Realisation of Holonomy Algebras on pseudo-Riemannain Manifolds by means of Manakov Operators

by

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Certificate of originality

This is to certify that I am responsible for the work submitted in this thesis, that the original work is my own except as specified in acknowledgments or in footnotes, and that neither the thesis nor the original work contained therein has been submitted to this or any other institution for a degree.

To my family for their love and unwavering support through the years

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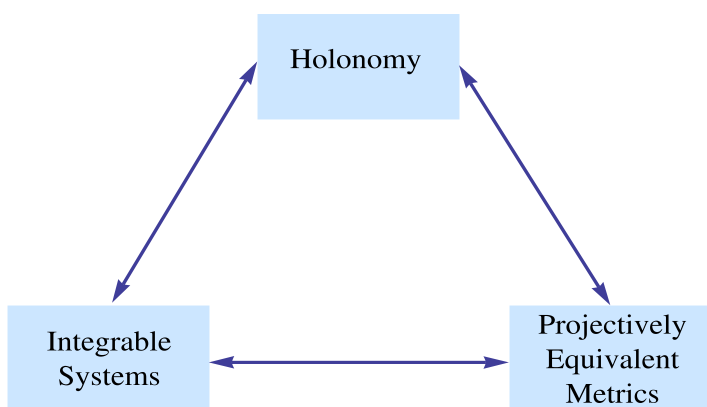
Abstract

In the present thesis we construct a new class of holonomy algebras in pseudo-Riemannian geometry. Starting from a smooth connected manifold M, we consider its $(1, 1)$-tensor fields acting on the tangent spaces. We then prove that there exists a class of pseudo-Riemannian metrics g on M such that the $(1, 1)$-tensor fields are g-self adjoint and their centralisers in the Lie algebra $\mathfrak{so}(g)$ are holonomy algebras for the Levi-Civita connection of g. Our construction is elaborated with the aid of Manakov operators and holds for any signature of the metric g.

Preface

A typical situation in modern mathematics is the following. A given problem can sometimes be better understood and eventually resolved using techniques and tools originating from an area of mathematics that at first glance is quite remote from the area the problem originated in. Thus, it will not be a surprise if this thesis goes precisely in such a direction. Broadly speaking, we can pictorially represent the content of this work with the following simple diagram



The three blocks represent the areas of mathematics to be touched on, while the arrows indicate the "passages" and/or "relations" between them. What we find exciting about this diagram is that while Holonomy and Projectively Equivalent Metrics are inherently present in the realm of Differential Geometry, the theory of Integrable Systems stands as a separate area in mathematics. Incredibly, it is the latter which proves to be of paramount importance for our investigations. It must be emphasised, however, that by no means do we attempt to exhaust any of these three rather vast areas. In this thesis, we shall only discuss a few remarkable relationships between them and show how they yield the solution to a particular problem.

Without further ado, let us briefly comment on the structure of the text. This thesis is, first and foremost, aimed at a broad mathematical audience. On the one hand, it is indeed our utter belief that our approach would catch the eye of an expert and hopefully would be of good use for further research. On the other hand, our desire to write an intelligible account of mathematics does not come secondary. The author has done his very best to find the balance, which would make this text both enjoyable and useful for readers of various backgrounds. The result of this effort is the following.

In Chapter 1 we give a description of the holonomy problem and state the main results to be proven. This chapter is concise and straight to the point. It will be the reader's discretion thereafter how to proceed further. Some readers might wish to skip certain sections, or even chapters, and head straight to the proofs of the genuine results, others would probably need to learn more about the objects involved. Bearing this in mind, as well as our intent to present as self-contained a text as possible, we gently start with the minimal prerequisites. These are briefly discussed in Chapter 2. It is the opinion of the author that this chapter completes the exposition and we also hope that it could be the straw that even a final year undergraduate could clutch at. In Chapter 3 we advance further the discussion on the actual background. Most of it concerns the three aforementioned mathematical areas, which allows us to break this chapter into a few virtually independent sections. Thus, the reader could only read the section(s) of interest. Nevertheless, we recommend some attention is paid to the opening and closing sections of this chapter. While in the former we define the leading character in our story, in the latter we discuss two important relationships between some of these three mathematical areas, which de facto motivates our approach. The content of the remaining chapters mostly constitute the original part of this dissertation. However, in two sections we inevitably include some known material. It is deliberately excluded from the background chapter as being exclusively relevant to those two specific parts in the text. We mean in particular the reduction to nilpotent g-symmetric operators and the covariantly constant linear operators discussed in Sections 4.3 and 5.1, respectively. In the remaining sections of both Chapters 4 and 5 we elaborate the proofs of the main results of our work. We conclude the text by appending a few concrete examples which illustrate the main theorem of Chapter 4. They are represented in such a way that a beautiful pattern is immediately recognisable.

Last, but by no means least, the fluency of the text remains a primary concern of ours. For this reason we have decided to write a brief summary to each chapter. We do hope that this will enhance the reader's navigation throughout the thesis.

Contents

"Truth is ever to be bound in the simplicity, and not in the multiplicity and confusion of things"

Sir Isaac Newton

Chapter 1

Introduction

The goal of this chapter is twofold. Firstly, the holonomy problem is discussed to the extent that an introductory chapter allows. In an attempt to familiarise the reader with the true character of this problem, the author embarks on describing some of its key features. However, by no means could this effort result in anything other than a sketchy survey on this matter. Secondly, and more importantly, the main theorems of the present thesis are stated. Thus, it is at this juncture for the reader to decide whether or not to skip the following Preliminaries and Background chapters.

1.1 The holonomy problem

The notion of *holonomy* has been pervading the realm of differential geometry for almost ninety years now and has had far reaching implications in both mathematics and physics. Beyond a shadow of a doubt, it is already a classical concept in differential geometry. Therefore, it is our first duty to bring into prominence the foremost results on holonomy as well as some of the recent achievements in the field. It should be noticed that, albeit our demand for a self-contained text, in the present section it will only be possible to outline the general framework and briefly trace the history of holonomy. Essentially, we shall only take a panoramic peek at the latter and shall not dwell on any precise definitions.

Some ideas will be discussed in detail later in Section 3.2 where we provide the necessary working knowledge on holonomy for the purposes of our work. It also deserves to be noticed, that we shall only refer to the key papers on holonomy and therefore cannot claim any bibliographical completeness. The reader may take care to consult the books [Ber2], [Bes], [Joy4] and [Sal] for detailed treatments and comprehensive bibliography.

To begin with, let us say a few words about the etymology of holonomy. It is of Greek origin and stems from the words *όλοζ* (pronounced 'olos' and meaning *whole*) and *νόμοζ* (pronounced 'nomos' and meaning *law*). Curiously, the term *holonomic* first appeared in 1894 in a posthumously published work of the German physicist Heinrich Hertz [Her]. His work was on classical mechanics and he spoke about *holonomic constraints* of a given mechanical system.

It was not until much later when the term *holonomy* was used by Élie Cartan in the context of differential geometry. In his works [Car1,Car2,Car3,Car4] dating back to the 1920s he pioneered the study of holonomy. Cartan considered a Riemannian manifold M with Levi-Civita connection ∇ such that $\mathrm{Hol}(\nabla) \subset \mathrm{O}(n)$. He was particularly interested in *symmetric spaces* which are characterised by the invariance of the curvature tensor R. Algebraically, this simply means that the action of the holonomy group on R is trivial. He then proved in [Car4] that for a given symmetric space the holonomy and isotropy group coincide up to connected components which enabled him to classify the irreducible symmetric spaces in the Riemannian case.

It was in the 1950s, however, when other people got interested in holonomy and most of the seminal results were obtained. In the early 1950s the first major contributions appeared in the works of Borel and Lichnerowicz [BL] and Ambrose and Singer [AS]. The upshot of the latter paper was the famous *Ambrose-Singer Holonomy Theorem*. This striking result asserts that the Lie algebra of the holonomy group is generated by the curvature of the connection. We shall come back to this theorem in Section 3.2. In 1952, de Rham [deRha] proved his famous splitting theorem for Riemannian manifolds, which was later generalised by H. Wu for arbitrary pseudo-Riemannian manifolds [Wu]. In order to state and understand this result we need the following brief discussion. Consider the product manifold $\mathrm{M}_1 \times \mathrm{M}_2$ of M_1 and M_2. Then at each point (p_1, p_2) we have the isomorphism $\mathrm{T}_{(p_1,p_2)}\mathrm{M} \cong \mathrm{T}_{p_1}\mathrm{M}_1 \oplus \mathrm{T}_{p_2}\mathrm{M}_2$. Let g_1 and g_2 be Riemannian metrics on M_1 and M_2, respectively. Due to the aforementioned isomorphism it is natural to define the product metric by means of the metric on $\mathrm{T}_{p_1}\mathrm{M}_1 \oplus \mathrm{T}_{p_2}\mathrm{M}_2$. Indeed, we define the product metric $g_1 \times g_2$ on $\mathrm{M}_1 \times \mathrm{M}_2$ by

$$g_1 \times g_2(\xi_1 \oplus \eta_1, \xi_2 \oplus \eta_2) = g_1(\xi_1, \xi_2) + g_2(\eta_1, \eta_2)$$

for all $\xi_1, \xi_2 \in \mathrm{T}_{p_1}\mathrm{M}_1$ and $\eta_1, \eta_2 \in \mathrm{T}_{p_2}\mathrm{M}_2$. We thus equip the product manifold $\mathrm{M}_1 \times \mathrm{M}_2$ with a metric, naturally call it a *Riemannian product* and write $(\mathrm{M}_1 \times \mathrm{M}_2, g_1 \times g_2)$. Now, a Riemannian manifold (M, g) is said to be *reducible* if it is isometric to a Riemannian product $(\mathrm{M}_1 \times \mathrm{M}_2, g_1 \times g_2)$. Further, (M, g) is called *locally reducible* if every point has a reducible open neighbourhood. Finally, we shall call (M, g) *irreducible* if it is not locally reducible. Then, the holonomy group of the product metric $g_1 \times g_2$ is given by the following proposition.

Proposition 1.1.1 *Let* (M_1, g_1) *and* (M_2, g_2) *be Riemannian manifolds. Then the product metric* $g_1 \times g_2$ *has holonomy group* $\mathrm{Hol}(g_1 \times g_2) = \mathrm{Hol}(g_1) \times \mathrm{Hol}(g_2)$.

This proposition is not difficult to proof and naturally motivates the following definition. We call the holonomy group $\mathrm{Hol}_p(M)$ *decomposable* if there is a $\mathrm{Hol}_p(M)$ - invariant decomposition of the tangent space

$$\mathrm{T}_p\mathrm{M} = V_1 \oplus \cdots \oplus V_r$$

with $r \geqslant 2$ and $V_j \neq 0$ for all j. If there is no such a decomposition we call $\mathrm{Hol}_p(M)$ *indecomposable*. Since the holonomy groups are conjugate we immediately observe that (in-)decomposability of the holonomy group is independent of the choice of the point $p \in \mathrm{M}$. We can now state the de Rham-Wu splitting theorem.

