connection between N and x becomes very clear.

It will be seen that the formula approaches the answer very quickly, but what happens with other starting points? How many steps?

Using a similar build up the pupils may be introduced to

$$N \simeq \left[\dfrac{x}{N^2} + 2N\right] \div 3,$$

and then be asked to rearrange so that the connection between N and x becomes very clear.

Does this require as many steps as

$$N \simeq \left[\frac{x}{N^2} + N \right] \div 2?$$

What could this new iterative method be used for?

Try to write down a similar iterative formula which might be used to find the fifth root of x.

(Terms such as 'converge' and 'recursive' may be introduced at the discretion of the teacher and the more able pupils may be shown the use of the more traditional notation used in many textbooks such as

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{70}{x_n} \right)$$

Advantages of this method include the fact that even if the first guess is a poor one, the correct answer will still emerge. Further, even if a mistake is made the correct answer will still emerge although, of course, after more steps than would have been necessary otherwise.

It may be desirable at some stage to encourage the use of the constant key on the calculator and then the use of the memory, but these should not be rushed or assumed to be understood by the pupils. Some pupils may well prefer to continue setting out each intermediate step as a check.

Pupils should appreciate that most iterative methods would be too lengthy and too complicated to continually carry out on a basic or scientific calculator - a programable calculator or computer would be more appropriate. However, by following a set of simple instructions, pupils may be able to work out what a particular iterative procedure does. Take, for example, the following set of instructions. Choose a number from 8, 64, 125 and call it x.

1) Start with 1

2) Multiply by your number x

3) Take the square root

4) And again!

5) Record your result but do not clear your calculator

6) Repeat steps 2, 3, 4, 5 over and over again until successive recorded results agree when corrected to three decimal places.

7) Write out your last result correct to 3 decimal places

8) Stop

What relationship has your final result with your number x?

The pupils have, in effect, been using the following iterative formula to produce a sequence of estimates to the cube root of x. It has been achieved using a calculator with a square root button.

$$E' = \sqrt{\sqrt{Ex}}$$

Rearrange $N = \sqrt{\sqrt{Ex}}$ to show the connection between x and N.

What might you expect these iterative formulae to lead to?

a) $E' = \sqrt{\sqrt{\sqrt{Ex}}}$?

b) $E = \sqrt{\sqrt{\frac{1}{E}x}\, x}$? Test your idea

An alternative form of the instructions given may be a flow chart.

```
                    ( Start )
                        |
                        v
                  +-----------+
                  |  Set E=1  |
                  +-----------+
                        |
                        v
              +-------------------+
              |   Calculate       |              +-----------+
              |      F            | <----------- |  Print E  |
              |   = √(Ex)         |              +-----------+
              +-------------------+                    |
                        |                              |
                        v                              |
                    /   Does  \                        |
                   /   F = E   \                 +-----------+
                  /  Correct    \      No        |   Set     |
                 <   to 3        >---------->     |   E = F   |
                  \  dec.       /                +-----------+
                   \  places ? /
                    \         /
                        |
                        v Yes
              +-------------------+
              |   Print F         |
              |   Correct to      |
              |   3 decimal       |
              |   places          |
              +-------------------+
                        |
                        v
                    ( Stop )
```
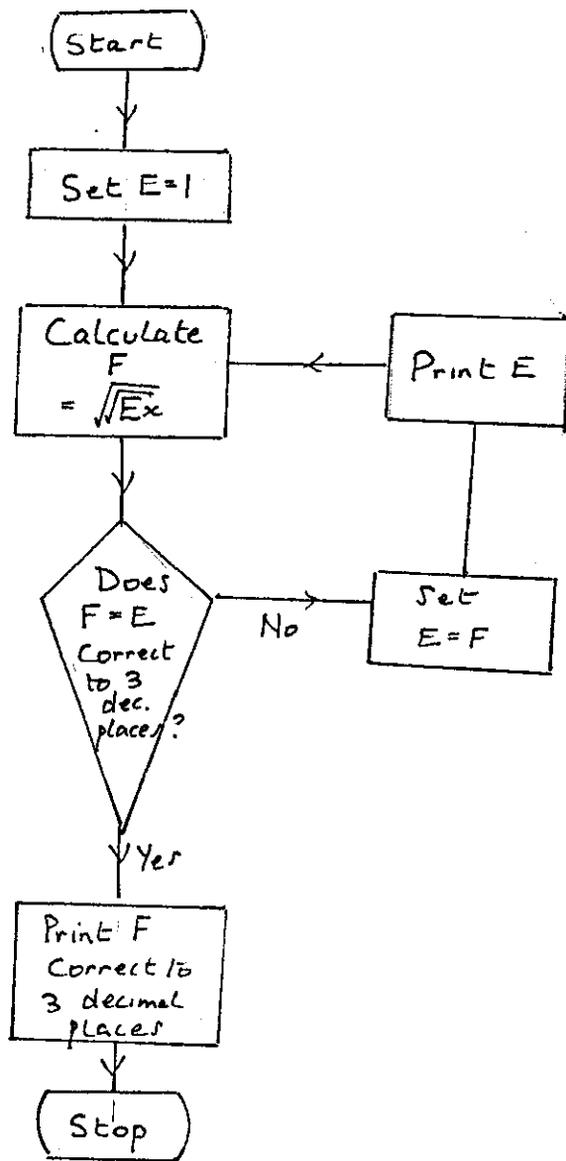
SOLUTION OF EQUATIONS

The work on iterative methods, sequences and limits may be
extended to the more able fifth year pupils and more traditional
notation used.

If we require the solution to a problem such as

$$x^2 - 2x - 2 = 0$$

we might rearrange as follows:

$$x^2 = 2x + 2$$
$$= \quad x = \pm\sqrt{2x + 2}$$

If the corresponding iterative process:

$$t_{n + 1} = \sqrt{(2t_n + 2)}$$

or alternatively $\quad x_{new} = \sqrt{2x + 2}$

results in a limit then it will be a solution of the original
equation. Putting $t_1 = 1$ will soon confirm that this leads to
2.732 as a solution of $x^2 - 2x - 2 = 0$
The process may be summarised as follows:
   1)   Rewrite the equation to be solved in the form
        $x = f(x)$ for some $f(x)$
   2)   Use $x_{new} = f(x)$ to generate a sequence of x-values
   3)   If a limit results, then this value satisfies
        $x = f(x)$ and so is a solution of the original
        equation.

Some important questions arise including:
        Will the processes always yield a solution no matter
how the equation is rearranged?

Is the starting point for the iterative process important? Well chosen examples will show that convergence may arise for some starting points but not others and that not all arrangements will, in fact, lead to a solution.

The situation may be confusing for the pupils and so a graphical approach can be employed to illustrate why some sequences diverge and others converge onto one particular solution of the related equation. Further, it should assist in deciding which particular rearrangement might be profitable.

We may adopt an investigatory approach to the solution of an equation in order to illustrate these main ideas.

Consider $x^2 - 7x - 3 = 0$

First rearrange in the form $x = f(x)$

There are many possibilities, for example

$$x^2 = 7x + 3 \text{ leads to } x = \sqrt{(7x + 3)} \text{ and } x = \frac{x^2 - 3}{7}$$

$$x(x - 7) - 3 = 0 \text{ leads to } x = \frac{3}{x - 7}$$

At this stage we could choose one of the above, together with a starting value, and see what emerged from the related iterative process, but this is rather hit-or-miss in approach to a complete solution. For example

$$x_{n + 1} = \sqrt{(7x_n + 3)}$$

with $x_1 = 10$ gives 7.405 as a limit but

$$x_{n + 1} = \frac{x_n^2 - 3}{7}$$

with $x_1 = 10$ does not give any limit.

