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CHARACTERSING THE TEACHING OF UNIVERSITY MATHEMATICS: A CASE OF LINEAR ALGEBRA

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This paper focuses on university mathematics teaching where the topic is linear algebra. The research team includes two mathematics educators and a mathematician who collaborate to study the teaching approach and the issues it raises for teaching-learning at university level. We see university mathematics to constitute a community of practice in which the practitioners are those who do mathematics. Such a perspective draws sociohistorically on established practices in doing, learning and teaching mathematics within a university. The paper offers an interpretation of these theoretical perspectives in relation to a first year course on Linear Algebra. We look at how teaching is constructed within the particular setting, with a critical eye on the learners, on learning outcomes and on the tensions experienced by the lecturer in satisfying student needs and mathematical values.

INTRODUCTION

The research reported in this paper is a case study within a broader project. The project focuses on mathematics teaching at university level and seeks to explore and characterise such teaching. The topic in focus is *Linear Algebra* which is taught as a year-long module in the first year of a three- or four-year undergraduate mathematics programme. In this research we seek to characterise the teaching of linear algebra particularly, and to use this topic to gain insight into university teaching of mathematics more broadly. The case study is collaborative between three researchers: a mathematician who teaches linear algebra and two mathematics educators who observe and analyse. It takes place within a *School of Mathematics* (SoM) which includes a *Mathematics Education Centre* (MEC). Mathematicians and educators (with considerable overlap) teach mathematics and undertake research into mathematics and into mathematics learning and teaching. Research questions include:

1) What is the nature of linear algebra teaching in this module?

2) What issues are raised when *linear algebra teaching* becomes a developmental focus?

THEORETICAL PERSPECTIVES

We draw on social practice theory (Lave & Wenger, 1991) to perceive university mathematics as a community of practice in which the practitioners are those who do mathematics at all levels including students and mathematicians (as in Hemmi, 2006). Within the community, participation varies according to the particular role of the participant. Thus an undergraduate student has a different role from a graduate student which differs from the role of a research mathematician. We draw

2009. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, C. (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 1, pp. XXX-YYY. Thessaloniki, Greece: PME.

particularly on the perspective offered by Wenger (1998) to characterise *identity* within a community of practice and the nature of *belonging* to such a community. The differing identities of mathematicians and students are especially relevant to seeing mathematics learning as a process of enculturation into established practices whose socio-historical dimension extends beyond the particular institution as well as being rooted in local traditions and cultural perspectives (Cole, 1996; Wertsch, 1991). In our case, the goals of a *Mathematics Education Centre* within a *School of Mathematics,* to engage in research into mathematics education at university level, are especially significant for the research we undertake. We seek to know more about the practices in which we and our colleagues engage in order to have the power to develop them in informed ways.

Surveying research into the teaching and learning of linear algebra, Dorier and Sierpinska (2001) write "It is commonly claimed … that linear algebra courses are badly designed and badly taught, and that no matter how it is taught, linear algebra remains a cognitively and conceptually difficult subject" (p. 255). Various authors offer suggestions for teaching in ways that encourage conceptual understanding (e.g. Berry, et al., 2008). We do not have space here, however, to review a considerable literature that recognises issues in linear algebra and ways of addressing the difficulties students face. Here our focus is on how teaching is constructed and how it recognises and takes into account the experiences of students. Unsurprisingly, we come up against student difficulties. Our aim is to characterise teaching and look towards approaches to understanding teaching better in order to address student difficulties and promote teaching development.

METHODOLOGY

Our methodology is *developmental* (Gravemeier, 1994). It includes traditional elements of case study and ethnography but goes further to establish a collaborative process in which insider and outsider researchers both study the practices, processes and issues in mathematics teaching-learning and use the research as a basis for understanding and reconsidering the practices involved (Jaworski, 2003). In this case relationships between insiders and outsiders may be seen to constitute a "clinical partnership" (Wagner, 1997, p. 15) involving inquiry in "jointly defined work" in which activity is open to scrutiny by all. In practice, research team members take differing roles such as the one who (mainly) designs and performs the teaching (the lecturer) and the ones who (mainly) observe and work with the data (the observers).