Theorem 1.1.2 (de Rham-Wu Splitting Theorem) *Let* (M, g) *be a (pseudo)- Riemannian manifold, and suppose that the holonomy group of its Levi-Civita connection is decomposable. Then locally,* (M, g) *is isometric to a product metric* $(\mathbb{R}^{k_1}, g_1) \times \cdots \times (\mathbb{R}^{k_r}, g_r)$ *with* $k_j = \dim V_j$*, and* $\mathrm{Hol}^0_p(\mathrm{M}) = H_1 \times \cdots \times H_r$ *with* $H_j \subset \mathrm{O}(V_j, g_j)$*. Moreover, if* M *is simply connected and* ∇ *is geodesically complete, then there is a splitting* $(\mathrm{M}, g) = (\mathrm{M}_1, g_1) \times \cdots \times (\mathrm{M}_r, g_r)$*, where the holonomy of* (M_j, g_j) *is* H_j*.*

The holonomy group of a Riemannian manifold (M, g) is always contained in the orthogonal group $\mathrm{O}(n)$ and is therefore compact. Since in this case the indecomposibility is equivalent to irreducibility of the group, the de Rham theorem along with prior works of Cartan necessitated the classification of all irreducible non-symmetric subgroups of $\mathrm{O}(n)$ which are holonomy groups for the manifold (M, g). It was Marcel Berger's pioneering work [Ber1] which not only gave the first classification theorem, but also and more importantly kindled an active quest on this matter. He proved the following theorem, now known as the *Berger's list*.

Theorem 1.1.3 (Berger) *Let* (M, g) *be an irreducible simply-connected Riemannian manifold of dimension* n *which is not locally a symmetric space. Then exactly one of the following cases holds.*

(i) $\mathrm{Hol}(g) = \mathrm{SO}(n)$,

(ii) $n = 2m$ *with* $m \geqslant 2$*, and* $\mathrm{Hol}(g) = \mathrm{U}(m) \subset \mathrm{SO}(2m)$,

(iii) $n = 2m$ *with* $m \geqslant 2$*, and* $\mathrm{Hol}(g) = \mathrm{SU}(m) \subset \mathrm{SO}(2m)$,

(iv) $n = 4m$ *with* $m \geqslant 2$*, and* $\mathrm{Hol}(g) = \mathrm{Sp}(m) \subset \mathrm{SO}(4m)$,

(v) $n = 4m$ *with* $m \geqslant 2$*, and* $\mathrm{Hol}(g) = \mathrm{Sp}(m)\mathrm{Sp}(1) \subset \mathrm{SO}(4m)$*,*

(vi) $n = 7$ *and* $\mathrm{Hol}(g) = G_2 \subset \mathrm{SO}(7)$*, or*

(vii) $n = 8$ *and* $\mathrm{Hol}(g) = \mathrm{Spin}(7) \subset \mathrm{SO}(8)$*.*

Note that this theorem only classified the possible holonomy groups for an irreducible simply-connected Riemannian manifold. It was shown later by others that the groups in the Berger list do occur as holonomy groups. However, this result was a key accomplishment because, above all else, its proof brought about a necessary criterion for a Lie group to be a holonomy group for a given Riemannian manifold. This criterion is a consequence of the Ambrose-Singer holonomy theorem and is presently known as the *Berger criterion.* It can be formulated as the following proposition.

Proposition 1.1.4 (Berger) *Let* $H \subset \mathrm{GL(V)}$ *be a Lie subgroup which occurs as the holonomy group of a torsion free affine connection on some manifold* M*. Then* H *must be a Berger group*[1]*. If the connection is not locally symmetric, then* H *must be a non-symmetric Berger group.*

In 1956, Hano and Ozeki [HO] showed that any (closed) Lie subgroup $H \subset \mathrm{Aut}(V)$ can be realised as the holonomy group of an affine connection (with torsion in general) on some manifold M and therefore no classification was possible. However, it was the *torsion freeness* condition that imposed the non-trivial flavour of the problem and sparkled the so called *Holonomy problem.*

The Holonomy problem: *Consider a finite dimensional vector space* V*. Then, what are the irreducible (closed) Lie subgroups* $H \subset \mathrm{Aut}(V)$ *that can occur as the holonomy group of a torsion free affine connection*?

[1] In this thesis we shall work with Berger algebras which are properly defined in Chapter 3 (see Definition 3.2.7).

Traditionally, this problem is split into two sub-problems. Using Berger's criterion, one first attempts to establish which subgroups $H \subset \mathrm{Aut}(V)$ are Berger groups. Albeit somewhat laborious, this algebraic part of the problem is usually not difficult. Next, it needs to be checked which Berger group can occur as a holonomy group. It is this part of the holonomy problem which is nontrivial. In this thesis we shall follow this approach and we shall deal with a class of Berger algebras in Chapter 4, whereas in Chapter 5 we shall prove that they do occur as holonomy algebras.

Some of the major achievements in the field in the past twenty five years are the following. The holonomy groups G^2 and Spin(7) are called exceptional holonomies as they only occur in dimensions 7 and 8, respectively. Robert Bryant proved locally the existence of metrics with exceptional holonomies [Bry1]. An example of complete metrics with exceptional holonomy [BS] followed shortly afterwards. The compact examples of exceptional holonomy were given by Dominic Joyce [Joy1, Joy2, Joy3]. The irreducible holonomy algebras of torsion free connections which are not necessarily compatible with a metric were classified by S. Merkulov and L. Schwachhöfer [MS, Sch]. In [Ber1], Berger also classified all connected irreducible Berger groups which are subgroups of $\mathrm{SO}(p, q)$. In other words, he gave a list with the candidates for the holonomy group of a pseudo-Riemannian manifold with metric of signature (p, q). The omission and errata in his list were corrected by Bryant [Bry2]. However, this only solved the first part of the holonomy problem as it remained to be shown that all the candidates do occur as holonomy groups. To the author's best knowledge, only the classification of holonomy algebras of Lorenztian manifolds has been settled. For major achievements in the Lorentzian case the reader is referred to the following papers [BI1, Bou, Gal2, Gal3, GL, Ike1]. A recent work by Thomas Leistner [Lei] is widely considered as the culmination of the classification of Lorentzian holonomy groups. While there are a number of results in the non-Lorentzian case, the classification of holonomy algebras of pseudo-Riemannian metrics of arbitrary signature (p,q) is not yet achieved. A striking recent result is the classification of Kählerian holonomies of complex signature $(1,n)$ (or of real signature $(2,2n)$) by Anton Galaev [Gal4]. Further results on signature $(2,n)$ may be found in [Gal1, Ike2] and on the signature (n,n) in [BI2]. The reader may also wish to consult the survey on the recent advances in the theory of holonomy by Bryant [Bry3]. We finish this section by stating a theorem which encompasses the current knowledge of the known classification results for holonomy algebras in the pseudo-Riemannian case.

Theorem 1.1.5 (Berger et al., Leistner) *Let $\mathfrak{g} \subset \mathfrak{so}(V,h)$ be an irreducible Berger algebra where (V,h) is a pseudo-Euclidean vector space. If $\mathfrak{g} \neq \mathfrak{so}(V,h)$ then $\mathfrak{g}$ is the holonomy representation of an irreducible pseudo-Riemannian symmetric space or given by the following list*

$$\mathfrak{u}(r,s), \mathfrak{su}(r,s) \subset \mathfrak{so}(2r,2s),$$

$$\mathfrak{sp}(1) \oplus \mathfrak{sp}(r,s), \mathfrak{sp}(r,s) \subset \mathfrak{so}(4r,4s),$$

$$\mathfrak{so}(r,\mathbb{C}) \subset \mathfrak{so}(r,r),$$

$$\mathfrak{sp}(r,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R}) \subset \mathfrak{so}(2r,2r),$$

$$\mathfrak{sp}(r,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C}) \subset \mathfrak{so}(4r,4r),$$

$$\mathfrak{g}_2 \subset \mathfrak{so}(7),$$

$$\mathfrak{g}_2^{\mathbb{C}} \subset \mathfrak{so}(7,\mathbb{C}) \subset \mathfrak{so}(7,7),$$

$$\mathfrak{g}_2^2 \subset \mathfrak{so}(4,3),$$

$$\mathfrak{spin}(7) \subset \mathfrak{so}(8),$$

$$\mathfrak{spin}(7,\mathbb{C}) \subset \mathfrak{so}(8,\mathbb{C}) \subset \mathfrak{so}(8,8),$$

$$\mathfrak{spin}(4,3) \subset \mathfrak{so}(4,4).$$

1.2 The main results of the thesis

In this brief section we only proclaim the main results of our work. The original ideas in this thesis stem from a joint work with Alexey Bolsinov which has been submitted to the Journal of Differential Geometry as the preprint [BT]. The present document constitutes an extended version of the latter preprint and culminates in the proof of the following theorem.

Theorem A *Let* M *be a smooth manifold,* $p \in \mathrm{M}$ *be a point and* g^0 *be a symmetric non-degenerate bilinear form on* $\mathrm{T}_p\mathrm{M}$ *and* $L_0 : \mathrm{T}_p\mathrm{M} \longrightarrow \mathrm{T}_p\mathrm{M}$ *be a* g^0*-symmetric operator. Then, in a local neighbourhood* U *of* p*, there exist a pseudo-Riemannian metric* g *and a* $(1,1)$*-tensor field* L *such that*
1)$g\mid_{\mathrm{T}_p\mathrm{M}} = g^0$,
2)$L\mid_{\mathrm{T}_p\mathrm{M}} = L_0$,
3)L *is* g*-symmetric,*
4) The centraliser $\mathfrak{g}_L$ *of* L *in the Lie algebra* $\mathfrak{so}(g)$ *is a holonomy algebra for the Levi-Civita connection of the metric* g*.*

Thus, the outcome of our work is an example of a new class of holonomy algebras in pseudo-Riemannian geometry. As the proof of this result is constructive, we end up with explicit pseudo-Riemannian metrics which do realise the Lie algebra $\mathfrak{g}_L$ as their holonomy algebra. We must strictly emphasise at this point that Theorem A is of a local character. Nonetheless, from the perspective of the metric signature it is a result of very general nature as it holds true in any signature of the metric g. The proof of this theorem is given in Chapter 5. Before settling Theorem A, however, we prove the following result.

Theorem B *Let (V, g) be a pseudo-Euclidean vector space and $L : V \longrightarrow V$ be a g-symmetric operator with centraliser $\mathfrak{g}_L$ in $\mathfrak{so}(g)$. Then $\mathfrak{g}_L$ is a Berger algebra.*

In contrast to the former result, Theorem B is of purely algebraic nature. It must be noticed that it is not merely an example of a new class of Berger algebras. As a matter of fact, its proof is of interest in the first place as it promotes techniques from the theory of integrable systems on semisimple Lie algebras. It should also be noted that some of these techniques are readily employed in the proof of Theorem A. Thus, it is this approach we consider the novelty in our work. The proof of Theorem B is elaborated in Chapter 4.

Chapter 2

Preliminaries

The purpose of this chapter is to mention the prerequisites for this thesis and to briefly recapitulate the classical notions such as affine connection, parallel transport, curvature, torsion and pseudo-Riemannian metric. We make no apology for writing it, since all these are fundamental concepts for this thesis and worth mentioning. More importantly, it is the author's belief that the following few pages make the text easily accessible for readers of different mathematical backgrounds.

2.1 The prerequisites

Despite our demand for writing as self-contained a text as possible, the prerequisites are inevitable. Thus, in the lines to follow we shall mention, but not properly define, the minimum prerequisites for this thesis. Apart from the notions to follow, everything else will be properly defined in due course.

- **Manifold.** In this thesis, the letter M (with the exception of Section 3.3) will denote a smooth manifold of dimension n. A local neighbourhood of M will traditionally be denoted U. The local coordinates in U will be denoted u^i for $1 \leqslant i \leqslant n$. $\mathrm{T}_p\mathrm{M}$ is the tangent space of M at the point p. The tangent bundle is $\mathrm{TM} = \cup_{p\in \mathrm{M}} \mathrm{T}_p\mathrm{M}$. We exclusively reserve ξ, η and ζ for the tangent vector fields on M. We shall sometimes think of them as sections of TM and shall write $\xi, \eta, \zeta \in \Gamma(\mathrm{TM})$. Finally, the dual space of $\mathrm{T}_p\mathrm{M}$ is the cotangent space of M at p, denoted $\mathrm{T}_p^*\mathrm{M}$. Then the cotangent bundle of M is denoted $\mathrm{T}^*\mathrm{M} = \cup_{p\in \mathrm{M}} \mathrm{T}_p^*\mathrm{M}$ and its sections are the differential 1-forms on M.