If, however, we first examine the graphs $y = x^2$ and $y = 7x + 3$ then we can see that there are two solutions to be found.
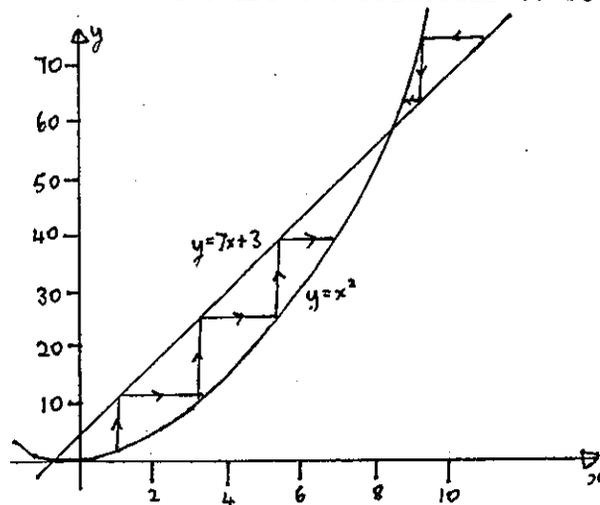
$$x = \sqrt{(7x + 3)}$$

gives rise to a path going 'vertical' to the line and 'horizontal' to the curve and thus any positive starting value leads to a larger solution as illustrated. However,

$$x = \frac{x^2 - 3}{7}$$

is effectively an inverse process but with the arrows reversed. Thus, a too large starting value (7.405) produces a divergent sequence, whereas for $x_1 < 7.405$ (and > -7.405) the resulting sequence converges to the other solution.
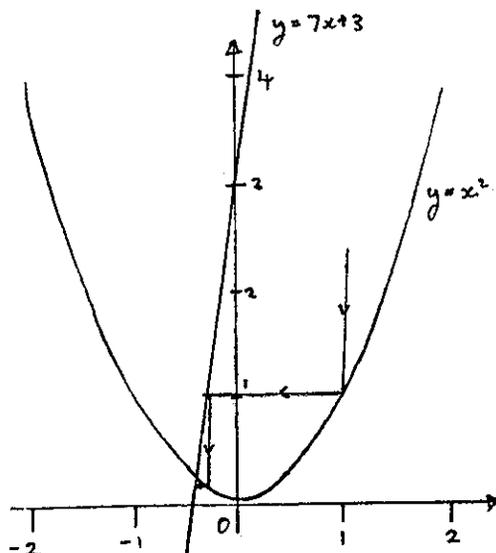
If we choose, say, $x_1 = 3$

$$x = \sqrt{(7x + 3)} \quad \text{yields } 7.405$$

$$\text{and } x = \frac{x^2 - 3}{7} \quad \text{yields } -0.405$$

which furnishes a complete solution of the equation.

Pupils appreciate visually the idea of spiralling towards a point of intersection but it is not so easy to understand how to work the corresponding numerical process. A step-by-step illustration is desirable and even then considerable teacher assistance will be necessary. Unless a graph-drawing exercise using the calculator is to be incorporated as part of the lesson, it would be useful to have the graphs already prepared by the teacher.

CHAPTER 3 – <u>THE CALCULATOR AS AN INFLUENCE ON CURRICULUM DESIGN</u>

1   The Effect of the Calculator on the Mathematics Curriculum

2   Some outstanding issues on the effective use of the Calculator

3   The Effect of the Calculator on Mathematical Content

"Look – if you have five pocket calculators and I take two away, how many have you got left?"

WITH ACKNOWLEDGEMENTS TO PUNCH

THE EFFECT OF THE CALCULATOR ON THE MATHEMATICS CURRICULUM

Over the past few years the introduction of the calculator into schools has become less of a contentious issue. However, it is doubtful whether the impact of this technological tool has been as dramatic as has been predicted and hoped for by many educators. This chapter will discuss some of the advantages the calculator could bring to the mathematics curriculum in general, the effects it might have on mathematics content in particular, and some outstanding issues in its effective use.

In brief, the calculator can be used:
1) To assist in UNDERSTANDING some mathematical CONCEPTS.
2) As an aid to PROBLEM-SOLVING.
3) To encourage discovery, exploration and creativity by introducing appropriate INVESTIGATIONS and mathematical ideas.
4) To introduce 'REAL WORLD' DATA into problems.
5) To MOTIVATE pupils by introducing more interesting and challenging topics.
6) As a SERVICE FUNCTION across the curriculum.
7) As an aid to COMPUTATION.
8) As an aid to more FLEXIBLE TEACHING programmes. For example, mixed ability teaching and individualised learning by 'freeing' the teacher and closing the gap between ability groups.

Add to the list a statement from Cockcroft(389):
·"We believe that there is one over-riding reason why all secondary pupils should, as part of their mathematics course, be taught and allowed to use a calculator. This arises from the increasing us which is being made of calculators both in employment and adult life".
and we see how strong the arguments for its use are.

THE CALCULATOR AS AN AID TO UNDERSTANDING SOME MATHEMATICAL CONCEPTS

The calculator is now invariably the device used by those people involved in regular day to day calculations and it has largely replace such aids as logarithm tables, slide rules and ready reckoners in shops, offices and factories. In school, however, the position is somewhat different for research has shown that the calculator has a specially important contribution to make in the teaching and learning of mathematics. The calculator is potentially a powerful teaching aid which can be used to considerable effect in helping pupils to acquire understanding of mathematical concepts.

In his excellent article 'Calculator : Abuses and Uses' David Johnson (1978) suggests that the pupil uses the calculator to generate output with the purpose that the output will demonstrate a concept or relationship, or that the actual generation of the output will serve to help reinforce a concept which has been taught previously.

In CONCEPT - DEMONSTRATION, the pupils are first asked to look for a pattern when the calculator is used to do some calculations the pupils have not yet learned how to do (discover a rule), then justify the result with a few, well-chosen examples, and then practise.

When a particular piece of mathematical content has already been introduced and the calculator activity is planned to provide an opportunity to practise or apply what has been learned, this can be thought of a Concept - reinforcement.

Chapter 2 attempts to demonstrate how certain basic topics, which have been identified by the C.S.M.S and S.E.S.M. research teams as presenting particular difficulties to many pupils, may be illuminated by using the calculator. These topics:

fractions, decimals and place value, and order of operations
are not the only ones identified which may be helped. Others
include percentage, proportion, negative number, and graphs,
where greater use of the unitary method may be made for the
understanding of proportion; greater emphasis on the link
between decimals and percentage rather than fractions and
percentage; easier plotting of points of a graph and testing
if points lie on a line; and so on.

The standard pencil and paper ALGORITHMS may, in time, be
removed from the curriculum but that does not mean that pupils
are introduced to fewer algorithms. There are many, more
powerful and more general, waiting to be explored. In the
words of Polya (1965) "we must teach 'guessing'". From
unstructured trial-and-error we can move to guided trial and
error, search methods and more general iterative procedures.
Blakeley (1980) suggests that pupils should be able to:

    a) follow an algorithm
    b) modify an algorithm to produce an alternative
       required result; and
    c) design algorithms, involving them in analysis of
       problems.

Iteration can be a powerful and widely applicable technique
and can be used to aid the understanding of such concepts as
square root and cube root as described in Chapter 2. Further
iteration can be used to attack all equations met in an o-level
course should the teacher so desire. Although traditional
methods of factorisation, completing the square and the formula
for quadratic equations may often be more easily applied at
this level, iterative methods may be more readily understood
by the pupils.

As an example of a possible style of algorithm we may ask a

pupil to DESCRIBE the method used with the calculator when
testing if certain numbers (say 901, 1009) are prime. We might
expect something like the following:

1) Enter the number

2) Divide by - smallest prime
- next odd number until this exceeds x
(x may be n, $\frac{n}{2}$ or even n depending
on the ability of the child)

3) Check for remainder (i.e. a decimal fraction in output)

4) If no remainder - Stop (not prime)

5) If remainder go back to 1

This style may be preferable to a flow chart for the majority
of pupils.

The emphasis will be on flexibility and clarity with real
discussion of pupils' suggestions, rather than the learning of
'standard' algorithms.

A better understanding of the concepts of square root and square
will naturally lead to a better understanding of Pythagoras;
an understanding of ratio and proportion leads to enlargement,
scale factors, lengths and areas of similar figures. Before,
circumference and area of circles can be understood, pupils
should have an appreciation of 'pi'. The Schools Council
(Calculators Count, 1983) provides a gradual approach by
considering the perimeter/diagonal relationship for a square,
then for a hexagon and finally the circumference/diameter
relationship for a circle. More ambitious pupils may like
to study the history of 'pi' and appreciate (but not necessarily
understand) the significance of the attempts made by such men
as Archimedes, Wallis, Vieta and Leibnitz to obtain approximations
for the elusive ratio. Turinese (Maths Teaching, No.89)
gives a brief outline of these. Rade and Kaufman (1980) too
give a fascinating account of the history of 'pi' as well as
other interesting and sometimes puzzling areas of real mathematics
via a pocket calculator.

