LOUGHBOROUGH UNIVERSITY

RANDOM DYNAMICS IN FINANCIAL MARKETS

by

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"In mathematics, you don't understand things. You just get used to them."

Johann von Neumann

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Abstract

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We study evolutionary models of financial markets. In particular, we study an evolutionary market model with short-lived assets and an evolutionary model with long-lived assets. In the long-lived asset market, investors are allowed to use general dynamic investment strategies. We find sufficient conditions for the Kelly portfolio rule to dominate the market exponentially fast. Moreover, when investors use simple strategies but have incorrect beliefs, we show that the strategy which is "closer" to the Kelly rule cannot be driven out of the market. This means that this strategy will either dominate or at least survive, i.e., the relative market share does not converge to zero. In the market with short-lived assets, we study the dynamics when the states of the world are not identically distributed. This marks the first attempt to study the dynamics of the market when the probability of success changes according to the relative shares of investors. In this problem, we first study a skew product of the random dynamical system associates with the market dynamics. In particular, we compute the Lyapunov exponents of the skew product. This enables us to produce a "surviving" investment strategy, i.e., the investor who follows this rule will dominate the market or at least survive. All the mathematical tools in the thesis lie within the framework of random dynamical systems.

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To my family

Chapter 1

Introduction

The theory of evolution has a long history in biological, physical and social sciences. In financial markets, evolutionary finance examines the dynamic interaction of investment strategies and their long-run performance. Evolutionary finance has been a very active area of research in financial mathematics for the past twenty years. Survival and extinction questions of investment strategies have been examined in Blume and Easley [15]. They generalised the pioneering work of Kelly [29]. Kelly showed that the investor who follows the principle of *betting your beliefs* ultimately accumulates total market wealth when the market composed only of Arrow securities. This principle prescribes dividing wealth amongst assets according to the probability of their success. The work of Kelly was inspired by ideas of his postdoctoral mentor Claude Shannon, the creator of information theory [19]. Afterwards, the work of Kelly was developed by Breiman [16] and Cover et. al. [1, 12, 13]. In all these papers, the authors assumed that the asset prices were exogenous.

At the beginning of this century, there has been a remarkable development in evolutionary ideas in finance. This development has been mainly carried by Evstigneev, Hens and Schenk-Hoppé [27]. Evstigneev et. al. [21] analysed an evolutionary market model with short-lived assets. In [21], the investors use constant investment strategies and the prices are endogenous in the model. It is proved that there is a unique survival investment strategy which accumulates the market wealth. After this paper, this evolutionary model was studied under different assumptions and from different points of view in [2, 26]. In the first paper, Amir et. al [2] used the homogenous discrete-time Markov process states of the world and general investment strategies were employed by the investors. In the last two papers, they used the ideas from the theory of random dynamical systems [5, 14]. In [10], we have examined the evolutionary market model with short-lived assets. However, we allowed the states of the world to be *not* identically distributed. They may depend on the amount of money invested in the assets. We have computed the Lyapunov exponents of the skew product associated with random market system and have applied these ideas to study wealth dynamics of investors. We have identified a portfolio rule similar to Kelly rule. We have shown, [10], that the investor who follows this rule cannot be driven out of the market. Our results in [10] marks the first attempt to study the market dynamics when the probability of success is allowed to change according to the relative shares of investors.

In [22, 23] Evstigneev et. al. introduced a model with long-lived asset. It is shown that the Kelly rule is evolutionary stable and it is single survivor in the market. Moreover, it is demonstrated in [7] that the Kelly rule forms a unique Nash equilibrium strategy. In the last three papers [7, 22, 23], the investors use simple portfolio rules. Also it is assumed that at least one of the investors uses the Kelly portfolio rule. Recently we have shown [11] that when all the investors do not have full information about the probability distribution of the assets and consequently none of them uses the exact Kelly rule, then the investor who is closer to the Kelly rule may dominate the market or at least survive. Our work in [11] also marks the first attempt to study the market model with long-lived assets where none of the investors have correct information about probability distribution of the states of the world.

Amir et. al. [3] analysed the evolutionary model with long-lived assets when investors use general, adaptive portfolio rules. The authors showed that the Kelly rule always survives. Their work suggested a very interesting question: Under what conditions does the Kelly rule dominate the market? In [9], we found sufficient conditions for the Kelly portfolio rule to dominate the market exponentially fast. Roughly speaking, we show that if the Kelly rule and the "market portfolio" deviate, then the Kelly rule will dominate the market.

The mathematical framework of all these models is given by random dynamical systems. Random dynamical systems appear in modelling of many phenomena in economics, biology, climatology, etc., when uncertainties or random influences are taken into consideration. These uncertainties or random influences are called noises. This mathematical framework was mainly developed by L. Arnold [5]. Random dynamical systems explain probabilistically how the dynamics is effected by noise. When the dynamical system is generated by a differential equation, then it is called continuous-time dynamical system. When the dynamical system is generated by a difference equation or a map, then it is called discrete-time dynamical system. In this thesis we will be concerned with the latter; i.e., discrete-time random dynamical systems. The concept of a random dynamical system is an extension of the deterministic concept of a dynamical system and it gets together the ideas and methods from the well developed areas of dynamical systems and probability theory. When applying results from dynamical systems to real life problems external noise is unavoidable. Therefore, it is essential to require the exact mathematical model to allow some small errors along orbits. This is archived in the framework of a random dynamical system: we allow to embed the randomness within the model to deal with this unavoidable uncertainty about the observed initial states and correct parameter values.

There are recent developments of random dynamical systems theory in economics. According to [33], the theory and application of random dynamical systems is at a cutting edge in both mathematics and economics. In K.R. Schenk-Hoppé [33], he studied and demonstrated the role and importance of dynamical systems theory in economics. Furthermore, he demonstrated that the theory of random dynamical systems for economic modelling and analysis is very useful with stochastic components. The work in [33] focused on stochastic dynamic models in economic growth. In particular, it was emphasised that random dynamical systems allows to examine stability properties of economic systems, random interactions and time dependent environments.

In Chapter 2 we give basic definitions and results from probability theory, stochastic processes and ergodic theory. Moreover we state definitions and results from finance and economics that we use in this thesis. In Chapter 3, we analyse random maps with constant and position dependent probabilities. In particular we provide formulae of Lyapunov exponents for certain position dependent random maps. This is part of our paper [10]. In Chapter 4, we introduce the evolutionary market model with short-lived assets. We first give a literature review for models with dynamic and constant investment strategies. Moreover, this chapter includes our result from [10] which provides a surviving portfolio criterion when the states of the world depend on the amount of money invested in the assets. In Chapter 5, the evolutionary market with long-lived assets is introduced. We first review notions and literature in evolutionary finance for models with long-lived assets. Section 5.3 includes our result from [9], where we have provided sufficient conditions for the Kelly portfolio rule to dominate the market exponentially fast. In Chapter 5 we also present our result from [11]. We have proved that when all the investors have partial or no information about the probability distribution of the assets, then the investor who is *closer* to the Kelly rule either dominates or at least survives. Namely, the relative market share of the investor does not converge to zero a.s. In Chapter 6, we conclude. In Appendix A we have proved auxiliary lemmas for the proof of the main results in Section 5.3 and Section 5.4.

Chapter 2

Background

2.1 Probability Theory

In this section we state basic definitions and results from probability theory. All these statements are need in this thesis. We have mostly used references [17, 18, 20, 34]. For more information on probability theory we refer the reader to [17, 18, 20, 34].

2.1.1 Probability Spaces

Definition 2.1.
Let Ω be a non-empty set. A *σ-field* $\mathcal{F}$ on Ω is a family of subsets of Ω such that

i) $\Omega \in \mathcal{F}$;

ii) If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$;

iii) If $A_1, A_2, \ldots$ is a sequence of sets in $\mathcal{F}$, then $\cup_{i=1}^{\infty} A_i \in \mathcal{F}$.

The elements of $\mathcal{F}$ are called *$\mathcal{F}$-measurable* sets. The pair $(\Omega, \mathcal{F})$ is called a *measurable space.*

Example 2.2.

- $\mathcal{F}_1 = \{\emptyset, \Omega\}$ *is the smallest σ-field.*

- $\mathcal{F}_2 = P(\Omega) = \{A : A \subseteq \Omega\}$ *is the largest* σ*-field of subsets of* Ω.

- *Let* $\Omega = \{0, 2, 4, 6, 8\}$ *and* $\mathcal{F}_3 = \{\emptyset, \Omega, \{2, 4\}, \{0, 6, 8\}\}$. $\mathcal{F}_3$ *is a* σ*-field. Indeed,*

  i) $\Omega \in \mathcal{F}_3$,

  ii) $\emptyset^c = \Omega \in \mathcal{F}_3$, $\Omega^c = \emptyset \in \mathcal{F}_3$,
  $\{2, 4\}^c = \{0, 6, 8\} \in \mathcal{F}_3$, $\{0, 6, 8\}^c = \{2, 4\} \in \mathcal{F}_3$,

  iii) $\emptyset \cup \{2, 4\} = \{2, 4\} \in \mathcal{F}_3$, $\emptyset \cup \{0, 6, 8\} = \{0, 6, 8\} \in \mathcal{F}_3$, $\emptyset \cup \Omega = \Omega \in \mathcal{F}_3$,
  $\{0, 6, 8\} \cup \{2, 4\} = \Omega \in \mathcal{F}_3$, $\Omega \cup \{2, 4\} = \Omega \in \mathcal{F}_3$, $\Omega \cup \{0, 6, 8\} = \Omega \in \mathcal{F}_3$.

An important definition in measure theory is given as follows.

Definition 2.3.
The *Borel field* on $\mathbb{R}$, $\mathcal{B}(\mathbb{R})$, is the σ-field generated by open intervals in $\mathbb{R}$. Subsets of $\mathbb{R}$ which belong to $\mathcal{B}(\mathbb{R})$ are called *Borel sets.*

Example 2.4.
For a topological space X*, the Borel algebra on* X *is the smallest* σ*-algebra containing all open or closed sets.*

Definition 2.5.
If $f : A \to B$ and $C \subset B$, we let

$$f^{-1}(C) = \{r \in A : f(r) \in C\}$$

and call $f^{-1}(C)$ the *inverse image* of C by f. Hence, $f^{-1}(C)$ contains all points in the domains of f mapped by f into C.

Proposition 2.6.
Let $X : \Omega \to \mathbb{R}$ *denote a real-valued function. The collection of sets* $X^{-1}(B)$*, where* B *ranges over the Borel subsets of* $\mathbb{R}$*, is a* σ*-field on* Ω*. We denote this* σ*-field by* $\mathcal{F}_X$ *and call it the* σ*-field generated by* X*.*

Definition 2.7.
A mapping $f : \Omega \to \mathbb{R}$, where $(\Omega, \mathcal{F})$ is a measurable space, is called $\mathcal{F}$*-measurable* if $f^{-1}(B) \in \mathcal{F}$ for every Borel subset $B \subset \mathbb{R}$.

Example 2.8.
Let $f : \mathbb{R} \to \mathbb{R}$*. The function* f *defined by* $f(r) = r$ *is measurable. Indeed,*

$$f^{-1}((a, b)) = \{r \in \mathbb{R} : a < f(r) < b\} = (a, b).$$

Since (a, b) is interval, thus it is measurable. Therefore, $f^{-1}((a, b))$ is measurable. So that f is measurable.

The properties of measurable functions are given by the following theorem.

Theorem 2.9 (**Properties of Measurable Functions**).

1) *If $f, g : \Omega \to \mathbb{R}$ are measurable functions, then $f+g, f-g, fg$ and $\frac{f}{g}\,(g \neq 0)$ are measurable.*

2) *If $f : \Omega \to \mathbb{R}$ is a measurable function, then f^2 is a measurable function.*

3) *If $f, g : \Omega \to \mathbb{R}$ are measurable functions and $\alpha, \beta \in \mathbb{R}$, then $\alpha f + \beta g$ is also measurable.*

4) *If $f, g : \mathbb{R} \to \mathbb{R}$ are measurable functions, then $(g \circ f)(r) = g(f(r))$ is measurable.*

5) *If $f, g : \Omega \to \mathbb{R}$ are measurable functions, then $\max\{f, g\}$ and $\min\{f, g\}$ are measurable.*

6) *If $f : \Omega \to \mathbb{R}$ is a measurable function, then the absolute value of f, $|f|$, is also measurable.*

7) *If $f, g : \Omega \to \mathbb{R}$ are two functions such that $f = g$ almost everywhere (a.e.) and f is a measurable function, then g is measurable.*

   Remark 2.10. $f = g$ almost everywhere means the set $\Omega = \{r \in \Omega : f(r) \neq g(r)\}$ is measure zero.

8) *Let A be a set, define*

$$\mathcal{X}_A(r) = \begin{cases} 1, & r \in A \\ 0, & r \notin A \end{cases}.$$

   If A is measurable, then $\mathcal{X}_A(r)$ is also a measurable function.

We now define a probability measure and then state its properties as a proposition.

Definition 2.11.
Let $\mathcal{F}$ be a σ-field on Ω. A *probability measure* P is a mapping

$$P : \mathcal{F} \to [0, 1]$$

such that

i) $P(\Omega) = 1$;

ii) If $A_1, A_2, \ldots, A_n, \ldots$ is any sequence of pairwise disjoint events in $\mathcal{F}$, then

$$P\left(\cup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P(A_n).$$

Proposition 2.12 (**Properties of Probability Measures**).

1) For any subset A of Ω we have $0 \leq P(A) \leq 1$.

2) $P(A^c) = 1 - P(A)$, where A^c is the complement of A.

3) If $A \subset B$, then $P(A) \leq P(B)$.

4) For any subsets A and B of Ω we have $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Definition 2.13.
A *probability space* is a triple $(\Omega, \mathcal{F}, P)$ where Ω is the sample space, $\mathcal{F}$ is a σ-field on Ω and P is the probability measure on $\mathcal{F}$.

Definition 2.14.
The events A and B are said to be *independent* if $P(A \cap B) = P(A)\, P(B)$.

Example 2.15.
Let us throw a die with all outcomes equally likely. Then $\Omega = \{1, 2, 3, 4, 5, 6\}$, $\mathcal{F} = 2^{\Omega}$ and $P\{i\}) = \frac{1}{6}$ for each i. Let $A = \{3, 5, 6\}$, $B = \{1, 4\}$, $C = \{1, 2, 3\}$. Then we see that $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{3}$ and $P(C) = \frac{1}{2}$. Also $B \cap C = \{1\}$ and $A \cap C = \{3\}$. Since $P(B \cap C) = \frac{1}{6} = \frac{1}{6} = P(B).P(C)$, events B and C are independent. However, since $P(A \cap C) = \frac{1}{6} \neq \frac{1}{4} = P(A).P(C)$, events A and C are not independent.

Definition 2.16.
Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X : \Omega \to \mathbb{R}$ be $\mathcal{F}$-measurable. Then X is said to be *random variable* on $(\Omega, \mathcal{F}, P)$.

Remark 2.17.
Proposition 2.6 and Theorem 2.9 can be applied to random variables since random variables are measurable functions.

Definition 2.18.
Let $X : (\Omega, \mathcal{F}, P) \to \mathbb{R}$ and $Y : \left(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P}\right) \to \mathbb{R}$ be random variables. Then X and Y are called *identically distributed* random variables if $P_X = P_Y$.

We now define independent random variables.

Definition 2.19.
The random variables X and Y on $(\Omega, \mathcal{F}, P)$ are said to be *independent* if the σ-fields they generate, $\mathcal{F}_X$ and $\mathcal{F}_Y$, are independent.

The following definition is one of the fundamental concepts in probability.

Definition 2.20.
Let $X : \Omega \to \mathbb{R}$ be a random variable. Suppose X can take values $r_1, r_2, \ldots, r_n$ with corresponding probabilities $\mathbf{p}(r_1), \mathbf{p}(r_2), \ldots, \mathbf{p}(r_n)$. Then the expected value of the random variable X is given by

$$E(X) = \sum_{k=1}^{n} r_k \mathbf{p}(r_k).$$

The expected value of a random variable demonstrates its adjusted average.

Example 2.21.
How many tails would you expect when you tossed a coin three times?
$X =$ *number of* $\in \{0, 1, 2, 3\}$ *and*
$p(0) = \frac{1}{8}$, $p(1) = \frac{3}{8}$, $p(2) = \frac{3}{8}$, $p(3) = \frac{1}{8}$. *Therefore, weighted average is*

$$0\frac{1}{8} + 1\frac{3}{8} + 2\frac{3}{8} + 3\frac{1}{8} = 1.5$$

In the following proposition we have listed the properties of expectation.

Proposition 2.22 (**Properties of Expectation**).
Let $(\Omega, \mathcal{F}, P)$ *be a probability space. Let* X, Y *be simple random variables on* $(\Omega, \mathcal{F}, P)$, $c \in \mathbb{R}$ *and* $A \in \mathcal{F}$. *Then*

1. $E[X \pm Y] = E[X] \pm E[Y]$;
2. *if* X *and* Y *are independent, then* $E[X.Y] = E[X].E[Y]$;
3. $E[cX] = cE[X]$;
4. *if* $X \geq 0$, *then* $E[X] \geq 0$;
5. *if* $X \geq Y$, *then* $E[X] \geq E[Y]$;

6. *if* $|X| \leq M$ *on* $A \in \mathcal{F}$*, then* $|E[X.\mathbf{1}_A]| \leq M.P(A)$.

To show the different components in the construction we need to introduce a new notation by

$$E[X] := \int_{\Omega} XdP. \tag{2.1}$$

When $X = \sum_{i=1}^{n} r_i \mathbf{1}_{A_i}$, $(A_i)_{i=1}^{n}$ is a partition of Ω into $\mathcal{F}$ measurable sets and $r_i = X(\omega_i)$ for any $\omega_i \in A_i$, by rewriting the integral, we obtain

$$E[X] = \sum_{i=1}^{n} X(\omega_i) P(A_i) = \int_{\Omega} XdP. \tag{2.2}$$

Example 2.23.

Let $\Omega = \{1, 2, \ldots, 7\}, \mathcal{F} = 2^{\Omega}, P(\{i\}) = \frac{1}{6}$ *for* $i = 1, 2, 3$ *and* $P(\{i\}) = \frac{1}{8}$ *for* $i = 4, 5, 6, 7$*. Let* $X(i) = 2i$ *for* $i \leq 4$*,* $X(i) = i$ *for* $i > 4$ *and let* $A = \{2, 4, 5, 7\}$*. Then*

$$\begin{aligned}
\int_A XdP = \int_{\{2,4,5,7\}} XdP &= \sum_{i \in \{2,4,5,7\}} X(i) P(\{i\}) \\
&= X(2) P(\{2\}) + X(4) P(\{4\}) + X(5) P(\{5\}) + X(7) P(\{7\}) \\
&= 4\frac{1}{6} + 8\frac{1}{8} + 5\frac{1}{8} + 7\frac{1}{8} \\
&= \frac{19}{6}.
\end{aligned}$$

We now talk about integrability and its properties for random variables. If X is a positive random variable on the probability space $(\Omega, \mathcal{F}, P)$, define

$$X^{[m]}(\omega) := \begin{cases} X(\omega) & if X(\omega) < m, \\ m & if X(\omega) \geq m. \end{cases}$$

Definition 2.24.

Let X denote a positive random variable on the probability space $(\Omega, \mathcal{F}, P)$. If $\lim_{m\to\infty} E[X^{[m]}] < \infty$, we call X an integrable random variable. If X is integrable, we let

$$E[X] = \lim_{m\to\infty} E[X^{[m]}].$$

Lemma 2.25.

Let X *and* Y *denote positive random variables on the probability space* $(\Omega, \mathcal{F}, P)$*. If* X *is integrable and* $X = Y$ *almost surely, then* Y *is integrable and* $E[X] = E[Y]$*.*

Definition 2.26.

A random variable X on a probability space $(\Omega, \mathcal{F}, P)$ is integrable if its positive and negative parts, X^+ and X^-, are both integrable. If X is integrable we let

$$E[X] := E[X^+] - E[X^-] = \int_\Omega X^+ dP - \int_\Omega X^- dP = \int_\Omega X dP.$$

$\int_\Omega X dP$ is called *the Lebesgue integral* of X over Ω with respect to P and let $L^1(\Omega, \mathcal{F}, P)$ denote the set of all integrable random variables on $(\Omega, \mathcal{F}, P)$.

Proposition 2.27.

Let X and Y be random variables on the probability space $(\Omega, \mathcal{F}, P)$.

1. *X is the pointwise limit of a sequence of $\mathcal{F}_X$ measurable simple random variables $(X_n)_{n=1}^\infty$ such that $|X_n| \leq |X|$ for all n.*
2. *X is integrable if and only if $|X|$ is integrable.*
3. *If X is integrable, then $|E[X]| \leq E[|X|]$.*
4. *If $|Y| \leq |X|$ and X is integrable, then Y is integrable.*
5. *If X and Y are integrable random variables and $c \in \mathbb{R}$, then $X \pm Y$ and cX are integrable.*
6. *If X and Y are integrable random variables on $(\Omega, \mathcal{F}, P)$, then $E[X+Y] = E[X] + E[Y]$.*

Proposition 2.28.

If $(\Omega, \mathcal{F}, P)$ is a probability space with $\Omega = (\omega_n)_{n=1}^\infty$ and $\mathcal{F} = 2^\Omega$, then $X : \Omega \to \mathbb{R}$ is integrable if and only if

$$\sum_{n=1}^\infty |X(\omega_n)| P(\{\omega_n\}) < \infty.$$

If X is integrable

$$E[X] = \int_\Omega X dP = \sum_{n=1}^\infty X(\omega) P(\{\omega_n\}).$$

2.1.2 Convex and Concave Functions

We now take a short break from probability theory and discuss certain properties of real functions. Such properties are often used when studying financial or economic problems. In particular, they are used quite often throughout this thesis.

Definition 2.29.
A function $\phi : (a, b) \subset \mathbb{R} \to \mathbb{R}$ is *convex* if for all r_1, r_2, $a < r_1 < r_2 < b$ and all t, $0 < t < 1$,

$$\phi (tr_1 + (1 - t) r_2) \leq t\phi (r_1) + (1 - t) \phi (r_2) . \tag{2.3}$$

A function ϕ is said to be (strictly) *concave* if $-\phi$ is (strictly) convex.

Remark 2.30.
For a twice differentiable function f, if the second derivative, $f^{''} (r)$, is positive, then the graph is convex; if $f^{''} (r)$ is negative, then the graph is concave.

Example 2.31.
Let $f_1 (r) = r^2$. We show that $f_1 (r)$ is convex. Following Remark 2.30 we have

$$f_1^{''} (r) = 2 > 0.$$

Hence, $f_1 (r)$ is convex.

Example 2.32.
Let $f_2 (r) = \ln r$, $r > 0$. We show that $f_2 (r)$ is concave. From Remark 2.30 we have

$$f_2^{''} (r) = -\frac{1}{r^2} < 0.$$

Hence, $f_2 (r)$ is concave.

Proposition 2.33.
If $\phi : (a, b) \subset \mathbb{R} \to \mathbb{R}$, then the following conditions are equivalent:

1. *ϕ is convex;*

2. *if $a < r_1 < r_2 < \ldots < r_n < b$, $0 < t_i < 1$ and $\sum_{i=1}^{n} t_i = 1$, then*

$$\phi \left(\sum_{i=1}^{n} t_i r_i \right) \leq \sum_{i=1}^{n} t_i \phi (r_i) ; \tag{2.4}$$

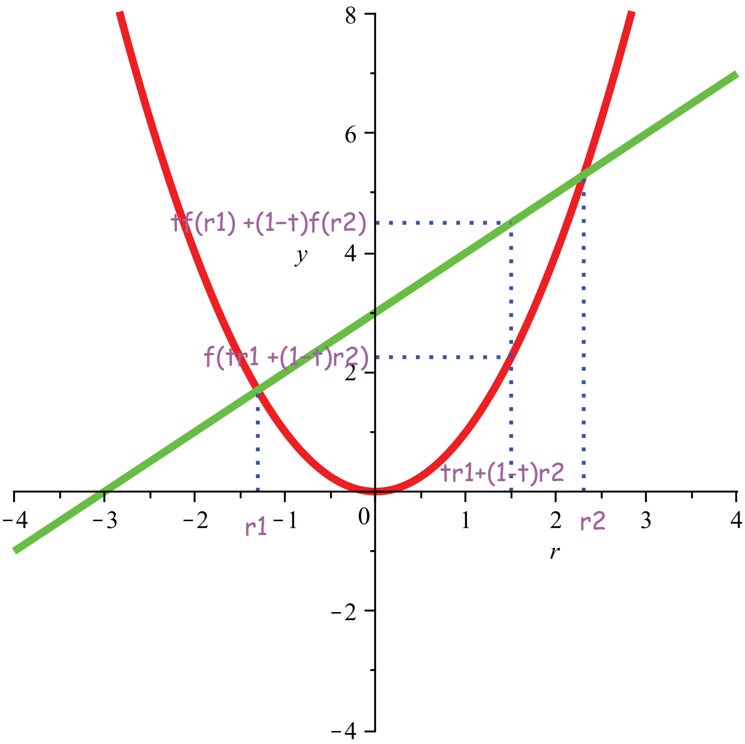


Figure 2.1: $f_1(r)$

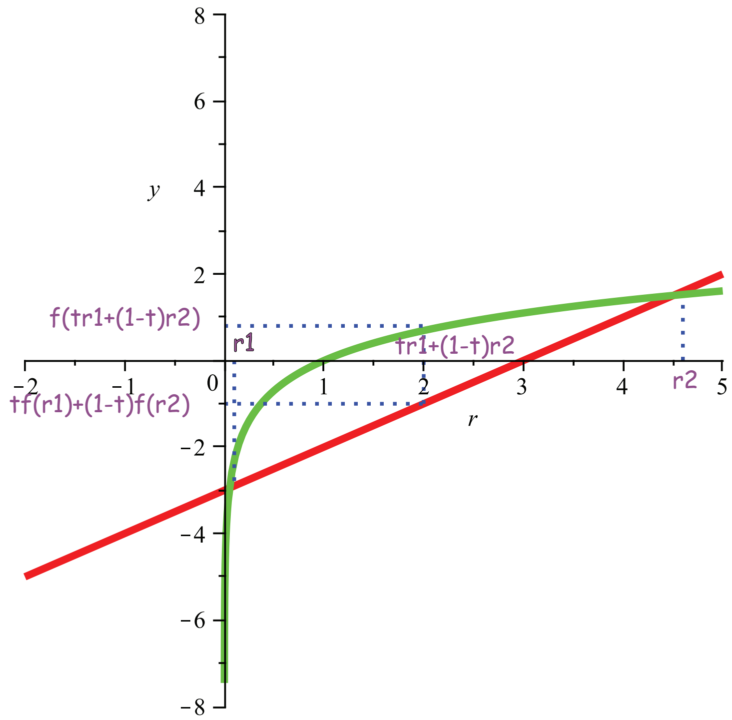


Figure 2.2: $f_2(r)$

3. *if* $a < r < y < z < b$, *then*

$$\frac{\phi(r) - \phi(y)}{r - y} \leq \frac{\phi(y) - \phi(z)}{y - z}. \tag{2.5}$$

Corollary 2.34.

1. *If* ϕ *is a twice continuously differentiable function defined on* (a, b), *then* ϕ *is convex if and only if* $\phi'' \geq 0$.