- **Tensor fields.** Let V be an n-dimensional vector space with dual V^*. Then the tensors of (r, s)-type are the (r, s)-linear functions

$$\underbrace{V^* \times \cdots \times V^*}_{r \text{ terms}} \times \underbrace{V \times \cdots \times V}_{s \text{ terms}} \longrightarrow \mathbb{R}.$$

Equivalently, they may be thought of as the elements of the tensor product

$$V_s^r = \underbrace{V \otimes \cdots \otimes V}_{r \text{ terms}} \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_{s \text{ terms}}.$$

If $\{e_i\}_{i=1}^n$ and $\{e^k\}_{k=1}^n$ are the dual bases for V and V^* respectively, then an (r, s)-type tensor A is uniquely expressed as

$$A = A_{k_1\ldots k_s}^{i_1\ldots i_r} e_{i_1} \otimes \cdots \otimes e_{i_r} \otimes e^{k_1} \otimes \cdots \otimes e^{k_s}.$$

- **Bi-vectors.** In this thesis we shall be constantly using the notion of a bi-vector. Recall that this is the anti-symmetric tensor of rank 2 denoted $X = X^{ij} u_i \wedge u_j$, where the $\wedge$ is the usual wedge product. We shall write $\Lambda^2 V$ for the vector space of bi-vectors. Let $e_1, ..., e_n$ be the standard basis of V. Then the standard basis of $\Lambda^2 V$ is the set of bi-vectors $\{e_i \wedge e_j \mid 1 \leqslant i < j \leqslant n\}$ and $\dim \Lambda^2 V = \binom{n}{2}$. We shall sometimes also make an use of 2-forms denoted $X = X_{ij} u^i \wedge u^j$.

- **Lie algebras.** We shall use the standard notation for Lie groups and Lie algebras. Thus, $\mathfrak{so}(n)$ denotes the Lie algebra of skew-symmetric matrices and $[\cdot, \cdot]$ its Lie bracket. For a given element $X \in \mathfrak{so}(n)$ we define the adjoint action of X on $\mathfrak{so}(n)$ as the map $\mathrm{ad}_X : \mathfrak{so}(n) \longrightarrow \mathfrak{so}(n)$ with $\mathrm{ad}_X(Y) = [X, Y]$ for all $Y \in \mathfrak{so}(n)$. Now, by means of the adjoint action we define the Killing form for the Lie algebra $\mathfrak{so}(n)$ by $\mathbf{B}(X, Y) = \mathrm{Tr}\,(\mathrm{ad}_X \mathrm{ad}_Y)$. Recall that the latter is a symmetric bilinear form defining an inner product on the Lie algebra. Amongst its other properties, the Killing form is adjoint-invariant in the sense that

$$\mathbf{B}\Big([X, Y], Z\Big) = -\mathbf{B}\Big(Y, [X, Z]\Big).$$

We shall make a particular use of this property in the Background chapter.

2.2 Affine connections, parallel transport and curvature

In this section we briefly recall the notions of affine connection, parallel transport and curvature and state their most important properties. By virtue of the first two we shall define, in Section 3.2, the notion of a holonomy algebra which is a central concept for this thesis. The notion of curvature, will pervade this thesis due to its intimate relationship with holonomy. In this, as well as in the subsequent section, we shall neither dwell on the details nor the proofs. For a more detailed treatment the reader may refer to the text books [CCL], [DNF] or [Cha].

Affine connections. To be able to develop differential calculus of all orders on a manifold M, one needs to know how to compare its tangent spaces at different points. This comparison is possible in the following sense. We define an *affine connection* on the tangent bundle of M as the map

$$\nabla : \Gamma(\mathrm{TM}) \longrightarrow \Gamma(\mathrm{T^*M} \otimes \mathrm{TM}),$$

such that for all $\xi, \eta_1, \eta_2 \in \Gamma(\mathrm{TM})$ and $\alpha \in C^\infty(\mathrm{M})$ it satisfies

$$\nabla(\eta_1 + \eta_2) = \nabla(\eta_1) + \nabla(\eta_2),$$
$$\nabla(\alpha\xi) = d\alpha \otimes \xi + \alpha \nabla \xi.$$

It is immediately seen that ∇ maps a zero tangent vector to a zero section and that $\nabla(-\xi) = -\nabla\xi$. Thus, ∇ is a linear operator from $\Gamma(\mathrm{TM})$ to $\Gamma(\mathrm{T}^*\mathrm{M} \otimes \mathrm{TM})$. Further, we wish to be able to differentiate the elements of $\Gamma(\mathrm{TM})$. For this purpose, we generalise the classical notion of a directional derivative in the following way. For a fixed vector field $\xi \in \Gamma(\mathrm{TM})$ and an arbitrary vector field $\eta \in \Gamma(\mathrm{TM})$, by means of the standard pairing between TM and T*M, we define

$$\nabla_\xi \eta = \langle \xi, \nabla \eta \rangle. \tag{2.2.1}$$

Clearly, $\nabla_\xi \eta \in \Gamma(\mathrm{TM})$. We call it the *covariant derivative* of the tangent vector η along the tangent vector field ξ. Now, it is not difficult to derive the following properties of the covariant derivative. For all $\xi, \eta \in \Gamma(\mathrm{TM})$, $\zeta, \zeta_1, \zeta_2 \in \Gamma(\mathrm{TM})$ and $f, h \in C^\infty(\mathrm{M})$ we have

$$\nabla_{(f\xi + h\eta)}\zeta = f\nabla_\xi \zeta + h \nabla_\eta \zeta,$$
$$\nabla_\xi(\zeta_1 + \zeta_2) = \nabla_\xi \zeta_1 + \nabla_\xi \zeta_2,$$
$$\nabla_\xi(f\zeta) = (\xi f)\zeta + f\nabla_\xi \zeta.$$

An affine connection is locally characterised through its values on the basis $\frac{\partial}{\partial u^i}$ on $\mathrm{T}_p\mathrm{M}$, that is

$$\nabla_{\frac{\partial}{\partial u^i}} \frac{\partial}{\partial u^j} = \Gamma_{ij}^k \frac{\partial}{\partial u^k}.$$

Note that we have adopted the Einstein summation convention, which will be much exploited throughout the text. The smooth functions Γ_{ij}^{k} are called *the components* of the connection ∇.

Parallel transport. *Parallel transport* is a very important concept in differential geometry. Above all else, it provides an isomorphism between the tangent spaces of M in the following manner. A tangent vector field $\eta \in \Gamma(\mathrm{TM})$ is called *parallel* if $\nabla \eta = 0$[1]. Consider further a parametrised curve $\gamma : [0,1] \longrightarrow \mathrm{M}$ with $\gamma(0) = p$ and $\gamma(1) = q$ on M. Let ξ be a tangent vector field along γ. Then the tangent vector η is called *parallel* along γ if $\nabla_{\xi}\eta = 0$. In a local coordinate neighbourhood U of M we have that $\gamma : u^i = u^i(t)$, $\eta = \lambda^{\alpha}\frac{\partial}{\partial u^{\alpha}}$ and $\xi = \frac{\mathrm{d}u^i}{\mathrm{d}t}\frac{\partial}{\partial u^i}$ for $1 \leqslant i, \alpha \leqslant m$. Then, the fact that η is parallel along the curve γ is tantamount to

$$\langle \xi, \nabla \eta \rangle = \left(\frac{\mathrm{d}\lambda^{\alpha}}{\mathrm{d}t} + \Gamma_{\beta i}^{\alpha}\frac{\mathrm{d}u^i}{\mathrm{d}t}\lambda^{\beta} \right) \frac{\partial}{\partial u^{\alpha}} = 0,$$

which implies

$$\frac{\mathrm{d}\lambda^{\alpha}}{\mathrm{d}t} + \Gamma_{\beta i}^{\alpha}\frac{\mathrm{d}u^i}{\mathrm{d}t}\lambda^{\beta} = 0, \quad 1 \leqslant \alpha \leqslant q.$$

Now, this is a system of ordinary differential equations and therefore possesses a unique solution for any given initial data. Therefore, any vector $v \in \mathrm{T}_p\mathrm{M}$ given at the point p on the curve γ determines uniquely a vector field parallel along γ. It is called the *parallel transport* of v along γ. In Section 3.2 we shall turn our attention back to parallel transport in order to define the notion of *holonomy*.

Curvature. The notion of curvature is intimately related to the one of connection. For any $\xi, \eta \in \Gamma(\mathrm{TM})$ we define $R(\xi, \eta) : \Gamma(\mathrm{TM}) \longrightarrow \Gamma(\mathrm{TM})$ such that

$$R(\xi, \eta) = \nabla_{\xi}\nabla_{\eta} - \nabla_{\eta}\nabla_{\xi} - \nabla_{[\xi,\eta]}. \tag{2.2.2}$$

[1] Notice that while the zero section is trivially parallel, a parallel non-zero section may not exist in general.

It is readily seen that the properties of the covariant derivative imply that $R(\xi,\eta)$ is linear, that is

$$R(\xi,\eta)\Big(f\zeta_1 + h\zeta_2\Big) = fR(\xi,\eta)\zeta_1 + hR(\xi,\eta)\zeta_2$$

for any tangent vector fields $\xi, \eta, \zeta_1, \zeta_2$ and any two smooth functions f and h. It is called the *curvature operator* of the connection ∇. The following properties are also easily derived. For any $\xi, \eta, \zeta \in \Gamma(\mathrm{TM})$ and $f \in C^\infty(\mathrm{M})$

$$R(\xi,\eta) = -R(\eta,\xi),$$

$$R(f\xi,\eta) = f\cdot R(\xi,\eta).$$

We shall say a bit more about curvature in the next section.

2.3 A hint of pseudo-Riemannian geometry

We have already mentioned that the general context of this thesis is pseudo-Riemannian. This necessitates a brief discussion on the basics of pseudo-Riemannian geometry. Let g be a smooth, everywhere non-degenerate symmetric $(0,2)$-type tensor on a smooth manifold M. Then M is called a *pseudo-Riemannian manifold* with metric tensor g and is denoted (M, g). Recall that if we require g be positive definite, then M is called a Riemannian manifold with metric tensor g. At this juncture it must be noted that not all theorems in Riemannian geometry have analogues in the pseudo-Riemannian context. However, the formulae to be discussed below are valid in both cases. Details and proofs may be found in [CCL] and/or [Cha]. Henceforth (M, g) is to be assumed a pseudo-Riemannian manifold.