Trigonometrical concepts, too, may be enhanced by the use of the calculator, and this section of work is now more accessible for pupils of moderate ability.

The ability to estimate an answer and to give an answer correct to a sensible number of significant figures is also highlighted in Chapter 2. Great importance is attached to these skills and the difficulties that pupils may have in understanding the concepts is not 'underestimated'.

However, although the use of the calculator in the classroom is becoming increasingly accepted, it is under this heading - 'Understanding of Concepts' - where least work has been done. There has been a steady trickle of articles and publications which have been designed to be easily usable in the classroom and related to the current mathematical curriculum. It is the need for these materials which is most pressing. To quote from Cockcroft:

"There is an urgent need for an increase in the limited amount of work which is at present being undertaken to develop classroom materials designed to develop understanding of fundamental principles". (392)

The Calculator as an aid to Problem-Solving

Mathematics is only 'useful' to the extent to which it can be applied to a particular situation and it is the ability to apply mathematics to a variety of situations to which we give the name 'problem-solving'.

In 1980, the first of N.C.T.M.'s Recommendations for School Mathematics of the 1980's states:
          "Problem solving must be the focus of school mathematics in the 1980's".
This recommendation not only indicates the importance of problem-solving, it also implies that a concentrated effort is needed in order to establish problem-solving as an integral part of the mathematics curriculum. Further, todays research, is concentrating on processes or the set of steps students use to find a solution rather than focussing on the actual solution to the problem or the answer to the exercises. It has been found that correct solutions to problems involve setting up a plan, however brief, for the solution (Kantowski, 1977, 1980) and that different students approach the same problem in a variety of ways, indicating the existence of a style or preference.

To many classroom teachers and other educators, a problem is simply a word problem or an exercise stated in verbal form. An example might be:
          "Clare bought a sandwich for 70p and a drink of squash for 30p. If the service charge is 10%, how much change would she receive from £2.00?"
Such problems are easily solved by most students by application of algorithms that are part of standard instruction.

To other educators a problem exists if a situation is non-routine, that is, if the student has no algorithm at hand that

will guarantee a solution. The student must put together
the available knowledge in a new way to find a solution. The
possibilities must be tabulated and some trial and error attempted.
Moreover, more than one solution may be possible. An example
might be:

"Clare has exactly £3.00 and would like to spend it all
on her lunch. The menu includes hot dogs at 80p, hamburgers
at 90p, onion rings at 60p, french fries at 50p and drinks at
30p, 40p, and 50p. The service charge is 10%. What would
Clare have for lunch?"

A third type of problem can be called <u>application</u> or 'real
problems'. A real problem involves a complex real-life situation
that must somehow be resolved. Often there is not an exact
solution, but one that is determined to be optimal to fit the
conditions. The Spode Group have produced a compilation of
such problems (Solving Real Problems, 1982) and include 'Easing
the Traffic Flow', 'Car-Park Layout', 'Where to place a Telephone
Box' as typical problems. Most of these problems include
substantial computation.

What part then can the calculator play in this new emphasis on
problem-solving in the curriculum? We must certainly be cautious
about claims that it is a panacea. Even with the verbal
question above, the pupil must still be able to select the correct
buttons to push. As Cockcroft states:

"We wish to stress that the availability of a calculator
in no way reduces the need for mathematical understanding on
the part of the person who is using it" (378)

Suydan (1981) points to evidence from research in America to
support the belief that problem-solving requires more than
computational skills. Several reports conclude that the use
of calculators does not affect problem-solving scores significantly.

However, there is also evidence to support the use of calculators
in problem-solving in the same Suydan review. Other findings
show that calculators are useful if the problems are within the
range of students' paper-and-pencil computational ability;
that students are less afraid to tackle difficult problems when
using calculators; and that students use more varied problem-solving
strategies when using calcuators.

The calculator then has a useful, if somewhat limited, role to
play. Polya (1973) considers four phases in the solution of
a problem, one of which is the computational skill. If the
calculator can considerably simplify this phase then the student
may focus his attention on the other phases. Further, the
student will have more confidence and less fear of the problem.

The calculator, too, may be useful in some of the thought
processes in both non-routine and real problems. If we consider
three stages of specialising (trying something out), conject-
uring (guessing a pattern) and verifying (explaining the pattern)
of a non-routine problem, various roles of the calculator may
be linked to these stages. These roles may be considered as

      1) providing the correct answer

      2) enabling attention to remain on the problem

      3) giving confidence to try specific cases.

Thus the calculator supports non-routine problems by allowing
the swift numerical exploration of particular cases as a basis
for 'guessing' and 'explaining'

The calculator overcomes the main objection to using realistic
problems in the syllabus. Knowing a calculator is at hand to
cope with the arithmetic is enough of a confidence booster to
get the pupils started. Often, if you can just get a feel for
what happens when a couple of specific cases are tried, this can
provide the impetus for more involvement in the problem. Interest

will be more likely to be maintained if the pupils know they will not get stuck on the arithmetic, and more time may be spent on interpreting the result. It is as a direct result of the use of calculators that statistics questions now tend to concentrate on interpretation of answers and less on calculations.

However, one note of caution should be heeded. Realistic numbers should not necessarily be introduced when a new concept is to be learned. It is still good teaching practice to use easy, manufactured and convenient numbers to illustrate a new concept.

"Problems involving 'messy' numbers rather than 'neat' numbers are not necessarily appropriate merely because of calculator availability. 'Messy' numbers may be as much a distraction with calculators as they are without them" (Kirst, 1980)

There is a further advantage that may be accrued from having the calculator available in the mathematics curriculum, although more indirect to problem solving. As Kantowski (1981) reflects:

"Too much emphasis on computational drill may be counter productive to the development of the flexibility needed for problem-solving. How can this demand for emphasis on basic skills be reconciled with the need for development of problem-solving ability in the limited time available for mathematics instruction".

Perhaps this time may now be more wisely spent. Kantowski sees the present decade as an exciting one with the curriculum promising to be more child-centred if the new technology (and she includes computers here) can be used to advantage in dealing with the diversity of problem-solving styles.

Problem-solving is a basic skill and we must teach problem-solving:

"Anyone who can execute standard routines but is incapable of solving a new problem <u>has no skill</u>" (Van Dormolen, 1976).

<u>To encourage discovery, exploration and creativity by introducing</u>
<u>appropriate investigations and mathematical ideas</u>

"The idea of investigation is fundamental both to the study of mathematics itself and also to an understanding of the ways in which mathematics can be used to extend knowledge and to solve problems in very many fields". - Cockcroft (250)

A mathematical investigation is often thought of as an extensive piece of work or 'project' which will take quite a long time to complete and will probably be undertaken individually or as a member of a small group. However, investigations need be neither lengthy nor difficult. An investigation may be planned by the teacher or it may arise spontaneously from a pupil's question such as "what happens if ....?" The teacher must be prepared to pursue the particular question which may have arisen from a routine lesson, and the result may be a brief, interesting discussion or may lead to a full investigation. Chapter 2 includes such an enquiry from a pupil which leads to an interesting and even exciting series of investigations on recurring decimals.

The teacher should not be afraid to veer from the syllabus nor to curtail the investigation because of lack of time, but must be aware of when the investigation has served its purpose and the pupils have lost interest.

The calculator may be thought of as having two significant roles in investigations. On the one hand it may prove an invaluable aid to computation in investigations which would otherwise have been too time-consuming and too difficult to pursue to any worthwhile length. Just as the ideas involved become interesting the arithmetic becomes boring and difficult. Watson's 'Exploring Numbers : Some Investigations with a

Calculator' (1979) contains activities concerning the usual
number patterns, arrow games, factors, Pascal's Triangle,
factors and questions are asked about rules, conjectures, tests
and proofs. The activities are interesting and worthwile,
but the use of the calculator is only incidental and, indeed,
the word does not appear in the text. Chapter 2 contains
investigations on the Fibonacci Series and sequences and limits,
where the calculator plays the same sort of useful but 'passive'
role.