2. *Convex functions are continuous.*

3. *If* $\phi : (a, b) \rightarrow \mathbb{R}$ *is convex, then* $\phi = \phi_1 + \phi_2$ *where* ϕ_1 *is convex and increasing and* ϕ_2 *is convex and decreasing.*

We now state Jensen's inequality for concave functions with positive weights.

Proposition 2.35.
For a real concave function ϕ, *numbers* $r_1, r_2, \ldots, r_n$ *in its domain, and positive weights* a_i, *Jensen's inequality can be stated as :*

$$\phi\left(\frac{\sum a_i r_i}{\sum a_j}\right) \geq \frac{\sum a_i \phi(r_i)}{\sum a_j}. \tag{2.6}$$

Let us give an example.

Example 2.36.
Let $\phi(r) = \sqrt{r}$. *First we show that* $\phi(r)$ *is concave. The second derivative of* $\phi(r)$ *is negative. Thus, from Remark 2.30 it is concave. Let* $r_1, r_2, \ldots, r_n$ *be positive numbers. Hence, they are in the domain of* $\phi(r) = \sqrt{r}$. *Let* a_i, $i = 1, \ldots, n$, *be positive numbers, then Proposition 2.35 implies that*

$$\sqrt{\frac{\sum a_i r_i}{\sum a_j}} \geq \frac{\sum a_i \sqrt{r_i}}{\sum a_j}$$

$$\Longleftrightarrow \frac{\sqrt{a_1 r_1 + \ldots + a_n r_n}}{\sqrt{a_1 + \ldots + a_n}} \geq \frac{a_1\sqrt{r_1} + \ldots + a_n\sqrt{r_n}}{a_1 + \ldots + a_n}.$$

Example 2.37.
Let $\phi(r) = \ln r$. *In Example 2.32 we have showed that* $\phi(r)$ *is concave. Let* $r_1, r_2, \ldots, r_n$ *be positive numbers. Hence, they are in the domain of* $\phi(r) = \ln r$.

Let a_i, $i = 1, \ldots, n$, be positive numbers, then Proposition 2.35 implies that

$$\ln\left(\frac{\sum a_i r_i}{\sum a_j}\right) \geq \frac{\sum a_i \ln(r_i)}{\sum a_j}.$$

This particular function is used frequently in this thesis. In particular this property is used in Chapter 5.

We now state the following lemma which is proved in [4]. It will be used in the proof of our main result in Section 4. This is a famous inequality which appears quite often in problems related to information theory.

Lemma 2.38.
For any vectors $(a_1, \ldots, a_K) > 0$ and $(b_1, \ldots, b_K) \geq 0$ satisfying $\sum a_k = \sum b_k = 1$, the following inequality holds

$$\sum_{k=1}^{K} a_k \ln a_k - \sum_{k=1}^{K} a_k \ln b_k \geq 0. \tag{2.7}$$

In particular, if $(a_1, \ldots, a_K) \neq (b_1, \ldots, b_K)$, then $\sum a_k \ln a_k > \sum a_k \ln b_k$.

Proof.
We have

$$\ln x \leq x - 1,$$

which implies

$$\frac{\ln x}{2} = \ln\sqrt{x} \leq \sqrt{x} - 1,$$

and so

$$-\ln x \geq 2 - 2\sqrt{x}.$$

By using this inequality, we get

$$\begin{aligned}\sum_{k=1}^{K} a_k(\ln a_k - \ln b_k) &= -\sum_{k=1}^{K} a_k \ln\frac{b_k}{a_k} \geq \sum_{k=1}^{K} a_k\left(2 - 2\frac{\sqrt{b_k}}{\sqrt{a_k}}\right) \\ &= 2 - 2\sum_{k=1}^{K}\sqrt{a_k b_k} = \sum_{k=1}^{K}\left(a_k - 2\sqrt{a_k b_k} + b_k\right) \\ &= \sum_{k=1}^{K}\left(\sqrt{a_k} - \sqrt{b_k}\right)^2 \geq 0.\end{aligned}$$

If $(a_1, \ldots, a_K) \neq (b_1, \ldots, b_K)$, then $\sum_{k=1}^{K} a_k(\ln a_k - \ln b_k) > 0$. □

2.1.3 Conditional Expectation

The expectation changes if new information becomes available.

Definition 2.39.
Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\mathcal{G}$ be a σ-field on Ω generated by a countable partition $(G_n)_{n=1}^{\infty}$ of Ω. Suppose $\mathcal{G} \subset \mathcal{F}$ and $P(G_n) > 0$ for all n. If X is an integrable random variable on $(\Omega, \mathcal{F}, P)$, let

$$E[X|\mathcal{G}](\omega) = \frac{1}{P(G_n)} \int_{G_n} X dP \tag{2.8}$$

for all n and all $\omega \in G_n$. $E[X|\mathcal{G}]$ is called the *conditional expectation of X given $\mathcal{G}$*. If $\mathcal{G}$ is generated by a random variable Y on $(\Omega, \mathcal{F}, P)$, we also write $E[X|Y]$ in place of $E[X|\mathcal{F}_Y]$.

Proposition 2.40.
Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\mathcal{G}$ be a σ-field on Ω generated by a countable partition $(G_n)_{n=1}^{\infty}$ of Ω. Suppose $\mathcal{G} \subset \mathcal{F}$ and $P(G_n) > 0$ for all n. If X is an integrable random variable on $(\Omega, \mathcal{F}, P)$, then $E[X|\mathcal{G}]$ is the unique $\mathcal{G}$ measurable integrable random variable on $(\Omega, \mathcal{F}, P)$ satisfying

$$\int_A E[X|\mathcal{G}] dP = \int_A X dP \tag{2.9}$$

for all $A \in \mathcal{G}$.

In particular, if $A = \Omega$, then

$$E[E[X|\mathcal{G}]] = \int_\Omega E[X|\mathcal{G}] dP = \int_\Omega X dP = E[X], \tag{2.10}$$

which means that the average of the averages is the average. The properties of conditional expectation can be listed by the following proposition.

Proposition 2.41 (Properties of Conditional Expectation).
Let X and Y be integrable random variables on the probability space $(\Omega, \mathcal{F}, P)$. Let c_1 and c_2 be real numbers and let $\mathcal{G}$ and $\mathcal{H}$ be σ-fields on Ω where $\mathcal{H} \subset \mathcal{G} \subset \mathcal{F}$.

1. *If Y is any version of $E(X|\mathcal{G})$ then $E(Y) = E(X)$.*

2. *If X is $\mathcal{G}$ measurable, then $E(X|\mathcal{G}) = X$ a.s.*

3. ***(Positivity)*** *If* $X \geq 0$*, then* $E(X|\mathcal{G}) \geq 0$*, a.s.*

4. ***(Linearity)*** $E[c_1X + c_2Y|\mathcal{G}] = c_1E[X|\mathcal{G}] + c_2E[Y|\mathcal{G}]$.

5. ***(Taking out what is known)*** *If* $X.Y$ *is integrable and* X *is* $\mathcal{G}$*-measurable, then*
$$E[X.Y|\mathcal{G}] = X.E[Y|\mathcal{G}].$$

6. ***(Indepence drops out)*** *If* X *and* $\mathcal{G}$ *are independent, then*
$$E[X|\mathcal{G}] = E[X].$$

7. ***(Tower Law)***
$$E[E[X|\mathcal{G}|\mathcal{H}]] = E[X|\mathcal{H}].$$

8. ***(Jensen's Inequality)*** *Let* $\phi : \mathbb{R} \to \mathbb{R}$ *be a convex function and let* X *be an integrable random variable on* $(\Omega, \mathcal{F}, P)$ *such that* $\phi(X)$ *is also integrable. Then*
$$\phi(E[X|\mathcal{G}]) \leq E[\phi(X)|\mathcal{G}], \tag{2.11}$$
for any σ*-field* $\mathcal{G}$ *on* Ω*.*

2.2 Stochastic Processes

We now define a stochastic process and a Markov process and then give an example.

Definition 2.42.
A *stochastic process* is a collection of random variables $(X_t)_{t\in T}$ defined on a probability space $(\Omega, \mathcal{F}, P)$, indexed by a subset T of the real numbers.

If subset T is countable, then the process X is called *discrete stochastic process.* If T is uncountable, then it is called *continuous stochastic process.* The index T represents time.

Definition 2.43.
For each fixed $\omega \in \Omega$, the mapping
$$t \to X_t(\omega),$$
defined on index set $t \in T$, is called a *sample path (or path).*

Definition 2.44.

Let $X = (X_t)_{t \in T}$ be a stochastic process and $(\mathcal{F}_t)_{t \in T}$ be a filtration on $(\Omega, \mathcal{F}, P)$. Then X is *adapted* to the filtration if X_t is $\mathcal{F}_t$- measurable for all $t \in T$.

Definition 2.45.

A random sequence $X_1, X_2, \ldots$ is called an *independent and identically distributed (i.i.d.) process*, if the sequence of random variables $X_1, X_2, \ldots$ is i.i.d..

Theorem 2.46 (**Law of Large Numbers**).

Let $X_1, X_2, \ldots$ be pairwise independent and identically distributed random variables, each having the same finite mean μ. Then

$$P\left(\lim_{n \to \infty} \frac{1}{n}\left(X_1 + X_2 + \ldots + X_n\right) = \mu\right) = 1.$$

In other words, the partial averages $\frac{1}{n}\left(X_1 + X_2 + \ldots + X_n\right)$ converge almost surely to μ.

The following theorem will be used in the proof of our main result in Chapter 5. The proof of the theorem can be found in [25].

Theorem 2.47.

Let $\{D_t, t \geq 1\}$ be a sequence of random variables and $\{\mathcal{F}_t, t \geq 1\}$ an increasing sequence of σ-fields with D_t measurable with respect to $\mathcal{F}_t$ for each t. Let D be a random variable and c be a constant such that $E|D| < \infty$ and $P\left(|D_t| > x\right) \leq cP\left(|D| > x\right)$ for each $x \geq 0$ and $t \geq 1$. Then

$$t^{-1} \sum_{i=1}^{t} \left[D_i - E\left(D_i | \mathcal{F}_{i-1}\right)\right] \xrightarrow{p} 0 \tag{2.12}$$

as $t \to \infty$. If $E\left(|D| \log^{+} |D|\right) < \infty$, or if the D_t are independent, or if $\{D_t, t \geq 1\}$ and $\{E\left(D_t | \mathcal{F}_{t-1}\right), t \geq 2\}$ are stationary sequences, then the convergence in probability in (2.12) can be strengthened to a.s. convergence.

Remark 2.48.

The above theorem is a weak version of law of large numbers unlike Theorem 2.46. In Theorem 2.47 the random variables are in general not required to be i.i.d. This result roughly says that the time average asymptotically equals to the "conditional"space average.

Definition 2.49.
A sequence of σ-fields $(\mathcal{F}_n)_{n=1}^{\infty}$ on Ω such that

$$\mathcal{F}_1 \subset \mathcal{F}_2 \subset \ldots \subset \mathcal{F}$$

is called a *filtration.*

A filtration means that as the time t increases, the information about an event increases.

Example 2.50.
For a sequence $\psi_1, \psi_2, \ldots$ of coin tosses we take $\mathcal{F}_t$ to be the σ-field generated by $\psi_1, \psi_2 \ldots, \psi_t$,

$$\mathcal{F}_t = \sigma\left(\psi_1, \psi_2, \ldots, \psi_t\right).$$

Let

$$A = \{\textit{the first } 7 \textit{ tosses produce at least one head and at least two tails}\}.$$

At time $t = 7$, we will be able to decide if A belongs to $\mathcal{F}_7$. Nevertheless at time $t = 6$ it is not possible to tell if A has occurred or not. We have to wait for the 7th toss to decide.

Definition 2.51.
A stochastic process $\{X_n\}$ where $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$ is called a *discrete-time finite state Markov chain* if

$$P\left(X_{n+1} = j | X_0 = c_0, \ldots, , X_{n-1} = c_{n-1}, X_n = i\right) = P\left(X_{n+1} = j | X_n = i\right), \quad (2.13)$$

for every $i, j, c_0, \ldots, c_{n-1}$ and for every n.
The matrix $P\left(X_{n+1} = j | X_n = i\right)_{j,i \in S}$ is called the *transition matrix* of the chain X_n.

Let us give an example.

Example 2.52.
Suppose that a gambler starts with £1 and at each game the gambler wins £1 with probability $\boldsymbol{p}$ or looses £1 with probability $1 - \boldsymbol{p}$. The game ends when the gambler has £7 or looses all his money. This is a Markov chain with its transition matrix

given by

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1-\boldsymbol{p} & 0 & \boldsymbol{p} & 0 & 0 & 0 & 0 & 0 \\ 0 & 1-\boldsymbol{p} & 0 & \boldsymbol{p} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1-\boldsymbol{p} & 0 & \boldsymbol{p} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-\boldsymbol{p} & 0 & \boldsymbol{p} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-\boldsymbol{p} & 0 & \boldsymbol{p} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1-\boldsymbol{p} & 0 & \boldsymbol{p} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

We now state a definition of Markov process with infinite state spaces.

Definition 2.53.
Let $(X, \mathcal{B}, \lambda)$ be a probability space. A function $\mathbb{P} : X \times \mathcal{B} \to [0,1]$ is called a stochastic transition function if it has the following properties:

i) for any $A \in \mathfrak{B}, \mathbb{P}(., A) : X \to [0,1]$ is a $\mathfrak{B}$-measurable function;

ii) for any $x \in X, \mathbb{P}(x, .) : \mathfrak{B} \to [0,1]$ is a measure.

A *Markov process* can be defined by a transition function $\mathbb{P}$. Let λ be a probabilistic measure on $\mathfrak{B}$ called initial probability. Then we define all probabilities related to the Markov process $\{X_n\}_{n\geq 0}$ using λ and $\mathbb{P}$:

$$P(\mathcal{X}_{n+1} \in A | \mathcal{X}_n = r) = \mathbb{P}(r, A);$$

$$P(\mathcal{X}_{n+1} \in A) = \underbrace{\int_X \cdots \int_X}_{n+1} d\lambda(r_0)\, \mathcal{P}(r_0, dr_1)\, \mathcal{P}(r_1, dr_2) \ldots \mathcal{P}(r_{n-1}, dr_n)\, \mathcal{P}(r_n, A).$$

2.2.1 Discrete Martingales

Definition 2.54.
Let $(\mathcal{F}_n)_{n=1}^{\infty}$ be a filtration on the probability space $(\Omega, \mathcal{F}, P)$. A sequence $(X_n)_{n=1}^{\infty}$ of random variables on $(\Omega, \mathcal{F}, P)$ is called a discrete martingale with respect to $(\mathcal{F}_n)_{n=1}^{\infty}$, if

- X_n is integrable, $n = 1, 2, \ldots$;

- $(X_n)_{n=1}^{\infty}$ is adapted to $(\mathcal{F}_n)_{n=1}^{\infty}$;
- $E[X_{n+1}|\mathcal{F}_n] = X_n$ a.s. for all $n = 1, 2, \ldots$.

Let us give an example for martingales.

Example 2.55.
Let $X_1, X_2, \ldots$ be a sequence of independent random variables with $E(X_s) = 0$, for each s. Define

$$S_0 := 0,$$

$$S_n := X_1 + X_2 + \ldots + X_n,$$

$$\mathcal{F}_0 := \{\emptyset, \Omega\},$$

$$\mathcal{F}_n := \sigma(X_1, X_2, \ldots, X_n).$$

Then, for $n \geq 1$, S_n is adapted to the filtration $\mathcal{F}_n$. Also it is integrable because

$$\begin{aligned} E(|S_n|) &= E(|X_1 + X_2 + \ldots + X_n|) \\ &\leq E(|X_1|) + E(|X_2|) + \ldots + E(|X_n|) \\ &< \infty. \end{aligned}$$

Moreover,

$$\begin{aligned} E(S_n|\mathcal{F}_{n-1}) &= E(S_{n-1} + X_n|\mathcal{F}_{n-1}) \\ &= E(S_{n-1}|\mathcal{F}_{n-1}) + E(X_n|\mathcal{F}_{n-1}) \\ &= S_{n-1} + E(X_n) = S_{n-1}, \end{aligned}$$

since S_{n-1} is $\mathcal{F}_{n-1}$-measurable and X_n is independent of $\mathcal{F}_{n-1}$. Thus S_n is a martingale with respect to $\mathcal{F}_{n-1}$.

Definition 2.56.
$(X_t)_{t\in\mathbb{Z}}$ is called a *martingale-difference sequence* with respect to the filtration $\{\mathcal{F}_t\}_{t\in\mathbb{Z}}$ if $E|X_t| < \infty$, X_t is $\mathcal{F}_t$-measurable and

$$E(X_t|\mathcal{F}_{t-1}) = 0, \quad \forall t \in \mathbb{Z}.$$

2.3 Ergodic Theory

In this section, we recall some definitions from ergodic theory. We follow [35] for this section.

2.3.1 Invariant Measures and Ergodicity

Definition 2.57.
Suppose $(X, \mathcal{F}, \mu)$ is a probability space. A transformation $\tau : X \to X$ is *measure-preserving* if τ is measurable and

$$\mu\left(\tau^{-1}(A)\right) = \mu(A),$$

for all $A \in \mathcal{F}$.

Let us give an example.

Example 2.58.
In this example we show that the doubling map $\tau(r) = 2r\ mod 1$ *preserves Lebesgue measure* m*, i.e.,* $\mu(\tau^{-1}(A)) = \mu(A)$*, for all* $A \in \mathcal{F}$*.*
First,

$$\begin{aligned}\tau^{-1}(a,b) &= \{r|\tau(r) \in (a,b)\}\\ &= \left(\frac{a}{2}, \frac{b}{2}\right) \cup \left(\frac{a+1}{2}, \frac{b+1}{2}\right).\end{aligned}$$

Therefore,

$$\begin{aligned}m\left(\tau^{-1}(A)\right) = m&\left(\left(\frac{a}{2}, \frac{b}{2}\right) \cup \left(\frac{a+1}{2}, \frac{b+1}{2}\right)\right)\\ &\frac{b}{2} - \frac{a}{2} + \frac{b+1}{2} - \frac{a+1}{2}\\ &b - a = m\left((a,b)\right).\end{aligned}$$

Definition 2.59.
Let $(X, \mathcal{F}, \mu)$ be a measure space. A measure-preserving transformation τ of $(X, \mathcal{F}, \mu)$ is called *ergodic* if the only members A of $\mathcal{F}$ with $\tau^{-1}A = A$ satisfy $\mu(A) = 0$ or $\mu(A) = 1$.

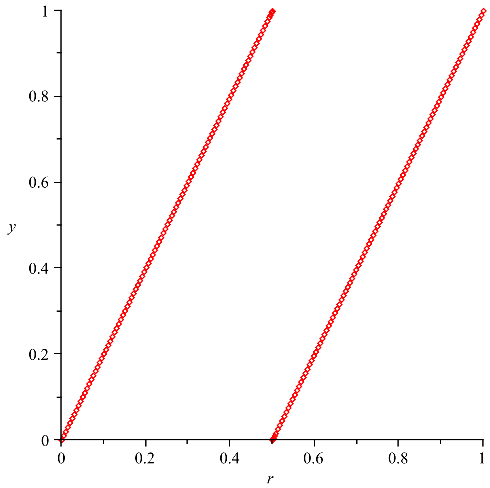


FIGURE 2.3: $\tau(r)$

Example 2.60.
Let us show that the doubling map $\tau(r) = 2r \bmod 1$ is ergodic with respect to Lebesgue measure m. Let N be a standard dyadic interval at scale n, i.e., an interval of type $\left(\frac{a}{2^n}, \frac{a+1}{2^n}\right)$, $a \in \mathbb{Z}$. If A is any measurable set, then $m(\tau^{-n}A \cap N) = 2^{-n}m(A)$. If A is τ-invariant, then $m(A \cap N) = 2^{-n}m(A)$. This means the relative density of A on N is $m(A)$. If $m(A)$ is not equal to 0 or 1, then this contradicts the Lebesgue density theorem.

We now state a major result in ergodic theory.

Theorem 2.61 (**Birkhoff Ergodic Theorem**).
Let $(X, \mathcal{F}, \mu)$ be a measure space and let $\tau : X \to X$ be a measure-preserving transformation. Then, for each $f \in L^1(\mu)$, there exists a function $f^* \in L^1(\mu)$ such that $f^*(\tau(r)) = f^*(r)$, $r \in X$ μ-a.e. and

$$\lim_{t\to\infty} \frac{1}{t} \sum_{i=0}^{t-1} f\left(\tau^i(r)\right) = f^*(r), \quad \forall r \in X, \quad \mu - a.e. \tag{2.14}$$

Furthermore, if $\mu(X) < \infty$, then $\int_X f^* d\mu = \int_X f d\mu$.
If τ is ergodic, then f^* is constant a.e. and so if $\mu(X) < \infty$ $f^* = \frac{1}{\mu(X)} \int f d\mu$ a.e.
If $(X, \mathcal{F}, \mu)$ is a probability space and τ is ergodic we have $\forall f \in L^1(\mu)$,

$$\lim_{t\to\infty} \frac{1}{t} \sum_{i=0}^{t-1} f\left(\tau^i(r)\right) = \int f d\mu \quad a.e.$$

The Birkhoff Ergodic Theorem shows that the asymptotic behaviour of the frequencies of the iterates $\tau^n(r)$ of r. The limit in (2.14) of the time average of f exists for a.e. r and the limit function f^* is integrable and measure-preserving.

2.3.2 Lyapunov Exponents

Definition 2.62.
A *manifold* is a topological space that is locally Euclidean (i.e., around every point, there is a neighbourhood that is topologically the same as the open unit ball in $\mathbb{R}^n$).

Euclidean space is the basic example of a manifold. Also, any smooth boundary of a subset of Euclidean space is a manifold, such as circle, sphere, etc. We can show that a circle is a manifold.

Example 2.63.
A circle is given by

$$C^1 = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 = 1\}$$

is a manifold of dimension one. A possible atlas is

$$\mathfrak{K} = \{(\mathfrak{H}_1, \alpha_1), (\mathfrak{H}_2, \alpha_2)\}$$

where

$$\mathfrak{H}_1 = C^1 \setminus \{(-1, 0)\}, \quad \alpha_1(x, y) = \arctan \frac{y}{x} \quad \textit{with} \quad -\pi < \alpha_1 < \pi$$

$$\mathfrak{H}_2 = C^1 \setminus \{(1, 0)\}, \quad \alpha_2(x, y) = \arctan \frac{y}{x} \quad \textit{with} \quad 0 < \alpha_2 < 2\pi.$$

Definition 2.64.
Suppose M is a C^s manifold ($s \geq 1$) and r is a point in M. Pick a chart $\Phi : U \to \mathbb{R}^n$ where U is an open subset of M containing r. Suppose two curves $Y_1 : (-1, 1) \to M$ and $Y_2 : (-1, 1) \to M$ with $Y_1 = Y_2 = r$ are given such that ΦY_1 and ΦY_2 are both differentiable at 0. Then Y_1 and Y_2 are called equivalent at 0 if the ordinary derivatives of ΦY_1 and ΦY_2 at 0 coincide. This defines an equivalence relation on such curves, and the equivalence classes are known as the *tangent vectors* of M at r. The equivalence class of the curve Y is written as $Y'(0)$. The *tangent space* of M at r, denoted by T_rM, is defined as the set of all tangent vectors.

We now define the Lyapunov exponents . We use [32] for the following definition and Oseledec's Multiplicative Ergodic Theorem.

Definition 2.65.
Let $f : M \to M$ be an endomorphism on a manifold M of dimension m. Let $|.|$ be the norm on tangent vectors induced by a Riemannian metric on M. For each $r \in M$ and $v \in T_r M$ let

$$l\left(r, v\right) = \lim_{t \to \infty} \frac{1}{t} \ln\left(|Df_r^t v|\right), \tag{2.15}$$

whenever the limit exists.

Let us give an example.

Example 2.66.
Let

$$\tau\left(r\right) = \begin{cases} 2r & 0 \leq r < \frac{1}{2}, \\ 2 - 2r & \frac{1}{2} \leq r \leq 1 \end{cases}$$

be a map. If r_0 *is such that* $r_j = T^j\left(r_0\right) = \frac{1}{2}$ *for some* j*, then* $l\left(r_0\right)$ *is not defined because its derivative does not exist. This kind of points make up a countable set. For other points* $r_0 \in [0, 1]$*,*

$$|f^{'}\left(r_j\right)| = 2 \quad for \;\; all \;\; j.$$

Therefore, the Lyapunov exponent is $l\left(r_0\right) = \log 2$*.*

We now state the following theorem.

Theorem 2.67 (**Oseledec's Multiplicative Ergodic Theorem**)**.**
Let M *be a compact manifold of dimension* m*,* $\mathcal{B}$ *be the* σ*-algebra generates by the Borel subsets of* M*, and* $f : M \to M$ *be a* C^2 *diffeomorphism. Then there is an invariant set* $B_f \in \mathcal{B}$ *of full measure for every* $\mu \in \mathcal{M}\left(f\right)$*, where* $\mu \in \mathcal{M}\left(f\right)$ *is the set of all invariant Borel probability measures for* f*, such that the Lyapunov exponents exist for all points* $r \in B_f$*.*
More precisely the following are true.

a) The set B_f *is*

i) invariant,

ii) of full measure, $\mu(B_f) = 1$ *for all* $\mu \in \mathcal{M}(f)$*.*

b) For each $r \in B_f$*, the tangent space at* r *can be written as an increasing set of subspaces*

$$\{0\} = V_r^0 \subset V_r^1 \subset \ldots V_r^{m(r)} = T_r M$$

such that

i) for $v \in V_r^j \; V_r^{j-1}$ *the limit defining* $l(r,v)$ *exists and* $l_j(r) = l(r,v)$ *is the same value for all such* v*,*

ii) the bundle of subspaces

$$\{V_r^j : r \in B_f \quad and \quad m(r) \geq j\}$$

are invariant in the sense that $Df_r V_r^j = V_{f(r)}^j$ *for all* $1 \leq j \leq m(r)$*.*

c) The function $m : B_f \to \{1, \ldots, s\}$ *is a measurable function and invariant,* $m \circ f = m$*.*

d) If $r \in B_f$*, the exponents satisfy*

$$-\infty \leq l_1(r) < l_2(r) < \ldots < l_{m(r)}(r).$$

For $1 \leq j \leq m$*, the function* $l_j(.)$ *is*

i) defined and measurable on the set

$$\{r \in B)f : s(r) \geq j\},$$

ii) is invariant, $l_j \circ f = l_j$*.*

2.4 Background from Finance and Economics

In this section we state definitions that we use in this thesis. We mostly have used the book [24] for the definitions.

Definition 2.68.
An *asset* is a resource controlled by the entity as a result of past events and from which future economic benefits are expected to flow to the entity. We use two types of assets in this thesis.

- A *short-lived asset* is an asset that the investor plans to hold it for short period of time, such as cash, securities, bank accounts etc.
- A *long-lived asset* is to be held for many years and are not intended to be disposed of in the near future, such as bonds, common stock etc.

2.4.1 Capital Asset Pricing Model (CAPM)

CAPM is a model which defines the relationship between expected return and risk. It is used to determine the pricing of risky securities.

2.4.1.1 Notation

- Asset prices are vectors denoted by

$$p_t = (p_{t,1}, \ldots, p_{t,K}) .$$

- Asset returns are random vectors denoted by

$$R = (R_1, \ldots, R_K) ,$$

where $R_k = \frac{p_{1,k} - p_{0,k}}{p_{0,k}}$. Here, 0 means the zero vector of dimension K. We use boldface font for $K + 1$ dimensional vector, for instance,

$$\mathbf{R} = (R_0, R_1, \ldots, R_K) .$$

$k = 0$ will be assumed to be risk-free, i.e., its return $R_0 = r > 0$ non-random number.