Data collection includes observation of lectures and tutorials, (audio-recorded), and subsidiary observation of small group tutorials (SGT - involving 6-9 students and tutored by other members of the SoM). A key feature of data collection, fitting with the collaborative spirit of the project, involves recorded team meetings between the three of us immediately after a lecture (standing outside the lecture theatre, walking through the campus, or over lunch), as well as more formal meetings where we sit in an office. These provide insight into established modes of planning and also allow us

to capture an immediacy of perception and to gain insight into issues as they arise. Analysis involves a data reduction process in which qualitative data are summarised, categorised and coded and transcriptions are made of data relating to key elements in categories on which research focuses. Detailed analysis of extracted key elements follows to address particular research questions which evolve alongside analysis. Student perceptions have been sought through two surveys and focus group meetings are planned. The research is ongoing and we report from preliminary analyses.

ELEMENTS OF A MACRO PERSPECTIVE ON TEACHING

We draw here on our data broadly, as well as on administrative information concerning organisation within the SoM and university regulations on teaching and examining. The linear algebra module is one of two year-long modules taken by all first year students in mathematics programmes (the other is calculus). The module is taught over 24 weeks (2 semesters) with two lectures and one tutorial each week and a cohort of 240 students, of which up to 180 attend lectures regularly. One of our team is the lecturer for the first semester; there is a different lecturer in the second semester. The two lecturers collaborate on the year-long design of the module and prepare a joint examination at the end of the year. The first semester offers an introduction to linear algebra and the second semester a more abstract treatment.

In the first semester, the lecturer prepares notes-with-gaps which are placed on LEARN (a virtual learning environment) for students to access in advance of a lecture. A purpose of the ‘gaps’ is to encourage students to attend lectures and complete the notes with solutions of key examples presented in a lecture. The lecturer’s design of the module includes choice, sequencing and written presentation of content, choice of examples, a weekly problem sheet, and preparation of assessment tasks which include on-line tests and tutor-marked coursework. Tutors of the SGT are sent problem sheets and are required to mark coursework. They are also personal tutors for students in their group, so they have access to student progress and student experiences of learning and teaching.

Issues emerging to date involve sequencing of content, choice of examples, emphasis on mathematical language and insight into student understanding. We focus on key elements of lectures and tutorials, including use of examples and metacommenting, and on perceptions of thinking with respect to the teaching of university mathematics and linear algebra in particular. Initial analysis suggests two significant features of this thinking which we refer to as *didactical challenge* and *didactic tension*.

ELEMENTS OF A MICRO PERSPECTIVE ON TEACHING

Analysis here is principally of data from project meetings, backed up by data from lectures and tutorials and minimally from SGTs and two student surveys. The meetings provide an opportunity for the lecturer to talk about his design of the module, his current teaching and perceptions of students’ learning and issues arising thereof. The two observers ask questions and offer observations or perceptions.

The lecturer talking in expository and didactic mode

Typically, in a lecture, the lecturer introduces material and works through examples, following the notes which students are asked to print and bring to a lecture. He *presents* some examples with *mathematical comments* and *metacomments* (see below). With other examples he invites students to tackle the example while he circulates and interacts with some students on the periphery of the lecture theatre. Our observations show that there is a buzz of talk while this happens: some students do not work on the example, rather seeming to wait for the lecturer to resume his exposition; others get involved with the example individually or in small groups.

Discussion in meetings has focused on responses of students to the examples and the lecturer's perception of students' understanding related to the material of the lecture. Often the nature of this discussion includes the lecturer talking about his own conceptions of the material of the lecture, of his didactical thinking with regard to this material, of his perceptions of students' activity and of his decision-making in constructing notes, examples and assessment tasks. The example below, of the lecturer's talk, shows *expository mode* (talking about his own conceptions of the material) in normal text and *didactic mode* (talking about his construction of the teaching of the material) in italic text.

> *Thursday is about defining the characteristic polynomial, understanding that its zeroes are the eigenvalues, and I'll show an example of an eigenvalue that has algebraic and geometric multiplicity 2.* Algebraic multiplicity, meaning this is the power with which the factor lamda minus eigenvalue appears in the characteristic polynomial, and geometric multiplicity is the number of linearly independent eigenvectors. *And these are the important concepts for determining if a matrix is diagonalisable because, for that, we need sufficiently many linearly independent eigenvectors.* Now if an eigenvalue has algebraic multiplicity larger than 1, that means there are correspondingly fewer eigenvalues. So, in principle, we can fail to find as many eigenvectors as we need in that case. On the other hand, if an eigenvector has algebraic multiplicity 3, the geometric multiplicity can be anywhere between 1 and 3. If it's 3, we are fine, if it's less than 3, we're missing out at least one linearly independent eigenvector. And in such a case the matrix would not be diagonalisable. *And that's the big observation that we need to get at next week, that a matrix is diagonalisable if and only if all the geometric multiplicities are equal to the algebraic multiplicities.*