The other role is when the investigation centres on the calculator
itself and its distinct features. The answer to the question:
        "In how may different ways can you carry out this
calculation on your calculator : which way requires the least
number of steps?"
depends on the particular model of calculator which is used,
and pupils who undertake an investigation of this kind will
produce a variety of answers, all of which are equally valid.

Keane and Fenby (Things to do with a Calculator) ask the reader
to investigate:
        "What is the biggest / smallest number you can show
on your calculator? What is the largest number you can square
and still get a result? Show 0.0123456 on the display by
pressing no more than 5 keys. Calculate powers of three on
your calculator and powers of 9 without one"

There are some purely artificial activities which rely on the
idiosyncrasies of the calculator. At a higher level, an
interesting exploration by John Clarke (Sept 1978) involves
reversing a random 4-digit number not ending in zero by
combining it using any of the four operations - though not
the same twice in succession - with any 2-digit numbers (the
first digit being non zero) made up from the digits of the
number being reversed. Digits after a decimal point produced

by division are omitted before the next operation.

Example: 1119 - 19 = 1100
                  1100 x 91 = 100100
             100100 - 11 = 9100
               9100 + 11 = 9111

Dr. Clarke says the maximum number of steps necessary so far
is 10, and there are some obvious variations.

David Fielker (M.T. number 89) begins some sessions with teachers
by saying:
          "I divided one whole number by another - each under
1000 - on my calculator and obtained 0.6 786 389: I can't
remember what the two numbers were; can you find out?"
After anything from 15 to 40 minutes there is usually a digress-
ionary discussion about what makes a good mathematical problem!
But sooner or later it is realised that the real problem is to
find an efficient algorithm rather than to find the two numbers.

Other calculator investigations of varying degrees of difficulty
abound and Haylock (M.T. 101) is even able to investigate the
mathematics of a dud calculator which had a defective zero key.

One other very significant value of the calculator is worth
mentioning at this stage. The less-able pupil is now able to
join in investigational work, to proceed at his own pace without
the distraction of difficult arithmetic, and discover the
pleasure of real mathematics. There still exists a minority
of teachers who believe that the calculator should be excluded
from this type of pupil, but this is surely a misguided view,
for it is the less-able pupil who must have more to gain from
the availability of the calculator.

Mathematical puzzles and games of various sorts also offer valuable opportunities for investigational work for these and other pupils, and their value will be discussed later.

It is perhaps appropriate to conclude this section with the thoughts of Brian Davies, I.L.E.A. (M.T. number 93) after two months of calculator activities with juniors:

"How exciting and interesting it has all been. How it has converted the teacher to an <u>investigational approach</u>. What a liberating force the calculators have been, enabling the children to tackle calculations that would have been tedious, time-consuming and error-prone without them. How important is the teacher's attitude, giving the children time, taking their ideas seriously, taking time for discussion, and being interested in the process as well as the answer".

<u>To Introduce 'real world' Data in Problems</u>

"How fast do you think James Hunt was going when he completed a lap at Silverstone in the British Grand Prix in 1 minute and 18.81 seconds?", the teacher asked the class of twelve year old children.

The overhead projector displayed the following information:

	<u>One Lap</u>	<u>Record Time</u>	<u>Driver</u>	<u>Year</u>
Silverstone	2.932 miles	1 min 18.81 secs	James Hunt	1976
Brands Harch	2.614 miles	1 min 18.60 secs	Niki Lauda	1978

First pupil: "Are those real numbers?"
Teacher : "Yes, they are the current records"
First pupil: "I'll bet it isn't as fast as Niki Lauda"
Second pupil: "You'r wrong! It's faster"
Teacher : "Let's use our calculator and figure the speed for both records"

Hutton (1980) quotes this lively discussion as typical of many during a course involving the use of calculators.

Bell et al (1978) argue that calculators, by allowing numbers of realistic size and actual measurements and observations to be handled, fulfil an important role in emphasising the usefulness of mathematics. Too often we restrict childrens' experience of the mathematical environment through the choice of convenient examples and problems. The children show more interest and committment to problems which they know contain real data.

Johnson (1978) uses applications of everyday living to illustrate

that the calculator can help us to become 'wise consumers'
e.g. comparative shopping; personal consumption of water,
gas, electricity. The calculator, too, helps us to make
decisions which have implications for benefiting society as
a whole e.g. population growth; conservation of resources;
health and health care; inflation; and so on.

Chapter 2 illustrates the use of realistic mortgages and compound
interest calculations, while statistics must be one of the
areas of mathematics which benefits most from the availability
of the calculator. Actual data from surveys may now be used;
the teacher will no longer have to use a convenient 30 when
taking readings from his class of 29; realistic pie-charts
may now be drawn, too.

Green (1981) invites his readers to examine and investigate
the incredible distances between the bodies in the solar system
and the sizes of these bodies, while the Spode Group (1982)
deal with actual attendances at soccer matches and transfer
fees.

However, as usual, some care must be taken. Reys (1980)
warns that merely displaying very large numbers on the calculator
may be falsely interpreted as understanding. Research has
shown, he says, that understanding the formulation of numbers
is a delicate cognitive process developed over a long period
of time, and the calculator may not bypass the development of
this concept.

It is with the use of very large, very small and realistic numbers,
generally, that the understanding of estimation and accuracy
really become of major importance.

To Motivate Pupils

During every mathematics lesson a child is not only learning,
or failing to learn mathematics as a result of the work he is
doing, but is also developing his attitude towards the subject.
Once attitudes are formed they can be very persistent and diff-
icult to change. Positive attitudes assist the learning of
mathematics; negative attitudes not only inhibit learning but
very often persist into adult life and affect choice of job.
The challenge for the teacher is to present mathematics in a
way which continues to be interesting and enjoyable and so
allows understanding to develop. The teacher must nurture
the young child's natural uninhibited enthusiasm and curiosity
in the primary school and, hopefully, this can be maintained
in the secondary school. Any aid to this difficult task is
very welcome. There can be little doubt of the motivating
effect which calculators have for very many children. As
Shumway (1981, pp. 169) emphasises:

> "One of the most powerful and consistently reported
> effects of student use of calculators is the high enthusiasm
> and valuing students have for calculator aided mathematics
> activities. Any device which causes so much pleasure to be
> associated with mathematics ...... deserves special note".

Shumway also believes that parental openness to calculator use
can be influenced significantly by the enthusiastic response
children exhibit to calculator aided mathematics.

Hutton (1980) too, reports that teachers in her study group
believed the calculators not only improved attitudes towards
the mathematics in general, but specifically improved interest
in problem-solving.

The calculator engenders confidence in the child by enabling
him to experience success. This allays the fear and anxiety

that Cockcroft emphasises, and especially in those less-able children who usually experience failure.

The calculator is an alternative source of the correct answer and is 'neutral'. As one young pupil said:

"I like the calculator because it is easy to work with. It is easy and it does not get angry with you when you get it wrong." (Open University, 1982)

The child's inquisitive nature is stimulated as soon as he encounters a calculator. However, Johnson (1978) warns that although there is no harm in pressing buttons to see what happens just for fun, there is concern if the child is asked for responses or procedures for which he lacks the necessary background for understanding.

There seems to be some disagreement in the educational world about using the calculator for checking answers. On the one hand some, and Johnson is one, believe that the calculator is the best device for doing the calculation in the first place, while others such as Cockcroft believe that the child can gain confidence by checking 'mental' answers or written answers.

The calculator, then, can motivate by not only increasing the child's chances of success but it can also allow the child to experience a much more exciting mathematical world. Amongst the most rewarding sources of calculator based activities are calculator <u>games and puzzles</u>.

<u>Games</u> need no justification as an aid to teaching as long as they have a mathematical objective. Children play games with great concentration and involvement and most enjoy the competitive nature of them. Games should be fun to play and should

have a logical reasoning component, but they should also
illustrate some mathematical idea or practice some useful
skill. The teacher should plan the type of game he uses,
where he introduces it into the curriculum and the reason for
doing so. It is too often used as an end of term treat or as
a casual addition to a lesson to relieve boredom without much
forethought. For maximum benefit the child should be encouraged
to keep records. The teacher should be able to exploit the
flexibility of the activity, for example by adjusting the
difficulty of the game. Further, the teacher must plan the
teaching sequence and decide how the game can be located within
the usual teaching of mathematical concepts. A motivating
and effective strategy is to start with a game and then teach
a relevant topic and finally let the children return to the game.