To comprehend the true character of the metric tensor, recall that as a $(0,2)$-type tensor on M it can be locally written in the form $g = g_{ij}du^i \otimes du^j$ for some $g_{ij} \in C^\infty(U)$. Then it is readily seen that at every point p on M, the metric tensor can in fact be thought of as the bilinear function $g : \mathrm{T}_p\mathrm{M} \times \mathrm{T}_p\mathrm{M} \longrightarrow \mathbb{R}$ defined by $g(\xi, \eta) = g_{ij}(p)\xi^i\eta^j$ for any two vectors $\xi, \eta \in \mathrm{T}_p\mathrm{M}$. We know from the general theory of bilinear forms that a necessary and sufficient condition for g to be non-degenerate at the point p is that $\det\Big(g_{ij}(p)\Big) \neq 0$. Thus, the non-degeneracy of the metric tensor naturally yields the existence of its inverse g^{-1}, which is a symmetric $(2, 0)$-type tensor. In coordinates, we simply write g^{ij} and immediately $g^{ik}g_{kj} = \delta^i_j$ holds true. Thus, the metric tensor and its inverse enable us to lower and raise tensorial indices. More concretely, for an arbitrary tensor $A^{j_1 \cdots j_s}_{i_1 \cdots i_r}$ these two operations are respectively given by

$$g_{lk}A^{kj_2 \dots j_s}_{i_1 \dots . i_r} = A^{j_2 \dots j_s}_{i_1 \dots \dots i_r l} \text{ and } g^{lk}A^{j_1 \dots j_s}_{i_1 \dots . i_{r-1}k} = A^{lj_1 \dots j_s}_{i_1 \dots \dots i_{r-1}}.$$

From these it is not difficult to observe that at each point p on M the metric tensor induces the canonical isomorphism $\mathrm{T}^{r,s}_p\mathrm{M} \cong T^{r+1,s-1}_p\mathrm{M}$. In particular, we have $\mathrm{T}_p\mathrm{M} \cong \mathrm{T}^*_p\mathrm{M}$ and therefore by means of g we can identify vectors with covectors. This fact will prove to be useful for our subsequent considerations.

At this juncture it is worth reminding the reader the notion of a *signature*. Let p, q and r be the number of positive, negative and zero eigenvalues of the metric tensor respectively. Clearly, $p + q + r = \dim\mathrm{M}$. For $q = r = 0$ the metric is called positive definite, or Reimannian. If $r > 0$ the metric is called *degenerate* and therefore is not of interest in this thesis. If both p and q are not zero then the metric signature is called *indefinite* or *pseudo-Riemannian*. The signature is traditionally denoted (p, q), while some authors prefer the more explicit $(+, +, \cdots, +, -, \cdots, -)$. An interesting particular example of pseudo-Riemannian metrics are the *Lorentzian metrics* which have signature $(1, q)$, or equivalently $(p, 1)$.

It has already been mentioned in the introductory chapter, that *torsion-freeness* makes the holonomy problem non-trivial. We therefore need to recollect the notion of a *torsion*.

It is a straightforward verification that the connection components Γ^j_{ik} do not transform as a tensor. Nevertheless, the difference $T^j_{ik} = \Gamma^j_{ki} - \Gamma^j_{ik}$ does transform as a $(1,2)$-type tensor and we can write $T = T^j_{ik}\frac{\partial}{\partial u^j} \otimes \mathrm{d}u^i \otimes \mathrm{d}u^k$. We obviously have that $T^j_{ik} = -T^j_{ki}$. Now, this $(1,2)$-type tensor is called the *torsion* of the connection ∇. Another way of thinking of T, and at times a better one, is as the map

$$T : \Gamma(\mathrm{TM}) \times \Gamma(\mathrm{TM}) \longrightarrow \Gamma(\mathrm{TM})$$

defined by

$$T(\xi, \eta) = \nabla_\xi \eta - \nabla_\eta \xi - [\xi, \eta],$$

for any $\xi, \eta \in \Gamma(\mathrm{TM})$. An affine connection is said to be *torsion free* whenever its torsion tensor is zero. Often torsion-free connections are called *symmetric connections* due to the obvious identity $\Gamma^j_{ki} = \Gamma^j_{ik}$ provided that the torsion tensor is zero.

We are now in a position to recall that the fundamental theorem of (pseudo)-Riemannian geometry asserts that on any (pseudo)-Riemannian manifold (M, g) there exists a unique torsion free connection ∇ which is metric compatible. The metric compatibility condition geometrically means that parallel transports with respect to the aforementioned connection preserve the metric. This "preferred" connection on the (pseudo)-Riemannian manifold is named after Levi-Civita. Henceforth, ∇ will always be assumed to be a Levi-Civita connection. The components Γ^k_{ij} of the Levi-Civita connection are called the *Christoffel symbols* and are nicely given by means of the metric as

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl} \left(\frac{\partial g_{il}}{\partial u^j} + \frac{\partial g_{jl}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^l} \right). \tag{2.3.1}$$

We shall make good use of this formula later in Chapter 5. Similarly, the curvature R of the Levi-Civita connection ∇ is locally written as

$$R\left(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}\right)\frac{\partial}{\partial u^k} = R^l_{ijk}\frac{\partial}{\partial u^l}, \tag{2.3.2}$$

where the components R^l_{ijk} are given by

$$R^l_{ijk} = \frac{\partial \Gamma^l_{ik}}{\partial u^j} - \frac{\partial \Gamma^l_{jk}}{\partial u^i} + \Gamma^m_{ik}\Gamma^l_{jm} - \Gamma^m_{jk}\Gamma^l_{im}. \tag{2.3.3}$$

Traditionally, the curvature tensor R of the Levi-Civita connection on a pseudo-Riemannian manifold (M, g), is called the *Riemann curvature tensor* of (M, g). Sometimes it is convenient to work with the purely covariant version of the Riemann curvature tensor, which is

$$R_{abcd} = g_{a\alpha}R^{\alpha}_{bcd}.$$

We then have the following foremost properties of R_{abcd}.

The skew symmetry : $R_{abcd} = -R_{bacd} = -R_{abdc}$,

The interchange symmetry : $R_{abcd} = R_{cdab}$,

The first Bianchi identity : $R_{abcd} + R_{acdb} + R_{adbc} = 0$.

We are closing this chapter with the following brief recollection. The *geodesics* of an affine connection are defined as the smooth curves whose tangent vectors are parallel along them. Recall that a smooth curve parametrised by the functions $x^{\lambda}(t)$ is a geodesic if and only if the following system of second order differential equations is satisfied

$$\frac{\mathrm{d}^2 u^{\lambda}(t)}{\mathrm{d}t^2} + \Gamma^{\lambda}_{\mu\nu}\frac{\mathrm{d}u^{\mu}(t)}{\mathrm{d}t}\frac{\mathrm{d}u^{\nu}(t)}{\mathrm{d}t} = 0.$$

Chapter 3

The background

The ultimate goal of this chapter is to provide a solid ground, without which this thesis could hardly claim to be self-contained. Fortunately, the first four sections are virtually independent and those already known to the reader can be readily skipped. In the first section we emphasise the practical advantages of pseudo-Euclidean linear algebra for our approach. In particular, we discuss the technicalities to be thoroughly exploited in due course. The principle object of interest, the Lie algebra $\mathfrak{g}_L$, is also introduced. Thus, the reader is strongly recommended to at least glance through this section. We then focus our attention on holonomy and discuss, amongst other things, the Berger algebras and the Berger criterion. These latter two are of seminal importance for this thesis. We next set our sights at the theory of integrable systems on semi-simple Lie algebras. More concretely, we swiftly arrive at the notion of a Manakov operator, which proves to be of foremost importance for our investigations. Incredibly, it is this notion which crucially determines the course of our quest. Finally, after a brief introduction to projectively equivalent metrics given in Section 3.4, we conclude the chapter with a blend of holonomy, integrability and projectively equivalent metrics. This last section constitutes the principal motivation for this work.

3.1 Pseudo-Euclidean linear algebra

To begin with, let us consider a pseudo-Riemannian manifold (M, g) and a linear operator $L : \mathrm{T}_p\mathrm{M} \longrightarrow \mathrm{T}_p\mathrm{M}$. Recall that we define its *g-adjoint* L^* via $g(L\xi, \eta) = g(\xi, L^*\eta)$. Further, we say that L is a *g-symmetric* (also *g-self adjoint*) operator whenever the identity $g(L\xi, \eta) = g(\xi, L\eta)$ holds true for all $\xi, \eta \in \mathrm{T}_p\mathrm{M}$. For the sake of brevity, we shall often just write $L^* = L$. For computational convenience, however, $L^\top g = gL$ will sometimes be preferable. In such a case L and g would stand for the matrices of the linear operator and the metric tensor respectively. Similarly, we define the *g-skew symmetry* property, which in this case reads $g(X\xi, \eta) = -g(\xi, X\eta)$, or $X^* = -X$, or in matrix notation $X^\top g = -gX$.

At this point, two remarks deserve to be stressed. Firstly, the aforementioned definitions remain valid in the more general context, i.e., for any vector space V endowed with a nondegenerate bilinear form $g : V \times V \longrightarrow V$ and a given linear operator $L : V \longrightarrow V$[1]. This, de facto justifies the algebraic flavour of this section. Secondly, we use the same notation for linear operators and their matrices as well as for bilinear forms and their matrices. However, this ambiguity is harmless due to the following result.

Proposition 3.1.1 *Let $L : V \to V$ be a g-symmetric operator. Then there exists a basis in V such that L and g simultaneously reduce to the following block diagonal matrix forms*

$$L = \begin{pmatrix} L_1 & & & \\ & L_2 & & \\ & & \ddots & \\ & & & L_k \end{pmatrix} \quad \textit{and} \quad g = \begin{pmatrix} g_1 & & & \\ & g_2 & & \\ & & \ddots & \\ & & & g_k \end{pmatrix} \tag{3.1.2}$$

[1]Throughout this thesis we shall use both V and $\mathrm{T}_p\mathrm{M}$ and shall bounce between the two assuming the relevant context. It should be evident for the reader that the latter will be used in a geometric context, whereas the former in the more general algebraic one.

where

$$L_i = \begin{pmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda & 1 \\ & & & & \lambda \end{pmatrix} \quad \text{and} \quad g_i = \pm \begin{pmatrix} & & & & 1 \\ & & & 1 & \\ & & \iddots & & \\ & 1 & & & \\ 1 & & & & \end{pmatrix}$$

are square matrices of size $n_i \times n_i$[2]*.*

Clearly reminiscent of the Jordan normal form theorem, this result will play an important role in our work. The reader may care to refer to [LR] and [Tho] for proofs as well as more general treatment on this matter. Henceforth, the special basis from Proposition 3.1.1 will be referred to as the "*canonical basis*", and for computational simplicity we shall assume that the g_is have +1 on their anti-diagonals.

Speaking about a pseudo-Riemannian metric, we fairly naturally think of and somewhat stay attached to its signature (p, q). However, it will prove very useful to our approach if we "forget" about the signature. By "forget" we mean the following. Firstly, since the metric can be thought of as a quadratic form, for our purposes it will suffice to consider the metric as a matrix. Secondly, we recall the following well-known fact from Linear algebra.

Proposition 3.1.3 *Let* $B(x, x) = b_{ij}x_ix_j$ *be a symmetric quadratic form on* n*-dimensional vector space* V*. Then there exists a basis in* V *such that* B *takes the form*

$$y_1^2 + \cdots + y_p^2 - y_{p+1}^2 - \cdots - y_q^2,$$

where $p + q \leqslant n$*. The equality holds true if and only if* $\det(b_{ij}) \neq 0$*.*

[2]Without loss of generality we shall always assume $1 \leq n_1 \leq n_2 \leq \cdots \leq n_k$. The particular case 1×1 matrices $L_i = 0$ and $g_i = \pm 1$ will be perfectly acceptable.

To exemplify the usefulness of this fact, let us consider the following example. Suppose we are given a metric tensor with matrix

$$g_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Write, for the sake of brevity, $g_1 = \textbf{antidiag}\{1,1,1,1,1\}$. This matrix clearly corresponds to the quadratic form $Q = 2x_1x_5 + 2x_2x_4 + x_3^2$. We now use the coordinate change

$$x_1 = \frac{1}{\sqrt{2}}(y_1+y_5);\;\; x_5 = \frac{1}{\sqrt{2}}(y_1-y_5);\;\; x_3 = y_3;\;\; x_4 = \frac{1}{\sqrt{2}}(y_2+y_4) \text{ and } x_2 = \frac{1}{\sqrt{2}}(y_2-y_4)$$

to see that with respect to the new basis we read off $Q' = y_1^2 - y_5^2 + y_3^2 + y_2^2 - y_4^2$. Evidently, we are dealing with metric of a signature $(3,2)$. We now invite the reader to verify that for $n = 4$, the metric tensor $g_2 = \textbf{antidiag}\{1,1,1,1\}$ is of signature $(2,2)$. Now, it is not difficult to perceive the truth of the following fact.