The distinction between expository mode and didactic mode is not clear cut. The sentence in italics in the middle of the quotation, might also be characterised as expository mode. However, it seems here that the lecturer is *meta*-commenting on the material: i.e. expressing his value judgment regarding important concepts that need to be appreciated, rather than just articulating mathematical relationships. This seems to relate to didactic judgments in terms of what needs to be emphasised for students. We observe that such statements in meetings correspond to what we have called meta-comments, or meta-mathematical comments in lectures. Such comments address what students need to attend to, either in terms of their work on the mathematical

content (meta-comments -- A) or of their understanding of the mathematical content (meta-mathematical-comments -- B). Examples A and B follow.

> A: First of all, … if I give you an equation system, this gives you a recipe to decide if that equation system is consistent or inconsistent. You transform it to echelon form and you check if there is such a special row that makes the system inconsistent.

> B: But it's important that you be able to understand the language that we're using and to use it properly. So please, pay attention to the new terms and the new ideas that we're going to introduce over this chapter.

We are emphasising this difference in modes of talk about the material of the module to contrast thinking about teaching (the didactic mode) with thinking about mathematics (expository mode). In meta-comment A, the lecturer draws students' attention to the nature of the mathematics and how they work with it. In met-mathematical comment B, he draws their attention to the processes of working with the mathematics and strategies that can lead to understanding. Both of these are "didactical" approaches on the part of the lecturer.

The lecturer reflecting and raising issues

In our meetings, we discussed frequently the kind of feedback that the lecturer received from students as to their understanding of the module material. In a lecture theatre with 180 students, feedback is not easy to recognise or interpret. For example,

> I do think, however, that didn't go very well because for many students, many students aren't sufficiently familiar with the idea of a linear transformation. We have discussed that many times, that a linear transformation is a function that is defined by a matrix. But my impression is that very many students haven't absorbed that idea of reading a matrix as a function. And whenever I talk about the transformation that is defined by a matrix in a small group tutorial, or when going around in class, quite often I get a blank stare. Now that being as it is there seems not much point in trying to express that function in a different basis, so that is … probably most students haven't really absorbed that section.

What is going on behind the "blank stare" is of course hard to interpret. The lecturer talks of the students having not "absorbed that section", referring to a section of the notes. This raises questions as to what it means for students to "absorb" material, how such absorption is thought to occur, and how the didactical process of module design relates to what students make of what they experience. However, the lecturer has to use whatever clues he can pick up from students. His impression is that students struggle with more conceptual material. Thus, his design of teaching has to take account of such difficulties and what is possible in the time allocation. For example,

> Usually people [lecturers at this level] do diagonalisation of matrices on the level of conjugation with an invertible matrix … and then we try to see if there is such a matrix. The disadvantage for that, I think, is that it's difficult to motivate. On the other hand, if the way to motivate it requires these abstract concepts that are so difficult to get across at this level, then it might well be that's the way to go. I don't know yet what I'm going to do about it next year. But the two alternatives I can see, either I can go back to leaving out the basis expansion stuff and just do conjugation with matrices, without providing

> much motivation at that point, and relying on [the other lecturer] to do more about basis expansions in his part of the course, which he will. The alternative is, I can do a lot more on linear transformations and try to get that concept across, which will require that, somehow, I find quite a bit of space in my module which will be very difficult to do.

Thus there are issues in what to offer and how to offer it that relate to the mathematics, to what the students need and what is possible in the available time. These are familiar issues in linear algebra (see Dorier & Sierpinska, 2001).

Issues in the lecturer's didactical decision-making

Discussion in the team has made clear that the lecturer tries out approaches to his teaching that he has described as "experimenting". An example of an experiment has been to give students some exploratory work to do in a lecture in order to get students to try their own approach before the lecturer offers a more formal explanation. Asked by one of the observers about this experiment, the lecturer replied:

> That's one of my experiments and I think largely it has gone well. At some points I realised I need to find different ways of phrasing the questions in order to make them more accessible. One example of that was the introductory example of, on subspaces, where I had asked students to find solutions to a homogeneous equation system with unknown coefficient matrix, given that they know a couple of solutions that I've given them. That was one question where I saw quite clearly that some of the students found it very easy, and some of the students didn't have the slightest idea even if they tried. And so at that point, because the concepts that come out of this example are so important for everything in Linear Algebra, because they lead to the ideas of linear combinations and linear independence, all that, because that is so important I think it would be good if I could come up with a way to make this example more accessible to students. As it were, to put in a couple of stepping stones for students who can't take it in the way in which I presented that. … And I am not sure how to do that.