The appendices of this project include games that have been
tried by the second year pupils of Collenswood School, Stevenage.
The games were from various sources such as F.R. Watson (1982)
and those drawn from the S.M.P. 11 - 16 syllabus. Those
included were considered by the pupils and staff to be the
most interesting, useful and appropriate of all the ones
attempted. Only a sample has been included.

The Calculator as a Service Function

There is one important function that the calculator offers that
is often neglected when the use of calculators is discussed. As
Kaner (1980) remarks:

> "It is in the service aspects of mathematics that
> the use of the calculator is such an enormous blessing. The
> barrier of poor arithmetic skills, that has kept so many
> children from effective study in the sciences, geography,
> home economics etc., has now been swept away. All that is
> needed is a small supply of calculators in the laboratory
> to be used whenever a calculation is to be carried out".

The science teacher should no longer be able to blame the mathe-
matics department for childrens lack of success in his subject.
Hart (1981) highlights many of the problems that exist between
the two departments but remarks that:

> "Calculators may serve a useful purpose in that children
> may be able to concentrate on the essence of the problem rather
> than the calculations involved".

However, it is the _duty_ of the mathematics teacher to do two
things. First he must make sure that his science colleagues
are fully aware of the help that a calculator can be to
ordinary children in learning science. Second, he must make
sure that _all_ his pupils are fully competent with the electronic
calculator.

It is important that, not only mathematics teachers, but other
teachers too, take advantage of in-service courses. Research
must be initiated into how the calculators can be used effect-
ively in other subjects. One such piece of recent research
is:

> 'An Experiment in Using Calculators in Teaching Home
> Economics' (Parry and Rothery 1983)

The Calculator as an Aid to Computation

It is well to note, obvious as it is, that calculators are the quickest, most accurate computational algorithm available to children today. In fact, the primary function of a calculator is to compute, and in the hands of children, the calculator serves the computational function better than any other technique or device in existence. They make logarithmic tables and slide-rules obsolete and have caused us to think hard about what written computational skills the children now need and the effect the calculator has had on these and their mental skills. (discussed at some length in Chapter 1)

Calculators are practical, convenient, and efficient. They remove the drudgery and save time on tedious calculations. They are less frustrating, especially for low achievers. They encourage speed and accuracy.

The point has been made in this chapter and others that the ability to compute can be very beneficial to various aspects of the mathematics curriculum such as problem-solving, inclusion of real data, motivation and so on. The calculator, therefore, narrows the gap between childrens' abilities and widens the scope of mathematics teaching. It widens the educational horizons of many pupils who have, until now, been held back by poor arithmetic skills. Further, Blakely (1980) remarks that the calculator puts numbers <u>back</u> into mathematics and also provides a portable number laboratory, which can be taken home and used there by the children.

However, there still exists some contentious issues surrounding the use of the calculator, and these are mainly concerned with not <u>if</u> they should be used but <u>how</u> and <u>when</u>.

OUTSTANDING ISSUES IN THE EFFECTIVE USE OF THE CALCULATOR

So far this chapter has been concerned with advantages attributed
to the use of the calculator in the secondary school curriculum.
To restore the balance of the argument it would seem fair to
consider any disadvantages that may arise. There are, perhaps,
two objections that have constantly been put forward to limit
their use, and even these seem to have been discredited over
the years.

Firstly, it has been argued that calculators encourage mental
laziness and that they are detrimental to the mastery of basic
facts and algorithms. It was this fear that prompted many
calculator investigations into the possible effects on comput-
ational skills. However, reviews and summaries of these
investigations, especially by Suydam (1977), Suydam (1978) and
again Suydam (1979) conclude that the calculator has no adverse
effect on these skills. A review by Roberts (1980) and year long
study by Moser (1979) agree with this conclusion. Further,
Shumway (1981) observed that :

     ".... children .... did not develop any of the feared
debilitations when tested without calculators because of
calculator use for instruction".

Seldom is research literature so clear that there is no cause
for concern or alarm about this particular calculator effect.

Secondly, it is argued that they are not available to all students
and so leave some at a disadvantage. It is evident that most
students now have access to a calculator either at home or
school and that more and more schools are able to offer their
pupils the use of one. For example, a survey (Saunders 1980)
of 100 'random' occupations reports that fully 98% of those
persons involved used a calculator.

Hopefully, therefore, we are able to avoid creating another

false dichotomy, as in new mathematics versus old mathematics,
discovery versus expository lessons. It is not appropriate to
consider calculator programmes versus non-calculator programmes.
The real question is not should calculators be used but when,
where and how they can be used most effectively.

Fielker (1979) believes that the cartoon at the beginning of
this chapter is only one step removed from reality because,
he says, judging by some of the literature that is being prod-
uced, it is sometimes a struggle to find the right sort of
mathematical questions to ask when calculators are readily
available to children. He believes that there are too many
books and articles which feature the word 'calculator' in the
title and yet use of the calculator is only incidental. More
research should be put into activities that use the power of
the calculator more directly.

The recent publication 'Calculators Count' (1983) demonstrates
various topics that can be taught more effectively with the
aid of the calculator. Perhaps, as important as the ideas
themselves, is the authors' suggestions where these topics may
be usefully placed into existing mathematics courses. The
appendices include their suggestions for both the S.M.P.
lettered books and the S.M.G. modern mathematics for schools
courses.

S.M.P. have produced their own calculator series of booklets
to complement their existing 'O' level course and the new
S.M.P. 11 - 16 course has been devised with the assumption
of calculator availability in mind.

Fielker, too, is one of the many educationalists outside the
school classroom who believe that the calculator should be
available to all children at all times (assuming correct use).

To those teachers who would advocate the learning of some
arithmetic before the calculator is used he would argue that
this is rather like not allowing anyone to pass their driving
test until they can run at 40 m.p.h.

"Children were born with brains, and they should use
them",
one teacher had said to him at a course. He replied that
she was born with legs, and had she walked the twenty miles
to the course? However, it might well have been pointed out
that walking is a beneficial exercise, and that many people
find it enjoyable and a challenge.

Perhaps a more appropriate analogy would concern the advent of
sewing machines. It was feared that many of the traditional
hand-sewing skills would disappear, and it is true they have,
but who could deny that the use of a sewing machine requires
new skills and, some would agree, better and more versatile
results.

The classroom teacher may well consider Fielker's view as
rather extreme. The ability to carry out simple calculations
by all children and more difficult ones for the more able
is still valued by parents, employers and the pupils themselves.
To them it is a matter of pride, and, to some, enjoyment.

It must be useful, at times, therefore to prohibit the use of
the calculator in order to test these skills. Mental arithmetic
is regarded as even more important in the calculator age. If
asked
"What is 8.16 x 25 ?,
most children, if they have access to a calculator, will use
it and quickly get the right answer, but it is very easy to
answer this question without one.

The ability to find short cuts and tricks (an interesting

and worthwhile activity in its own right) may well disappear if
some limitation, however slight, is not put on calculator use.
Also, well-worn arguments such as the calculator breaking down
or forgetting your calculator, do have a ring of truth.

It is unlikely that any external examination in mathematics
now forbids the use of calculators on all of its papers.
Cockcroft recommends that:
       "Examination Boards should design their syllabus   .
and examinations on the assumption that all candidates will
have access to a calculator by 1985". (395)
However, despite strong views that calculators should be allowed
at all times, most examinations like to include a section
which forbids its use. This usually involves questions of
a simple, computational nature.

Using calculators to check paper and pencil calculations is
often cited as an important use for pupils. In 1979
P.R.I.S.M. reported that 93% of professional people sampled
by them gave 'checking answers' as the best use of calculators.
It is not surprising that this use is cited, but it is sur-
prising that it was quoted as the most important use by so
many. Reys (1979), like many other educators, believes
there is little point in checking computations done without
a calculator when the calculator could have been used in the
first place.

It soon becomes obvious that the mere possession of a calculator
by a pupil is not enough. There is still a good deal of
convincing to do that the calculator offers more than the ability
to ease the burden of computation. Great claims have been
made as to the effect the calculator will have on the actual
content of the mathematics curriculum and some of these claims
will be examined briefly in the next section.

EFFECT OF THE CALCULATOR ON MATHEMATICS CONTENT

In considering the impact of the calculator on the mathematics
syllabus at secondary level, we may ask three questions:

> "What comes in?"
> "What goes out?"
> "What do we do about what is left?"