Proposition 3.1.4 *Let (V, g) be an n-dimensional pseudo-Euclidean vector space with metric tensor $g = \textbf{antidiag}\{1,1,...,1\}$. Then the signature of g is given by*

$$(p,q) = \begin{cases} \left(\frac{n}{2}, \frac{n}{2}\right) & \text{if } n \text{ is even,} \\ \left(\frac{n+1}{2}, \frac{n-1}{2}\right) & \text{if } n \text{ is odd.} \end{cases} \tag{3.1.5}$$

One can easily generalise the present situation as follows. Consider two pseudo-Euclidean vector spaces (V_1, g_1) and (V_2, g_2). Suppose that g_1 is of signature (p_1, q_1) and that g_2 is of signature (p_2, q_2). Then we can construct the bigger pseudo-Euclidean space $V = V_1 \oplus V_2$

with metric tensor

$$g = \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix},$$

which will be of signature (p_1+p_2, q_1+q_2). Thus, by "forgetting" the metric signature we simply mean that we shall only consider particular matrix representations for the metric tensor g.

We define the special orthogonal algebra associated to the nondegenerate bilinear form g as the set

$$\mathfrak{so}(g) = \{X \in \mathfrak{gl}(V) \mid X^* = -X\}.$$

In other words, this is the set of all g-skew symmetric endomorphisms of V. This algebra, and especially its elements, will be much exploited within this thesis. For this reason, we shall need some working knowledge of this object. By straightforward computations we summarise it in the following proposition.

Proposition 3.1.6 *The matrix representation of $\mathfrak{so}(g)$ with respect to the canonical basis (Proposition 3.1.1) is given by the block matrix*

$$X = \begin{pmatrix} X_{11} & X_{12} & \cdots & X_{1k} \\ X_{21} & X_{22} & \cdots & X_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ X_{k1} & \cdots & \cdots & X_{kk} \end{pmatrix}$$

with entries satisfying $X_{ji} = -g_j X_{ij}^{\top} g_i$.

Evidently, the relation $X_{ji} = -g_j X_{ij}^{\top} g_i$ readily implies that the diagonal blocks of X are skew-symmetric matrices with respect to their anti-diagonal. As for the off-diagonal entries, we easily perceive the following relation

$$X_{ij} = \begin{pmatrix} x_{11} & \cdots & x_{1n_j} \\ \vdots & \ddots & \vdots \\ x_{n_i1} & \cdots & x_{n_in_j} \end{pmatrix} \Longleftrightarrow \quad X_{ji} = \begin{pmatrix} -x_{n_in_j} & \cdots & -x_{1n_j} \\ \vdots & \ddots & \vdots \\ -x_{n_i1} & \cdots & -x_{11} \end{pmatrix}. \qquad (3.1.7)$$

The moment is now ripe for the following remark. If we bear in mind the signature of the metric we would rather have spoken about the Lie algebra

$$\mathfrak{so}(p,q) = \{X \in \mathfrak{gl}(n,\mathbb{R}) \mid X^\top E_{p,q} + E_{p,q}X = 0\},$$

where $E_{p,q} = \mathrm{diag}(\underbrace{1,...,1}_{p},\underbrace{-1,...,-1}_{q})$. However, this does not represent any different situation as it is not difficult to comprehend the isomorphism $\mathfrak{so}(p,q) \cong \mathfrak{so}(g)$. Yet again our choice to "forget" the metric's signature is being justified. Henceforth we shall only think of and work with $\mathfrak{so}(g)$.

We are now ready to define the principal object of investigation of this thesis. Let L be a linear operator acting on a pseudo-Euclidean vector space (V,g). Then the *centraliser* of L in the Lie algebra $\mathfrak{so}(g)$ is defined as the set

$$\mathfrak{g}_L = \{X \in \mathfrak{so}(g) \mid XL - LX = 0\}.$$

It is not difficult to observe that $\mathfrak{g}_L$ is a Lie subalgebra of $\mathfrak{so}(g)$. This fact along with Proposition 3.1.6 bring about the following proposition.

Proposition 3.1.8 *The matrix representation of $\mathfrak{g}_L$ with respect to the canonical basis*

(Proposition 3.1.1) is given by

$$\begin{pmatrix} 0 & A_{12} & \cdots & A_{1k} \\ A_{21} & 0 & & \vdots \\ \vdots & & \ddots & A_{k-1,k} \\ A_{k1} & \cdots & A_{k,k-1} & 0 \end{pmatrix} \quad with \quad A_{ij} = \begin{pmatrix} 0 & \cdots & 0 & \alpha_1 & \alpha_2 & \cdots & \alpha_{n_i} \\ 0 & \cdots & 0 & 0 & \alpha_1 & \ddots & \vdots \\ \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \alpha_2 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \alpha_1 \end{pmatrix},$$

for all $i < j$ and $a_i \in \mathbb{R}$.

Clearly, $A_{ji} = -g_j A_{ij}^\top g_i$ is satisfied and A_{ij} is a upper-triangular square matrix whenever $n_i = n_j$. Further, we fix i and j, so that $i < j$, and write $\mathfrak{m}_{ij}$ for the subspace of $\mathfrak{g}_L$ consisting of matrices with only non-zero block entries A_{ij} and A_{ji} for fixed $i < j$. Then assuming $n_i < n_j$, Proposition 3.1.8 has the following corollary.

Corollary 3.1.9 *The subspace $\mathfrak{m}_{ij}$ is a commutative subalgebra of $\mathfrak{g}_L$ $(i < j)$ and is of dimension n_i. Furthermore, as a vector space, $\mathfrak{g}_L$ is the direct sum $\sum_{i<j} \mathfrak{m}_{ij}$. In particular,* $\dim \mathfrak{g}_L = \sum_{i=1}^{k} (k - i) n_i$.

This, elementary at first glance, corollary will later play an important role in our considerations.

At this juncture the following remark needs to be stressed. A linear operator L is called *regular* if and only if each of its eigenvalues corresponds to a unique Jordan block. Otherwise, L will be called *singular*. Notice that Proposition 3.1.8 makes sense only for singular g-symmetric operators. If L is a regular g-symmetric operator, then its centraliser in $\mathfrak{so}(g)$ is trivial, i.e., $\mathfrak{g}_L = \{0\}$. Let us demonstrate this fact in one simple example.

Assume that we are given

$$L = \left(\begin{array}{cc|cc} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ \hline 0 & 0 & \mu & 1 \\ 0 & 0 & 0 & \mu \end{array}\right) \text{ and } g = \left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right).$$

Let $X \in \mathfrak{so}(g)$. Then, in order to compute $\mathfrak{g}_L$, we need to solve the matrix equation

$$XL = LX.$$

Using Proposition 3.1.6 we write

$$X = \left(\begin{array}{cc|cc} a & 0 & c & d \\ 0 & -a & e & f \\ \hline -f & -d & b & 0 \\ -e & -c & 0 & -b \end{array}\right)$$

and immediately observe that to find $\mathfrak{g}_L$ we only need to solve, in terms of a, b, c, d, e and f, the following system of linear equations.

$$\begin{cases} \lambda a = a\lambda \\ -a = a \\ \lambda c + e = c\mu \\ \lambda d + f = c + d\mu \end{cases} \quad \begin{cases} 0 = 0 \\ -\lambda a = -\lambda a \\ \lambda e = e\mu \\ \lambda e + f\mu \end{cases} \quad \begin{cases} -f\mu - e = -f\lambda \\ -d\mu - c = -f - d\lambda \\ b\mu = b\mu \\ -b = b \end{cases} \quad \begin{cases} -e\mu = -e\lambda \\ -\mu c = -e - \lambda c \\ 0 = 0 \\ -b\mu = -b\mu \end{cases}$$

Evidently, $a = b = 0$, regardless the values of λ and μ. Furthermore, it is readily seen that $\lambda \neq \mu$ immediately implies that $c = d = e = f = 0$. Thus, from now on we shall only consider singular operators.

Bearing Propositions 3.1.6 and 3.1.8 in mind, we set up the following convenient notation. From now on, by the (2; 2) - *case* we shall understand that L, g, $\mathfrak{so}(g)$ and $\mathfrak{g}_L$ have the following matrix representations.

$$L = \left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad g = \left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$$\mathfrak{so}(g) = \left(\begin{array}{cc|cc} x_1 & 0 & x_2 & x_3 \\ 0 & -x_1 & x_4 & x_5 \\ \hline -x_5 & -x_3 & x_6 & 0 \\ -x_4 & -x_2 & 0 & -x_6 \end{array}\right) \qquad \mathfrak{g}_L = \left(\begin{array}{cc|cc} 0 & 0 & a & b \\ 0 & 0 & 0 & a \\ \hline -a & -b & 0 & 0 \\ 0 & -a & 0 & 0 \end{array}\right)$$

For brevity, we shall occasionally write $L^{(2;2)}$ whenever dealing with this case. Similarly, we shall speak about the $(k;n)$-case, $(k;n;l)$-case and so on. Thus, we shall respectively use the shorthand $L^{(k;n)}$, $L^{(k;n;l)}$, *etc.* Similarly, for the corresponding centralisers we shall write $\mathfrak{g}_L^{(k;n)}$, $\mathfrak{g}_L^{(k;n;l)}$, *etc.*

It is a well-known fact that, for a given vector space V, we naturally identify $\Lambda^2 V$ with $\mathfrak{so}(n)$. We extend it to the following proposition.

Proposition 3.1.10 *For any pseudo-Euclidean vector space (V, g) we have the identification $\Lambda^2 V \cong \mathfrak{so}(g)$.*

Proof. One can easily check that for all $u_i, v_i \in V, i = 1, 2$

$$[u_1 \wedge u_2, v_1 \wedge v_2] = g(u_2, v_1)u_1 \wedge v_2 + g(u_1, v_2)u_2 \wedge v_1 - g(u_2, v_2)u_1 \wedge v_1 - g(u_1, v_1)u_2 \wedge v_2,$$

defines a Lie bracket on $\Lambda^2 V$. Let us consider the map $\varphi : \Lambda^2 V \longrightarrow \mathfrak{so}(g)$ defined by

$u \wedge v \longmapsto u \otimes g(v) - v \otimes g(u)$ for any $u, v \in V$. To prove that it is well-defined it is sufficient to show that $\Big(\varphi(u \wedge v)\Big)^* = -\varphi(u \wedge v)$. This relation is by definition rewritten as $g\Big(\varphi(u \wedge v)a, b\Big) = g\Big(a, -\varphi(u \wedge v)b\Big)$ for all $u, v, a, b \in V$. Thus, writing for brevity $X = \varphi(u \wedge v)$, we compute

$$\begin{aligned} g(Xa, b) &= g\Bigg(\Big(u \otimes g(v) - v \otimes g(u)\Big)a, b\Bigg) = g\Bigg(\Big(u \cdot g(v, a) - v \cdot g(u, a)\Big), b\Bigg) \\ &= g\Big(u \cdot g(v, a), b\Big) - g\Big(v \cdot g(u, a), b\Big) = g(v, a)g(u, b) - g(u, a)g(v, b). \end{aligned}$$

Similarly, $g(Xb, a) = g(v, b)g(u, a) - g(u, b)g(v, a)$ and therefore our map is well-defined. Finally, we compute $\varphi\Big([u_1 \wedge u_2, v_1 \wedge v_2]\Big) = \Big[\varphi(u_1 \wedge u_2), \varphi(v_1 \wedge v_2)\Big]$ and since φ is bijective by definition, we conclude that $\Lambda^2 V \cong \mathfrak{so}(g)$. □

Note that this identification will play a profound role in the sequel. Namely by its virtue, we shall be able to view Manakov operators as formal curvature operators.