- Investor's portfolio is characterised by a vector

$$x = (x_1, \ldots, x_K) ,$$

$$\mathbf{x} = (x_0, x_1, \ldots, x_K) .$$

- The vector of expected returns on the assets

$$\mathfrak{m} = (\mathfrak{m}_1, \ldots, \mathfrak{m}_K) ,$$

where $\mathfrak{m}_k = ER_k$,

$$\mathbf{m} = (r, \mathfrak{m}_1, \ldots, \mathfrak{m}_K).$$

- The $K \times K$ covariance matrix $\mathcal{V} = (\sigma_{kj})$, where

$$\begin{aligned}\sigma_{kj} &= Cov\,(R_k, R_j)\\ &= E\,(R_k R_j) - E\,(R_k)\,E\,(R_j).\end{aligned}$$

  The matrix $\mathcal{V}$ is positive definite. Thus, its inverse $\mathcal{W} := \mathcal{V}^{-1}$ exists.

- The return on a normalized portfolio $\mathbf{x}$ is denoted by $\mathbf{R}_{\mathbf{x}} = \langle \mathbf{R}, \mathbf{x} \rangle$.
- The expected return on $\mathbf{x}$ is denoted by $\mathbf{m}_{\mathbf{x}}$.
- The variance of a portfolio return is denoted by $\sigma^2_{\mathbf{x}}$.
- $\eta \geq 0$ is the risk tolerance of the investor.

2.4.1.2 CAPM Equation

The Markowitz portfolio selection problem

$$\max_{\mathbf{x} \in \mathbf{R}^{K+1}} \{2\eta E\mathbf{R}_{\mathbf{x}} - Var\mathbf{R}_{\mathbf{x}}\} \tag{2.16}$$

subject to

$$\langle \mathbf{e}, \mathbf{x} \rangle = 1$$

has a unique solution $\mathbf{x}^*_\eta$ given by

$$\mathbf{x}^*_\eta = \mathbf{x}^{MIN} + \eta \mathbf{y}^*,$$

where $\mathbf{x}^{MIN} = (1, 0, \ldots, 0)$, $\mathbf{y}^* = (y^*_0, y^*)$, $y^*_0 = rC - A$, $y^* = \mathcal{W}\,(\mathfrak{m} - re)$, where $A = \langle e, \mathcal{W}\mathfrak{m} \rangle$, $B = \langle \mathfrak{m}, \mathcal{W}\mathfrak{m} \rangle$, $C = \langle e, \mathcal{W}e \rangle$ and $D = BC - A^2$ ([30]).

Theorem 2.69.
Let $\boldsymbol{x}^*_\eta$ be a solution of (2.16). Then for each $k = 1, \ldots, K$, the following equation holds:

$$ER_k - r = \frac{Cov\,(R_k, \boldsymbol{R}_{\boldsymbol{x}})}{Var\boldsymbol{R}_{\boldsymbol{x}}}\,(E\boldsymbol{R}_{\boldsymbol{x}} - r). \tag{2.17}$$

The equation (2.17) is called capital asset pricing model (CAPM) ([28]).

2.4.2 Game Theory

Game theory focuses on situations in which a group of people interact. It was found by the mathematician John von Neumann. The first book was *The Theory of Games and Economic Behaviour*. Game theory is firstly used in economics to understand a collection of economic behaviour which includes markets, firms and consumers. Then it is extended to psychology, political science, biology and logic. It is the formal study of decision-making where many decision-makers must make choices which affect the other players' interests. The games analysed in game theory are well-defined mathematical tools. It consists of players, payoffs and strategies of players. Let us start by describing the game in a formal language.

Definition 2.70.
A *game* is a formal model of an interactive situation which consists of players, actions, payoffs and information. These are known as the rules of the game.

Definition 2.71.

i) *Players* are the individuals who make decisions. Their aim is to maximise utility by choice of actions.

ii) The *payoff* of a player i is the expected utility he/she receives as a function of the strategies chosen by himself/herself and the other players.

iii) A *strategy* of player i is a rule that tells him/her which action to choose at each instant of the game, given his/her information set.

iv) A *strategy profile* $\left(\lambda_t^1, \ldots, \lambda_t^I\right)$ is a list consisting of one strategy for each of the I players in the game.

Let us give an example of game. This game is called *stag hunt* in game theory. Also this game is an example of a symmetric game. The following Examples 2.72, 2.75, 2.76 and 2.77 are well-known in game theory. Details can be found in [24, 31].

Example 2.72 (**Stug Hunt**).
There are two hunters decide to go out for hunting. Each can individually choose to hunt a stag or hunt a hare. Each hunter does not know the other hunter's choice while they choose what to hunt. If one of the hunters hunts a stag, then he must have the cooperation of the other hunter in order to succeed. A hunter can chooses

a hare by himself, but stag's meat is much better than a hare's meat. The payoff matric is given as the following.

$$\begin{array}{c} \\ Hunter\ 1\ (stag) \\ Hunter\ 1\ (hare) \end{array} \begin{array}{cc} Hunter\ 2\ (stag) & Hunter\ 2\ (hare) \\ \left(\begin{array}{cc} (9,9) & (0,6) \\ (6,0) & (4,4) \end{array} \right. & \left. \right) \end{array}$$

In the above payoff matrix we see that if the both hunters hunt a stag both will get a payoff 9. But if the Hunter 1 hunts a stag and Hunter 2 hunts a hare, then the first one gets a payoff 0 and second hunter gets a payoff 6. If they both hunt a hare they will both get a payoff 4.

Definition 2.73.
The *normal-form representation* of an I-player game specifies the players' strategy spaces $\lambda_t^1, \ldots, \lambda_t^I$ and their payoff functions $u_1, \ldots, u_I$. We denote this game by $G = \{\lambda_t^1, \ldots, \lambda_t^I; u_1, \ldots, u_I\}$.

We now define a Nash equilibrium.

Definition 2.74.
In the I-player normal-form game $G = \{\lambda_t^1, \ldots, \lambda_t^I; u_1, \ldots, u_I\}$, the strategies $\left(\bar{\lambda}_t^1, \ldots, \bar{\lambda}_t^I\right)$ are a *Nash equilibrium* if, for each player i, $\bar{\lambda}_t^i$ is player i's best response to the strategies specified for the $I-1$ other players, $\left(\bar{\lambda}_t^1, \ldots, \bar{\lambda}_t^{i-1}, \bar{\lambda}_t^{i+1}, \ldots, \bar{\lambda}_t^I\right)$:

$$u_i\left(\bar{\lambda}_t^1, \ldots, \bar{\lambda}_t^{i-1}, \bar{\lambda}_t^{i+1}, \ldots, \bar{\lambda}_t^I\right) \geq u_i\left(\bar{\lambda}_t^1, \ldots, \bar{\lambda}_t^{i-1}, \bar{\lambda}_t^{i}, \bar{\lambda}_t^{i+1}, \ldots, \bar{\lambda}_t^I\right) \tag{2.18}$$

for every feasible strategy λ_t^i; that is, $\bar{\lambda}_t^i$ solves

$$\max_{\lambda_t^i} u_i\left(\bar{\lambda}_t^1, \ldots, \bar{\lambda}_t^{i-1}, \bar{\lambda}_t^{i}, \bar{\lambda}_t^{i+1}, \ldots, \bar{\lambda}_t^I\right). \tag{2.19}$$

Let us give a famous example in game theory which is called *prisoner's dilemma*, where it has a unique Nash equilibrium.

Example 2.75 (Prisoner's Dilemma).
Each prisoner obtains a higher payoff when he betrays the other prisoner, no matter what the other prisoner decides. If both prisoner cooperate and stay silent then they will be charged for 1 *year. If both prisoners confess then this time they will be charged for* 5 *years. However, if the prisoner, say A, betrays the other and*

prisoner B stays silent, then the prisoner A will be free and the prisoner B will be charged for 10 years, or vice versa. The payoff matrix is represented as below.

$$\begin{array}{c} \\ Prisoner\ A\ (cooperate) \\ Prisoner\ A\ (confess) \end{array} \begin{array}{c} \begin{array}{cc} Prisoner\ B\ (cooperate) & Prisoner\ B\ (confess) \end{array} \\ \begin{pmatrix} (1,1) & & & (10,0) \\ (0,10) & & & (5,5) \end{pmatrix} \end{array}$$

Therefore both way, confessing makes sense, as it would for the other prisoner. So, the Nash equilibrium is formed at 5 *years for each.*

Let us show an other example where the game has two Nash equilibrium. This game is known as *battle of the sexes* in game theory.

Example 2.76 (**Battle of the Sexes**).
A girl and a boy want to go out together for a meeting this evening. There are two activities in town; dancing and cricket match. However, they will not communicate before the meeting. The girl wants to go for dancing. The boy wants to go to the cricket match. But they prefer being together rather that being alone. The payoff matrix of the game is the following.

$$\begin{array}{c} \\ Girl\ (dance) \\ Girl\ (cricket) \end{array} \begin{array}{c} \begin{array}{cc} Boy\ (dance) & Boy\ (cricket) \end{array} \\ \begin{pmatrix} (4,3) & & (0,0) \\ (0,0) & & (3,4) \end{pmatrix} \end{array}$$

This game has two pure strategy Nash equilibrium. One is both going out for dancing and the other is both going out to watch the cricket match.

We can now illustrate an example where the game has no Nash equilibrium. This game is known as *matching pennies* in game theory.

Example 2.77 (**Matching Pennies**).
This game is played between two players; Player 1 and Player 2. Each player has a penny and must secretly turn the penny to heads or tails. Then the players show their choices simultaneously. If the pennies match (both heads or tails), then Player 1 wins the Player 2's penny. If the pennies do not match (one heads and one tails), then Player 2 wins thePlayer 1's penny. The payoff matrix of this game

is given as the following.

$$\begin{array}{cc} & \begin{array}{cc} Player\ 2\ (heads) & Player\ 2\ (tails) \end{array} \\ \begin{array}{c} Player\ 1\ (heads) \\ Player\ 2\ (tails) \end{array} & \left(\begin{array}{cc} (+1,-1) & (-1,+1) \\ (-1,+1) & (+1,-1) \end{array} \right) \end{array}$$

In this game there is no pure strategy Nash equilibrium. Because there is no pair of pure strategies such that neither player would want to switch if told the player what the other player would do. Furthermore, this game is an example of a zero-sum game. Because one of the players' gain is equal to the other player's loss.

Chapter 3

Random Dynamical Systems

In this section, we recall some definitions from the theory of random dynamical systems. In this section we mainly follow [5].

Definition 3.1.
A measurable dynamical system $(\theta(t))_{t\in T}$ on a probability space $(\Omega, \mathcal{F}, P)$ for which each $\theta(t)$ is an endomorphism is called a *measure preserving* or *metric dynamical system* and is denoted by $\Sigma = \left(\Omega, \mathcal{F}, P, (\theta(t))_{t\in T}\right)$ or, for short, by $\theta(.)$.

Definition 3.2.
A *measurable random dynamical system* on the measurable system $(X, \mathcal{B})$ over (or covering or extending) a metric dynamical system $\left(\Omega, \mathcal{F}, P, (\theta(t))_{t\in T}\right)$ with time T is a mapping

$$\phi : T \times \Omega \times X \to X, \quad (t, \omega, r) \mapsto \phi(t, \omega, r),$$

with the following properties:

i) *Measurability*: ϕ is $\mathcal{B}(T) \otimes \mathcal{F} \otimes \mathcal{B}, \mathcal{B}$-measurable.

ii) *Cocycle property*: The mappings $\phi(t, \omega) := \phi(t, \omega, .) : X \to X$ form a cocycle over $\theta(.)$, i.e. they satisfy

$$\phi(0, \omega) = id_r \quad \text{for all} \quad \omega \in \Omega \quad (\text{if} \quad 0 \in T),$$

$$\phi(t+s, \omega) = \phi(t, \theta(s)\omega) \circ \phi(s, \omega) \quad \text{for all} \quad s, t \in T \quad \omega \in \Omega,$$

where "$\circ$ "means composition.

3.1 Random Maps With Constant Probabilities

We now define the notion of a random map (also called an iterated function system with probabilities) and state some results from [8] for a certain class of iterated function systems. Random maps are a special type of random dynamical systems. Let S be a finite set, $S = \{s_1, \dots, s_L\}$. Let $\mathbf{p}$ be a probability distribution on S such that $\mathbf{p}_s > 0$ for all $s \in S$ and $\tau_s : X \to X$ be measurable. The collection

$$F = \{\tau_1, \tau_2, \dots, \tau_L; \mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_L\}$$

is called a *random map* or an *iterated function system* (IFS) with probabilities. We denote the space of sequences $\omega = \{s_1, s_2, \dots\}$, $s_l \in S$, by Ω. The topology on Ω is defined as the product topologies on S. The Borel measure on Ω is defined as the product measure $\mathbf{p}^{\mathbb{N}}$. Moreover, we write

$$s^t := (s_1, s_2, \dots, s_t)$$

for the history (information) up to time t.
Formally F is understood as a Markov process with a transition function

$$\mathbb{P}(r, A) = \sum_{s=1}^{L} \mathbf{p}_s \mathcal{X}_A (\tau_s (r)),$$

where $A \in \mathcal{B}$ and $\mathcal{X}_A$ is the characteristic function of the set A. Intuitively, this means that at each time step

$$F(r) = \tau_s(r)$$

with probability $\mathbf{p}_s$; i.e., at each time step, one transformation τ_s is selected with probability $\mathbf{p}_s$ and applied to the process.

Example 3.3.
Let $S = \{1, 2\}$, i.e., S consists of two symbols. Then $F = \{\tau_1, \tau_2; \boldsymbol{p}_1, \boldsymbol{p}_2\}$. Let us consider a specific example where $X = [0, 1]$. Suppose that the formulae of τ_1, τ_2 are given by

$$\tau_1(r) = 4r(1 - r),$$

and

$$\tau_2(r) = \begin{cases} 2r & 0 \leq r < \frac{1}{2}, \\ 2-2r & \frac{1}{2} \leq r \leq 1, \end{cases}$$

and $\boldsymbol{p}_1 = \boldsymbol{p}_2 = \frac{1}{2}$.

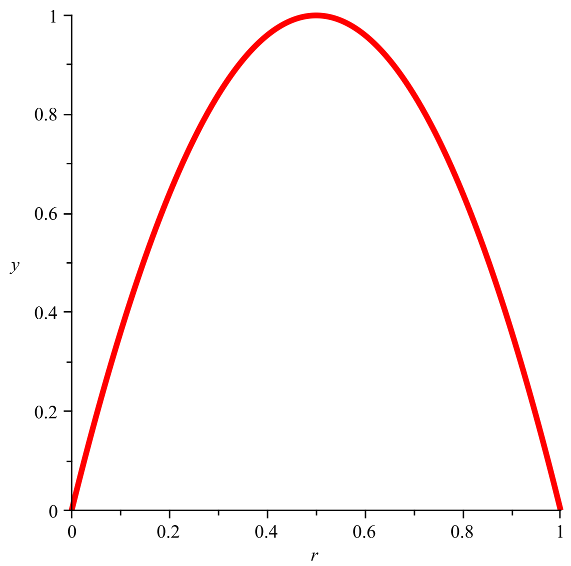


FIGURE 3.1: $\tau_1(r)$

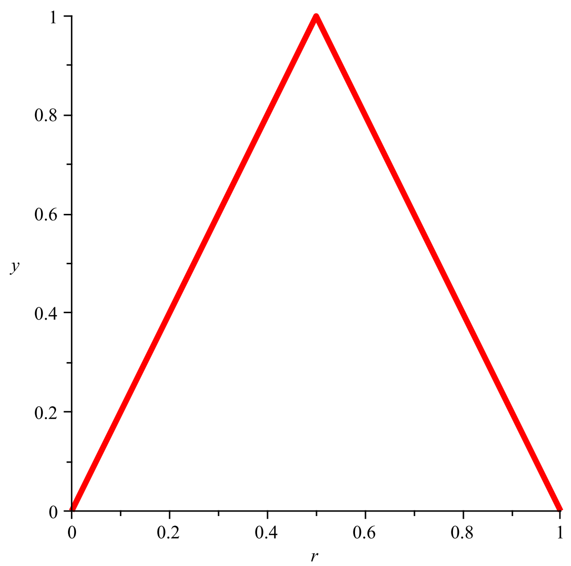
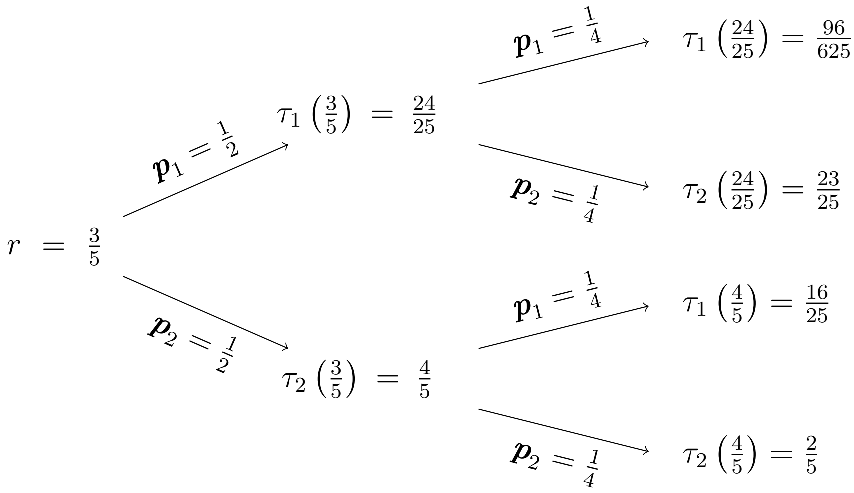


FIGURE 3.2: $\tau_2(r)$

For instance, when we take $r = \frac{3}{5}$, *then we can see how a random orbit looks like at time* $t = 1, 2$ *by the following diagram.*



3.1.1 Skew Products

A random map can be realised as a *skew product (deterministic map)* on the extended phase space:

$$\mathfrak{R}\left(r, w\right) : X \times \Omega \to X \times \Omega$$

given by

$$\mathfrak{R}\left(r, w\right) = \left(\tau_{w_1} r, \sigma w\right),$$

where Ω is the space of one-sided sequences $w = \{w_1, w_2, \ldots\}$, $w_i \in \{1, \ldots, K\}$ and $\left(\sigma w\right)_i = w_{i+1}$ is the left-shift map.

3.1.2 Random Homeomorphisms on $[0, 1]$

In [8], the following random map was studied. Let

$$F = \{\tau_1, \tau_2, \ldots, \tau_L; \mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_L\}$$

such that

i) $\tau_s : [0, 1] \to [0, 1]$,

ii) τ_s is continuous and increasing,

iii) $\tau_s\left(0\right) = 0$ and $\tau_s\left(1\right) = 1$.

Let $r_t(s^t)$ denote

$$r_t\left(s^t\right) := \tau_{s_t} \circ \tau_{s_{t-1}} \circ \ldots \circ \tau_{s_1}(r_0),$$

where $s^t := (s_1, s_2, \ldots, s_t), s_i \in S$.

Example 3.4.
Let $S = \{1, 2\}$, i.e., S is the space of two symbols. Then $F = \{\tau_1, \tau_2; \boldsymbol{p}_1, \boldsymbol{p}_2\}$. Suppose that the formulae of τ_1, τ_2 are given by

$$\tau_1(r) = r^2,$$

and

$$\tau_2(r) = \sqrt{r},$$

with $\boldsymbol{p}_1 = \frac{1}{3}$ and $\boldsymbol{p}_2 = \frac{2}{3}$.

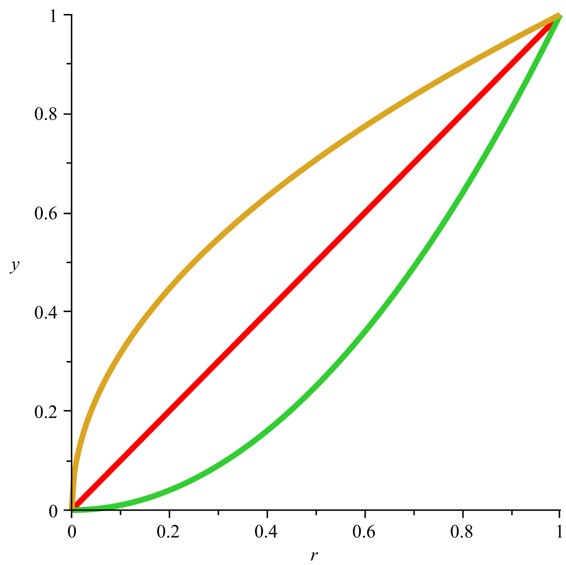


FIGURE 3.3: $\tau_s(r), \quad s = 1, 2$

One can easily check that the transformations $\tau_1(r)$ and $\tau_2(r)$ satisfy the properties of the random map in [8].

3.1.3 Representation of Random Homeomorphisms

For such random map, one can easily observe (see Lemma 4.2. [8]) that each constituent map of the random map can be represented as follows:

$$\tau_s(r) = r^{\beta_s(r)},$$

with $\beta_s(r)$ satisfying;

1. $\beta_s(r) > 0$ *in* $(0, 1)$;
2. $(\ln r)\,\beta_s(r)$ *increasing*;
3. $\lim_{r\to 0}(\ln r)\,\beta_s(r) = -\infty$;
4. $\lim_{r\to 1}(\ln r)\,\beta_s(r) = 0$.

Among other results the following proposition can be found in [8]. Let

$$\alpha_t := \beta_s(r_{t-1}) \text{ with probability } \mathbf{p}_s, t = 1, 2, \ldots.$$

Proposition 3.5.
Let $F = \{\tau_s; \boldsymbol{p}_s\}_{s\in S}$ *be a random map such that* $\tau_s(r) = r^{\beta_s(r)}$. *Assume that* $0 < b_s \le \beta_s(r) \le B_s < \infty$ *for all* $r \in [0, 1]$. *If* $E(\ln \alpha_t \mid s^{t-1}) \le 0$ *a.s., then* $\lim_{t\to\infty} r_t(s^t) \ne 0$ *a.s.*

With Proposition 3.5 we say that the investor will dominate the market or at least survive. This proposition will be used in our main result in Section 5.4.

3.2 Random Maps With Position Dependent Probabilities

$(X, \mathfrak{B}(X), \mu)$ will denote a measure space, where $\mathfrak{B}(X)$ is a σ-algebra of subsets of X and μ is a probability measure on $(X, \mathfrak{B})$. In particular, $(I, \mathfrak{B}(I), m)$ will be the unit interval $I = [0, 1)$, with $\mathfrak{B}(I)$ the Borel σ-algebra on I and m being Lebesgue measure on $(I, \mathfrak{B}(I))$. For $s = 1, \ldots, L$, let $\tau_s : X \to X$ be measurable transformations and $\mathbf{p}_s : X \to I$ be measurable functions such that $\sum_{s=1}^{L} \mathbf{p}_s(r) = 1$, that is, a measurable partition of unity.

3.2.1 Position dependent random maps

We let $F = \{\tau_1, \ldots, \tau_L; \mathbf{p}_1, \ldots, \mathbf{p}_L\}$ denote the associated random map. A random map is more precisely a discrete time Markov process with transition function

$$\mathbb{P}(r, A) = \sum_{s=1}^{L} \mathbf{p}_s(r) \mathcal{X}_A(\tau_s(r)),$$

where $\mathcal{X}_A$ denotes the characteristic function of a measurable set A.

3.2.2 A Deterministic Representation For Position Dependent Random Maps

Following [6] we can represent the position dependent random map by a skew product as follows. This deterministic representation represents the position dependent random map on the extended phase space $X \times I$ where I is the unit interval which accounts for the noise space. We make use of the following simple lemma:

Lemma 3.6. *Let Y and Z be a measurable spaces and let $(J_s)_{s\in\kappa}$ be a finite (or countable), measurable partition of Y. For each $s \in \kappa$, assume that F_s is a measurable map from J_s to Z. Then the piecewise-defined map $F : Y \to Z$ defined by $F(r) = F_s(r)$ if $r \in J_s$ is measurable.*

In our construction, $Y = Z = X \times I$ and the set J_s will be given by $J_s = \{(r, w) : \sum_{i<s} \mathbf{p}_i(r) \leq w \leq \sum_{i\leq s} \mathbf{p}_i(r)\}$. We define maps $\varphi_s : J_s \to I$ by

$$\varphi_s(r, w) = \frac{1}{\mathbf{p}_s(r)} w - \frac{\sum_{l=1}^{s-1} \mathbf{p}_l(r)}{\mathbf{p}_s(r)}. \tag{3.1}$$

The maps F_s are defined on J_s by $F_s(r, w) = (\tau_s(r), \varphi_s(r, w))$. By

$$\varphi_r(w) : [0, 1] \to [0, 1]$$

we will denote the piecewise linear expanding map, whose L branches are given by

$$\varphi_s(r, w), \quad s = 1, \ldots, L.$$

It is well known ([35]) that for such a piecewise linear expanding map Lebesgue measure is invariant and ergodic. Define the skew product transformation $\mathfrak{R} : X \times I \to X \times I$ by

$$\mathfrak{R}(r, w) = (\tau_s(r), \varphi_{s,r}(w)), \tag{3.2}$$

for $(r, w) \in J_s$. $\mathfrak{R}$ is then $\mathfrak{B}(X) \times \mathfrak{B}(I)$-measurable.
We give the following example.

Example 3.7. *Let T be a random map which is given by $\{\tau_1, \tau_2; \boldsymbol{p}_1(r), \boldsymbol{p}_2(r)\}$ where*

$$\tau_1(r) = \begin{cases} 2r & 0 \le r \le \frac{1}{2}, \\ r & \frac{1}{2} < r \le 1, \end{cases}$$

$$\tau_2(r) = \begin{cases} r + \frac{1}{2} & 0 \le r \le \frac{1}{2}, \\ 2r - 1 & \frac{1}{2} < r \le 1; \end{cases}$$

and

$$\boldsymbol{p}_1(r) = \begin{cases} \frac{2}{3} & 0 \le r \le \frac{1}{2}, \\ \frac{1}{3} & \frac{1}{2} < r \le 1, \end{cases}$$

$$\boldsymbol{p}_2(r) = \begin{cases} \frac{1}{3} & 0 \le r \le \frac{1}{2}, \\ \frac{2}{3} & \frac{1}{2} < r \le 1. \end{cases}$$

Then, $\mathfrak{R}(r, w)$ is given by:

$$\mathfrak{R}(r, w) = \begin{cases} \left(2r, \frac{3}{2}w\right) & for\,(r, w) \in [0, \frac{1}{2}] \times [0, \frac{2}{3}] \\ (r, 3w) & for\,(r, w) \in (\frac{1}{2}, 1] \times [0, \frac{1}{3}] \\ \left(2r - 1, \frac{3}{2}w - \frac{1}{2}\right) & for\,(r, w) \in (\frac{1}{2}, 1] \times (\frac{1}{3}, 1] \\ \left(r + \frac{1}{2}, 3w - 2\right) & for\,(r, w) \in [0, \frac{1}{2}] \times (\frac{2}{3}, 1]. \end{cases}$$

3.2.3 Lyapunov Exponents of an Endomorphism

In this subsection, we recall the definition of Lyapunov exponents of an endomorphism [32].