Pollak (1977) writes of two partial orderings of the curriculum,
one supplied by the discipline of mathematics (the partial
ordering of prerequisites) and the other by society (the
partial ordering of importance). He argues that both will
be affected in a major way by the calculator. Certainly
both will be borne in mind when considering the needs of our
pupils when they are 10 or 20 years into their adult life.

The following suggestions do not purport to be a comprehensive
or definitive list, and they will certainly cause some dis-
agreement. However, it is clear that due consideration must
be given to the syllabus now. Many factors, of course, have
to be considered when devising a syllabus for the future; the
calculator is only one of them.

What goes out?

There seems to be general agreement that the calculator will
make logarithm tables and the slide rule redundant as calculating
aids. Trigonometrical tables, too, will soon be superceded by
the scientific calculators. The exclusion of the long division
algorithm for many pupils and more difficult fractions may
cause a little more controvesy but, hopefully, not too much.

Other minor areas, such as the use of coding for calculation
of the mean and standard deviation, and other less obvious areas

will need careful consideration. In the latter category,
Blakeley (1980) looked at the S.M.P. 'O' level course which,
as he comments, was devised before calculators were taken
into account, and he was able to eliminate 22 chapters out of
124 by adding to the list:

>   linear programming,
>   latitude and longitude,
>   and the matrices/transformations link.

He does not reason strongly for these and suggests other topics
could be argued against by other people. His 'hatchet job'
is really to create room in an already overcrowded syllabus.

For a start, a conservative list might be:

>   LOGARITHMIC TABLES
>   TRIGONOMETRICAL TABLES
>   SLIDE RULES
>   FRACTIONS IN DEPTH
>   DIVISION ALGORITHMS FOR MOST PUPILS

What comes in?

A long list of additions to the syllabus might appear rather
daunting to teachers, but few of these are entirely new.
Some of them, such as patterns, estimation and accuracy have
been on the fringe for some time while others, such as iterative
methods, may replace more traditional techniques. Further,
it must be assumed that more time will be available with less
emphasis on the standard pencil-and-paper algorithms.

MATHEMATICAL MODELLING involves translating a real situation
into a mathematical equation which can be used to explain

known phenomena and even predict. Excessive calculations have,
in the past, hampered the inclusion of this topic in the syllabus.

ESTIMATION OF ANSWERS AND ACCURACY take on a new importance
in the calculator/computer age, and more emphasis can be
placed on PROBLEM-SOLVING. The concept of ALGORITHM and the use
of ITERATIVE PROCEDURES, too, are basic in today's computer age.
Children need to be given the opportunity to design and modify
new algorithms, and iterative methods may be given as an alter-
native or even replace such traditional techniques as solving
quadratic equations by factorising or by the formula. Certainly,
more interesting and higher order equations could be investigated,
as well as more interesting graphs.

Studying more practical applications of geometry and trig-
onometry, such as TECHNICAL DRAWING AND SURVEYING, has been
reported by Wilson (1977) to be much more successful for pupils
of moderate ability, provided calculators were used.

GROWTH AND DECAY RATES should gain greater prominance on the
new syllabus. Compound Interest is a vital topic to us all
ranging from personal overdrafts and hire purchase and mortgages
to major economic problems of the National Debt.

The exploration of PATTERN has already become a popular topic
in new books and courses related to the calculator. However,
if it is to be worthwhile, it must be used to promote mathe-
matical thinking. While the main purpose of the calculator
is to generate the pattern, the main purpose of the activity
should be the search for the pattern, with some possible
explanation of the pattern, a generalisation and then the
calculator may be used again to verify the result. Johnson
(1978) examines patterns in more detail in his article
'Calculators : Use and Abuses'.

Most of the books on calculators have given prominence to

GAMES AND PUZZLES, APPLICATIONS of all types and to INVESTIGATIONS, and the calculator allows much greater scope for these. Arguments for the inclusion of SEQUENCES, SERIES AND LIMITS, would be strong.

We must not forget what might be thought of as the two most important additions to the syllabus: the PROPER USE OF THE CALCULATOR and MENTAL TECHNIQUES as suggested by Plunkett to replace the standard written algorithms.

The list might look like this:
>   MATHEMATICAL MODELLING
>   ESTIMATION OF ANSWERS
>   ACCURACY AND ERRORS
>   ALGORITHMS
>   ITERATIVE METHODS
>   POLYNOMIALS
>   PROBLEM-SOLVING TECHNIQUES
>   GROWTH AND DECAY APPLICATIONS
>   PRACTICAL GEOMETRIC AND TRIGONOMETRIC APPLICATIONS
>   (Surveying and technical drawing)
>   PATTERNS
>   APPLICATION TO THE 'REAL WORLD' PROBLEMS WITH REAL DATA
>   INVESTIGATIONS
>   GAMES AND PUZZLES
>   SEQUENCES, SERIES AND LIMITS
>   PROPER USE OF THE CALCULATOR
>   NEW MENTAL TECHNIQUES TO REPLACE STANDARD WRITTEN ALGORITHMS

No doubt, a strong case could be made for several other topics and it is inevitable that the 'inclusion' list will far exceed the 'exclusion' list.

<u>What do we do about what is left</u>?

It is up to all educators to examine each topic in the syllabus
and decide if and how the calculator can make a real contribution
to the UNDERSTANDING and LEARNING of that topic. It is up to
every teacher to exploit its very special properties. However,
the application of calculators should not be forced if better
aids, and methods exist.

Suggestions for ways in which the calculator can aid the
understanding of some topics, have already been described.
Many other topics can also benefit, including the promotion
of all pupils 'numeracy' skills.

The ORDER in which topics are introduced into the syllabus
will also need rethinking although other factors, such as
the work of PIAGET and C.S.M.S., will have to be taken into
consideration. For example, decimals may be introduced before
fractions, negative numbers may be taught earlier.

CONCLUSION

If we have any doubts about the tremendous interest that the
advent of the calculator has had in educational research, then
we should heed the words of Suydam (1979, pp. 3) In the
second of her state-of-the-art reviews she asserted that:

"Almost 100 studies on the effect of calculator use
have been conducted during the past four or five years. This
is more investigation than almost any other topic or tool or
technique in mathematics instruction during this century".

Reports of calculator use in school settings have continued
to be released, more or less unabated, ever since.

However, research is not an end in itself. We should do
something as a result of research. The teacher should
incorporate the conclusion of research (after taking into account
his/her own unique situation and circumstances) into his/her
daily teaching. We may not be inclined to take the dramatic
action some educators strongly recommend by allowing the use
of the calculator at all times. When Shumway (1981) thinks
of his own children, for example, he wants to say:

"Don't waste their time doing trivia! Give them a
calculator and get on to teaching them the mathematics they
cannot now do".

The classroom teacher may have to take a more conservative
line but recent support for the use of the calculator by
Cockcroft has given the impetus to take action. The Head of
the Mathematics Department has an important role to play here:

1) He has to decide where effective calculator based
activities can be fitted into his existing syllabus, and alter

aspects of the curriculum with the calculator in mind.

2) He should rethink his philosophy about the standard pencil-and-paper algorithms. Whatever his decision, he should liase closely with the primary school and employers. He should convince parents and colleagues, too, that the calculator has no detrimental effects on the pupils own skills.

3) He should initiate closer co-operation between mathematics and other departments by emphasising the service aspect of calculators. As Green (1980) reports:

" ..... there is a need for closer co-operation. Maybe the calculator can be the catalyst".

The L.E.A's must provide the finance and in-service training, and the teachers and other educators must carry on with the research, especially for material where the calculator can be used as a teaching and learning aid.

There is an increasing danger, however, that the calculator will be overshadowed by its technological 'big-brother' — the microcomputer. In his article 'Remember the Calculator', David E. Williams (1983) reports on a conversation between two mathematics supervisors:

"I'm putting all my money into microcomputers this year .... "
"Have you ordered any calculators lately?"
"Calculators are dead! Nobody cares about calculators any more. Everyone wants to order microcomputers .... "
"But even with micros being purchased, calculators are still valuable tools and aids for students and teachers to use".
"Don't waste your time and money on calculators. Microcomputers are going to revolutionise mathematics education. Get on with it — you're really behind the times".