3.2 A glimpse of holonomy

This section is aimed at acquainting the reader with the concept of *holonomy*. Alas, by no means could this notion be unveiled in all of its glory within the scope of a doctoral dissertation. Thus, we unavoidably endeavour a rather brief discussion on the minimum background required for our inquiry. More specifically, we define the notion of a *holonomy group* and sketch some of its foremost properties. We thenceforth conclude this section with a definition of Berger algebras, which are of paramount importance for our investigation. The exposition herein is mostly influenced by the monographs [Bes] and [Joy4] to which the reader is referred for a more thorough treatment on this matter. The well-known text books [KN] and [Sal] are highly recommended as well.

Although, the principal inquiry of this thesis is primarily interested in a smooth connected pseudo-Riemannian manifold (M, g) equipped with a Levi-Civita connection ∇, we shall adopt, in this section, the more general language of vector bundles. This, we believe, will not lead to any confusion as the discussion we are about to embark on can be easily given in terms of the tangent bundle. Thus, by the end of this section we shall consider a vector bundle $\mathrm{E} \longrightarrow \mathrm{M}$ over a smooth connected manifold M with connection ∇^{E}. Let $\gamma : [0,1] \longrightarrow \mathrm{M}$ be a smooth curve in M. We know from the theory of vector bundles that the *pull-back* $\gamma^*(\mathrm{E})$ of E to the interval $[0,1]$ is a vector bundle over $[0,1]$ with fibre $\mathrm{E}_{\gamma(t)}$ over the points $t \in [0,1]$, where E_x is the fibre of E over $x \in \mathrm{M}$. To add rigour, note that the connection on the bundle $\gamma^*(\mathrm{E})$ over $[0,1]$ is the pull back of the connection ∇^{E}. Nevertheless, we shall not use different notation for the pull-back connection as it should be clear from the context which connection is being used. We shall write s for the sections of the vector bundle $\gamma^*(\mathrm{E}) \longrightarrow [0,1]$. The values of s lie on the fibres and are denoted $s(t) \in \mathrm{E}_{\gamma(t)}$ for each $t \in [0,1]$. The section s is called *parallel* if for all $t \in [0,1]$ we have $\nabla^{\mathrm{E}}_{\dot{\gamma}} s(t) = 0$, where $\dot{\gamma}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\gamma(t) \in \mathrm{T}_{\gamma(t)}\mathrm{M}$. Now, assuming that $\gamma(0) = x$ and $\gamma(1) = y$, we have that for each $e \in \mathrm{E}_x$ there exists a unique smooth section s of $\gamma^*(\mathrm{E})$ satisfying $\nabla^{\mathrm{E}}_{\dot{\gamma}} s(t) = 0$ for all $t \in [0,1]$ and with $s(0) = e$[3]. In this language the *parallel transport map* along the curve γ is defined as $P_\gamma : \mathrm{E}_x \longrightarrow \mathrm{E}_y$ with $P_\gamma(e) = s(1)$. We are now in a position to establish that the parallel transport map along any piecewise smooth curve is invertible and that a composition of parallel transports along a concatenation of two piecewise smooth curves is a parallel transport as well. For this purpose we assume that for $x, y, z \in \mathrm{M}$, α and β are two piecewise smooth curves in M such that $\alpha(0) = x$, $\alpha(1) = y = \beta(0)$ and $\beta(1) = z$. Then, we define the inverse of the piecewise smooth curve α as

$$\alpha^{-1}(t) = \alpha(1-t),$$

[3] It is indeed obvious as $\nabla^{\mathrm{E}}_{\dot{\gamma}} s(t) = 0$ is a system of first order ordinary differential equations for $s(t)$ and the uniqueness of the solution to its initial value problem is a well-known fact.

and the composition of α and β as

$$\beta\alpha(t) = \begin{cases} \alpha(2t) & \text{if} \quad 0 \leqslant t \leqslant \frac{1}{2}, \\ \beta(2t-1) & \text{if} \quad \frac{1}{2} \leqslant t \leqslant 1. \end{cases}$$

Clearly, α^{-1} and $\beta\alpha$ are piecewise smooth curves in M with $\alpha^{-1}(0) = y$, $\alpha^{-1}(1) = x$, $\beta\alpha(0) = x$ and $\beta\alpha(1) = z$. We can now prove that the parallel transport map is invertible. To do so, we suppose that $e_x \in \mathrm{E}_x$ and $P_\alpha(e_x) = e_y \in \mathrm{E}_y$. There exists a unique parallel section s of $\alpha^{-1}(\mathrm{E})$ with $s(0) = e_x$ and $s(1) = e_y$. It is not difficult to observe that $s'(t) = s(1-t)$ is a parallel section of $(\alpha^{-1})^*(\mathrm{E})$ and since $s'(0) = e_y$ and $s'(1) = e_x$ it immediately follows that $P_{\alpha^{-1}}(e_y) = e_x$. Clearly, the latter justifies the fact that P_α and $P_{\alpha^{-1}}$ are inverse maps. By analogy, we establish the law of composition of parallel transports, namely $P_{\alpha\beta} = P_\alpha \circ P_\beta$. We further write

$$\mathcal{L}_p(\mathrm{M}) = \{\gamma : [0,1] \to \mathrm{M} \mid \gamma(0) = \gamma(1) = p\}$$

for the set of all smooth loops γ on M based at a point p. It is then apparent that $\alpha^{-1}, \alpha\beta \in \mathcal{L}_p(\mathrm{M})$ provided $\alpha, \beta \in \mathcal{L}_p(\mathrm{M})$. From above we know that $P_{\alpha^{-1}} = P_\alpha^{-1}$ and $P_{\alpha\beta} = P_\alpha \circ P_\beta$. Furthermore, the existence of the identity parallel transport as well as the fact that the associativity of "$\circ$" is naturally inherited from the one in $\mathrm{GL}(\mathrm{E}_p)$ is obvious. Thus, these observations bring about the following definition. The set of all parallel transports along all smooth loops based at a point p, is called the *holonomy group* of the connection ∇^{E} at p. Formally, we write

$$\mathrm{Hol}_p(\nabla^{\mathrm{E}}) = \{P_\gamma \mid \gamma \in \mathcal{L}_p(\mathrm{M})\} \subset \mathrm{GL}(\mathrm{E}_p).$$

It naturally raises the question of how does the holonomy change at different points on our manifold? To answer this question we consider a piecewise smooth path $\gamma : [0,1] \longrightarrow \mathrm{M}$ with $\gamma(0) = p$ and $\gamma(1) = q$. Let $P_\gamma : \mathrm{E}_p \longrightarrow \mathrm{E}_q$ be the parallel transport map along γ. Then, if α is a loop based at the point p, then clearly $\gamma\alpha\gamma^{-1}$ is a loop based at the point q. In our notation this simply means that $P_{\gamma\alpha\gamma^{-1}} = P_\gamma \circ P_\alpha \circ P_\gamma^{-1}$. Hence, if $P_\alpha \in \mathrm{Hol}_p(\nabla^{\mathrm{E}})$ then $P_\gamma \circ P_\alpha \circ P_\gamma^{-1} \in \mathrm{Hol}_p(\nabla^{\mathrm{E}})$. We thus perceive that the holonomy groups at different points are isomorphic by conjugation, that is

$$P_\gamma \circ \mathrm{Hol}_p(\nabla^{\mathrm{E}}) \circ P_\gamma^{-1} = \mathrm{Hol}_q(\nabla^{\mathrm{E}}). \tag{3.2.1}$$

Furthermore, suppose that the fibres of E are n-dimensional vector spaces over $\mathbb{R}$. It is then clear that the identification $\mathrm{E}_x \cong \mathbb{R}^n$ induces the isomorphism $\mathrm{GL(E)} \cong \mathrm{GL}(n,\mathbb{R})$. Thus, the holonomy group $\mathrm{Hol}_x(\nabla^{\mathrm{E}})$ may be thought of as a subgroup H of $\mathrm{GL}(n,\mathbb{R})$. If, in addition, we choose another identification of the fibre E_x with $\mathbb{R}^n$, we instead end up with the group aHa^{-1} for some $a \in \mathrm{GL}(n,\mathbb{R})$. We have therefore proven the following important property of the holonomy group.

Proposition 3.2.2 *Let* M *be a connected manifold,* E *a vector bundle over* M *with fibre* $\mathbb{R}^n$*, and* ∇^{E} *a connection on* E*. For each* $p \in \mathrm{M}$*, the holonomy group* $\mathrm{Hol}_p(\nabla^{\mathrm{E}})$ *may be regarded as a subgroup* $\mathrm{GL}(n,\mathbb{R})$ *defined up to conjugation in* $\mathrm{GL}(n,\mathbb{R})$.

This property of the holonomy group tells us that, up to conjugation of groups, $\mathrm{Hol}_p(\nabla^{\mathrm{E}})$ is *independent of the choice of base point* p. Thus, in this sense, the holonomy group may be regarded as a global invariant of the connection. By virtue of this proposition, we may also disregard the subscript x and simply denote the holonomy group of ∇^{E} by $\mathrm{Hol}(\nabla^{\mathrm{E}}) \subset \mathrm{GL}(n,\mathbb{R})$, presuming that two subgroups of $\mathrm{GL}(n,\mathbb{R})$ are equivalent provided they are conjugate in $\mathrm{GL}(n,\mathbb{R})$. Next, assuming that M is simply connected, we have the following result.

Proposition 3.2.3 *Let* M *be a simply-connected manifold,* E *a vector bundle over* M *with fibre* $\mathbb{R}^n$*, and* ∇^{E} *a connection on* E*. Then* $\mathrm{Hol}(\nabla^{\mathrm{E}})$ *is a connected Lie subgroup of* $\mathrm{GL}(n,\mathbb{R})$*.*

We shall not dwell on the proof of this statement, which the reader can find in [Joy4]. Nevertheless, the power of this fact will be used immediately. Indeed, the first question in mind now should be what happens when M is not simply connected? In this case it is convenient to consider the *restricted holonomy group.* To define it, we fix $x \in \mathrm{M}$ and recall that $\gamma \in \mathcal{L}_x(\mathrm{M})$ is called *null-homotopic* if it can be contracted to the constant loop at x, which is the point x itself. We shall denote the set of all null-homotopic loops in M based at the point x as $\mathcal{L}^0_x(\mathrm{M})$. We then define the *restricted holonomy group* $\mathrm{Hol}^0_x(\nabla^{\mathrm{E}})$ of the connection ∇^{E} to be

$$\mathrm{Hol}^0_x(\nabla^{\mathrm{E}}) = \{P_\gamma \,|\, \gamma \in \mathcal{L}^0_x(\mathrm{M})\} \subseteq \mathrm{GL}(\mathrm{E}_x).$$

It immediately follows from this definition that $\mathrm{Hol}^0_x(\nabla^{\mathrm{E}}) \subseteq \mathrm{Hol}_x(\nabla^{\mathrm{E}})$. Also, as earlier, we may regard $\mathrm{Hol}^0_x(\nabla^{\mathrm{E}})$ as a subgroup of $\mathrm{GL}(n,\mathbb{R})$ which is independent of the base point x, and may omit the subscript x and write $\mathrm{Hol}^0(\nabla^{\mathrm{E}})$. The most important properties of the restricted holonomy group are given in the following proposition.