Definition 3.8.
Let $f : M \to M$ be an endomorphism on a manifold M of dimension m. Let $|.|$ be the norm on tangent vectors induced by a Riemannian metric on M. For each $r \in M$ and $v \in T_r M$ let

$$l(r, v) = \lim_{t \to \infty} \frac{1}{t} \ln\left(|Df_r^t v|\right), \tag{3.3}$$

whenever the limit exists.

Remark 3.9.
The multiplicative ergodic theorem of Oseledec [32] says that, for almost all $r \in M$:

i) the limit in (3.3) exists for all tangent vectors $v \in T_r M$, and

ii) there are at most m distinct values of $l(r, v)$ for one point r.

Let $m(r)$ be the number of distinct values of $l(r, v)$ at r for $v \in T_r M$, with tangent vectors $v^j \in T_r M$ for $1 \leq j \leq m(r)$ giving distinct values:

$$l_j(r) = l\left(r, v^j\right)$$

with

$$l_1(r) < l_2(r) \ldots < l_{m(r)}(r).$$

These distinct values are called the Lyapunov exponents at r.

3.3 Computing the Lyapunov exponents of the skew product

In this section, we are going to assume that $X = [0, 1]$ and that

$$\tau_s : [0, 1] \to [0, 1]$$

are differentiable maps. In the next proposition, we will obtain formulae for the Lyapunov exponents of the skew product $\mathfrak{R}$.

Proposition 3.10.
Let $\mathfrak{R} : [0, 1] \times [0, 1] \to [0, 1] \times [0, 1]$ *be defined as in* (3.2)*. Then*

1.
$$l_1(r, w) \geq \lim_{t \to \infty} \frac{1}{t} \sum_{i=0}^{t-1} \ln \left|\tau_{s_i}^{'}\left(\tau_{s_{i-1}}(r)\right)\right|,$$

2.
$$l_2(r, w) = \lim_{t \to \infty} \frac{1}{t} \ln \frac{1}{\boldsymbol{p}_{s_{t-1}}\left(\tau_{s_{t-2}} \circ \tau_{s_{t-3}} \ldots \tau_{s_0}(r)\right)} \cdots \frac{1}{\boldsymbol{p}_{s_0}(r)}.$$

Proof.
For $(r, w) \in J_s$ we have

$$\mathfrak{R}(r, w) = (\tau_s(r), \varphi_{s,r}(w)).$$

Let us first compute the derivative matrix of $\mathfrak{R}$. We have

$$A(r, w) = \begin{bmatrix} \tau_{s_0}^{'}(r) & 0 \\ \frac{\partial}{\partial r}\varphi_{s_0,r}(w) & \frac{\partial}{\partial w}\varphi_{s_0,r}(w) \end{bmatrix}.$$

Let $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, then

$$A(r, w)\, v_1 = \begin{pmatrix} \tau_{s_0}^{'}(r) \\ \frac{\partial}{\partial r}\varphi_{s_0,r}(w) \end{pmatrix}. \tag{3.4}$$

Also

$$A(\mathfrak{R}(r, w)) = \begin{bmatrix} \tau_{s_1}^{'}(\tau_{s_0}(r)) & 0 \\ \frac{\partial}{\partial r}\varphi_{s_1,\tau_{s_0}(r)}(\varphi_{s_0,r}(w)) & \frac{\partial}{\partial w}\varphi_{s_1,\tau_{s_0}(r)}(\varphi_{s_0,r}(w)) \end{bmatrix}. \tag{3.5}$$

Therefore, by (3.4) and (3.5) we have

$$A(\mathfrak{R}(r, w))\, A(r, w)\, v_1 = \begin{pmatrix} \tau_{s_1}^{'}(\tau_{s_0}(r))\, \tau_{s_0}^{'}(r) \\ \frac{\partial}{\partial r}\varphi_{s_1,\tau_{s_0}(r)}(\varphi_{s_0,r}(w))\, \tau_{s_0}^{'}(r) + \frac{\partial}{\partial w}\varphi_{s_1,\tau_{s_0}(r)}(\varphi_{s_0,r}(w))\, \frac{\partial}{\partial r}\varphi_{s_0,r}(w) \end{pmatrix}. \tag{3.6}$$

Thus, in general, we have

$$A\left(\mathfrak{R}^{t-1}(r, w)\right) \ldots A(\mathfrak{R}(r, w))\, A(r, w)\, v_1 = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}, \tag{3.7}$$

where

$$A_1 = \tau_{s_{t-1}}^{'}\left(\tau_{s_{t-2}}(r)\right) \ldots \tau_{s_1}^{'}(\tau_{s_0}(r))\, \tau_{s_0}^{'}(r)$$

and A_2 includes terms which are analogous to the second component in (3.6). Therefore, using (3.7) we obtain

$$\begin{aligned} &\|A\left(\mathfrak{R}^{t-1}(r, w)\right) \ldots A(\mathfrak{R}(r, w))\, A(r, w)\, v_1\| \\ &\qquad \geq |\tau_{s_{t-1}}^{'}\left(\tau_{s_{t-2}}(r)\right) \ldots \tau_{s_1}^{'}(\tau_{s_0}(r))\, \tau_{s_0}^{'}(r)|. \end{aligned} \tag{3.8}$$

Since ln is an increasing function, using (3.8), we obtain

$$
\begin{aligned}
l_1(r,w) &\geq \lim_{t\to\infty}\frac{1}{t}\ln\left|\tau'_{s_{t-1}}\left(\tau_{s_{t-2}}(r)\right)\dots\tau'_{s_1}\left(\tau_{s_0}(r)\right)\tau'_{s_0}(r)\right| \\
&= \lim_{t\to\infty}\frac{1}{t}\sum_{i=0}^{t-1}\ln\left|\tau'_{s_i}\left(\tau_{s_{i-1}}(r)\right)\right|,
\end{aligned}
$$

where we have used the notation $\tau_{s_{-1}}(r) = r$. For $l_2(r,w)$, we first compute $A(r,w)\,v_2$, where $v_2 = \begin{pmatrix}0\\1\end{pmatrix}$. We have

$$
A(r,w)\,v_2 = \begin{pmatrix} 0 \\ \frac{\partial}{\partial w}\varphi_{s_0,r}(w)\end{pmatrix},
$$

and by (3.5)

$$
A(\mathfrak{R}(r,w))\,A(r,w)\,v_2 = \begin{pmatrix} 0 \\ \frac{\partial}{\partial w}\varphi_{s_1,\tau_{s_0}(r)}\left(\varphi_{s_0,r}(w)\right)\frac{\partial}{\partial w}\varphi_{s_0,r}(w)\end{pmatrix}.
$$

Thus, in general, we have

$$
\begin{aligned}
&A\left(\mathfrak{R}^{t-1}(r,w)\right)\dots A(\mathfrak{R}(r,w))\,A(r,w)\,v_2 = \\
&\begin{pmatrix} 0 \\ \frac{\partial}{\partial w}\varphi_{s_{t-1},\tau_{s_{t-2}}\circ\dots\circ\tau_{s_0}(r)}\left(\varphi_{s_{t-2},\tau_{s_{t-3}}\circ\dots\circ\tau_{s_0}}(w)\right)\dots\frac{\partial}{\partial w}\varphi_{s_0,r}(w)\end{pmatrix}.
\end{aligned}
\tag{3.9}
$$

Moreover, by the definition of $\varphi_{s,r}(w)$ (see (3.1)), we have

$$
\frac{\partial}{\partial w}\varphi_{s,r}(w) = \frac{1}{\mathbf{p}_s(r)}. \tag{3.10}
$$

Therefore, by (3.9) and (3.10) we have

$$
l_2(r,w) = \lim_{t\to\infty}\frac{1}{t}\ln\frac{1}{\mathbf{p}_{s_{t-1}}\left(\tau_{s_{t-2}}\circ\dots\circ\tau_{s_0}(r)\right)}\dots\frac{1}{\mathbf{p}_{s_0}(r)}.
$$

□

The next proposition shows that at common fixed points of maps τ_s, the Lyapunov exponents have more precise formulae.

Proposition 3.11.

Let r_0 be a common fixed point for all the constituent maps τ_s; i.e., $\tau_s(r_0) = r_0$ for all $s \in S$. Then

1. $l_1(r_0, w) \geq \sum_{s=1}^{L} \boldsymbol{p}_s(r_0) \ln |\tau_s^{'}(r_0)|$
2. $l_2(r_0, w) = -\sum_{s=1}^{L} \boldsymbol{p}_s(r_0) \ln \boldsymbol{p}_s(r_0)$

Proof.
By Proposition 3.10, since $\tau_s(r_0) = r_0$, we have

$$l_1(r_0, w) \geq \lim_{t\to\infty} \frac{1}{t} \sum_{i=0}^{t-1} \ln |\tau_{s_i}^{'}(r_0)|, \tag{3.11}$$

where $s_0 \to s_1 \to \ldots \to s_{t-1}$ is the orbit of s_0 generated by the map $\varphi_{r_0}(w)$. Since Lebesgue measure m is φ_{r_0}-invariant and ergodic, we can use the Birkhoff Ergodic Theorem and (3.11) to obtain that

$$\begin{aligned} l_1(r_0, w) &\geq \lim_{t\to\infty} \frac{1}{t} \sum_{i=0}^{t-1} \ln |\tau_{s_i}^{'}(r_0)| \\ &= \int_0^1 \ln |\tau_w^{'}(r_0)| dm(w). \end{aligned} \tag{3.12}$$

Write $m := m_{r_0}$ to associate $[0, 1]$ with the particular partition of φ_{r_0}. Then, using the partition J_s defined in Subsection 3.2.2 and (3.12) we get

$$\begin{aligned} l_1(r_0, w) &\geq \int_0^1 \ln |\tau_w^{'}(r_0)| dm_{r_0}(w) \\ &= \sum_{s=1}^{L} \int_{J_s} \ln |\tau_w^{'}(r_0)| dm_{r_0}(w) \\ &= \sum_{s=1}^{L} \mathbf{p}_s(r_0) \ln |\tau_s^{'}(r_0)|. \end{aligned}$$

Similarly, for $l_2(r_0, w)$, we use Proposition 3.10 and Birkhoff Ergodic Theorem to obtain that

$$\begin{aligned} l_2(r_0, w) &= -\int_0^1 \ln \mathbf{p}_w(r_0)\, dm_{r_0} \\ &= -\sum_{s=1}^{L} \mathbf{p}_s(r_0) \ln \mathbf{p}_s(r_0). \end{aligned}$$

$\square$

Remark 3.12.
Obviously, we always have $l_2(r_0, w) > 0$. Thus, if $l_1(r_0, w) > 0$ then nearby points cannot be attracted to (r_0, w) under the dynamics of $\mathfrak{R}$. In particular, this will mean that nearby points $r \in [0, 1]$ cannot be attracted to r_0 under the dynamics of the random map F.

Chapter 4

An Evolutionary Market Model With Short-lived Assets

We first recall the model of [21].

4.1 The Model

Let S is finite set and $s_t \in S$, $t = 1, 2, \ldots$, be the states of the world at date t. Let $\mathbf{p}$ be a probability distribution on S such that for all $s \in S$, $\mathbf{p}_s > 0$. $s_1, s_2, \ldots$ are independent but their probability distribution will change at each time step according to the money invested in the assets. This will be made more explicit later in the model.

There are $i = 1, \ldots, I$ investors initially endowed with wealth $w_0^i > 0$ and K short-lived assets $k = 1, \ldots, K$ live for one period only and reborn in every period. They yield the non-negative return $D_k(s)$ at state s, and we assume that $D_k(s) \neq 0$ for at least one s. Moreover, we assume that $\sum_k D_k(s) > 0$ for all s.

At each time t, every investor i selects a portfolio

$$x_t^i\left(s^t\right) = \left(x_{t,1}^i\left(s^t\right), \ldots, x_{t,K}^i\left(s^t\right)\right) \in \mathbb{R}_+^K,$$

where $x_{t,k}^i$ is the number of units of asset k in the portfolio $x_t^i = x_t^i(s^t)$, $(s^t) = (s_1, \ldots, s_t)$. We assume that for each moment of time $t \geq 1$ and each random situation s^t, the market for every asset k clears:

$$\sum_{i=1}^{I} x_{t,k}^{i}\left(s^{t}\right) = 1. \tag{4.1}$$

Each investor is endowed with initial wealth $w_0^i > 0$. Wealth w_{t+1}^i of investor i at time $t+1$ can be computed as follows:

$$w_{t+1}^{i} = \sum_{k=1}^{K} D_k\left(s_{t+1}\right) x_{t,k}^{i}. \tag{4.2}$$

Total market wealth at time $t+1$ is equal to

$$W_{t+1} = \sum_{i=1}^{I} w_{t+1}^{i} = \sum_{k=1}^{K} D_k\left(s_{t+1}\right). \tag{4.3}$$

Investment strategies are characterised in terms of investment proportions:

$$\Lambda^i = \{\lambda_0^i, \lambda_1^i, \ldots\}$$

of K-dimensional vector functions $\lambda_t^i = \left(\lambda_{t,1}^i, \ldots, \lambda_{t,K}^i\right)$, $\lambda_{t,k}^i = \lambda_{t,k}^i\left(s^t\right)$, $t \geq 0$, satisfying $\lambda_{t,k}^i \geq 0$, $\sum_{k=1}^{K} \lambda_{t,k}^i = 1$. Here, $\lambda_{t,k}^i$ stands for the share of the budget w_t^i of investor i that is invested into asset k at time t. In general $\lambda_{t,k}^i$ may depend on $\left(s^t\right) = \left(s_1, \ldots, s_t\right)$. Given strategies $\Lambda^i = \{\lambda_0^i, \lambda_1^i, \ldots\}$ of investors $i = 1, \ldots, I$, the equation

$$p_{t,k}.1 = \sum_{i=1}^{I} \lambda_{t,k}^{i} w_t^i \tag{4.4}$$

determines the market clearing price $p_{t,k} = p_{t,k}\left(s^t\right)$ of asset k. The number of units of asset k in the portfolio of investor i at time t is equal to

$$x_{t,k}^{i} = \frac{\lambda_{t,k}^{i} w_t^i}{p_{t,k}}. \tag{4.5}$$

Therefore

$$x_{t,k}^{i} = \frac{\lambda_{t,k}^{i} w_t^i}{\sum_{j=1}^{I} \lambda_{t,k}^{j} w_t^j}. \tag{4.6}$$

By using (4.6) and (4.2), we get

$$w_{t+1}^{i} = \sum_{k=1}^{K} D_k\left(s_{t+1}\right) \frac{\lambda_{t,k}^{i} w_t^i}{\sum_{j=1}^{I} \lambda_{t,k}^{j} w_t^j}. \tag{4.7}$$

Since $w_0^i > 0$, we obtain $w_t^i > 0$ for each t. The main focus of the model is on the analysis of the dynamics of the market shares of the investors

$$r_t^i = \frac{w_t^i}{W_t}, \quad i = 1, \dots, I.$$

Using (4.7) and (4.3), we obtain

$$r_{t+1}^i = \sum_{k=1}^{K} R_k\left(s_{t+1}\right) \frac{\lambda_{t,k}^i r_t^i}{\sum_{j=1}^{I} \lambda_{t,k}^j r_t^j}, \quad i = 1, \dots, I, \tag{4.8}$$

where

$$R_k\left(s_{t+1}\right) = \frac{D_k\left(s_{t+1}\right)}{\sum_{m=1}^{K} D_m\left(s_{t+1}\right)}$$

are the relative (normalised) payoffs of the assets $k = 1, 2, \dots, K$. We have $R_k\left(s\right) \geq 0$ and $\sum_k R_k\left(s\right) = 1$.

Define

$$\lambda_k^* := ER_k\left(s_t\right), \quad k = 1, \dots, K, \tag{4.9}$$

and put

$$\lambda^* = \left(\lambda_1^*, \dots, \lambda_K^*\right),$$

where $E\left(.\right)$ is the expectation with respect to the underlying probability on P. The portfolio rule λ^* is called the **Kelly rule**.

Assume $I = 2$. Then the random dynamical system (4.8) reduces to

$$r_{t+1}^1 = \sum_{k=1}^{K} R_k\left(s_{t+1}\right) \frac{\lambda_{t,k} r_t^1}{\lambda_{t,k} r_t^1 + \bar{\lambda}_{t,k}\left(1 - r_t^1\right)}, \tag{4.10}$$

where $\lambda_{t,k}$ is the strategy of investor 1 and $\bar{\lambda}_{t,k}$ is the strategy of investor 2. We will assume that the probability function on S is a function of the relative wealth, r, that the states $s_1, \dots, s_L$ are independent.

Lemma 4.1.
The random dynamical system in (4.10) can be represented by a random map $F = \{\tau_s; \boldsymbol{p}_s\left(r\right)\}_{s=1}^{L}$, where

$$\tau_s\left(r\right) = \sum_{k=1}^{K} R_k\left(s\right) \frac{\lambda_{t,k} r}{\lambda_{t,k} r + \bar{\lambda}_{t,k}\left(1 - r\right)}$$

such that:

1. $\tau_s : [0,1] \to [0,1]$,
2. $\tau_s(0) = 0$ *and* $\tau_s(1) = 1$,
3. τ_s *is differentiable.*

The proof of Lemma 4.1-(3) is in Appendix A.1.

4.1.1 Notions in Evolutionary Finance

Definition 4.2.
In the theory of evolutionary finance there are three possibilities for investor i :

1. Extinction; i.e., $\lim_{t\to\infty} r_t^i = 0$ a.s.
2. Survival; i.e., $\limsup_{t\to\infty} r_t^i > 0$ a.s. but $\liminf_{t\to\infty} r_t^i < 1$ a.s.
3. Domination; i.e., $\lim_{t\to\infty} r_t^i = 1$ a.s.

We have used [22] for the following two definitions.

Definition 4.3.
A portfolio rule λ^i is called *evolutionarily stable*, if for every portfolio rule $\lambda^j \neq \lambda^i$ there is a random variable $\epsilon > 0$ such that $\lim_{t\to\infty} \varphi^i(t,\omega,r) = 1$ for all $r^i \geq 1 - \epsilon(\omega)$ almost surely.

Definition 4.4.
A portfolio rule λ^i is called *locally evolutionarily stable*, if there exists a random variable $\delta(\omega) > 0$ such that λ^i is evolutionarily stable for all portfolio rules $\lambda^j \neq \lambda^i$ with $\|\lambda^i(\omega) - \lambda^j(\omega)\| < \delta(\omega)$ for all ω.

4.2 Literature Review

In this section we review some recent results from evolutionary finance with short-lived assets. Firstly, we start with the general case where all the investors use dynamic strategies.

4.2.1 Dynamic Investment Strategies

4.2.1.1 CAPM Decision Rule

Amir et. al [2] analysed the model when the investors use general investment strategies. Moreover, it is assumed that the states of the world are homogenous discrete-time Markov process. The main result is given by the following theorem in [2].

Theorem 4.5.
Investor 1 using the strategy $\lambda^*_{t,k}(s_t)$ is a single survivor in the market selection process, and moreover, dominates the others exponentially, if and only if the following condition is fulfilled:

- *There exists a random variable $\kappa > 0$ such that*

$$\liminf_{T\to\infty} \frac{1}{T} \#\{t \in \{0, \dots, T\} : |\lambda^*(s_t) - \xi_t(s^t)| \geq \kappa\} > 0$$

  with probability 1, where

$$\begin{aligned}\xi_t = (\xi_{t,1}, \dots, \xi_{t,K}) &= f\left(r_t, \lambda_t^2, \dots, \lambda_t^I\right) \\ &= \sum_{j=2}^{I} \frac{r_t^j}{1 - r_t^1} \lambda_t^j.\end{aligned}$$

  Here, f is termed as the CAPM decision rule.

The Theorem 4.5 tells us the investor who follows the portfolio rule λ^* eventually accumulates the total market wealth. This portfolio rule λ^* is asymptotically distinct from CAPM rule. When the investor uses the CAPM rule then the relative market wealth remains constant. Hence he/she neither dominates nor gets extinct from out of the market.

4.2.1.2 Asset Market Games of Survival

Recently, in [4] by using general, adaptive portfolio rules Amir et. al. studied that the investor who employs the identified portfolio rules survives in the market. These strategies depend on observed history of the game and the exogenous states of the world. Furthermore, it is indicated that this kind of strategy is essentially unique. The following theorems are proved in [4].

Theorem 4.6.
The portfolio rule $\Lambda^* = (\lambda_t^*)$ *is a survival strategy.*

Theorem 4.6 means that the relative market share is positive, bounded away from zero over an infinite time horizon.

Theorem 4.7.
If $\Lambda = (\lambda_t)$ *is a basic survival strategy; i.e., the investment strategy depends only on the history* s^t *of the process of states of the world, and does not depend on the market history, then*

$$\sum_{t=1}^{\infty} \|\lambda_t^* - \lambda_t\|^2 < \infty \quad a.s.,$$

where $\|.\|$ *is the Euclidean norm in a finite-dimensional space.*

The above theorem shows us the strategy λ_t^* is essentially unique, i.e., any other this type of strategy is asymptotically similar to the λ_t^*.

4.2.2 Constant Investment Strategies

In this section, it is supposed that there are no redundant assets, i.e. the relative payoffs of the assets $R_1(s), \ldots, R_K(s)$ are linearly independent and the states of the world are independent, identically distributed.

4.2.2.1 Evolutionary Stability

In [26], Hens and Schenk-Hoppé studied the evolution of market shares of portfolio rules with short-lived assets. The market is incomplete and the prices are given endogenously. They found necessary and sufficient conditions for the evolutionary stability of portfolio rules. The random dynamical system theory was used during the analysis. The main results in this paper are given by the following theorems.

Theorem 4.8.
Let the state of nature be determined by an ergodic process. Suppose investors

only employ simple strategies, i.e., $\lambda(w) \equiv \lambda \in \Delta^K$*. Then the simple strategy* λ^* *defined by,*

$$\lambda_k^* = ER_k(w)$$

for $k = 1, \ldots, K$ *is evolutionarily stable, and no other strategy is locally evolutionarily stable.*

Theorem 4.9.
Let the state of nature be determined by an i.i.d. process. Then $\lambda_k^* = ER_k$*,* $k = 1, \ldots, K$*, is the only evolutionarily stable portfolio rule. Moreover, if* $\mathcal{S}$ *is the power set of the set of states* S*, then we find that all other completely mixed adapted strategies are not even locally evolutionarily stable.*

The technical assumption that $\mathcal{S}$ is the power set of the set of states S is fulfilled. For example, if S is countable (or finite) and $\mathcal{S}$ is the Borel σ-field. This condition is necessary to guarantee measurability of a strategy.

Theorem 4.10.
Let the state of nature be determined by a Markov process (with transition probability P*). Then the adaptive strategy* λ^* *defined by*

$$\lambda_k^*(w_0) = E\left(R_k(w_1)\,|w_0\right) = \int_S R_k(s)\, P(ds, w_0)\,,$$

for $k = 1, \ldots, K$ *is the only evolutionarily stable portfolio rule. Moreover, if* $\mathcal{S}$ *is the power set of the set of states* S*, then we find that all other completely mixed adapted strategies are not even locally evolutionarily stable.*

The above three theorems tell us the local stability conditions conduct a simple portfolio rule in the case of ergodic, i.i.d., and Markov process, respectively. This portfolio rule is the unique evolutionarily stable strategy.

4.2.2.2 Domination

In [21], Evstigneev et. al. studied the model in Section 4.1 when the investors use simple investment strategies in an incomplete market; i.e., the number of securities is less than the number of states. The prices are endogenous. The following theorem which is proved in [21] tells us there is a unique survival investment strategy that accumulates the total market wealth.

Theorem 4.11.
Let investor i use the strategy $\lambda^i = \lambda^*$, while all the other investors $j \neq i$ use strategies $\lambda^j \neq \lambda^*$. Then investor i is the single survivor.

From Definition 4.2 - (3), we say that the investor is a single survivor if the market share of this investor tends to 1 a.s.

4.2.2.3 Investors With Partial Information on Probability p

In the model used in [8] with short-lived assets, the investors use constant (simple) investment strategies. The Kelly rule requires the full knowledge of the probability distribution from the investor which is more difficult. In [8], Bahsoun et. al. used an IFS representation of (4.8) and Proposition 3.5 and found another successful strategy which does not require full knowledge of the probability distribution.

Theorem 4.12.
If for each $k \in \{1, \ldots, K\}$ λ_k^1 lies between ER_k and λ_k^2, then investor 1 cannot be driven out of the market; i.e., he/she either dominates or at least survives.

In Theorem 4.12, λ_k^1 and λ_k^2 are investment strategies of investor 1 and investor 2, respectively. It means that by using Proposition 3.5 a rule is provided with partial information on $\mathbf{p}$ for investors. As long as the investor employs this rule he/she either dominates or at least survives in the market. More details can be found in [8].

4.3 Betting Games and The Probabilities of Success

This section is one of our main results [10]. We will apply our ideas (Proposition 3.10 and Proposition 3.11) in Section 3.2 to study the wealth dynamics of investors, where the states of the world are not identically distributed. In particular, they will depend on the amount of money invested in the assets. To simplify this idea, let us first reconsider the Kelly model [29] in a more *realistic* setting. In particular, let us consider a horse race model where the odds of the outcomes of the events depend on the amount wagered on them. For example, in the case of a horse race between two horses, say *black* and *white*, this means that the probability that *black* wins is a function of the amount bet on *black*. Such a setting is common in real-life betting games. We will investigate this situation in a more general setting. In particular in the financial model of [21] introduced in Section 4.1 when the probabilities of success of assets depend on the amount invested in them. Investors are allowed to use simple investment strategies and states of the world are not identically distributed. Our main result is showed by the following theorem.

Theorem 4.13.
Suppose that investor 1 *uses the strategy* $\lambda_k = \sum_{s=1}^{L} \boldsymbol{p}_s(0) R_k(s)$. *If investor* 2 *uses a different strategy; i.e.,* $\bar{\lambda} \neq \lambda$, *then investor* 1 *will survive.*

Proof.
First find $\tau_s^{'}(0)$ for any s. We get

$$\tau_s^{'}(0) = \sum_{k=1}^{K} R_k(s) \frac{\lambda_k}{\bar{\lambda}_k},$$

where λ_k is the strategy of investor 1 and $\bar{\lambda}_k$ is the strategy of investor 2. Using Proposition 3.10, we find that the first Lyapunov exponent at 0.

$$\begin{aligned} l_1(0, w) &\geq \sum_{s=1}^{L} \mathbf{p}_s(0) \ln |\tau_s^{'}(0)| \\ &= \sum_{s=1}^{L} \mathbf{p}_s(0) \ln \sum_{k=1}^{K} R_k(s) \frac{\lambda_k}{\bar{\lambda}_k}. \end{aligned}$$

By Jensen's inequality,

$$\begin{aligned} &\geq \sum_{s=1}^{L} \mathbf{p}_s(0) \sum_{k=1}^{K} R_k(s) \ln \frac{\lambda_k}{\bar{\lambda}_k} \\ &= \sum_{k=1}^{K} \left(\sum_{s=1}^{L} \mathbf{p}_s(0) R_k(s) \right) \ln \frac{\lambda_k}{\bar{\lambda}_k} \\ &= \sum_{k=1}^{K} \lambda_k \ln \frac{\lambda_k}{\bar{\lambda}_k} > 0, \end{aligned}$$

where the last inequality follows from Lemma 2.38. Since the Lyapunov exponent of $l_1(0, w) > 0$ at 0 for all w, we conclude that nearby orbits of the skew product map representing the random map of the market cannot converge to zero. Consequently investor 1 survives. □

Chapter 5

An Evolutionary Market Model With Long-lived Assets

We first recall the model of [3].