Mathematics educators are rushing to purchase computer hardware, and yet it is curious that many who questioned the educational value of the role of the calculator in the mathematics education of primary school children have no reservations about putting microcomputers in the primary classroom. (The reader might enjoy 'The Discovery of the Mathematics Textbook', an amusing, tongue-in-cheek article by Michael Cornelius, 1982). We must, of course, continue to develop ways to use both calculators and computers in our classrooms.

Others may see the danger coming from another source. On a more frivolous note Watson (1980) draws a picture of what may be in store for us. Some have concluded that, just as with the invention of the motor car, mankind is losing the ability to walk, so we shall lose all computational facility in the calculator age.
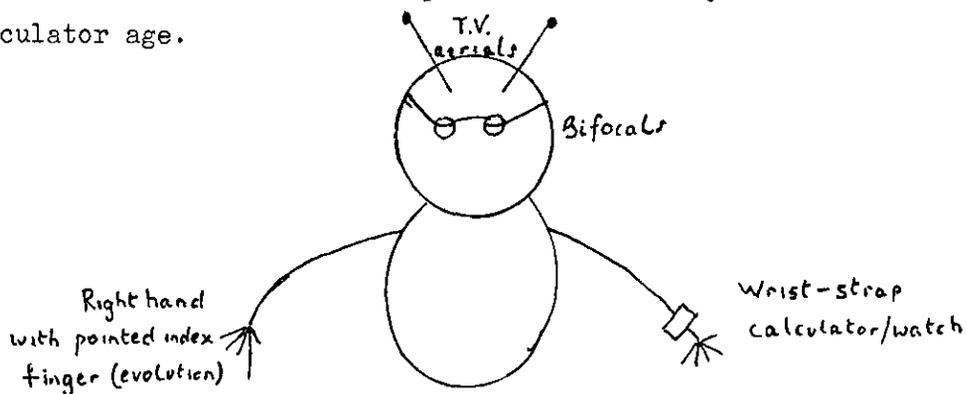
You will notice that the pupil has no legs (car travel) and that his television aerials are 'built in'. The pointed index finger is for pressing the tiny buttons of his wrist-watch calculator, and the bifocals are for reading the tiny displays.

However, Watson does not share this pessimistic view

Weaver, (1980), too, is confident that we can embark on school mathematics programmes with "freedom from fear" - freedom from fear that calculator use will have harmful or debilitating effects on students' mathematical achievement. This can no longer be

used as a <u>reason</u>, nor <u>excuse</u>, for not welcoming and including
calculators among the instructional aids and materials that
have potential contributions to make in connection with school
mathematics programmes.

He goes on to conclude:

"Now, as we enter the 1980's, we are in a position
to reformulate school mathematics programmes in a manner
that will free them from the shackles of the attainment of
computational skills with pencil-and-paper algorithms as the
basis upon which instruction is initiated, organised and
sequenced at the pre-secondary level; and will have analagous
reorganisational implications for programmes at the secondary
level. THE CALCULATOR IS THE KEY. Now is the time to turn
that key in all earnestness".

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APPENDIX 1

S.M.G. Modern Mathematics for Schools

Possible fit of some of the project materials to this series of textbooks.

	PROJECT MATERIAL
BOOK 1	8 Understanding the (x^2) key
	15 Odd numbers and their sum
BOOK 2 ARITHMETIC - Chapter 1	2 Multiplication of decimals
	3 Division by a decimal
	6 Fractions and decimals
	20 Counting and estimation
BOOK 2 ALGEBRA - Chapter 4	4 Introduction to the distributive law
BOOK 2 GEOMETRY - Chapter 2	5 Areas of rectangles and
BOOK 3 GEOMETRY - Chapter 2	decimal multiplication
BOOK 3 ARITHMETIC - Chapter 1	7 Fractions, decimals and percentages
	39 Electricity meters and electricity bills
BOOK 4 ARITHMETIC - Chapter 2	11 Introduction to cube roots
	13 An iterative method for finding the square roots
BOOK 4 ARITHMETIC - Chapter 3	31 Introduction to the number 'pi'
	32 Introduction to the area of a circle
BOOK 4 ARITHMETIC - Chapter 4	21 Word length
	22 Letter frequency
BOOK 4 GEOMETRY - Chapter 2	30 Introduction to Pythagoras' rule
BOOK 5 TRIGONOMETRY - Chapter 1	33 Introduction to trigonometric ratios
BOOK 6 ALGEBRA - Chapter 3	37 Gutter design
	38 Box making
BOOK 6 TRIGONOMETRY - Chapter 6	34 The sine rule
BOOK 7 ARITHMETIC - Chapter 1	27 Errors and accuracy: areas
BOOK 7 ALGEBRA - Chapter 4	19 Introduction to rates of change for non-linear relationships
BOOK 8 CALCULUS - Chapter 1	
BOOK 9 ALGEBRA - Chapter 3	41 How money grows
	42 Growth factors and decay factors
	43 Mathematics and the guitar

APPENDIX 2

S.M.P. Lettered Books

BOOK/CHAPTER	PROJECT MATERIAL
After A1	15 Odd numbers and their sum
	16 Whole numbers and their sum
After A8	17 Triangle numbers and their sum
B5	6 Fractions and decimals
After B3	23 Errors and accuracy : length
(or as late	24 Errors and accuracy : weight
as X5)	25 Errors and accuracy : time
	26 Errors and accuracy : angles
B9 or C11	22 Letter frequency
	2 Multiplication of decimals
	3 Division by a decimal
C4	5 Areas of rectangles and
	decimal multiplication
With C4	39 Electricity meters and
	electricity bills
	18 The jumping frog
After C4	40 Conversion between metric
	and imperial units
After C4 (or X5)	27 Errors and accuracy : area
C11	20 Counting and estimation
	21 Word length
D2	35 Lengths of similar figures
With D2 (or X5)	28 Errors and accuracy :
	comparing lengths
D5 or earlier	1 Number messages
D5	4 Introduction to the
	distributive law
D8	8 Understanding the (x^2) key
D10	7 Fractions, decimals and
	percentages
E1	30 Introduction to Pythagoras' rule
E5	9 Introduction to square roots
	10 Square roots by decimal search
E9	37 Gutter design
	38 Box making
After E9	11 Introduction to cube roots
	12 Cube roots by decimal search
E10	33 Introduction to trigonometric
	ratios
E11	31 Introduction to the number 'pi'
	32 Introduction to the area of a circle

APPENDIX 2 (Continued)

Calculator Games

(1)   For two players and one calculator.

Player A enters any number.

Player B <u>multiplies</u> by any number he or she chooses to
try to make the display show
                    1.0******
where the asterisks may be any digits. If B succeeds
he or she wins.

Player A <u>multiplies</u> the number <u>now</u> in the display by any
number he or she chooses to try to make the display show
                    1.0******
They continue until either A or B succeeds.

For a harder game try 1.00*****
                    or 1.000****

PRACTICE: Estimation and showing that multiplication does
not necessarily increase a number.

(2)   For any number of players, 1 calculator per player, pencil
and paper.

Choose a two-digit number as the target.

Now choose four digits as the ammunition.

Try to hit the target using only the specified ammunition and
the operations +, -, x, -.

Score 10 for a direct hit
      7 for an "inner" (within 2 of target)
      3 for an "outer" (within 5 of target)

<u>Example</u> - Target 58 Ammunition 4, 9, 3, 7.

       Player A : (4x9) + (3x7)   57
             B : (9+3) x 4 + 7   55
             C : 7x(4+3) + 9   58

PRACTICE : Ordering of Operations

(3)   <u>Twenty Questions</u>

A thinks of a whole number < 10000
B tries to guess it in 20 questions
Both players have calculators; answers to questions can
only be YES/NO.

Questions which involve the words "less than", "more
than" count <u>double</u>!
(Easier versions for younger pupils:
Number must be less than 1000 (100)
Remove the restriction on "less than"/"more than")

PRACTICE : Properties of Numbers

(4)   <u>Calculator Snooker</u>

Player A enters any two-digit number, B takes a 'shot'
by performing a multiplication sum. To 'pot' a ball,
the first digit of the answer must be correct according
to the table shown. (The degree of accuracy can be varied
according to experience.)