Proposition 3.2.4 *Let* M *be a connected manifold,* E *a vector bundle over* M *with fibre* $\mathbb{R}^n$*, and* ∇^{E} *a connection on* E*. Then* $\mathrm{Hol}^0(\nabla^{\mathrm{E}})$ *is a connected Lie subgroup of* $\mathrm{GL}(n,\mathbb{R})$*. It is the connected component of the identity of* $\mathrm{Hol}(\nabla^{\mathrm{E}})$*. Moreover,* $\mathrm{Hol}^0(\nabla^{\mathrm{E}})$ *is a normal subgroup of* $\mathrm{Hol}(\nabla^{\mathrm{E}})$ *and there is a natural, surjective group homomorphism*

$$\phi : \pi_1(\mathrm{M}) \longrightarrow \mathrm{Hol}(\nabla^{\mathrm{E}})/\mathrm{Hol}^0(\nabla^{\mathrm{E}}),$$

where $\pi_1(\mathrm{M})$ *is the fundamental group of* M*.*

Proof. The fact that $\mathrm{Hol}^0(\nabla^E)$ is a connected Lie subgroup of $\mathrm{GL}(n, \mathbb{R})$ is immediately guaranteed by Proposition 3.2.3. Now, fix $x \in \mathrm{M}$ and let $\alpha \in \mathcal{L}_x(\mathrm{M})$, $\beta \in \mathcal{L}^0_x(\mathrm{M})$. It is then not difficult to observe that $\alpha\beta\alpha^{-1} \in \mathcal{L}^0_x(\mathrm{M})$. This ensures that $P_{\alpha\beta\alpha^{-1}} = P_\alpha P_\beta P_{\alpha^{-1}}$ lies in $\mathrm{Hol}^0(\nabla^E)$ for $P_\alpha \in \mathrm{Hol}(\nabla^E)$ and $P_\beta \in \mathrm{Hol}^0(\nabla^E)$. Thus, the restricted holonomy group $\mathrm{Hol}^0(\nabla^E)$ is a normal subgroup of $\mathrm{Hol}(\nabla^E)$. Next we consider $\gamma \in \mathcal{L}_x(\mathrm{M})$ and write $[\gamma]$ for its corresponding element of the fundamental group $\pi_1(\mathrm{M})$. We then define the map

$$\begin{aligned} \phi : \pi_1(\mathrm{M}) &\longrightarrow \mathrm{Hol}(\nabla^{\mathrm{E}}) \big/ \mathrm{Hol}^0(\nabla^{\mathrm{E}}) \\ [\gamma] &\mapsto P_\gamma \cdot \mathrm{Hol}^0(\nabla^{\mathrm{E}}). \end{aligned} \tag{3.2.5}$$

It is clear that (3.2.5) is by definition surjective. Since $P^{-1}_{\gamma_2} \cdot \mathrm{Hol}^0(\nabla^{\mathrm{E}}) \cdot P_{\gamma_2} = \mathrm{Hol}^0(\nabla^{\mathrm{E}})$, for any two $[\gamma_1], [\gamma_2] \in \pi_1(\mathrm{M})$ we compute

$$\begin{aligned} \phi([\gamma_1][\gamma_2]) &= \phi([\gamma_1])\phi([\gamma_2]) = P_{\gamma_1} \cdot \mathrm{Hol}^0(\nabla^{\mathrm{E}}) \cdot P_{\gamma_2} \cdot \mathrm{Hol}^0(\nabla^{\mathrm{E}}) \\ &= P_{\gamma_1} \cdot P_{\gamma_2} \cdot P^{-1}_{\gamma_2} \cdot \mathrm{Hol}^0(\nabla^{\mathrm{E}}) \cdot P_{\gamma_2} \cdot \mathrm{Hol}^0(\nabla^{\mathrm{E}}) \\ &= P_{\gamma_1} \cdot P_{\gamma_2} \cdot \mathrm{Hol}^0(\nabla^{\mathrm{E}}) = P_{\gamma_1\gamma_2} \cdot \mathrm{Hol}^0(\nabla^{\mathrm{E}}), \end{aligned}$$

and therefore (3.2.5) is a group homomorphism. At this point we recall the fact that the fundamental group $\pi_1(\mathrm{M})$ is countable. Then, the surjective homomorphism (3.2.5) implies that the quotient group $\mathrm{Hol}(\nabla^{\mathrm{E}}) \big/ \mathrm{Hol}^0(\nabla^{\mathrm{E}})$ is countable too. Hence the restricted holonomy group $\mathrm{Hol}^0(\nabla^{\mathrm{E}})$ is the connected component of $\mathrm{Hol}(\nabla^{\mathrm{E}})$ containing the identity. □

Notice that as an immediate corollary we have that if M is a simply-connected manifold then $\mathrm{Hol}(\nabla^{\mathrm{E}}) = \mathrm{Hol}^0(\nabla^{\mathrm{E}})$. However, the most important part of this proposition for the purposes of this section is the fact that $\mathrm{Hol}^0(\nabla^{\mathrm{E}})$ is the connected component of the identity of $\mathrm{Hol}(\nabla^{\mathrm{E}})$. This enables us to pass from Lie groups to Lie algebras. We define the *holonomy algebra* $\mathfrak{hol}_x(\nabla^{\mathrm{E}})$ of the connection ∇^{E} to be the Lie algebra of $\mathrm{Hol}^0(\nabla^{\mathrm{E}})$. It is a Lie subalgebra of the endomorphisms $\mathrm{End}(\mathrm{E}_x)$ of the fibre E_x. Nevertheless, in analogy to holonomy groups, it will be convenient to assume $\mathrm{E}_x \cong \mathbb{R}^n$ and therefore the holonomy algebra $\mathfrak{hol}(\nabla^{\mathrm{E}})$ of the restricted holonomy group will be regarded as a Lie subalgebra of $\mathfrak{gl}(n, \mathbb{R})$ defined up to the adjoint action of $\mathrm{GL}(n, \mathbb{R})$. It deserves to be noticed at this juncture that the Lie algebras of $\mathrm{Hol}^0(\nabla^{\mathrm{E}})$ and $\mathrm{Hol}(\nabla^{\mathrm{E}})$ coincide since the former is the identity component of the latter. For computational reasons, in this thesis holonomy algebras will be preferred. It now raises the question: *How do we compute holonomy*? The answer is given by the famous Ambrose-Singer holonomy theorem [AS]. We need the following brief discussion before stating the theorem. The definition of curvature for a connection of a vector bundle is not any different than the one we already mentioned for the case of a tangent bundle (see Chapter 2). Namely, for a connection ∇^{E} on a vector bundle $\mathrm{E} \longrightarrow \mathrm{M}$ there exists a unique 2-form $R(\nabla^{\mathrm{E}})$ such that it defines the multilinear map $R(\mathrm{E}) : \Gamma(\mathrm{TM}) \times \Gamma(\mathrm{TM}) \times \Gamma(\mathrm{E}) \longrightarrow \Gamma(\mathrm{E})$ given by $R(\nabla^{\mathrm{E}})(\xi, \eta)\sigma = \nabla_\xi \nabla_\eta \sigma - \nabla_\eta \nabla_\xi \sigma - \nabla_{[\xi,\eta]}\sigma$. Notice that the values of $R(\nabla^{\mathrm{E}})$ are in the endomorphism bundle $\mathrm{End}(\mathrm{E}) = \mathrm{E} \otimes \mathrm{E}^*$, that is $R(\nabla^{\mathrm{E}}) \in \mathrm{End}(\mathrm{E}) \otimes \Lambda^2\mathrm{TM}^*$. Now, for a given point $x \in \mathrm{M}$ we consider a piecewise smooth curve $\gamma : [0,1] \longrightarrow \mathrm{M}$ with $\gamma(0) = x$ and $\gamma(1) = y$ and the parallel transport map $P_\gamma : \mathrm{E}_x \longrightarrow \mathrm{E}_y$. For $\xi, \eta \in \mathrm{T}_x\mathrm{M}$ we set $R_\gamma(\xi, \eta) = P_\gamma^{-1} \circ R(\nabla^{\mathrm{E}})\Big(P_\gamma \xi, P_\gamma \eta\Big) \circ P_\gamma$ which is clearly an endomorphism of the fibre E_x. With this notation in mind we can now state the Ambrose-Singer holonomy theorem.

Theorem 3.2.6 (Ambrose - Singer Holonomy Theorem) *Let* M *be a manifold,* E *a vector bundle over* M*, and* ∇^{E} *a connection on* E*. Fix* $x \in \mathrm{M}$*, so that* $\mathfrak{hol}_x(\nabla^{\mathrm{E}})$ *is a Lie subalgebra of* $\mathrm{End}(\mathrm{E}_x)$*. Then* $\mathfrak{hol}_x(\nabla^{\mathrm{E}})$ *is the vector subspace of* $\mathrm{End}(\mathrm{E}_x)$ *spanned by the elements of* $R_\gamma(\xi, \eta)$ *for all piecewise smooth curves* γ*.*

This theorem tells us that the curvature $R(\nabla^{\mathrm{E}})$ determines the holonomy algebra $\mathfrak{hol}_x(\nabla^{\mathrm{E}})$

and hence the restricted holonomy group $\mathrm{Hol}^0(\nabla^{\mathrm{E}})$. For instance, if ∇^{E} is flat, then $R(\nabla^{\mathrm{E}}) = 0$. This implies $\mathfrak{hol}_x(\nabla^{\mathrm{E}}) = \{0\}$, which is $\mathrm{Hol}^0(\nabla^{\mathrm{E}}) = \{1\}$.

At this juncture we arrive at the definition of a Berger algebra. Let us first remind the reader that a map $R : \Lambda^2 V \to \mathfrak{gl}(V)$ is called a *formal curvature tensor* if it satisfies the Bianchi identity, which is

$$R(u \wedge v)w + R(v \wedge w)u + R(w \wedge u)v = 0 \qquad \text{for all } u, v, w \in V.$$

This definition simply means that R, viewed as a tensor of type $(1, 3)$, satisfies all the algebraic properties of a curvature tensor of a torsion free connection. According to the Ambrose-Singer theorem the Lie algebra of the holonomy group is generated by the operators of the form $R(u \wedge v)$. This motivates the following definition.

Definition 3.2.7 *Let $\mathfrak{h} \subset \mathfrak{gl}(V)$ be a Lie subalgebra. Consider the set of all formal curvature tensors $R : \Lambda^2 V \to \mathfrak{gl}(V)$ such that $\operatorname{Im} R \subset \mathfrak{h}$:*

$$\mathcal{R}(\mathfrak{h}) = \{R : \Lambda^2 V \to \mathfrak{h} \mid R(u \wedge v)w + R(v \wedge w)u + R(w \wedge u)v = 0,\ u, v, w \in V\}.$$

We say that $\mathfrak{h}$ is a Berger algebra if it is generated as a vector space by the images of the formal curvature tensors $R \in \mathcal{R}(\mathfrak{h})$, which is $\mathfrak{h} = \mathrm{span}\{R(u \wedge v) \mid R \in \mathcal{R}(\mathfrak{h}),\ u, v \in V\}$.

The Berger's criterion stated in Chapter 1 in this case can be reformulated as follows. Let ∇ be a Levi-Civita connection on M. Then the Lie algebra $\mathfrak{hol}\,(\nabla)$ of its holonomy group $\mathrm{Hol}\,(\nabla)$ is a *Berger algebra.*

3.3 A scent of integrability: Manakov operators

In this section we shall only take a peek at the vast field of integrable systems. Our goal is to acquaint the reader with *Manakov operators* which play a profound role in this dissertation. We shall give one particular example of a Manakov operator that will be much exploited in the next two chapters.