5.1 The Model

We consider a market where $K \geq 2$ long-lived assets. The market is influenced by random factors modeled in terms of independent, identically distributed random elements $s_1, s_2, \ldots$ in a finite space S. At each date $t = 1, 2, \ldots$ assets $k = 1, 2, \ldots, K$ pay dividends $D_k(s_t) \geq 0$ depending on the "state of the world" s_t at date t. The functions $D_k(s_t)$ are measurable and satisfy

$$\sum_{k=1}^{K} D_k(s) > 0 \quad \text{for all } s.$$

This condition means that in each random situation at least one asset yields a strictly positive dividend. The total volume (the number of units) of asset k traded in the market at date t is $V_{t,k} = V_{t,k}(s^t) > 0$, where $s^t := (s_1, \ldots, s_t)$ is the history of the process (s_t) from time 1 to time t. For $t = 0, V_{t,k}$ is a constant number, and for $t \geq 1$, $V_{t,k}(s^t)$ is a measurable function of s^t.

We denote by $p_t \in \mathbb{R}_+^K$ the vector of market prices of the assets. For each $k = 1, \ldots, K$, the coordinate $p_{t,k}$ of $p_t = (p_{t,1}, \ldots, p_{t,K})$ stands for the price of one unit of asset k at date t. There are $I \geq 2$ *investors (traders)* acting in the market. A *portfolio* of investor i at date $t = 0, 1, \ldots$ is specified by a vector $x_t^i = \left(x_{t,1}^i, \ldots, x_{t,K}^i\right) \in \mathbb{R}_+^K$ where $x_{t,k}^i$ is the amount (the number of units) of asset k in the portfolio x_t^i. The scalar product $\langle p_t, x_t^i \rangle = \sum_{k=1}^K p_{t,k} x_{t,k}^i$ expresses the value of the investor i's portfolio x_t^i at date t in terms of the prices $p_{t,k}$. At date $t = 0$, the investors have initial endowments $w_0^i > 0$ $(i = 1, 2, \ldots, I)$ that form their budgets at date 0. Investor i's *wealth (budget)* at date $t \geq 1$ is

$$w_t^i := \langle D_t + p_t, x_{t-1}^i \rangle, \tag{5.1}$$

where

$$D_t := D\left(s_t\right) := \left(D_1\left(s_t\right), \ldots, D_K\left(s_t\right)\right).$$

It consists of two components: the dividends $\langle D_t, x_{t-1}^i \rangle$ paid by the portfolio x_{t-1}^i and the market value $\langle p_t, x_{t-1}^i \rangle$ of the portfolio x_{t-1}^i expressed in terms of the today's prices p_t. A fraction $\mu_t = \mu_t\left(s^{t-1}\right)$ of the budget is invested into assets. We suppose that the *investment rate* $0 < \mu_t^i\left(s^{t-1}\right) < 1$ is the same for all the investors, although it may depend on time and random factors. We assume that μ_t is predictable: it depends on the history s^{t-1} of the process (s_t) up to time $t-1$ (not t). The number $1-\mu_t$ represents the *consumption rate*. The assumption that $1-\mu_t$ is essential since we focus in this work on the analysis of the long-term performance of trading strategies. Without this assumption, an analysis of this kind does not make sense: a seemingly worse performance of a portfolio rule in the long run might be simply due to a higher consumption rate of the investor [3]. We shall suppose that the function $\mu_t\left(s^{t-1}\right)$ is measurable (for $t = 0, 1$ it is constant) and satisfies the following condition:

$$\mu_t\left(s^{t-1}\right) < V_{t,k}\left(s^t\right) / V_{t-1,k}\left(s^{t-1}\right). \tag{5.2}$$

This condition holds, in particular, when the total mass $V_{t,k}\left(s^t\right)$ of each asset k does not decrease, i.e., when the right-hand side of (5.2) is not less than one. But (5.2) does not exclude the situation when $V_{t,k}$ decreases at some rate, not faster than μ_t.

An *investment strategy (portfolio rule)* of investor $i = 1, 2, \ldots, I$ is specified by a vector of *investment proportions* $\lambda_t^i = \left(\lambda_{t,1}^i, \ldots, \lambda_{t,K}^i\right)$ according to which he/she plans to distribute the available budget between assets at each date t. Vectors λ_t^i

belong to the unit simplex

$$\Delta^K := \{(a_1, \ldots, a_K) \geq 0 : a_1 + \ldots + a_K = 1\}.$$

In this model it is assumed that the market clears (asset supply is equal to asset demand), which makes it possible to determine the equilibrium price $p_{t,k}$ of each asset k from the equations

$$p_{t,k} V_{t,k} = \mu_t \sum_{i=1}^{I} \lambda_{t,k}^i w_t^i, \quad k = 1, \ldots, K. \tag{5.3}$$

On the left-hand side of (5.3) we have the total value $p_{t,k}V_{t,k}$ of all the assets of the type k in the market (recall that the amount of each asset k at date t is $V_{t,k}$). The right-hand side represents the total wealth invested in asset k by all the investors. Equilibrium implies the equality in (5.3). The investment proportions $\lambda_1^i, \ldots, \lambda_K^i$ chosen by the investors determine their portfolios $x_t^i = \left(x_{t,1}^i, \ldots, x_{t,K}^i\right)$ at date t by the formula

$$x_{t,k}^i = \frac{\mu_t \lambda_{t,k}^i w_t^i}{p_{t,k}}, \quad k = 1, \ldots, K, \quad i = 1, \ldots, I. \tag{5.4}$$

Note that for $t \geq 1$, the price vector p_t is determined implicitly as the solution to the system of equations (5.3), which can be written

$$p_{t,k} V_{t,k} = \mu_t \sum_{i=1}^{I} \lambda_{k,t}^i \langle D_t + p_t, x_{t-1}^i \rangle, \quad k = 1, \ldots, K. \tag{5.5}$$

Given a strategy profile $\left(\lambda_t^1, \ldots, \lambda_t^I\right)$ of investors and their initial endowments $w_0^1, \ldots, w_0^I$, we can generate a path

$$\left(p_t; x_t^1, \ldots, x_t^I\right), \tag{5.6}$$

of market dynamics, by defining the price vectors $p_t = p_t\left(s^t\right)$ and the portfolios $x_t^i = x_t^i\left(s^t\right)$ recursively according to equations (5.3) - (5.4). Equations (5.4) make sense only if $p_{t,k} > 0$, or equivalently, if the aggregate demand for each asset (under the equilibrium prices) is strictly positive. Those strategy profiles $\left(\lambda_t^1, \ldots, \lambda_t^I\right)$ which guarantee that the recursive procedure described above leads at each step to strictly positive equilibrium prices will be called *admissible*. In what follows, we will deal only with such strategy profiles. The hypothesis of admissibility guarantees that the random dynamical system under consideration is well-defined. Under this hypothesis, we obtain by induction that on the equilibrium path, all the portfolios $x_t^i = \left(x_{t,1}^i, \ldots, x_{t,K}^i\right)$ are non-zero and the wealth $w_t^i = \langle D_t + p_t, x_{t-1}^i \rangle$ of each investor is strictly positive. Further, by summing up equations (5.4) over $i = 1, \ldots, I$, we find that

$$\sum_{i=1}^{I} x_{t,k}^i = \frac{\sum_{i=1}^{I} \mu_t \lambda_{t,k}^i w_t^i}{p_{t,k}} = \frac{p_{t,k} V_{t,k}}{p_{t,k}} = V_{t,k} \tag{5.7}$$

(the market clears) for every asset k and each date $t \geq 1$. Thus for every equilibrium state of the market $\left(p_t, x_t^1, \ldots, x_t^I\right)$, we have $p_t > 0$, $x_t^i \neq 0$ and (5.7). Assume that the total mass $V_{t,k}$ of each asset k grows (or decreases) at the same rate $\gamma_t = \gamma_t\left(s^{t-1}\right) > 0$:

$$V_{t,k}/V_{t-1,k} = \gamma_t \; (t \geq 1). \tag{5.8}$$

Thus

$$V_{t,k}\left(s^{t-1}\right) = \gamma_t\left(s^{t-1}\right) \ldots \gamma_2\left(s^1\right) \gamma_1 V_k, \tag{5.9}$$

where $V_k > 0$ $(k = 1, 2, \ldots, K)$ are the initial amounts of the assets. The growth rate process γ_t (like the investment rate process μ_t) is predictable: γ_t depends only on the history s^{t-1} of the states of the world up to time $t-1$. In the case of dividend-paying assets involving investments in the real economy, assumption (5.8) means that the economic system under consideration is on a *balanced growth path*. Define the *relative dividends* of the assets $k = 1, \ldots, K$ by

$$R_k\left(s_t\right) = \frac{D_k\left(s_t\right) V_k}{\sum_{m=1}^{K} D_m\left(s_t\right) V_m}. \tag{5.10}$$

It follows from (5.8) that

$$R_{t,k} = \frac{D_{t,k} V_{t-1,k}}{\sum_{m=1}^{K} D_{t,m} V_{t-1,m}},$$

where $R_{t,k} = R_k\left(s_t\right)$ and $D_{t,k} = D_k\left(s_t\right)$.

Define

$$\lambda_k^* := E R_k\left(s_t\right), \quad k = 1, \ldots, K, \tag{5.11}$$

and put

$$\lambda^* = \left(\lambda_1^*, \ldots, \lambda_K^*\right),$$

where $E(.)$ is the expectation with respect to the underlying probability on P. The portfolio rule λ^* is called the **Kelly rule**. The portfolio rule specified by (5.11) prescribes to distribute wealth across assets in accordance with the proportions of the expected relative dividends (which do not depend on t because the random elements s_t are i.i.d.). Assume that the following conditions hold:

(A1) For each k, the expectation $ER_k(s_t)$ is strictly positive.

(A2) The functions $R_1(s), \ldots, R_K(s)$ are linearly independent with respect to the probability distribution of s_t, i.e., the equality $\sum \beta_k R_k(s_t) = 0$ holding a.s. for some constants β_k implies that $\beta_1 = \ldots = \beta_K = 0$.

(A3) There exist constants $0 < \sigma' < \sigma'' < 1$ such that the process

$$\sigma_t\left(s^{t-1}\right) := \mu_t\left(s^{t-1}\right) / \gamma_t\left(s^{t-1}\right)$$

satisfies $\sigma' \leq \sigma_t\left(s^{t-1}\right) \leq \sigma''$.

Condition (A1) implies that the vector λ^* has strictly positive coordinates. Hypothesis (A2) can be interpreted as the absence of *redundant assets*. Condition (A3) states that the discount factor σ_t cannot be too close to 0 and 1.

The *market share* of investor i is defined by

$$r_t^i := \frac{w_t^i}{W_t},$$

where $W_t := \sum_{i=1}^I w_t^i$ is the *total market wealth*. Consider the path (5.6) of the random dynamical system generated by $\left(\lambda_t^1, \ldots, \lambda_t^I\right)$ and the sequence of vectors $r_t = \left(r_t^1, \ldots, r_t^I\right)$ of the market shares of the investors at date t. The following proposition is proved in [3]. We include its proof for the sake of completeness.

Proposition 5.1.
The following equations hold:

$$r_{t+1}^i = \sum_{k=1}^K \left[\rho_{t+1}\langle\lambda_{t+1,k}, r_{t+1}\rangle + (1-\rho_{t+1}) R_{t+1,k}\right] \frac{\lambda_{t,k}^i r_t^i}{\langle\lambda_{t,k}, r_t\rangle}, \quad i = 1, \ldots, I,\ t \geq 0. \tag{5.12}$$

Proof.
From (5.3) and (5.4) we get

$$p_{t,k} = V_{t,k}^{-1}\mu_t \sum_{i=1}^{I} \lambda_{t,k}^i \langle p_t + D_t, x_{t-1}^i\rangle =$$

$$\mu_t V_{t,k}^{-1} \sum_{i=1}^{I} \lambda_{t,k}^i w_t^i = \mu_t V_{t,k}^{-1}\langle \lambda_{t,k}, w_t\rangle, \tag{5.13}$$

$$x_{t,k}^i = \frac{V_{t,k}\lambda_{t,k}^i w_t^i}{\langle \lambda_{t,k}, w_t\rangle}, \tag{5.14}$$

where $t \geq 1$, $w_t := \left(w_t^1, \dots, w_t^I\right)$ and $\lambda_{t,k} := \left(\lambda_{t,k}^1, \dots, \lambda_{t,k}^I\right)$. The analogous formulas for $t = 0$,

$$p_{0,k} = \mu_0 V_{0,k}^{-1}\langle \lambda_{0,k}, w_0\rangle, \quad x_{0,k}^i = \frac{V_{0,k}\lambda_{0,k}^i w_0^i}{\langle \lambda_{0,k}, w_0\rangle}. \tag{5.15}$$

Consequently, we have

$$w_{t+1}^i = \sum_{k=1}^{K} \left(p_{t+1,k} + D_{t+1,k}\right) x_{t,k}^i =$$

$$\sum_{k=1}^{K} \left(\mu_{t+1}\frac{\langle \lambda_{t+1,k}, w_{t+1}\rangle}{V_{t+1,k}} + D_{t+1,k}\right) \frac{V_{t,k}\lambda_{t,k}^i w_t^i}{\langle \lambda_{t,k}, w_t\rangle} =$$

$$\sum_{k=1}^{K} \left(\mu_{t+1}\frac{\langle \lambda_{t+1,k}, w_{t+1}\rangle V_{t,k}}{V_{t+1,k}} + D_{t+1,k}V_{t,k}\right) \frac{\lambda_{t,k}^i w_t^i}{\langle \lambda_{t,k}, w_t\rangle}, \quad t \geq 0. \tag{5.16}$$

By summing up these equations over $i = 1, \dots, I$, we obtain

$$W_{t+1} = \sum_{k=1}^{K} \left(\mu_{t+1}\frac{\langle \lambda_{t+1,k}, w_{t+1}\rangle V_{t,k}}{V_{t+1,k}} + D_{t+1,k}V_{t,k}\right) \frac{\sum_{i=1}^{I} \lambda_{t,k}^i w_t^i}{\langle \lambda_{t,k}, w_t\rangle} =$$

$$\sum_{k=1}^{K} \left(\mu_{t+1}\frac{\langle \lambda_{t+1,k}, w_{t+1}\rangle V_{t,k}}{V_{t+1,k}} + D_{t+1,k}V_{t,k}\right).$$

As long as

$$V_{t+1,k}/V_{t,k} = \gamma_{t+1} > 0 \tag{5.17}$$

(see (5.8)), we have

$$\begin{aligned}
W_{t+1} &= \sum_{k=1}^{K} \left(\mu_{t+1}\gamma_{t+1}^{-1}\langle \lambda_{t+1,k}, w_{t+1}\rangle + D_{t+1,k}V_{t,k}\right) \\
&= \sum_{k=1}^{K} \left(\mu_{t+1}\gamma_{t+1}^{-1}\langle \lambda_{t+1,k}, w_{t+1}\rangle\right) + \sum_{k=1}^{K} D_{t+1,k}V_{t,k} \\
&= \mu_{t+1}\gamma_{t+1}^{-1}\sum_{k=1}^{K}\sum_{i=1}^{I} \lambda_{t+1,k}^{i} w_{t+1}^{i} + \sum_{k=1}^{K} D_{t+1,k}V_{t,k} \\
&= \mu_{t+1}\gamma_{t+1}^{-1}\sum_{i=1}^{I}\left(\sum_{k=1}^{K} \lambda_{t+1,k}^{i}\right) w_{t+1}^{i} + \sum_{k=1}^{K} D_{t+1,k}V_{t,k} \\
&= \mu_{t+1}\gamma_{t+1}^{-1}\sum_{i=1}^{I} w_{t+1}^{i} + \sum_{k=1}^{K} D_{t+1,k}V_{t,k} \\
&= \mu_{t+1}\gamma_{t+1}^{-1} W_{t+1} + \sum_{k=1}^{K} D_{t+1,k}V_{t,k}.
\end{aligned}$$

This implies the formula

$$W_{t+1} = \frac{1}{1-\mu_{t+1}\gamma_{t+1}^{-1}} \sum_{m=1}^{K} D_{t+1,m}V_{t,m}, \tag{5.18}$$

where $\mu_{t+1}\gamma_{t+1}^{-1} := \rho_{t+1}$. From (5.16) and (5.17), we find

$$w_{t+1}^{i} = \sum_{k=1}^{K} \left(\rho_{t+1}\langle \lambda_{t+1,k}, w_{t+1}\rangle + D_{t+1,k}V_{t,k}\right) \frac{\lambda_{t,k}^{i} w_t^{i}}{\langle \lambda_{t,k}, w_t\rangle}, \quad t \geq 0.$$

Dividing both sides of this equation by W_{t+1} and using (5.18), we get

$$r_{t+1}^{i} = \sum_{k=1}^{K} \left[\rho_{t+1}\langle \lambda_{t+1,k}, r_{t+1}\rangle + (1-\rho_{t+1}) \frac{D_{t+1,k}V_{t,k}}{\sum_{m=1}^{K} D_{t+1,m}V_{t,m}}\right] \frac{\lambda_{t,k}^{i} w_t^{i}/W_t}{\langle \lambda_{t,k}, w_t\rangle / W_t},$$

which yields (5.12) by virtue of (5.8) and (5.10). □

5.1.1 Notions in Evolutionary Finance

Definition 5.2.
In the theory of evolutionary finance there are three possibilities for investor i :

1. Extinction; i.e., $\lim_{t\to\infty} r_t^i = 0$ a.s.

2. Survival; i.e., $\limsup_{t\to\infty} r_t^i > 0$ a.s. but $\liminf_{t\to\infty} r_t^i < 1$ a.s.

3. Domination; i.e., $\lim_{t\to\infty} r_t^i = 1$ a.s.

Definition 5.3.
A portfolio rule λ^i is called *evolutionarily stable*, if for every portfolio rule $\lambda^j \neq \lambda^i$ there is a random variable $\epsilon > 0$ such that $\lim_{t\to\infty} \varphi^i(t, \omega, r) = 1$ for all $r^i \geq 1 - \epsilon(\omega)$ almost surely.

Definition 5.4.
A portfolio rule λ^i is called *locally evolutionarily stable*, if there exists a random variable $\delta(\omega) > 0$ such that λ^i is evolutionarily stable for all portfolio rules $\lambda^j \neq \lambda^i$ with $\|\lambda^i(\omega) - \lambda^j(\omega)\| < \delta(\omega)$ for all ω.

5.2 Literature Review

In this section we review some recent results from evolutionary finance. We first start with the general case where all investors use dynamic strategies.

5.2.1 Dynamic Investment Strategies

Amir et al. [3] studied the above model when general, adaptive portfolio rules are used by the investors. The following theorem is proved in [3]. It shows that the Kelly rule λ^* survives; i.e., keeps relative wealth bounded away from zero a.s., but it does not necessarily dominate the market.

Theorem 5.5.
The portfolio rule λ^* is a survival strategy.

Theorem 5.5 means that the investor using λ^* cannot be driven out of the market.

5.2.2 Constant Investment Strategies

In this section we first assume: All investors use constant strategies: for all $i = 1, 2, ..., I$ and $t \geq 0$

$$\lambda_t^i := \lambda^i = (\lambda_1^i, ..., \lambda_K^i).$$

Strategies of this kind are called *fixed-mix*, or *constant proportions*, portfolio rules: they prescribe to select investment proportions at time 0 and keep them fixed over the whole infinite time horizon. Thus the random dynamical system (5.12) reduces to

$$r^i_{t+1} = \sum_{k=1}^{K} [\rho_{t+1}\langle \lambda_k, r_{t+1}\rangle + (1-\rho_{t+1})R_{t+1,k}] \frac{\lambda^i_k r^i_t}{\langle \lambda_k, r_t\rangle}, \tag{5.19}$$

$i = 1, ..., I$, $t \geq 0$.

5.2.2.1 The game theoretic setting

In [7] the model of Section 5.1 was studied from a game theoretic point of view. Given a strategy profile $(\lambda^1, ..., \lambda^I)$, the performance of a strategy λ^i used by investor i will be characterised by the following random variable

$$\xi^i := \limsup_{t\to\infty} \frac{1}{t} \ln \frac{w^i_t}{\sum_{j\neq i} w^j_t}. \tag{5.20}$$

The expression $w^i_t / \sum_{j\neq i} w^j_t$ is the relative wealth of player/investor i and the group $\{j : j \neq i\}$ of i's rivals. The random variable $\xi^i = \xi^i(s^\infty; \lambda^1, ..., \lambda^I)$ depends on the strategy profile $(\lambda^1, ..., \lambda^I)$ and on the whole history $s^\infty := (s_1, s_2, ...)$ of states of the world from time 1 to ∞. In the game under consideration, ξ^i plays the role of the (random) *payoff function* of player i.

We shall say that a strategy $\bar\lambda$ forms a *symmetric Nash equilibrium almost surely (a.s.)* if

$$\xi^i(s^\infty; \bar\lambda, ..., \bar\lambda) \geq \xi^i(s^\infty; \bar\lambda, ..., \lambda, ..., \bar\lambda) \text{ (a.s.)} \tag{5.21}$$

for every i, each strategy λ of investor i and each set of initial endowments $w^1_0 > 0, ..., w^I_0 > 0$. The Nash equilibrium is called *strict* if the inequality in (5.21) is strict.

The following theorem is proved in [7]:

Theorem 5.6.
The portfolio rule λ^* is a unique strategy forming a symmetric Nash equilibrium a.s. This equilibrium is strict.

In game theory, Nash equilibrium is a solution concept of a game. There are at least two investors in the game. It is assumed that each investor knows the equilibrium strategies of the other investors. And also, if the investor changes his/her strategy while the other investors keep their strategies same then, this investor does not gain anything.

5.2.2.2 Domination

In an earlier work, Evstigneev et al. [23] proved that the Kelly rule dominates when all investors use constant strategies. Namely they proved:

Theorem 5.7.
The investor who follows the Kelly rule λ^* dominates the market.

5.2.3 Local and Global Stability

Results on the local and global stability (see definitions 5.3 and 5.4) of investment strategies and the proof of the following main result for local and global stability can be found in [22].
The market selection process is given by the following random dynamical system:

$$r_{t+1} = \frac{\lambda_0}{D_{t+1}\left(w^{t+1}\right)} f_t\left(w^{t+1}, w_t\right), \tag{5.22}$$

where

$$f_t\left(w^{t+1}, w_t\right) =$$

$$\left[\mathbf{I}d - \left[\frac{\lambda^i_{t,k}\left(w^t\right) w^i_t}{\lambda_{t,k}\left(w^t\right) w_t}\right]_{i,k} \Lambda_{t+1}\left(w^{t+1}\right)\right]^{-1} \left[\sum_{k=1}^{K} D^k_{t+1}\left(w^{t+1}\right) \frac{\lambda^i_{t,k}\left(w^t\right) w^i_t}{\lambda_{t,k}\left(w^t\right) w_t}\right]_i$$

and $\Lambda_{t+1}\left(w^{t+1}\right)^T = \left(\lambda_{t+1,1}\left(w^{t+1}\right)^T, \ldots, \lambda_{t+1,K}\left(w^{t+1}\right)^T\right) \in \mathbb{R}^{I\times K}$ is the matrix of portfolio rules. To analyse evolutionarily stability of a portfolio rule, one has to consider the random dynamical system (5.22) with an incumbent (with market share r^1_t) and a mutant (with market share $r^2_t = 1 - r^1_t$). The resulting one-dimensional system governing the market selection process for two stationary portfolio rules is given by

$$r^1_{t+1} = \frac{\lambda_0}{\delta_{t+1}} \left(\left[1 - \sum_{k=1}^{K} \lambda^2_{t+1,k} x^2_{t,k} \right] \sum_{k=1}^{K} d^k_{t+1} x^1_{t,k} + \left[\sum_{k=1}^{K} \lambda^2_{t+1,k} x^1_{t,k} \right] \sum_{k=1}^{K} d^k_{t+1} x^2_{t,k} \right) \tag{5.23}$$

where $\lambda^i_{t,k} = \lambda^i_k \left(w^t\right)$, $d^k_{t+1} = d^k \left(w_{t+1}\right)$, and

$$\begin{aligned} \delta_{t+1} = & \left[1 - \sum_{k=1}^{K} \lambda^1_{t+1,k} x^1_{t,k} \right] \left[1 - \sum_{k=1}^{K} \lambda^2_{t+1,k} x^2_{t,k} \right] \\ & - \left[\sum_{k=1}^{K} \lambda^2_{t+1,k} x^1_{t,k} \right] \left[\sum_{k=1}^{K} \lambda^1_{t+1,k} x^2_{t,k} \right]. \end{aligned}$$

The portfolio of the incumbent and the mutant , respectively, are given by $x^1_{t,k} = \frac{\lambda^1_{t,k} r^1_t}{\lambda^1_{t,k} r^1_t + \lambda^2_{t,k}\left(1 - r^1_t\right)}$ and $x^2_{t,k} = \frac{\lambda^2_{t,k}\left(1 - r^1_t\right)}{\lambda^1_{t,k} r^1_t + \lambda^2_{t,k}\left(1 - r^1_t\right)}$. Denote the right-hand side of (5.23) by $h\left(w^{t+1}, r^1_t\right)$. The variational equation $v_{t+1} = \left[\partial h \left(w^{t=1}, r^1_t\right) / \partial r^1_t |_{r^1_t = 1}\right] v_t$ governs the stochastic dynamics of the linearisation of (5.23) at the fixed point $r^1_t \equiv 1$. It is derived from the derivative of (5.23)'s right-hand side with respect to r^1_t evaluated at $r^1_t = 1$. This derivative can be equated as

$$\frac{\partial h\left(w^{t+1}, r^1_t\right)}{\partial r^1_t}|_{r^1_t = 1} = \sum_{k=1}^{K} \left(\lambda^1_k \left(w^{t+1}\right) + \lambda_0 d^k \left(w_{t+1}\right) \right) \frac{\lambda^2_k \left(w^t\right)}{\lambda^1_k \left(w^t\right)}. \tag{5.24}$$

From (5.24) it can be read off the exponential growth rate of portfolio rule λ^2's market share in a small neighborhood of $r^1 = 1$, i.e. the state in which portfolio rule λ^1 owns total market wealth. The following assumption is made on the process that governs the state of nature and in turn determines the asset payoffs.

- The state of nature follows a Markov process with strictly positive transition probabilities, i.e. $\pi_{s\bar{s}} > 0$ for all $s, \bar{s} \in S$.

The exponential growth rate of portfolio rule λ^2's wealth share in a small neighborhood of $r^1 = 1$ is given by the Lyapunov exponent of the above variational equation. It is given by

$$g_{\lambda^1}\left(\lambda^2\right) = \int_{S^N} \sum_{s \in S} \pi_{w_0 s} \ln \left[\sum_{k=1}^{K} \left(\lambda^1_k \left(w^0, s\right) + \lambda_0 d^k \left(s\right) \right) \frac{\lambda^2_k \left(w_0\right)}{\lambda^1_k \left(w_0\right)} \right] \mathbb{P}\left(dw^0\right)$$

where $\mathbb{P}$ denotes the stationary probability measure on histories w^t induced by the Markov chain. The main result in [22] is given by the following theorem.

Theorem 5.8.
Define the portfolio rule λ^* by $\lambda_0^* = \lambda_0$ and

$$\lambda^* = \lambda_0 \sum_{m=1}^{\infty} (1-\lambda_0)^m \pi^m d, \tag{5.25}$$

using the matrix notation $\lambda^* = (\lambda_k^*(s))_{s,k}$ and $d = \left(d^k(s)\right)_{s,k}$.