This statement indicates that the lecturer is experimenting and (sometimes) being satisfied with the outcomes of his experiments. Experimenting has the additional effect of drawing the lecturer's attention to other issues with which it is not always clear how to deal. He asked the observers on one occasion "have you got any ideas?" thus opening opportunity for didactic discussion and wider consideration of issues.

One area of issues has emerged in recognition of students who can engage with *abstraction* and appreciate *concepts* in abstract relationships and students who remain at a more *computational* level.

> In effect, I'm saying they have mastered the material on that computational level. I am very happy that they have. But of course I would want them to be able to go further than that and put things into context a little bit more. … I had asked students to check if a given number is an eigenvalue of a matrix and if so, find the eigenvectors. And they look up how I did that in the examples and then they know they have to write down 'A minus lamda-I' and put the zero next to it. On the computational level most students can do that but then, of course, the way I would like them to think about it is, I do this calculation because I've got the eigenvalue equation and this is what it means for a number to be an

> eigenvalue, that I check if there is a non-zero vector that satisfies that. My suspicion is that for most students it's 'I do this calculation because the lecturer did that calculation in a similar example', which is again, on the computational level they can do it but most are probably not thinking about that material the way I would like them to think about it.

"The way I would like them to think about it" is on a conceptual or abstract level rather than at the more instrumental or "computational" level. So a didactic challenge for the lecturer is what approach might result in students thinking conceptually. He expressed a tension in thinking about his approach:

> I wouldn't want to go to a system that means we are in effect only teaching the top half of the class that's coming in. On the other hand, I do think we should challenge our students to adopt, well, more abstract, certainly more conceptual, views of things because that's where the power of mathematics comes from, … if you solve the coursework problems and exam problems on the level of 'if the question says this, that's the calculation you do', you have no way of adapting that to even slightly different circumstances than what the standard set of circumstances is, to which the typical exam problem is geared.

In the first sentence above, he made reference to a system in which a course presented at a highly abstract level overall resulted in the drop-out of half the students. Nevertheless, while it seems inappropriate to introduce linear algebra through a largely abstract/conceptual approach, it also seems unsatisfactory if students achieve only computational facility without appreciating conceptual relationships. It may be that the conceptual understanding comes in the second semester, but nevertheless there are issues here for the two lecturers to consider further in their overall design.

The tension highlighted here might be called a *didactic tension* (Mason, 2002), emphasising outcomes in which teaching approaches lead to form rather than substance. Students learn to walk the talk rather than talking the walk. They achieve a form of instrumental facility rather than a relational understanding (Skemp, 1976. These are concerns recognised widely in mathematics education. How they are dealt with at this level however is not well understood.

Concluding remarks

Linear algebra is generally a basic topic in an undergraduate degree programme and there are expectations in university mathematical culture as to what a linear algebra course will achieve in terms of students' knowledge and facility. Conversations within the SoM reveal such expectations and teachers designing a course do so as part of such a cultural position. Teachers and students form identities in relation to established cultures and mathematics learning for students can be seen as enculturation into established practices and ways of being mathematical.

We are trying to make sense of what it means to design and teach at this level, and how design of teaching relates to outcomes for students. Revealing issues as in our examples above enables us to address how such issues are tackled and to open up

dialogue – a teaching discourse – within the mathematical culture. Within our particular environment, a School of Mathematics with a Mathematics Education Centre, we see an essential part of our community of practice to be to encourage such a discourse to enable us to address collectively how we work with students and to develop practice in mathematics teaching. Awareness of didactical challenge and a didactic tension can illuminate practice more broadly. Research such as we describe here starts to open up an inquiry process into the teaching of university mathematics. It brings practitioners together to inquire into issues of mutual concern. Its outcomes both inform and contribute to the development of an inquiry community where research into teaching becomes a regular part of teaching practice and the community becomes more knowledgeable about its teaching (Jaworski, 2008).

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