BALL	Red	Yellow	Green	Brown	Blue	Pink	Black
RESULT NEEDED	1..	2..	3..	4..	5..	6..	7..
SCORE	1	2	3	4	5	6	7

Otherwise the rules are similar to 'real' snooker. There are 10
(or 15) reds and one of each of the six 'colours'. A player must
score in the order red, colour, red, colour, and so on, until all
the reds have gone. (Note that the colours are replaced but the
reds are not.) When the last red has gone, the colours are potted
'in order' and are not replaced.

For example, one sequence of plays was

Player	Enters	Display	Comments
Jimmy	69	69	
Peter	x 2 =	138	Peter pots the first red, He elects to go for blue ..
	x 5 =	690	and misses.
Karen	x 2 =	1380	Karen pots the second red, She elects to go for black..
	x 5.5 =	7590	... and pots it.
	x 2.7 =	20493	She misses the next red.

PRACTICE : Estimation skills

(5)   Target Practice (2 players)

The players agree to aim for a 'target area' which is
a range of decimal numbers (say 0.2 to 0.5 inclusive).
Each player in turn must then choose two whole numbers
which are then divided on the calculator. If the result
falls into the target area, then they score a hit.

Sample Play

Player	Numbers chosen	Key Sequence	Answer	Score
A	8, 4	8 ÷ 4 =	2	0
B	4, 8	4 ÷ 8 =	0.5	1
A	1, 4	1 ÷ 4 =	0.25	1
B	3, 5	3 ÷ 5 =	0.6	0
etc	etc	etc	etc	etc

Note

Keeping a record in a table is slow and rather
unexciting. Much better is to record the play on
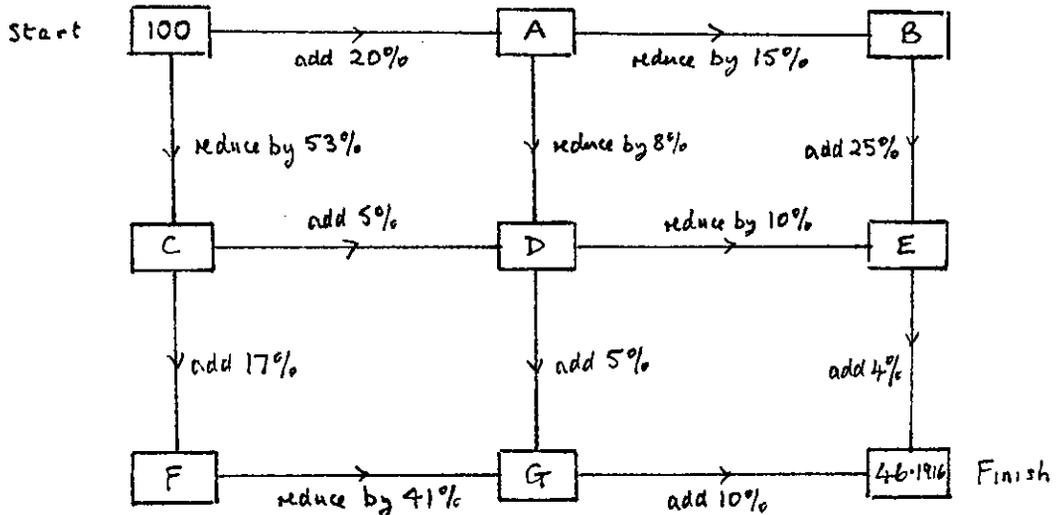graph paper, thus:
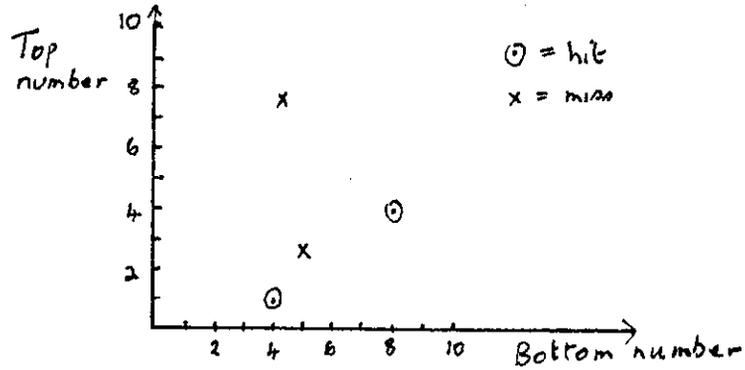
PRACTICE : Decimals and Estimation

(6)   Percentage Maze

Work through this maze, starting with 100 on your
calculator. Find which route gets to 46.1916 (the finish)
Choose from :

ABE, ADE, ADG, CDE, CDG, CFG.

PRACTICE : Percentage

145

(7)   Going down

You need a calculator. | 1 2 3 4 5 6 7 8 9 |

Write down the digits 1, 2, 3, ..., 9.
This is the check list.
Enter 1 into the calculator.
Players take it in turns to subtract a number from the
display.
The number you subtract can have 2 digits at most
You can use 0 as many times as
you like in the numbers you
subtract.
But each number in the check list
can only be used once in the game
Cross off the digits in the check list as you use them

The winner is the first person to get 0.01 in the display

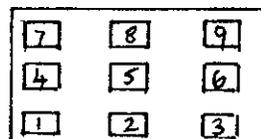The game is a draw if no-one can get 0.01.
If you get a negative number in the display, you lose
straight away.

PRACTICE : Place Value and Decimals

(8)   The Keyboard Game

In this game you only
use the numbers from 1 to 9
Start with 31 on the display.

The first player picks any number from 1 to 9
They subtract this off the display.

The second player picks a number.
Their number must be next to the key the first player used.
The second player subtracts their number.

Then the first player picks a number next to the second
player's, and so on.

The first player to get a negative answer loses.

PRACTICE : Mainly a game of strategy

140

(9)   Target 100

Similar to Game Number 1 but the target is 100.****
by multiplication.

For harder games the target becomes 100.0***
                                    or 100.00** etc.

Now try playing "target" when you are only allowed to press
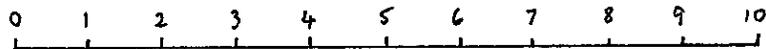the $\div$ button!

PRACTICE : Estimation

(10)   4-in-a-row

This is a calculator game for two players.

You need one calculator, some graph paper and a pencil.

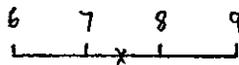Copy the number line below onto your graph paper. (Let
2 cm represent 1 unit)

```
     0   1   2   3   4   5   6   7   8   9   10
     |___|___|___|___|___|___|___|___|___|___|
```

Player 1 chooses two numbers from the table below, and an
operation (either x or +)

103	0.4	0.07
25	1.5	5
11	0.5	8.3

Player 1 writes down this sum and works out the answer
using a calculator (for example: 25 x 0.3 = 7.5)

He marks his answer on the number line with a cross (x)

```
     6   7   8   9
     |___|_x_|___|
```

Player 2 then chooses two more numbers and an operation
x or $\div$

He marks his answer with a blob (•)

If the point cannot be marked, (off the edge or already taken), then the player misses a go.

The first player to get 4 of his points in a consecutive row wins!

i.e. ─•─x───x─•─x─•─•─•─x── is a win for player 2.

PRACTICE : Use of scale and estimation.

(11)   <u>Home</u>

2 Players (more-able children)
One calculator with $\left.\begin{array}{c}\sqrt{X} \\ X^2\end{array}\right\}$ keys

1. A sets any number in a calculator display - 8 or 9 is a reasonable choice.

2. B adds any number Z, presses $\boxed{=}$ and $\boxed{\sqrt{X}}$ and then <u>subtracts</u> Z, presses $\boxed{=}$, $\boxed{X^2}$

3. A does the same as B, (choosing a different no. Z), and so on.

(The display is NOT cleared at each stage, so each player carries on where the other leaves off).

First to get 10.0**** in the display is the winner.

NOTE   i) The number Z can be NEGATIVE (i.e. subtract 1st and <u>then</u> add)

ii) **** can be <u>any</u> digits, but the display must <u>start</u> with 10.0 ...

Harder version "Home" is 10.00**

"You are the commander of a space craft, which is malfunctioning. In order to ensure safe return to Earth (before the motor goes critical) you have to set the quasor-warp-force display at 10.0 .. "

PRACTICE : Behaviour of functions $\sqrt{X}$ and X^2