Stability:

i) Suppose $\left[\lambda_k^*(s) + \lambda_0 d^k(s)\right]_{s,k}$ has full rank. Then for every portfolio rule $\lambda \neq \lambda^*$, one has $g_{\lambda^*}(\lambda) < 0$. Thus λ^* is evolutionarily stable.

ii) For every λ, one has $g_{\lambda^*}(\lambda) \leq 0$. Thus λ^* is never evolutionarily unstable.

Instability:

iii) For every $\lambda \neq \lambda^*$ there exist arbitrarily close portfolio rules $\mu \neq \lambda$ such that $g_\lambda(\mu) > 0$. Thus every $\lambda \neq \lambda^*$ is locally evolutionarily unstable and, in particular, evolutionarily unstable.

Theorem 5.8 means that the investors who employ the Kelly portfolio rule drive the investor who does not employ the Kelly portfolio rule out of the market. This is known as the property of evolutionarily stability of λ^* in evolutionary finance. For more details on the existing literature on evolutionary finance we refer to [22].

5.3 The Kelly Portfolio Rule Dominates

Amir et al. [3] studied the model in Section 5.1 when investors employ general, adaptive portfolio rules. It was shown in [3] that the Kelly rule λ^* survives; i.e., keeps relative wealth bounded away from zero a.s., but it does not necessarily dominate the market. The result of [3] suggests a very interesting question: Suppose that investors are allowed to use general adaptive portfolio rules, can the Kelly rule dominate? Or, more precisely, under what conditions does the Kelly dominate?

Obviously, in general, without any restrictions, one cannot expect the Kelly rule to dominate. To see this, we first describe a recursive method of constructing strategies which allows the investor who follows this recursive method to have constant relative wealth at all times. Consequently, if other investors employ the Kelly rule, they will not be able to accumulate the total relative wealth of the market; i.e., they will not be able to dominate. This will be shown in the following proposition.

Proposition 5.9.
If investor 1 uses the portfolio rule

$$\lambda_{t,k}^1 = \frac{\sum_{j=2}^I \lambda_{t,k}^j r_t^j}{1 - r_t^1}, \tag{5.26}$$

for all $t = 0, 1, \ldots,$ and all $k = 1, \ldots, K$, then her/his relative wealth remains constant at all times; i.e., for all $t = 0, 1, \ldots,$ $r_{t+1}^1 = r_t^1$.

Proof.
From (5.12) we have

$$r_{t+1}^1 = \sum_{k=1}^K \left[\rho_{t+1} \sum_{j=1}^I \lambda_{t+1,k}^j r_{t+1}^j + (1 - \rho_{t+1}) R_{t+1,k} \right] \frac{\lambda_{t,k}^1 r_t^1}{\sum_{j=1}^I \lambda_{t,k}^j r_t^j}. \tag{5.27}$$

Observe that the portfolio rule (5.26) implies that

$$\lambda_{t,k}^1 = \sum_{j=1}^I \lambda_{t,k}^j r_t^j. \tag{5.28}$$

Therefore, from (5.27) and (5.28), we obtain

$$\sum_{k=1}^K \left[\rho_{t+1} \sum_{j=1}^I \lambda_{t+1,k}^j r_{t+1}^j + (1 - \rho_{t+1}) R_{t+1,k} \right] = \frac{r_{t+1}^1}{r_t^1}. \tag{5.29}$$

The left hand side of the above equation (5.29) is equal to 1. Indeed,

$$\sum_{k=1}^K \left(\rho_{t+1} \sum_{j=1}^I \lambda_{t+1,k}^j r_{t+1}^j \right) + 1 - \rho_{t+1} = 1.$$

This completes the proof of the proposition. □

The above observation gives us the following. If one of the investors $2, \dots, I$ uses the portfolio rule λ^*, then he/she cannot be a single survivor, as long as investor 1 uses the decision rule (5.26). Consequently, the relative market share of investor 1 remains constant.

Remark 5.10. The portfolio rule (5.26) has an economic interpretation. By (5.3) and (5.4), the portfolio of investor 1 is given by

$$x^1_{t,k} = \frac{\mu_t \lambda^1_{t,k} w^1_t}{p_{t,k}} = \frac{\lambda^1_{t,k} w^1_t \cdot V_{t,k}}{\sum_{i=1}^I \lambda^i_{t,k} w^i_t}. \tag{5.30}$$

If investor 1 uses the portfolio rule (5.26), then by (5.28) and (5.30), we obtain that

$$x^1_{t,k} = r^1_t \cdot V_{t,k}.$$

Thus the vector x^1_t will be proportional to *market portfolio*; i.e. to the $(V_1, \dots, V_K)$ whose components indicate the amounts of assets $k = 1, \dots, K$ traded in the market. Following [2] we call such portfolios CAPM portfolios. This is due to the fact that portfolios having this structure result from the mean-variance optimisation in the Capital Asset Pricing Model (CAPM).

In the following section we will state sufficient conditions for the Kelly rule to dominate the market even when other investors are allowed to use dynamic adaptive strategies.

5.3.1 The main result.

Define

$$\xi_t = (\xi_{t,1}, \dots, \xi_{t,K}) = f_t\left(r_t, \lambda^2_t, \dots, \lambda^I_t\right), \tag{5.31}$$

where f is the decision rule (5.26); i.e., for all $t = 1, 2, \dots$,

$$\xi_t(s^t) := \frac{\sum_{j=2}^I \lambda^j_{t,k} r^j_t}{1 - r^1_t}.$$

Following [2] (see Remark 5.10 above) we call the portfolio rule in (5.31) *CAPM strategy*. Our first main result in this chapter is given by the following theorem [9].

Theorem 5.11.
If investor 1 *employs the Kelly strategy* (5.11)*, then he/she dominates the market exponentially fast provided the following conditions are satisfied:*

(C1) $E\left(\ln \xi_{t,k}\left(s^t\right)|s^{t-1}\right) = \ln \xi_{t-1,k}(s^{t-1})$ *a.s.,*

(C2) There exists a strictly positive random variable $\kappa > 0$*, such that, almost surely,*

$$|\lambda^* - \xi_t(s^t)| \geq \kappa \tag{5.32}$$

for t *large enough.*

(C3) The coordinates $\lambda_{t,k}\left(s^t\right)$ *of the vectors* $\lambda_t\left(s^t\right)$ *are bounded away from zero by a strictly positive non-random constant* ϱ *(that might depend on the strategy* λ*, but not on* k, t *and* s^t*), i.e.* $\inf_{i,k,t,s^t} \lambda^i_{t,k}\left(s^t\right) > \varrho > 0$.

The proof of Theorem 5.11 is in section 5.3.2.

Remark 5.12.
Assumption (C1) means that the CAPM strategy $\xi_t(s^t)$ forms a martingale with respect to the filtration generated by (s^t).

Remark 5.13.
Since $\xi_t(s^t)$ is a convex combination of $\lambda^2_t, \ldots, \lambda^I_t$, assumption (C2) is certainly satisfied if there exist a random variable κ, and a $T > 0$, such that for $t \geq T$, with probability one, the distance between the vector λ^* and the convex hull of the vectors $\lambda^2_t, \ldots, \lambda^I_t$ is at least κ.

5.3.2 Proofs

We start this section by proving two lemmas which are needed in the proof of Theorem 5.11.

Lemma 5.14.
We have

$$1 - r^1_{t+1} = \sum_{k=1}^{K} \left\{\rho_{t+1}\left[\lambda^1_{t+1,k} r^1_{t+1} + \left(1 - r^1_{t+1}\right)\xi_{t+1,k}\right] + \left(1 - \rho_{t+1}\right) R_{t+1,k}\right\} \times \frac{\xi_{t,k}\left(1 - r^1_t\right)}{\lambda^1_{t,k} r^1_t + \left(1 - r^1_t\right)\xi_{t,k}},$$

and

$$r^1_{t+1} = \sum_{k=1}^{K} \left\{\rho_{t+1}\left[\lambda^1_{t+1,k} r^1_{t+1} + \left(1 - r^1_{t+1}\right)\xi_{t+1,k}\right] + \left(1 - \rho_{t+1}\right) R_{t+1,k}\right\} \times \frac{\lambda^1_{t,k} r^1_t}{\lambda^1_{t,k} r^1_t + \left(1 - r^1_t\right)\xi_{t,k}}.$$

Proof.

By using (5.12), we have

$$\begin{aligned}
\frac{1 - r^1_{t+1}}{1 - r^1_t} &= \frac{\sum_{i=2}^{I} r^i_{t+1}}{1 - r^1_t} \\
&= \frac{\sum_{i=2}^{I}\left\{\sum_{k=1}^{K}\left[\rho_{t+1}\langle\lambda_{t+1,k}, r_{t+1}\rangle + \left(1 - \rho_{t+1}\right) R_{t+1,k}\right]\frac{\lambda^i_{t,k} r^i_t}{\langle\lambda_{t,k}, r_t\rangle}\right\}}{1 - r^1_t} \\
&= \frac{\sum_{i=2}^{I}\left\{\sum_{k=1}^{K}\left[\rho_{t+1}\sum_{j=1}^{I}\lambda^j_{t+1,k} r^j_{t+1} + \left(1 - \rho_{t+1}\right) R_{t+1,k}\right]\frac{\lambda^i_{t,k} r^i_t}{\sum_{j=1}^{I}\lambda^j_{t,k} r^j_t}\right\}}{1 - r^1_t} \\
&= \frac{\sum_{k=1}^{K}\left[\rho_{t+1}\sum_{j=1}^{I}\lambda^j_{t+1,k} r^j_{t+1} + \left(1 - \rho_{t+1}\right) R_{t+1,k}\right]\frac{\sum_{i=2}^{I}\lambda^i_{t,k} r^i_t}{\sum_{j=1}^{I}\lambda^j_{t,k} r^j_t}}{1 - r^1_t} \\
&= \sum_{k=1}^{K}\left[\rho_{t+1}\psi_{t+1,k} + \left(1 - \rho_{t+1}\right) R_{t+1,k}\right]\frac{\xi_{t,k}}{\psi_{t,k}},
\end{aligned}$$

where

$$\begin{aligned}
\psi_{t,k} &= \sum_{j=1}^{I}\lambda^j_{t,k} r^j_t = \lambda^1_{t,k} r^1_t + \left(1 - r^1_t\right)\frac{\sum_{i=2}^{I}\lambda^i_{t,k} r^i_t}{1 - r^1_t} \\
&= \lambda^1_{t,k} r^1_t + \xi_{t,k}\left(1 - r^1_t\right).
\end{aligned}$$

Therefore,

$$1 - r^1_{t+1} = \sum_{k=1}^{K} \left\{\rho_{t+1}\left[\lambda^1_{t+1,k} r^1_{t+1} + \left(1 - r^1_{t+1}\right)\xi_{t+1,k}\right] + \left(1 - \rho_{t+1}\right) R_{t+1,k}\right\} \times \frac{\xi_{t,k}\left(1 - r^1_t\right)}{\lambda^1_{t,k} r^1_t + \left(1 - r^1_t\right)\xi_{t,k}},$$

$$r_{t+1}^1 = \sum_{k=1}^{K} \left\{ \rho_{t+1} \left[\lambda_{t+1,k}^1 r_{t+1}^1 + \left(1 - r_{t+1}^1\right) \xi_{t+1,k} \right] + \left(1 - \rho_{t+1}\right) R_{t+1,k} \right\} \times \frac{\lambda_{t,k}^1 r_t^1}{\lambda_{t,k}^1 r_t^1 + \left(1 - r_t^1\right) \xi_{t,k}}.$$

□

Lemma 5.15.

For $t = 1, 2, \ldots,$ *let*

$$D_t = \ln \frac{r_t^1 \left(r_{t-1}^1\right)^{-1}}{\left(1 - r_t^1\right) \left(1 - r_{t-1}^1\right)^{-1}}.$$

D_t *are uniformly bounded random variables.*

Proof.

We have

$$\begin{aligned} r_{t+1}^i &= \sum_{k=1}^{K} \left\{ \rho_{t+1} \sum_{j=1}^{I} \lambda_{t+1,k}^j r_{t+1}^j + \left(1 - \rho_{t+1}\right) R_{t+1,k} \right\} \frac{\lambda_{t,k}^i r_t^i}{\sum_{j=1}^{I} \lambda_{t,k}^j r_t^j} \\ &\geq \sum_{k=1}^{K} \left(1 - \rho_{t+1}\right) R_{t+1,k} \frac{\lambda_{t,k}^i r_t^i}{\sum_{j=1}^{I} \lambda_{t,k}^j r_t^j} \\ &\geq r_t^i \left(1 - \rho_{t+1}\right) \sum_{k=1}^{K} R_{t+1,k} \frac{\min_k \lambda_{t,k}^i}{1} \\ &= r_t^i \left(1 - \rho_{t+1}\right) \min_k \lambda_{t,k}^i. \end{aligned}$$

By assumption (C3), $\inf \lambda_{t,k}^i \geq \varrho$. Then, we have

$$\varrho \leq \frac{r_{t+1}^i}{r_t^i}.$$

For the upper bound we have

$$
\begin{aligned}
\frac{r_{t+1}^i}{r_t^i} &= \sum_{k=1}^{K}\left\{\rho_{t+1}\sum_{j=1}^{I}\lambda_{t+1,k}^j r_{t+1}^j + (1-\rho_{t+1})\,R_{t+1,k}\right\}\frac{\lambda_{t,k}^i}{\sum_{j=1}^{I}\lambda_{t,k}^j r_t^j}\\
&\leq \sum_{k=1}^{K}\left\{\rho_{t+1}\sum_{j=1}^{I}\lambda_{t+1,k}^j r_{t+1}^j + (1-\rho_{t+1})\,R_{t+1,k}\right\}\frac{1}{\min_k \lambda_{t,k}^i}\\
&= \left\{\sum_{k=1}^{K}\left(\rho_{t+1}\sum_{j=1}^{I}\lambda_{t+1,k}^j r_{t+1}^j\right) + (1-\rho_{t+1})\right\}\frac{1}{\min_k \lambda_{t,k}^i}\\
&= \frac{1}{\min_k \lambda_{t,k}^i} \leq \varrho^{-1}.
\end{aligned}
$$

Therefore, $\varrho \leq \frac{r_{t+1}^i}{r_t^i} \leq \varrho^{-1}$ and this implies, because $1 - r_t^1 = \sum_{m=2}^{I} r_t^m$, that the random variables D_t are uniformly bounded. □

Proof of Theorem 5.11.
By Lemma 5.14, it is sufficient to consider the case of two investors 1 and 2, using the strategies λ^* and ξ, and whose market relative shares are given by

$$
\begin{aligned}
r_{t+1}^1 = \sum_{k=1}^{K}\left\{\rho_{t+1}\left[\lambda_{t+1,k}^1 r_{t+1}^1 + \left(1-r_{t+1}^1\right)\xi_{t+1,k}\right] + (1-\rho_{t+1})\,R_{t+1,k}\right\}\times\\
\frac{\lambda_{t,k}^1 r_t^1}{\lambda_{t,k}^1 r_t^1 + \left(1-r_t^1\right)\xi_{t,k}},
\end{aligned}
\tag{5.33}
$$

$$
\begin{aligned}
1 - r_{t+1}^1 = \sum_{k=1}^{K}\left\{\rho_{t+1}\left[\lambda_{t+1,k}^1 r_{t+1}^1 + \left(1-r_{t+1}^1\right)\xi_{t+1,k}\right] + (1-\rho_{t+1})\,R_{t+1,k}\right\}\times\\
\frac{\xi_{t,k}\left(1-r_t^1\right)}{\lambda_{t,k}^1 r_t^1 + \left(1-r_t^1\right)\xi_{t,k}}.
\end{aligned}
\tag{5.34}
$$

We consider the ratio $z_t = \frac{r_t^1}{1-r_t^1}$ of the market shares of investors 1 and 2. Then the dynamics of z_t are described by the following equation

$$
z_t = z_{t-1}\frac{\sum_{k=1}^{K}\left[\rho_t\xi_{t,k} + (1-\rho_t)\,R_{t,k}\right]\frac{\lambda_k^*}{\lambda_k^* z_{t-1}+\xi_{t-1,k}}}{\sum_{k=1}^{K}\left[\rho_t\lambda_k^* + (1-\rho_t)\,R_{t,k}\right]\frac{\xi_{t-1,k}}{\lambda_k^* z_{t-1}+\xi_{t-1,k}}}
$$

$$\Leftrightarrow$$

$$
\frac{z_t}{z_{t-1}} = \frac{\sum_{k=1}^{K}\left[\rho_t\xi_{t,k} + (1-\rho_t)\,R_{t,k}\right]\frac{\lambda_k^*}{\lambda_k^* r_{t-1}^1+\xi_{t-1,k}\left(1-r_{t-1}^1\right)}}{\sum_{k=1}^{K}\left[\rho_t\lambda_k^* + (1-\rho_t)\,R_{t,k}\right]\frac{\xi_{t-1,k}}{\lambda_k^* r_{t-1}^1+\xi_{t-1,k}\left(1-r_{t-1}^1\right)}}.
\tag{5.35}
$$

The derivation of equation (5.35) can be found in Appendix A.2.

To prove the theorem, our goal is to show that, with probability 1,

$$\liminf_{T\to\infty} \frac{1}{T} \ln \frac{r_T^1}{1-r_T^1} > 0. \tag{5.36}$$

For this purpose, we define for $t = 1, 2, \ldots$

$$D_t := \ln \frac{r_t^1 \left(r_{t-1}^1\right)^{-1}}{\left(1-r_t^1\right)\left(1-r_{t-1}^1\right)^{-1}} = \ln \frac{z_t}{z_{t-1}}. \tag{5.37}$$

Observe that

$$D_1 + D_2 + \ldots + D_T = \ln \frac{r_T^1}{1-r_T^1} - \ln \frac{r_0^1}{1-r_0^1}. \tag{5.38}$$

Hence, (5.36) holds if and only if

$$\liminf_{T\to\infty} \frac{1}{T} \left(D_1 + D_2 + \ldots + D_T\right) > 0 \quad a.s. \tag{5.39}$$

We have the following identity:

$$\frac{1}{T}\sum_{t=1}^{T} D_t = \frac{1}{T}\sum_{t=1}^{T} E(D_t|s^{t-1}) + \frac{1}{T}\sum_{t=1}^{T}\left(D_t - E(D_t|s^{t-1})\right).$$

Let $G_t := D_t - E(D_t|s^{t-1})$. By Lemma 5.15, the random variables D_t are uniformly bounded. Therefore, by the law of large numbers, (see Theorem 2.47 proved in [25]), we have

$$\frac{1}{T}\left(G_1 + \ldots + G_T\right) \to 0$$

with probability 1. It follows that

$$\liminf_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} D_t = \liminf_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} E(D_t|s^{t-1}). \tag{5.40}$$

Therefore, (5.36) is equivalent to

$$\liminf_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} E(D_t|s^{t-1}) > 0 \quad a.s. \tag{5.41}$$

By using (5.35), we have

$$E(D_t|s^{t-1}) = E\left[\ln \frac{z_t}{z_{t-1}}|s^{t-1}\right] =$$
$$E\left[\ln \frac{\sum_{k=1}^{K}\left[\rho_t \xi_{t,k}\left(s_t\right) + \left(1-\rho_t\right) R_{t,k}\right] \frac{\lambda_k^*}{\lambda_k^* r_{t-1}^1 + \xi_{t-1,k}\left(1-r_{t-1}^1\right)}}{\sum_{k=1}^{K}\left[\rho_t \lambda_k^* + \left(1-\rho_t\right) R_{t,k}\right] \frac{\xi_{t-1,k}}{\lambda_k^* r_{t-1}^1 + \xi_{t-1,k}\left(1-r_{t-1}^1\right)}}|s^{t-1}\right]. \tag{5.42}$$

We will show that (5.42) is bounded away from zero. When we apply Jensen's inequality for conditional expectations (see properties of conditional expectation 8) to equation (5.42), we obtain

$$\begin{aligned}
&E\left(\ln \sum_{k=1}^{K}\left[\rho_t \xi_{t,k} + \left(1-\rho_t\right) R_{t,k}\right] \frac{\lambda_k^*}{\lambda_k^* r_{t-1}^1 + \xi_{t-1,k}\left(1-r_{t-1}^1\right)}|s^{t-1}\right) \\
&\qquad - E\left(\ln \sum_{k=1}^{K}\left[\rho_t \lambda_k^* + \left(1-\rho_t\right) R_{t,k}\right] \frac{\xi_{t-1,k}}{\lambda_k^* r_{t-1}^1 + \xi_{t-1,k}\left(1-r_{t-1}^1\right)}|s^{t-1}\right) \\
&\geq \rho_t E\left(\ln \sum_{k=1}^{K} \xi_{t,k} \frac{\lambda_k^*}{\lambda_k^* r_{t-1}^1 + \xi_{t-1,k}\left(1-r_{t-1}^1\right)}|s^{t-1}\right) \\
&\qquad + \left(1-\rho_t\right) E\left(\ln \sum_{k=1}^{K} R_{t,k} \frac{\lambda_k^*}{\lambda_k^* r_{t-1}^1 + \xi_{t-1,k}\left(1-r_{t-1}^1\right)}|s^{t-1}\right) \\
&\qquad\qquad - E\left(\ln \sum_{k=1}^{K}\left[\rho_t \lambda_k^* + \left(1-\rho_t\right) R_{t,k}\right] \frac{\xi_{t-1,k}}{\lambda_k^* r_{t-1}^1 + \xi_{t-1,k}\left(1-r_{t-1}^1\right)}|s^{t-1}\right) \\
&= \rho_t E\left(\ln \sum_{k=1}^{K} \xi_{t,k} \frac{\lambda_k^*}{\lambda_k^* r_{t-1}^1 + \xi_{t-1,k}\left(1-r_{t-1}^1\right)}|s^{t-1}\right) \\
&\qquad + \left(1-\rho_t\right) E\left(\ln \sum_{k=1}^{K} R_{t,k} \frac{\lambda_k^*}{\lambda_k^* r_{t-1}^1 + \xi_{t-1,k}\left(1-r_{t-1}^1\right)}|s^{t-1}\right) \\
&\qquad\qquad - \rho_t E\left(\ln \sum_{k=1}^{K}\left[\rho_t \lambda_k^* + \left(1-\rho_t\right) R_{t,k}\right] \frac{\xi_{t-1,k}}{\lambda_k^* r_{t-1}^1 + \xi_{t-1,k}\left(1-r_{t-1}^1\right)}|s^{t-1}\right) \\
&\qquad\qquad\quad - (1-\rho_t) E\left(\ln \sum_{k=1}^{K}\left[\rho_t \lambda_k^* + \left(1-\rho_t\right) R_{t,k}\right] \frac{\xi_{t-1,k}}{\lambda_k^* r_{t-1}^1 + \xi_{t-1,k}\left(1-r_{t-1}^1\right)}|s^{t-1}\right) \\
&:= \rho_t \mathbf{A} + (1-\rho_t)\mathbf{B},
\end{aligned}$$

where

$$\mathbf{A} := E\left[\ln \frac{\sum_{k=1}^{K} \xi_{t,k} \frac{\lambda_k^*}{\lambda_k^* r_{t-1}^1 + \xi_{t-1,k}\left(1-r_{t-1}^1\right)}}{\sum_{k=1}^{K} \tilde{R}_{t,k} \frac{\xi_{t-1,k}}{\lambda_k^* r_{t-1}^1 + \xi_{t-1,k}\left(1-r_{t-1}^1\right)}}|s^{t-1}\right],$$

$$\mathbf{B} := E\left[\ln \frac{\sum_{k=1}^{K} R_{t,k} \frac{\lambda_k^*}{\lambda_k^* r_{t-1}^1 + \xi_{t-1,k}\left(1-r_{t-1}^1\right)}}{\sum_{k=1}^{K} \tilde{R}_{t,k} \frac{\xi_{t-1,k}}{\lambda_k^* r_{t-1}^1 + \xi_{t-1,k}\left(1-r_{t-1}^1\right)}} | s^{t-1}\right],$$

and

$$\tilde{R}_{t,k} := \rho_t \lambda_k^* + (1-\rho_t) R_{t,k}.$$

Therefore, to show that (5.42) is bounded away from zero, it is enough to show that $\mathbf{A} \geq 0$ and $\mathbf{B}$ is bounded away from zero. For $\mathbf{A}$, noticing that $E(\tilde{R}_{t,k}|s^{t-1}) = \lambda_k^*$, and using Jensen's inequality for conditional expectations (see properties of conditional expectation 8), we obtain

$$\begin{aligned}\mathbf{A} &= E\left(\ln \sum_{k=1}^{K} \xi_{t,k} \frac{\lambda_k^*}{\lambda_k^* r_{t-1}^1 + \xi_{t-1,k}\left(1-r_{t-1}^1\right)} | s^{t-1}\right) \\ &\qquad\qquad - \ln \sum_{k=1}^{K} \xi_{t-1,k} \frac{\lambda_k^*}{\lambda_k^* r_{t-1}^1 + \xi_{t-1,k}\left(1-r_{t-1}^1\right)} \\ &= E\left[\ln \frac{\sum_{k=1}^{K} \frac{\xi_{t,k}}{\xi_{t-1,k}} \frac{\lambda_k^* \xi_{t-1,k}}{\lambda_k^* r_{t-1}^1 + \xi_{t-1,k}\left(1-r_{t-1}^1\right)}}{\sum_{k=1}^{K} \frac{\lambda_k^* \xi_{t-1,k}}{\lambda_k^* r_{t-1}^1 + \xi_{t-1,k}\left(1-r_{t-1}^1\right)}} | s^{t-1}\right].\end{aligned} \tag{5.43}$$

Define $a_k := \frac{\lambda_k^* \xi_{t-1,k}}{\lambda_k^* r_{t-1}^1 + \xi_{t-1,k}\left(1-r_{t-1}^1\right)}$. Notice that $a_k > 0$ and that a_k is measurable with respect to information generated by s^{t-1}. Then, by applying the finite form of Jensen's inequality for concave functions (see Proposition 2.35) to the right hand side of (5.43) we obtain

$$\begin{aligned}\mathbf{A} = E\left(\ln \frac{\sum_{k=1}^{K} a_k \frac{\xi_{t,k}}{\xi_{t-1,k}}}{\sum_{k=1}^{K} a_k} | s^{t-1}\right) &\geq E\left[\frac{\sum_{k=1}^{K} a_k \ln \frac{\xi_{t,k}}{\xi_{t-1,k}}}{\sum_{k=1}^{K} a_k} | s^{t-1}\right] \\ &= \frac{1}{\sum_{k=1}^{K} a_k} E\left(\sum_{k=1}^{K} a_k \ln \frac{\xi_{t,k}}{\xi_{t-1,k}} | s^{t-1}\right) \\ &= \frac{\sum_{k=1}^{K} a_k E\left[\ln \xi_{t,k} - \ln \xi_{t-1,k} |\, s^{t-1}\right]}{\sum_{k=1}^{K} a_k} = 0,\end{aligned} \tag{5.44}$$

by using condition (C1). For $\mathbf{B}$, by Lemma A.2 (see Appendix A.3), we have

$$\mathbf{B} \geq \delta_{\varrho}(|\lambda^* - \xi_{t-1}(s^{t-1})|), \tag{5.45}$$

where ϱ is the strictly positive constant bounding away from zero the coordinates of λ_t^i. Therefore, by (5.44) and (5.45) we obtain

$$E(D_t|s^{t-1}) \geq \delta_\varrho \left(|\lambda^* - \xi_{t-1}\left(s^{t-1}\right)|\right). \tag{5.46}$$

Consequently,

$$\liminf_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} E(D_t|s^{t-1}) \geq \liminf_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} \delta_\varrho \left(|\lambda^* - \xi_{t-1}\left(s^{t-1}\right)|\right) \geq \delta_\varrho(\kappa) > 0.$$

The last inequality follows from our assumption (5.32). Therefore, we obtained (5.41) which implies (5.36); i.e.,

$$\liminf_{T\to\infty} \frac{1}{T} \ln \frac{r_T^1}{1-r_T^1} > 0 \text{ a.s.}$$

Consequently, for large T, there exists a strictly positive random variable η such that

$$\frac{r_T^1}{1-r_T^1} > e^{\eta T}.$$

Thus, investor one dominates the market and its relative wealth converges to 1 exponentially fast. □

5.4 Absence of Correct Beliefs

In Section 5.4, we use Proposition 3.5 to show that the investor who is closer to the Kelly rule cannot be driven out of the market (see Theorem 5.17). We have the *relative wealth* of the investors given by

$$r_{t+1}^i = \sum_{k=1}^{K} \left[\rho\langle\lambda_k, r_{t+1}\rangle + (1-\rho)R_k\left(s_{t+1}\right)\right] \frac{\lambda_k^i r_t^i}{\langle\lambda_k, r_t\rangle}, \qquad i = 1, \cdots, I. \tag{5.47}$$

Here, we only consider the case when $I = 2$. From equation (5.47), we obtain

$$r_{t+1} = \sum_{k=1}^{K} \left\{\rho\left[\overline{\lambda}_k(1-r_{t+1}) + \lambda_k r_{t+1}\right] + (1-\rho)R_k(s_{t+1})\right\} \frac{\lambda_k r_t}{\overline{\lambda}_k(1-r_t) + \lambda_k r_t}, \tag{5.48}$$

where $\lambda = (\lambda_k)_{k=1}^K$ is the strategy of investor 1 whose relative wealth is r_{t+1} and $\overline{\lambda} = (\overline{\lambda}_k)_{k=1}^K$ is the strategy of investor 2 whose relative wealth is $1 - r_{t+1}$. Now from equation (5.48), we obtain

$$r_{t+1}\Big(1-\sum_{k=1}^{K}\rho(\lambda_k-\overline{\lambda}_k)\frac{\lambda_k r_t}{\overline{\lambda}_k(1-r_t)+\lambda_k r_t}\Big)$$
$$=\sum_{k=1}^{K}\Big(\rho\overline{\lambda}_k+(1-\rho)R_k(s_{t+1})\Big)\frac{\lambda_k r_t}{\overline{\lambda}_k(1-r_t)+\lambda_k r_t}. \tag{5.49}$$

Note that R.H.S. of the equation (5.49) is positive for all t. Then, L.H.S. of this equation is positive for all t. Since $r_{t+1}>0$ for all t, we have

$$1-\sum_{k=1}^{K}\rho(\lambda_k-\overline{\lambda}_k)\frac{\lambda_k r_t}{\overline{\lambda}_k(1-r_t)+\lambda_k r_t}>0. \tag{5.50}$$

Therefore, we can divide both sides of the equation (5.49) by (5.50) and we obtain

$$r_{t+1}=\frac{\sum_{k=1}^{K}\Big(\rho\overline{\lambda}_k+(1-\rho)R_k(s_{t+1})\Big)\frac{\lambda_k r_t}{\overline{\lambda}_k(1-r_t)+\lambda_k r_t}}{1-\sum_{k=1}^{K}\rho(\lambda_k-\overline{\lambda}_k)\frac{\lambda_k r_t}{\overline{\lambda}_k(1-r_t)+\lambda_k r_t}}. \tag{5.51}$$

In conclusion, the above random dynamical system (5.51) can be represented by the random map

$$F=\{\tau_s,\mathbf{p}_s\}_{s\in S} \tag{5.52}$$

where

$$\tau_s(r)=\frac{\sum_{k=1}^{K}\left(\rho\overline{\lambda}_k+(1-\rho)\,R_k(s)\right)\frac{\lambda_k r}{\overline{\lambda}_k(1-r)+\lambda_k r}}{1-\sum_{k=1}^{K}\rho\left(\lambda_k-\overline{\lambda}_k\right)\frac{\lambda_k r}{\overline{\lambda}_k(1-r)+\lambda_k r}}:=\frac{A}{B}, \tag{5.53}$$

and $\mathbf{p}=(\mathbf{p}_s)$ is the distribution on S. We now state our main result. From now on, we impose the following condition.

Assumption 5.16.
We assume that for $k\in\{1,\dots,K\}$

$$\begin{cases}\text{either} & \overline{\lambda}_k\le\lambda_k\le\lambda_k^*,\\ \text{or} & \lambda_k^*\le\lambda_k\le\overline{\lambda}_k.\end{cases} \tag{5.54}$$

Assumption 5.16 means that the investment strategy of investor 1 is closer (coordinatewise) than that of investor 2 to the Kelly rule. Our second main result in this chapter is given by the following theorem [11].

Theorem 5.17.
Let $I = 2$. Under assumption (5.16) investor 1 cannot be driven out of the market. He/she will either dominate or at least survive.

5.4.1 Proofs

To analyse the performance of investment strategies in the absence of "correct beliefs", i.e., in the absence of an investor using the Kelly rule, we invoke the theory of random dynamical systems (see Section 3 for definition of RDS). In this section we define the notion of a random dynamical system (RDS) and state some results from [8]. Our ideas are inspired by [8], where techniques from RDS were applied to the model of short-lived assets of Evstigneev et al. [21].

We verify that the evolution of the relative market wealth (5.12) can be represented by a random map whose constituent maps satisfy the assumptions of [8]. We refer the reader to sections 3.1, 3.1.2 and 3.1.3 of Chapter 3.

Lemma 5.18.

1. $\tau_s(0) = 0$, $\tau_s(1) = 1$.
2. τ_s *is an increasing function which maps* $[0,1]$ *into itself.*
3. τ_s *is a continuous function on* $[0,1]$, *moreover it is differentiable.*

Proof.

1. $\tau_s(0) = 0$ is obvious.

$$\begin{aligned}
\tau_s(1) &= \frac{\sum_{k=1}^{K}\left(\rho\overline{\lambda}_k + (1-\rho)R_k(s)\right)\frac{\lambda_k}{\lambda_k}}{1-\sum_{k=1}^{K}\rho\left(\lambda_k - \overline{\lambda}_k\right)\frac{\lambda_k}{\lambda_k}} \\
&= \frac{\rho\sum_{k=1}^{K}\overline{\lambda}_k + (1-\rho)\sum_{k=1}^{K}R_k(s)}{1-\sum_{k=1}^{K}\rho\left(\lambda_k - \overline{\lambda}_k\right)} \\
&= \frac{\rho + 1 - \rho}{1} = 1.
\end{aligned}$$

2. Let $g_s(r) = \dfrac{\tau_s(r)}{1-\tau_s(r)}$. Note that $g_s(r)$ is increasing $\Longleftrightarrow$ $\tau_s(r)$ is increasing. Thus, it is enough to show that $g_s(r)$ is increasing.

$$\begin{aligned}
g_s(r) &= \frac{\sum_{k=1}^{K}\left(\rho\overline{\lambda}_k + (1-\rho)R_k(s)\right)\frac{\lambda_k r}{\overline{\lambda}_k(1-r)+\lambda_k r}}{1-\sum_{k=1}^{K}\rho\left(\lambda_k-\overline{\lambda}_k\right)\frac{\lambda_k r}{\overline{\lambda}_k(1-r)+\lambda_k r}} \\
&\qquad \times \frac{1}{1-\frac{\sum_{k=1}^{K}\left(\rho\overline{\lambda}_k+(1-\rho)R_k(s)\right)\frac{\lambda_k r}{\overline{\lambda}_k(1-r)+\lambda_k r}}{1-\sum_{k=1}^{K}\rho\left(\lambda_k-\overline{\lambda}_k\right)\frac{\lambda_k r}{\overline{\lambda}_k(1-r)+\lambda_k r}}} \\
&= \frac{\sum_{k=1}^{K}\left(\rho\overline{\lambda}_k + (1-\rho)R_k(s)\right)\frac{\lambda_k r}{\overline{\lambda}_k(1-r)+\lambda_k r}}{1-\sum_{k=1}^{K}\left(\rho\lambda_k+(1-\rho)R_k(s)\right)\frac{\lambda_k r}{\overline{\lambda}_k(1-r)+\lambda_k r}} \\
&= \frac{f_s}{h_s}.
\end{aligned}$$

Observe that

$$\begin{aligned}
f_s(r) &= \sum_{k=1}^{K}\left(\rho\overline{\lambda}_k + (1-\rho)R_k(s)\right)\frac{\lambda_k r}{\overline{\lambda}_k(1-r)+\lambda_k r} \\
&= \sum_{k=1}^{K}\left(\rho\overline{\lambda}_k + (1-\rho)R_k(s)\right)\frac{\lambda_k}{\overline{\lambda}_k\frac{(1-r)}{r}+\lambda_k}
\end{aligned}$$

increases as r increases. Moreover,

$$h_s(r) = 1-\sum_{k=1}^{K}\left(\rho\lambda_k + (1-\rho)R_k(s)\right)\frac{\lambda_k r}{\overline{\lambda}_k(1-r)+\lambda_k r}.$$

Since $\left(\rho\lambda_k+(1-\rho)R_k(s)\right)\frac{\lambda_k r}{\overline{\lambda}_k(1-r)+\lambda_k r}$ increases, $h_s(r)$ decreases. Therefore, $g_s(r)$ increases.

3. The proof of (3) is standard but long. Therefore, we have added it as an appendix at the end of the paper.

$\square$

Lemma 5.19.

Let

$$\tau(r) = \frac{\sum_{k=1}^{K}\left(\rho\overline{\lambda}_k + (1-\rho)R_k\right)\frac{\lambda_k r}{\overline{\lambda}_k(1-r)+\lambda_k r}}{1-\sum_{k=1}^{K}\rho\left(\lambda_k-\overline{\lambda}_k\right)\frac{\lambda_k r}{\overline{\lambda}_k(1-r)+\lambda_k r}}, \qquad r\in[0,1],$$

and

$$\tau\left(r\right)=r^{\beta\left(r\right)}.$$

Then, for any $r\in\left[0,1\right]$, $\ln\left(\beta\left(r\right)\right)$ *is bounded.*

Proof.
We have

$$\tau\left(r\right)=r^{\beta\left(r\right)}=\exp\left(\ln\left(r\right)\beta\left(r\right)\right).$$

Consequently,

$$\beta\left(r\right)=\frac{\ln\left(\tau_{r}\right)}{\ln\left(r\right)},\ \text{ for any }\ 0<r<1\ \text{ and }\ 0<\tau\left(r\right)<1.$$

Notice that for any $0<r<1$ and $0<\tau\left(r\right)<1$, $\beta\left(r\right)>0$. The minimum and maximum of $\beta\left(r\right)$ can be attained at $r=0$, $r=1$ or at a point of local extremum. We apply De L'Hospital rule to find the $\lim_{r\to0^{+}}\beta\left(r\right)$ and $\lim_{r\to1^{-}}\beta\left(r\right)$.

$$\begin{aligned}\lim_{r\to0^{+}}\frac{\ln\left(\tau_{r}\right)}{\ln\left(r\right)}&=\lim_{r\to0^{+}}\frac{\left\{\sum_{k=1}^{K}\left(\rho\overline{\lambda}_{k}+\left(1-\rho\right)R_{k}\right)\frac{\lambda_{k}\overline{\lambda}_{k}}{\left\{\overline{\lambda}_{k}\left(1-r\right)+\lambda_{k}r\right\}^{2}}\right\}B}{AB}r\\&+\frac{\left\{\sum_{k=1}^{K}\rho\left(\lambda_{k}-\overline{\lambda}_{k}\right)\frac{\lambda_{k}\overline{\lambda}_{k}}{\left\{\overline{\lambda}_{k}\left(1-r\right)+\lambda_{k}r\right\}^{2}}\right\}A}{AB}r\\&=\frac{\sum_{k=1}^{K}\left(\rho\overline{\lambda}_{k}+\left(1-\rho\right)R_{k}\right)\frac{\lambda_{k}}{\overline{\lambda}_{k}}}{\sum_{k=1}^{K}\left(\rho\overline{\lambda}_{k}+\left(1-\rho\right)R_{k}\right)\frac{\lambda_{k}}{\overline{\lambda}_{k}}}=1.\end{aligned}$$

$$\begin{aligned}\lim_{r\to1^{-}}\frac{\ln\left(\tau_{r}\right)}{\ln\left(r\right)}&=\lim_{r\to1^{-}}\frac{\left\{\sum_{k=1}^{K}\left(\rho\overline{\lambda}_{k}+\left(1-\rho\right)R_{k}\right)\frac{\lambda_{k}\overline{\lambda}_{k}}{\left\{\overline{\lambda}_{k}\left(1-r\right)+\lambda_{k}r\right\}^{2}}\right\}B}{AB}r\\&+\frac{\left\{\sum_{k=1}^{K}\rho\left(\lambda_{k}-\overline{\lambda}_{k}\right)\frac{\lambda_{k}\overline{\lambda}_{k}}{\left\{\overline{\lambda}_{k}\left(1-r\right)+\lambda_{k}r\right\}^{2}}\right\}A}{AB}r\\&=\frac{\sum_{k=1}^{K}\left(\rho\lambda_{k}+\left(1-\rho\right)R_{k}\right)\frac{\overline{\lambda}_{k}}{\lambda_{k}}}{\sum_{k=1}^{K}\left(\rho\overline{\lambda}_{k}+\left(1-\rho\right)R_{k}\right)}\\&=\sum_{k=1}^{K}\left(\rho\lambda_{k}+\left(1-\rho\right)R_{k}\right)\frac{\overline{\lambda}_{k}}{\lambda_{k}}.\end{aligned}$$

$\square$

Lemma 5.20.
The function

$$G(r) = \frac{\sum_{k=1}^{K}\left(\rho\overline{\lambda}_k + (1-\rho)\lambda_k^*\right)\frac{\lambda_k}{\overline{\lambda}_k(1-r)+\lambda_k r}}{1-\sum_{k=1}^{K}\rho\left(\lambda_k-\overline{\lambda}_k\right)\frac{\lambda_k r}{\overline{\lambda}_k(1-r)+\lambda_k r}} \geq 1, \tag{5.55}$$

for $r \in [0,1]$.

Proof.

$$\begin{aligned}
&G(r) \geq 1 \\
\iff &\sum_{k=1}^{K}\left(\rho\overline{\lambda}_k + (1-\rho)\lambda_k^*\right)\frac{\lambda_k}{\overline{\lambda}_k(1-r)+\lambda_k r} \geq \\
&\qquad 1-\sum_{k=1}^{K}\rho\left(\lambda_k-\overline{\lambda}_k\right)\frac{\lambda_k r}{\overline{\lambda}_k(1-r)+\lambda_k r} \\
\iff &H(r) := 1-\sum_{k=1}^{K}\left\{\rho\left(\lambda_k-\overline{\lambda}_k\right)r + \rho\overline{\lambda}_k + (1-\rho)\lambda_k^*\right\}\frac{\lambda_k}{\overline{\lambda}_k(1-r)+\lambda_k r} \leq 0.
\end{aligned}$$

Since $H(1) = 0$, it is enough to show that $H(r)$ is increasing. We have

$$H'(r) = \sum_{k=1}^{K}\frac{(1-\rho)\lambda_k^*\lambda_k\left(\lambda_k-\overline{\lambda}_k\right)}{\left\{\overline{\lambda}_k(1-r)+\lambda_k r\right\}^2}. \tag{5.56}$$

- For $\overline{\lambda}_k \leq \lambda_k \leq \lambda_k^*$, we have

$$\begin{aligned}
\frac{\lambda_k^*\lambda_k}{\left\{\overline{\lambda}_k(1-r)+\lambda_k r\right\}^2} &\geq \frac{\lambda_k^*\lambda_k}{\left(\max\{\lambda_k,\overline{\lambda}_k\}\right)^2} \\
&\geq \frac{\lambda_k^*\lambda_k}{(\lambda_k)^2} = \frac{\lambda_k^*}{\lambda_k} \geq 1
\end{aligned}$$

$$\frac{\lambda_k^*\lambda_k\left(\lambda_k-\overline{\lambda}_k\right)}{\left\{\overline{\lambda}_k(1-r)+\lambda_k r\right\}^2} \geq \lambda_k-\overline{\lambda}_k. \tag{5.57}$$

- For $\lambda_k^* \leq \lambda_k \leq \overline{\lambda}_k$, we have

$$\begin{aligned}
\frac{\lambda_k^*\lambda_k}{\left\{\overline{\lambda}_k(1-r)+\lambda_k r\right\}^2} &\leq \frac{\lambda_k^*\lambda_k}{\left(\min\{\lambda_k,\overline{\lambda}_k\}\right)^2} \\
&= \frac{\lambda_k^*\lambda_k}{(\lambda_k)^2} \leq 1
\end{aligned}$$

$$\frac{\lambda_k^* \lambda_k \left(\lambda_k - \overline{\lambda}_k\right)}{\left\{\overline{\lambda}_k (1-r) + \lambda_k r\right\}^2} \geq \lambda_k - \overline{\lambda}_k. \tag{5.58}$$

From (5.57) and (5.58), for all k, we have

$$\frac{\lambda_k^* \lambda_k \left(\lambda_k - \overline{\lambda}_k\right)}{\left\{\overline{\lambda}_k (1-r) + \lambda_k r\right\}^2} \geq \lambda_k - \overline{\lambda}_k.$$

Consequently,

$$\sum_{k=1}^{K} \frac{(1-\rho)\, \lambda_k^* \lambda_k \left(\lambda_k - \overline{\lambda}_k\right)}{\left\{\overline{\lambda}_k (1-r) + \lambda_k r\right\}^2} \geq 0.$$

So, the function $H(r)$ is increasing and therefore, the function $H(r) \leq 0$. □

Proof of Theorem 5.17.

Let us consider the expression

$$\begin{aligned}
&\sum_{s=1}^{L} \mathbf{p}_s \ln\left(\beta_s(r)\right) \leq \ln\left(\sum_{s=1}^{L} \mathbf{p}_s \beta_s(r)\right) = \ln\left(\sum_{s=1}^{L} \mathbf{p}_s \frac{\ln(\tau_s(r))}{\ln(r)}\right) \\
&\leq \ln\left(\frac{1}{\ln r} \ln\left(\sum_{s=1}^{L} \mathbf{p}_s \tau_s(r)\right)\right) \\
&= \ln\left(\frac{1}{\ln r} \ln\left(\sum_{s=1}^{L} \mathbf{p}_s \frac{\sum_{k=1}^{K}\left(\rho\overline{\lambda}_k + (1-\rho)\, R_k(s)\right) \frac{\lambda_k r}{\overline{\lambda}_k(1-r)+\lambda_k r}}{1 - \sum_{k=1}^{K} \rho\left(\lambda_k - \overline{\lambda}_k\right) \frac{\lambda_k r}{\overline{\lambda}_k(1-r)+\lambda_k r}}\right)\right) \\
&= \ln\left(\frac{1}{\ln r}\left(\ln r + \ln\left(\frac{\sum_{k=1}^{K}\sum_{s=1}^{L} \mathbf{p}_s \left[\rho\overline{\lambda}_k + (1-\rho)\, R_k(s)\right] \frac{\lambda_k}{\overline{\lambda}_k(1-r)+\lambda_k r}}{1 - \sum_{k=1}^{K} \rho\left(\lambda_k - \overline{\lambda}_k\right) \frac{\lambda_k r}{\overline{\lambda}_k(1-r)+\lambda_k r}}\right)\right)\right) \\
&= \ln\left(\frac{1}{\ln r}\left(\ln r + \ln\left(\frac{\sum_{k=1}^{K}\left[\rho\overline{\lambda}_k + (1-\rho)\, \lambda_k^*\right] \frac{\lambda_k}{\overline{\lambda}_k(1-r)+\lambda_k r}}{1 - \sum_{k=1}^{K} \rho\left(\lambda_k - \overline{\lambda}_k\right) \frac{\lambda_k r}{\overline{\lambda}_k(1-r)+\lambda_k r}}\right)\right)\right) \\
&= \ln\left(\frac{1}{\ln r}\left(\ln r + \ln\left(G(r)\right)\right)\right) \\
&= \ln\left(1 + \frac{1}{\ln r} \ln\left(G(r)\right)\right) \leq 0.
\end{aligned} \tag{5.59}$$

In the last inequality we used the fact that which was proved in Lemma 5.20 $G(r) \geq 1$. Since the stochastic process s_t is an independent, identically distributed process, we have, by (5.59),

$$E\left(\ln \alpha_t \mid s^{t-1}\right) = \sum_{s=1}^{L} \mathbf{p}_s \ln\left(\beta_s\left(r_{t-2}\right)\right) \leq 0.$$

Therefore, by Proposition 3.5, $\lim_{t\to\infty} r_t\left(s^t\right) \neq 0$ a.s. This means that investor 1 either dominates or at least survives. $\square$

Chapter 6

Conclusion

We have studied survival and extinction problems in evolutionary finance. Firstly, we have examined an evolutionary market model with short-lived assets. We presented our results when the states of the world are *not* identically distributed. They may depend on the amount of money invested in the assets. We have computed the Lyapunov exponents of the skew product associated with random market system [10]. We used the Lyapunov exponents to study wealth dynamics of investors. We have found that the investor who employs a particular portfolio rule cannot be driven out of the market [10]. Then, we analysed the market model with long-lived assets. In [9], we have found sufficient conditions for an investor using the Kelly rule to be a single survivor. Moreover, we showed that this investor dominates the others exponentially fast. The investors were allowed to use dynamic investment strategies. Finally, in [11], we analyse the long-lived asset model when the exact probability distribution of the states of the world is not available for investors. We have shown that in the absence of correct beliefs, the investor who is *closer* to the Kelly rule may dominate the market or at least survive; i.e., this investor cannot be driven out of the market. Our techniques are borrowed from the theory of random dynamical systems.

Our results are based on many assumptions that can be extended in future research. For instance, we showed the strategy that is closer to the Kelly rule cannot be driven out of the market in case of two investors. One can extend this result in case of $I > 2$ investors in the market. Moreover, we have assumed that the investors use constant strategies. We would like to study this problem when the investors are allowed to use dynamic investment strategies. The solution of this problem would be a remarkable result in the field.

Appendix A

Appendix

A.1 Appendix

Proof of Lemma 4.1-(3).

To show that the function

$$\tau_s(r) = \sum_{k=1}^{K} R_k(s) \frac{\lambda^*_{t,k} r}{\lambda^*_{t,k} r + \bar{\lambda}_{t,k}(1-r)}$$

is differentiable we need to show that the following limit

$$\lim_{h\to 0}\left\{\sum_{k=1}^{K} R_k(s) \frac{\lambda^*_{t,k}(a+h)}{\lambda^*_{t,k}(a+h)+\bar{\lambda}_{t,k}(1-a-h)} - \sum_{k=1}^{K} R_k(s) \frac{\lambda^*_{t,k} a}{\lambda^*_{t,k} a + \bar{\lambda}_{t,k}(1-a)}\right\}\frac{1}{h} \tag{A.1}$$

exists. From (A.1) we have

$$\begin{aligned}
&\lim_{h\to 0}\left\{\sum_{k=1}^{K} R_k(s) \frac{\lambda^*_{t,k}(a+h)\left[\lambda^*_{t,k} a + \bar{\lambda}_{t,k}(1-a)\right]}{\left[\lambda^*_{t,k}(a+h)+\bar{\lambda}_{t,k}(1-a-h)\right]\left[\lambda^*_{t,k} a + \bar{\lambda}_{t,k}(1-a)\right]}\right.\\
&\qquad\left. - \sum_{k=1}^{K} R_k(s) \frac{\lambda^*_{t,k} a\left[\lambda^*_{t,k}(a+h)+\bar{\lambda}_{t,k}(1-a-h)\right]}{\left[\lambda^*_{t,k}(a+h)+\bar{\lambda}_{t,k}(1-a-h)\right]\left[\lambda^*_{t,k} a + \bar{\lambda}_{t,k}(1-a)\right]}\right\}\frac{1}{h}\\
&= \lim_{h\to 0}\left\{\sum_{k=1}^{K} R_k(s) \frac{\lambda^*_{t,k}\bar{\lambda}_{t,k}(a+h)(1-a) - \lambda^*_{t,k}\bar{\lambda}_{t,k} a(1-a-h)}{\left[\lambda^*_{t,k}(a+h)+\bar{\lambda}_{t,k}(1-a-h)\right]\left[\lambda^*_{t,k} a + \bar{\lambda}_{t,k}(1-a)\right]}\right\}\frac{1}{h}\\
&= \lim_{h\to 0}\left\{\sum_{k=1}^{K} R_k(s) \frac{\lambda^*_{t,k}\bar{\lambda}_{t,k} h}{\left[\lambda^*_{t,k}(a+h)+\bar{\lambda}_{t,k}(1-a-h)\right]\left[\lambda^*_{t,k} a + \bar{\lambda}_{t,k}(1-a)\right]}\right\}\frac{1}{h}
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{h\to 0}\sum_{k=1}^{K} R_k\left(s\right)\frac{\lambda^*_{t,k}\bar{\lambda}_{t,k}}{\left[\lambda^*_{t,k}\left(a+h\right)+\bar{\lambda}_{t,k}\left(1-a-h\right)\right]\left[\lambda^*_{t,k}a+\bar{\lambda}_{t,k}\left(1-a\right)\right]}\\
&= \sum_{k=1}^{K} R_k\left(s\right)\frac{\lambda^*_{t,k}\bar{\lambda}_{t,k}}{\left[\lambda^*_{t,k}a+\bar{\lambda}_{t,k}\left(1-a\right)\right]^2}
\end{aligned}
$$

We showed that the limit exists. Therefore, the function $\tau_s\left(r\right)$ is differentiable. □

A.2 Appendix

We make use the following proposition in Proof of Theorem 5.11. This proposition can be found in [3].

Proposition A.1.
The process z_t is governed by the following random dynamical system:

$$
\frac{z_t}{z_{t-1}} = \frac{\sum_{k=1}^{K}\left[\rho_t\lambda^2_{t,k}+\left(1-\rho_t\right)R_{t,k}\right]\frac{\lambda^1_{t,k}}{\lambda^1_{t,k}z_{t-1}+\lambda^2_{t-1,k}}}{\sum_{k=1}^{K}\left[\rho_t\lambda^1_{t,k}+\left(1-\rho_t\right)R_{t,k}\right]\frac{\lambda^2_{t-1,k}}{\lambda^1_{t-1,k}z_t+\lambda^2_{t-1,k}}}. \tag{A.2}
$$

Proof. By using (5.12) with $I=2$, we obtain

$$
r^i_t = \sum_{k=1}^{K}\left[\rho_t\left(\lambda^i_{t,k}r^i_t+\lambda^j_{t,k}\left(1-r^i_t\right)\right)+\left(1-\rho_t\right)R_{t,k}\right]\frac{\lambda^i_{t-1,k}r^i_{t-1}}{\lambda^i_{t-1,k}r^i_{t-1}+\lambda^j_{t-1,k}r^j_{t-1}},
$$

where $i,j\in\{1,2\}$ and $i\neq j$. Setting $C^{ij}_{t-1,k} := \frac{\lambda^i_{t-1,k}r^i_{t-1}}{\lambda^i_{t-1,k}r^i_{t-1}+\lambda^j_{t-1,k}r^j_{t-1}}$, we obtain

$$
r^i_t\left[1+\rho_t\sum_{k=1}^{K}\left(\lambda^j_{t,k}-\lambda^i_{t,k}\right)C^{ij}_{t-1,k}\right] = \sum_{k=1}^{K}\left[\rho_t\lambda^j_{t,k}+\left(1-\rho_t\right)R_{t,k}\right]C^{ij}_{t-1,k}.
$$

Thus

$$
\frac{r^i_t}{r^j_t} = \frac{A^{ij}_t/B^{ij}_t}{A^{ji}_t/B^{ji}_t},
$$

where

$$
A^{ij}_t := \sum_{k=1}^{K}\left[\rho_t\lambda^j_{t,k}+\left(1-\rho_{t+1}\right)R_{t,k}\right]C^{ij}_{t-1,k}
$$

,

$$
B^{ij}_t := 1+\rho_t\sum_{k=1}^{K}\left(\lambda^j_{t,k}-\lambda^i_{t,k}\right)C^{ij}_{t-1,k}.
$$

Observe that $B_t^{ij} = B_t^{ji}$. Indeed,

$$\begin{aligned} B_t^{ij} - B_t^{ji} &= \rho_t \sum_{k=1}^{K} \left[\left(\lambda_{t,k}^j - \lambda_{t,k}^i\right) C_{t-1,k}^{ij} - \left(\lambda_{t,k}^i - \lambda_{t,k}^j\right) C_{t-1,k}^{ji} \right] \\ &= \rho_t \sum_{k=1}^{K} \left(\lambda_{t,k}^j - \lambda t, k^i\right) = 0 \end{aligned}$$

because $C_{t-1,k}^{ij} + C_{t-1,k}^{ji} = 1$. Consequently,

$$\frac{r_t^1}{r_t^2} = \frac{A_t^{12}}{A_t^{21}} = \frac{r_{t-1}^1}{r_{t-1}^2} \frac{\sum_{k=1}^{K} \left[\rho_t \lambda_{t,k}^2 + (1-\rho_t) R_{t,k}\right] \frac{\lambda_{t-1,k}^1}{\lambda_{t-1,k}^1 r_{t-1}^1 / r_{t-1}^2 + \lambda_{t-1,k}^2}}{\sum_{k=1}^{K} \left[\rho_t \lambda_{t,k}^1 + (1-\rho_t) R_{t,k}\right] \frac{\lambda_{t-1,k}^2}{\lambda_{t-1,k}^1 r_{t-1}^1 / r_{t-1}^2 + \lambda_{t-1,k}^2}},$$

which yields (A.2).

□

A.3 Appendix

Let S be a finite set and for each $s \in S$, $p(s) \geq 0$ be a probability distribution on S. Let $R(s) = (R_1(s), \ldots, R_K(s))$ be a vector in the simplex Δ^K satisfying (5.11) and (A2). Let $\tilde{R}_k(s) := \rho\lambda_k^* + (1-\rho) R_k(s)$, where $0 < \rho < 1$. Let $\varrho > 0$ be a number, such that $\lambda_k^* > \varrho$. Denote by Δ_ϱ^K the set of those vectors $(b_1, \ldots, b_K)$ in Δ^K that satisfy $b_k \geq \varrho$, $k = 1, \ldots, K$. Consider the function

$$\begin{aligned} \Theta(s, \kappa, \mu) = \ln \sum_{k=1}^{K} R_k(s) \frac{\lambda_k^*}{\lambda_k^* \kappa + (1-\kappa) \mu_k} \\ - \ln \sum_{k=1}^{K} \tilde{R}_k(s) \frac{\mu_k}{\lambda_k^* \kappa + (1-\kappa) \mu_k} \end{aligned} \tag{A.3}$$

of $s \in S$, $\kappa \in [0,1]$ and $\mu = (\mu_k) \in \Delta_\varrho^K$.

Lemma A.2.
There exists a function $\delta_\varrho(\gamma) \geq 0$ of $\gamma \in [0, \infty)$ satisfying the following conditions:

1. *The function $\delta(.)$ is non-decreasing, and $\delta_\varrho(\gamma) \geq 0$ for all $\gamma > 0$.*

2. For any $\kappa \in [0,1]$ and $\mu = (\mu_k) \in \Delta_\varrho^K$, we have

$$E\left[\Theta\left(s,\kappa,\mu\right)\right] \geq \delta_\varrho\left(\left|\lambda^* - \mu\right|\right). \tag{A.4}$$

Proof.
The proof of this lemma is based on the proofs of Lemma 3.1 of [21] and Lemma 1 of [2]. The only difference is that the second summand we have $\tilde{R}_k$ as defined above and not R_k. But since $\sum_{s\in S} p\left(s\right) R_k(s) = \sum_{s\in S} p\left(s\right) \tilde{R}_k(s) = \lambda_k^*$. Moreover, since $R_k(s)$ satisfies (A2), $\tilde{R}_k(s)$ satisfies (A2) too. Let us take the expectation of equation (A.3). Then from Lemma 3.1 of [21], for all $s \in S$, $\kappa \in [0,1]$ and any $\mu \in \Delta_\varrho^K$, $\mu \neq \lambda^*$, the value of $E\left[\Theta\left(s,\kappa,\mu\right)\right]$ is strictly positive.

$$E\ln\sum_{k=1}^{K} R_k\left(s\right)\frac{\lambda_k^*}{\lambda_k^*\kappa + \mu_k\left(1-\kappa\right)} - E\ln\sum_{k=1}^{K} \tilde{R}_k\left(s\right)\frac{\mu_k}{\lambda_k^*\kappa + \mu_k\left(1-\kappa\right)} \geq 0. \tag{A.5}$$

Indeed, if $\mu = \lambda^*$, inequality (A.5) turns into an equality. We now show that the expression on the left-hand side of (A.5) - which is denoted by $E\left[\Theta\left(s,\kappa,\mu\right)\right]$ - is strictly positive for all $\kappa \in [0,1]$ and $\mu \neq \lambda^*$. By applying Jensen's inequality, we find

$$E\ln\sum_{k=1}^{K} R_k\left(s\right)\frac{\lambda_k^*}{\lambda_k^*\kappa + \mu_k\left(1-\kappa\right)} \geq \sum_{k=1}^{K}\lambda_k^*\ln\frac{\lambda_k^*}{\lambda_k^*\kappa + \mu_k\left(1-\kappa\right)}, \tag{A.6}$$

$$E\ln\sum_{k=1}^{K} \tilde{R}_k\left(s\right)\frac{\mu_k}{\lambda_k^*\kappa + \mu_k\left(1-\kappa\right)} \leq \ln E\sum_{k=1}^{K} \tilde{R}_k\left(s\right)\frac{\mu_k}{\lambda_k^*\kappa + \mu_k\left(1-\kappa\right)} \tag{A.7}$$

and so

$$E\left[\Theta\left(s,\kappa,\mu\right)\right] \geq \sum_{k=1}^{K} a_k\ln\frac{a_k}{a_k\kappa + \mu_k\left(1-\kappa\right)} - \ln\sum_{k=1}^{K} a_k\frac{\mu_k}{a_k\kappa + \mu_k\left(1-\kappa\right)}, \tag{A.8}$$

where $a_k = \lambda_k^*$.
Let $\kappa = 0$. Then the right-hand side of (A.8) reduces to

$$\sum a_k\ln a_k - \sum a_k\ln\mu_k.$$

This difference is strictly positive, since $(a_k) \neq (\mu_k)$.
If $\kappa \in (0,1]$, then we have a strict inequality in (A.7). To prove this it suffices to

show that the function

$$\phi(s) = \sum_{k=1}^{K} \tilde{R}_k(s)\, \mu_k \left[\lambda_k^* \kappa + (1-\kappa)\,\mu_k\right]^{-1}, \quad s \in S,$$

is not a constant. Suppose $\phi(s)$ is constant, i.e. $\phi(s) \equiv \beta$. Then

$$\sum_{k=1}^{K} \tilde{R}_k(s) \left(\mu_k \left[\lambda_k^* \kappa + (1-\kappa)\,\mu_k\right]^{-1} - \beta\right) = 0, \quad s \in S,$$

which implies $\mu_k = \beta\,(\lambda_k^* \kappa + (1-\kappa)\,\mu_k)$, since the functions $\tilde{R}_k(.)$, $k = 1, 2, \ldots, K$, are linearly independent. We can see that $\beta = 1$, and so $\kappa\,(\lambda_k^* - \mu_k) = 0$. Since $\kappa > 0$, this implies $\lambda_k^* = \mu_k$, $k = 1, 2, \ldots, K$, which, however, is ruled out by our assumptions.

It remains to prove that the expression on the right-hand side of (A.8) is non-negative. It is equal to zero if $\kappa = 1$. If $\kappa < 1$, we can write it in the form

$$g(u) = \sum_{k=1}^{K} a_k \ln \frac{a_k}{a_k u + \mu_k} - \ln \sum_{k=1}^{K} a_k \frac{\mu_k}{a_k u + \mu_k}, \tag{A.9}$$

where $u = \kappa\,(1-\kappa)^{-1}$. We can see that $g(u) \to 0$ as $u \to \infty$. Thus it remains to prove the inequality $g'(u) \leq 0$ for all $u > 0$. We write

$$g'(u) = -\sum_{k=1}^{K} a_k^2 \left(a_k u + \mu_k\right)^{-1} + \frac{\sum_{k=1}^{K} a_k^2 \mu_k \left(a_k u + \mu_k\right)^{-2}}{\sum_{k=1}^{K} a_k \mu_k \left(a_k u + \mu_k\right)^{-1}}.$$

The sign of $g'(u)$ is the same as the sign of the expression

$$J := -\left[\sum_{k=1}^{K} a_k^2 \left(a_k u + \mu_k\right)^{-1}\right] \sum_{k=1}^{K} a_k \mu_k \left(a_k u + \mu_k\right)^{-1} + \sum_{k=1}^{K} a_k^2 \mu_k \left(a_k u + \mu_k\right)^{-2}.$$

By setting $w_k = a_k u + \mu_k$, we find $\mu_k = w_k - a_k u$ and

$$\begin{aligned}
J &= -\left[\sum_{k=1}^{K} a_k^2 w_k^{-1}\right] \sum_{k=1}^{K} a_k \left(w_k - a_k u\right) w_k^{-1} + \sum_{k=1}^{K} a_k^2 \left(w_k - a_k u\right) w_k^{-2} \\
&= -\left[\sum_{k=1}^{K} a_k^2 w_k^{-1}\right] \left[1 - \sum_{k=1}^{K} a_k^2 u \omega_k^{-1}\right] + \sum_{k=1}^{K} a_k^2 w_k^{-1} - \sum_{k=1}^{K} a_k^3 u w_k^{-2} \\
&= u \left[\left(\sum_{k=1}^{K} a_k \nu_k\right)^2 - \sum_{k=1}^{K} a_k \nu_k^2\right],
\end{aligned}$$

where $\nu_k = a_k w_k^{-1}$. The last expression is non-positive by virtue of the Schwartz inequality.

By following Lemma 1 of [2], fix some $\gamma_0 > 0$ for which the set $W(s,\gamma) = \{\mu \in \Delta_\varrho^K : |\lambda^* - \mu| \geq \gamma\}$ is non-empty for all $s \in S$, $\gamma \in [0,\gamma_0]$ and define

$$\delta_\varrho(s,\gamma) = \inf\{E[\Theta(s,\kappa,\mu)] : \kappa \in [0,1], \mu \in W(s,\gamma)\}$$

if $\gamma \in [0,\gamma_0]$ and $\delta_\varrho(s,\gamma) = \delta_\varrho(s,\gamma_0)$ if $\gamma > \gamma_0$. Since $E[\Theta(s,\kappa,\mu)]$ is continuous and strictly positive on the compact set $[0,1] \times W(s,\gamma)$ $(\gamma > 0)$, the function $\delta_\varrho(s,\gamma)$ takes on strictly positive values for $\gamma > 0$. Clearly this function is non-decreasing in γ. Fix some s, consider any $\mu \in \Delta_\varrho^K$ and define $\gamma = |\lambda^* - \mu|$. Then we have $\mu \in W(s,\gamma)$, and so

$$E[\Theta(s,\kappa,\mu)] \geq \delta_\varrho(s,\gamma) = \delta_\varrho(s,|\lambda^* - \mu|).$$

□

A.4 Appendix

Proof of Lemma 5.18-(3).

We rewrite the function τ as

$$\tau(r) = \frac{\sum_{k=1}^K \delta_k f_k(r)}{1 - \sum_{k=1}^K \beta_k f(r)},$$

where $\delta_k := \rho\overline{\lambda}_k + (1-\rho) R_k$, $f_k(r) := \frac{\lambda_k r}{\overline{\lambda}_k(1-r)+\lambda_k r}$ and $\beta_k := \rho\left(\lambda_k - \overline{\lambda}_k\right)$. We need to show that

$$\lim_{h\to 0}\left[\frac{\sum_{k=1}^K \delta_k f_k(a+h)}{1-\sum_{k=1}^K \beta_k f_k(a+h)} - \frac{\sum_{k=1}^K \delta_k f_k(a)}{1-\sum_{k=1}^K \beta_k f_k(a)}\right]\frac{1}{h} \tag{A.10}$$

exists. The limit of (A.10) is equivalent to

$$\lim_{h\to 0}\left\{\frac{\sum_{k=1}^K \delta_k f_k(a+h)\left[1-\sum_{k=1}^K \beta_k f_k(a)\right]}{\left[1-\sum_{k=1}^K \beta_k f_k(a+h)\right]\left[1-\sum_{k=1}^K \beta_k f_k(a)\right]}\right.$$

$$- \frac{\sum_{k=1}^{K} \delta_k f_k(a) \left[1 - \sum_{k=1}^{K} \beta_k f_k(a+h)\right]}{\left[1 - \sum_{k=1}^{K} \beta_k f_k(a+h)\right] \left[1 - \sum_{k=1}^{K} \beta_k f_k(a)\right]} \Bigg\} \frac{1}{h}$$

$$= \lim_{h\to 0} \Bigg\{ \frac{\sum_{k=1}^{K} \delta_k \left[f_k(a+h) - f_k(a)\right] + \sum_{k=1}^{K} \delta_k f_k(a) \sum_{k=1}^{K} \beta_k f_k(a+h)}{\left[1 - \sum_{k=1}^{K} \beta_k f_k(a+h)\right] \left[1 - \sum_{k=1}^{K} \beta_k f_k(a)\right]}$$

$$- \frac{\sum_{k=1}^{K} \delta_k f_k(a+h) \sum_{k=1}^{K} \beta_k f_k(a)}{\left[1 - \sum_{k=1}^{K} \beta_k f_k(a+h)\right] \left[1 - \sum_{k=1}^{K} \beta_k f_k(a)\right]} \Bigg\} \frac{1}{h}.$$

When we substitute the function f_k in the above equation, we obtain

$$= \lim_{h\to 0} \Bigg\{ \frac{\sum_{k=1}^{K} \frac{\delta_k \lambda_k \overline{\lambda}_k h}{\left[\overline{\lambda}_k(1-a-h)+\lambda_k(a+h)\right]\left[\overline{\lambda}_k(1-a)+\lambda_k(a)\right]}}{\left[1 - \sum_{k=1}^{K} \beta_k \frac{\lambda_k(a+h)}{\overline{\lambda}_k(1-a-h)+\lambda_k(a+h)}\right] \left[1 - \sum_{k=1}^{K} \beta_k \frac{\lambda_k a}{\overline{\lambda}_k(1-a)+\lambda_k a}\right]}$$

$$+ \frac{\sum_{k=1}^{K} \delta_k \frac{\lambda_k a}{\overline{\lambda}_k(1-a)+\lambda_k a} \sum_{k=1}^{K} \beta_k \frac{\lambda_k(a+h)}{\overline{\lambda}_k(1-a-h)+\lambda_k(a+h)}}{\left[1 - \sum_{k=1}^{K} \beta_k \frac{\lambda_k(a+h)}{\overline{\lambda}_k(1-a-h)+\lambda_k(a+h)}\right] \left[1 - \sum_{k=1}^{K} \beta_k \frac{\lambda_k a}{\overline{\lambda}_k(1-a)+\lambda_k a}\right]}$$

$$- \frac{\sum_{k=1}^{K} \delta_k \frac{\lambda_k(a+h)}{\overline{\lambda}_k(1-a-h)+\lambda_k(a+h)} \sum_{k=1}^{K} \beta_k \frac{\lambda_k a}{\overline{\lambda}_k(1-a)+\lambda_k a}}{\left[1 - \sum_{k=1}^{K} \beta_k \frac{\lambda_k(a+h)}{\overline{\lambda}_k(1-a-h)+\lambda_k(a+h)}\right] \left[1 - \sum_{k=1}^{K} \beta_k \frac{\lambda_k a}{\overline{\lambda}_k(1-a)+\lambda_k a}\right]} \Bigg\} \frac{1}{h}$$

$$= \lim_{h\to 0} \Bigg\{ \frac{\sum_{k=1}^{K} \frac{\delta_k \lambda_k \overline{\lambda}_k h}{\left[\overline{\lambda}_k(1-a-h)+\lambda_k(a+h)\right]\left[\overline{\lambda}_k(1-a)+\lambda_k(a)\right]}}{\left[1 - \sum_{k=1}^{K} \beta_k \frac{\lambda_k(a+h)}{\overline{\lambda}_k(1-a-h)+\lambda_k(a+h)}\right] \left[1 - \sum_{k=1}^{K} \beta_k \frac{\lambda_k a}{\overline{\lambda}_k(1-a)+\lambda_k a}\right]}$$

$$+ \frac{\sum_{k=1}^{K} \sum_{k=1}^{K} \frac{\delta_k \lambda_k a}{\overline{\lambda}_k(1-a)+\lambda_k a} \frac{\beta_k \lambda_k(a+h)}{\overline{\lambda}_k(1-a-h)+\lambda_k(a+h)}}{\left[1 - \sum_{k=1}^{K} \beta_k \frac{\lambda_k(a+h)}{\overline{\lambda}_k(1-a-h)+\lambda_k(a+h)}\right] \left[1 - \sum_{k=1}^{K} \beta_k \frac{\lambda_k a}{\overline{\lambda}_k(1-a)+\lambda_k a}\right]}$$

$$+ \frac{\sum_{k,j=1}^{K} \frac{\delta_k \lambda_k a}{\overline{\lambda}_k(1-a)+\lambda_k a} \frac{\beta_j \lambda_j(a+h)}{\overline{\lambda}_j(1-a-h)+\lambda_j(a+h)}}{\left[1 - \sum_{k=1}^{K} \beta_k \frac{\lambda_k(a+h)}{\overline{\lambda}_k(1-a-h)+\lambda_k(a+h)}\right] \left[1 - \sum_{k=1}^{K} \beta_k \frac{\lambda_k a}{\overline{\lambda}_k(1-a)+\lambda_k a}\right]}$$

$$- \frac{\sum_{k=1}^{K} \sum_{k=1}^{K} \frac{\delta_k \lambda_k a}{\overline{\lambda}_k(1-a)+\lambda_k a} \frac{\beta_k \lambda_k(a+h)}{\overline{\lambda}_k(1-a-h)+\lambda_k(a+h)}}{\left[1 - \sum_{k=1}^{K} \beta_k \frac{\lambda_k(a+h)}{\overline{\lambda}_k(1-a-h)+\lambda_k(a+h)}\right] \left[1 - \sum_{k=1}^{K} \beta_k \frac{\lambda_k a}{\overline{\lambda}_k(1-a)+\lambda_k a}\right]}$$

$$- \frac{\sum_{k,j=1}^{K} \frac{\delta_j \lambda_j a}{\overline{\lambda}_j(1-a)+\lambda_j a} \frac{\beta_k \lambda_k(a+h)}{\overline{\lambda}_k(1-a-h)+\lambda_k(a+h)}}{\left[1 - \sum_{k=1}^{K} \beta_k \frac{\lambda_k(a+h)}{\overline{\lambda}_k(1-a-h)+\lambda_k(a+h)}\right] \left[1 - \sum_{k=1}^{K} \beta_k \frac{\lambda_k a}{\overline{\lambda}_k(1-a)+\lambda_k a}\right]} \Bigg\} \frac{1}{h}$$

$$= \lim_{h\to 0}\left\{ \frac{\sum_{k=1}^{K} \frac{\delta_k \lambda_k \overline{\lambda}_k h}{\left[\overline{\lambda}_k(1-a-h)+\lambda_k(a+h)\right]\left[\overline{\lambda}_k(1-a)+\lambda_k(a)\right]}}{\left[1-\sum_{k=1}^{K}\beta_k \frac{\lambda_k(a+h)}{\overline{\lambda}_k(1-a-h)+\lambda_k(a+h)}\right]\left[1-\sum_{k=1}^{K}\beta_k \frac{\lambda_k a}{\overline{\lambda}_k(1-a)+\lambda_k a}\right]}\right.$$

$$+ \frac{\sum_{k,j=1}^{K} \frac{[\delta_k\lambda_k\beta_j\lambda_j a(a+h)]\left[\left(\overline{\lambda}_k(1-a-h)+\lambda_k(a+h)\right)\left(\overline{\lambda}_j(1-a)+\lambda_j a\right)\right]}{\left(\overline{\lambda}_k(1-a-h)+\lambda_k(a+h)\right)\left(\overline{\lambda}_j(1-a)+\lambda_j a\right)\left(\overline{\lambda}_k(1-a)+\lambda_k a\right)\left(\overline{\lambda}_j(1-a-h)+\lambda_j(a+h)\right)}}{\left[1-\sum_{k=1}^{K}\beta_k \frac{\lambda_k(a+h)}{\overline{\lambda}_k(1-a-h)+\lambda_k(a+h)}\right]\left[1-\sum_{k=1}^{K}\beta_k \frac{\lambda_k a}{\overline{\lambda}_k(1-a)+\lambda_k a}\right]}$$

$$\left. - \frac{\sum_{k,j=1}^{K} \frac{[\delta_k\lambda_k\beta_j\lambda_j a(a+h)]\left[\left(\overline{\lambda}_k(1-a)+\lambda_k a\right)\left(\overline{\lambda}_j(1-a-h)+\lambda_j(a+h)\right)\right]}{\left(\overline{\lambda}_k(1-a-h)+\lambda_k(a+h)\right)\left(\overline{\lambda}_j(1-a)+\lambda_j a\right)\left(\overline{\lambda}_k(1-a)+\lambda_k a\right)\left(\overline{\lambda}_j(1-a-h)+\lambda_j(a+h)\right)}}{\left[1-\sum_{k=1}^{K}\beta_k \frac{\lambda_k(a+h)}{\overline{\lambda}_k(1-a-h)+\lambda_k(a+h)}\right]\left[1-\sum_{k=1}^{K}\beta_k \frac{\lambda_k a}{\overline{\lambda}_k(1-a)+\lambda_k a}\right]}\right\}\frac{1}{h}$$

$$= \lim_{h\to 0}\left\{ \frac{\sum_{k=1}^{K} \frac{\delta_k \lambda_k \overline{\lambda}_k h}{\left[\overline{\lambda}_k(1-a-h)+\lambda_k(a+h)\right]\left[\overline{\lambda}_k(1-a)+\lambda_k(a)\right]}}{\left[1-\sum_{k=1}^{K}\beta_k \frac{\lambda_k(a+h)}{\overline{\lambda}_k(1-a-h)+\lambda_k(a+h)}\right]\left[1-\sum_{k=1}^{K}\beta_k \frac{\lambda_k a}{\overline{\lambda}_k(1-a)+\lambda_k a}\right]}\right.$$

$$\left. + \frac{\sum_{k,j=1}^{K} \frac{\delta_k\lambda_k\beta_j\lambda_j a(a+h)h\left(\lambda_k\overline{\lambda}_j-\overline{\lambda}_k\lambda_j\right)}{\left(\overline{\lambda}_k(1-a-h)+\lambda_k(a+h)\right)\left(\overline{\lambda}_j(1-a)+\lambda_j a\right)\left(\overline{\lambda}_k(1-a)+\lambda_k a\right)\left(\overline{\lambda}_j(1-a-h)+\lambda_j(a+h)\right)}}{\left[1-\sum_{k=1}^{K}\beta_k \frac{\lambda_k(a+h)}{\overline{\lambda}_k(1-a-h)+\lambda_k(a+h)}\right]\left[1-\sum_{k=1}^{K}\beta_k \frac{\lambda_k a}{\overline{\lambda}_k(1-a)+\lambda_k a}\right]}\right\}\frac{1}{h}$$

$$= \lim_{h\to 0}\left\{ \frac{\sum_{k=1}^{K} \frac{\delta_k \lambda_k \overline{\lambda}_k}{\left[\overline{\lambda}_k(1-a-h)+\lambda_k(a+h)\right]\left[\overline{\lambda}_k(1-a)+\lambda_k(a)\right]}}{\left[1-\sum_{k=1}^{K}\beta_k \frac{\lambda_k(a+h)}{\overline{\lambda}_k(1-a-h)+\lambda_k(a+h)}\right]\left[1-\sum_{k=1}^{K}\beta_k \frac{\lambda_k a}{\overline{\lambda}_k(1-a)+\lambda_k a}\right]}\right.$$

$$\left. + \frac{\sum_{k,j=1}^{K} \frac{\delta_k\lambda_k\beta_j\lambda_j a(a+h)\left(\lambda_k\overline{\lambda}_j-\overline{\lambda}_k\lambda_j\right)}{\left(\overline{\lambda}_k(1-a-h)+\lambda_k(a+h)\right)\left(\overline{\lambda}_j(1-a)+\lambda_j a\right)\left(\overline{\lambda}_k(1-a)+\lambda_k a\right)\left(\overline{\lambda}_j(1-a-h)+\lambda_j(a+h)\right)}}{\left[1-\sum_{k=1}^{K}\beta_k \frac{\lambda_k(a+h)}{\overline{\lambda}_k(1-a-h)+\lambda_k(a+h)}\right]\left[1-\sum_{k=1}^{K}\beta_k \frac{\lambda_k a}{\overline{\lambda}_k(1-a)+\lambda_k a}\right]}\right\}.$$

When we take the limit we obtain

$$\tau'(a) = \frac{\sum_{k=1}^{K}\delta_k \frac{\lambda_k\overline{\lambda}_k}{\left[\overline{\lambda}_k(1-a)+\lambda_k a\right]^2} + \sum_{k=1}^{K}\delta_k \frac{\lambda_k a}{\overline{\lambda}_k(1-a)+\lambda_k a}\sum_{k=1}^{K}\beta_k \frac{\lambda_k\overline{\lambda}_k}{\left[\overline{\lambda}_k(1-a)+\lambda_k a\right]^2}}{\left[1-\sum_{k=1}^{K}\beta_k \frac{\lambda_k a}{\overline{\lambda}_k(1-a)+\lambda_k a}\right]^2}$$

$$- \frac{\sum_{k=1}^{K}\delta_k \frac{\lambda_k\overline{\lambda}_k}{\left[\overline{\lambda}_k(1-a)+\lambda_k a\right]^2}\sum_{k=1}^{K}\beta_k \frac{\lambda_k a}{\overline{\lambda}_k(1-a)+\lambda_k a}}{\left[1-\sum_{k=1}^{K}\beta_k \frac{\lambda_k a}{\overline{\lambda}_k(1-a)+\lambda_k a}\right]^2}.$$

The limit in (A.10) exists. Hence τ is differentiable. □

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