First and foremost, it deserves to be noticed that not only are Manakov operators interesting in their own right, but they also play an important role in the theory of integrable systems. For this reason we briefly mention a few words about their origin. It all goes back to classical mechanics and the problem of describing the motion of a three-dimensional rigid body around a fixed point. The dynamics of such a body is governed by a system of six first order ordinary differential equations, which are collectively known as the *Euler-Poisson equations.* Their integration turned out to be rather nontrivial and this resulted in the birth of the theory of Hamiltonian dynamical systems and Liouville integrability[4]. We have neither the time nor the room for a thorough discussion and to cut a long story short, let us consider the special case of motion of a three-dimensional rigid body fixed at its centre of mass. Now, the dynamics is described by the system of differential equations

$$\dot{x} = \frac{\lambda_1 - \lambda_2}{\lambda_1 + \lambda_2} yz, \quad \dot{y} = \frac{\lambda_3 - \lambda_1}{\lambda_3 + \lambda_1} xz, \quad \dot{z} = \frac{\lambda_2 - \lambda_3}{\lambda_2 + \lambda_3} xy, \tag{3.3.1}$$

known as the *Euler equations.* It is a remarkable fact that the system (3.3.1) has an intimate relationship with the three dimensional Lie algebra $\mathfrak{so}(3)$ of the rotation group SO(3). Indeed, we observe that by identifying the vectors $(x, y, z) \in \mathbb{R}^3$ with the skew-symmetric matrices

$$X = \begin{pmatrix} 0 & x & y \\ -x & 0 & z \\ -y & -z & 0 \end{pmatrix},$$

we comprehend the isomorphism $\mathbb{R}^3 \cong \mathfrak{so}(3)$. Clearly, by virtue of this observation one

[4]Not to be confused with the notion of Frobenius integrability which deals with overdetermined differential systems.

may neatly rewrite the Euler equations (3.3.1) in a matrix form. More importantly, this situation naturally generalises to the n-dimensional case and initiates the study of Hamiltonian systems on suitable Lie algebras. Indeed, following the famous two-page paper of Sergey Manakov [Man], we consider a linear operator $A : \mathfrak{so}(n) \longrightarrow \mathfrak{so}(n)$. Then, it can be shown that writing $A\Omega = \mathbf{M}$ for $\Omega \in \mathfrak{so}(n)$, the Euler equations are given by the following matrix equation

$$\dot{\mathbf{M}} = [\mathbf{M}, \Omega]. \tag{3.3.2}$$

Manakov confined himself to the case of free rotation of a multidimensional body. In that case we have that

$$\mathbf{M} = J\Omega + \Omega J,$$

where J is symmetric positive-definite matrix and is called the *inertia tensor*. Thus, the equation (3.3.2) can be rewritten in the form

$$J\dot{\Omega} + \dot{\Omega}J = [J, \Omega^2]. \tag{3.3.3}$$

Under these assumptions, Manakov proved that for any finite dimension n the equation (3.3.3) has

$$N(n) = \frac{1}{2}\left[\frac{n}{2}\right] + \frac{n(n-1)}{2}$$

single-valued integrals of motion and that its general solution is expressible in terms of θ-functions on Riemann surfaces. This was a remarkable result published in a remarkably short paper. For our purposes, however, the consequences of this paper are more important. The work of Manakov swiftly resulted in the introduction of one important class of operators - *the Manakov operators*. We say that a linear map $R : \mathfrak{so}(n) \longrightarrow \mathfrak{so}(n)$ is a *Manakov operator*, if R is self-adjoint with respect to the Killing form and satisfies the algebraic identity

$$[R(X), L] = [X, M], \tag{3.3.4}$$

for all $X \in \mathfrak{so}(n)$ and some fixed nonzero symmetric matrices L and M.[5] Shortly after his paper, Manakov's ideas were further developed by A. Mischenko and A. Fomenko [MF] who proved the following important result. Before stating it, we wish to remind the reader that $\mathbf{B}(\cdot,\cdot)$ denotes the Killing form of the Lie algebra $\mathfrak{so}(n)$.

Theorem 3.3.5 (Manakov, Mischenko & Fomenko) *Let $R : \mathfrak{so}(n) \longrightarrow \mathfrak{so}(n)$ be a Manakov operator and let $H = \frac{1}{2}\mathbf{B}(R(X), X)$ be a Hamiltonian on $\mathfrak{so}(n)$. Then the Euler equations on $\mathfrak{so}(n)$ have the form*

$$\frac{\mathrm{d}X}{\mathrm{d}t} = [R(X), X], \tag{3.3.6}$$

admit the following Lax representation with a spectral parameter λ

$$\frac{\mathrm{d}}{\mathrm{d}t}(X + \lambda L) = [R(X) + \lambda M, X + \lambda L]$$

and therefore possess first integrals of the form $\mathrm{Tr}\,(X + \lambda L)^k$. These integrals commute and, if L is regular, form a complete family in involution so that the Euler equations (3.3.6) *are completely integrable.*

The following remark needs to be noticed at this point. The form of the Euler equations (3.3.6) is not accidental. A general picture in the theory of Hamiltonian dynamics on semi-simple Lie algebras is the following. A real valued function $H : \mathfrak{g} \longrightarrow \mathbb{R}$ on a semisimple Lie algebra $\mathfrak{g}$ is called a *Hamiltonian*. Since the Killing form $\mathbf{B}(\cdot,\cdot)$ on $\mathfrak{g}$ induces an isomorphism $\mathfrak{g} \cong \mathfrak{g}^*$, we have that $\mathrm{d}H(X) \in \mathfrak{g}$ for all $X \in \mathfrak{g}$. Then, it can be shown that the Euler equations have the form

[5]Notice that M and $\mathbf{M}$ bear absolutely different meanings.

$$\dot{X} = [\mathrm{d}H(X), X]. \tag{3.3.7}$$

More importantly, Theorem 3.3.5 was generalised for arbitrary semisimple Lie algebras by Mischenko and Fomenko [MF]. We now know from their work that if $\mathfrak{so}(n)$ is replaced by $\mathfrak{so}(p,q)$ then the construction above essentially remains the same. Thus, for the purposes of the present work, we shall think of Manakov operators as the maps $R : \mathfrak{so}(g) \longrightarrow \mathfrak{so}(g)$, which are self-adjoint with respect to the Killing form on $\mathfrak{so}(g)$ and obeying the algebraic identity (3.3.4). We shall not go any further or deeper into the theory of integrable systems. The reader may consult the books [FT] and [Fom] for more details.

Looking yet again at the algebraic identity (3.3.4), we now wish to derive an explicit formula for $R(X)$. We first observe that the adjoint invariance of the Killing form on $\mathfrak{so}(g)$ and the identity (3.3.4) imply $[M, L] = 0$. This is immediately seen from the following simple computation. For all $X \in \mathfrak{so}(n)$ we have

$$\begin{aligned}
\mathbf{B}\Big([M,L],X\Big) &= -\mathbf{B}\Big(L,[M,X]\Big) = \mathbf{B}\Big(L,[X,M]\Big) \\
&= \mathbf{B}\Big(L,[R(X),L]\Big) = -\mathbf{B}\Big(L,[L,R(X)]\Big) \\
&= \mathbf{B}\Big([L,L],R(X)\Big) \equiv 0.
\end{aligned}$$

Thus, M can be represented as a polynomial of L. Writing $M = p(L)$ we are motivated to define the map $R : \mathfrak{so}(g) \longrightarrow \mathfrak{so}(g)$ given by the formula[6]

$$R(X) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} p\big(L + tX\big). \tag{3.3.8}$$

We shall now show that this map satisfies the identity (3.3.4). To see this, it is sufficient

[6]Notice that this formula represents only one particular solution of the algebraic identity (3.3.4).

to consider the obvious formula

$$[p(L + tX), L + tX] = 0.$$

Differentiating it with respect to t we quickly get

$$\left[\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} p(L + tX), L\right] + [p(L), X] = 0,$$

which is $[R(X), L] + [M, X] = 0$, as required. We are thus in possession of an algebraic formula defining a Manakov operator. It is this formula (3.3.8) which introduces to holonomy theory a new method of constructing pseudo-Riemannian metrics of arbitrary signature with a given holonomy algebra $\mathfrak{g}_L$.

At this juncture, as we leave this section, we shall re-enter the field of differential geometry but from a rather unusual perspective. We wish to remind the reader that the formal curvature tensor for the Lie algebra $\mathfrak{so}(g)$ is the map $R : \Lambda^2 V \longrightarrow \mathfrak{so}(g)$ satisfying the Bianchi identity. Now, due to the identification of $\Lambda^2 V$ with $\mathfrak{so}(g)$, we can think of the formal curvature tensor as the map $R : \mathfrak{so}(g) \longrightarrow \mathfrak{so}(g)$. Further, the symmetry property $R_{ij,kl} = R_{kl,ij}$ immediately implies that R is self-adjoint with respect to the Killing form on $\mathfrak{so}(g)$. This means that there is a good chance that Manakov operators relate to curvature in a nice way. We thus naturally raise the question: *Do sectional operators satisfy the Bianchi identity*? In Section 4.2 we shall see that the Manakov operator defined by (3.3.8) does satisfy the Bianchi identity.

3.4 Projectively equivalent metrics

In this section we shift our attention to the theory of projectively equivalent metrics. Although these will not explicitly be used in the subsequent chapters, they deserve a brief mention. The reason for this is that the principle motivation for this work stems from

Theorem 3.5.8 (see next section), which blends together projectively equivalent metrics with Manakov operators.

To begin our discussion we let (M, g) be a (pseudo)-Riemannian manifold of dimension $n \geqslant 2$. Let us also choose another metric $\tilde{g}$ on M and look at the geodesics of both metrics g and $\tilde{g}$ as unparametrised curves. If the geodesics of these two metrics coincide, then the metric $\tilde{g}$ is said to be *projectively equivalent* to g. Furthermore, we shall call g and $\tilde{g}$ *affinely equivalent* whenever their geodesics coincide as parametrised curves. To secure a greater clearness of view, we offer one basic example due to Beltrami [Bel]. It possibly stands as the very first example known in this area. Let us consider the half-sphere S^2 and the Euclidean plane E^2, respectively defined by

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1,\ z < 0\} \text{ and } E^2 = \{(x, y, z) \in \mathbb{R}^3 \mid z = -1\}.$$

Traditionally, S^2 is centred at the origin $O = (0, 0, 0)$ of some Cartesian coordinate system in $\mathbb{R}^3$. We define, in the usual manner, a stereographic projection $f : S^2 \longrightarrow E^2$ with respect to the origin O. Geometrically speaking, every point $\mathbf{x} \in S^2$ is taken to a point $f(\mathbf{x}) \in E^2$ by means of a straight line passing through $O, \mathbf{x}$ and $f(\mathbf{x})$. It now must be intuitive, if not obvious, that f is a diffeomorphism sending the great circles in S^2 to the straight lines in E^2. In other words, the round metric on S^2 is projectively equivalent to the Euclidean metric in E^2. It is well-known that this example can be generalised for all dimensions as well as for hyperbolic spaces. Hence, it naturally raises the following general question.

Problem 1 (Beltrami) *Describe all possible projectively equivalent metrics.*

Notice that in the example above S^2 is a manifold of constant curvature. As a matter of fact, it has been known for a long time that all spaces of constant curvature are locally projectively equivalent. The Beltrami problem for Riemannian metrics of non-constant curvature was answered in 1896 by a famous theorem of Levi-Civita [Lev]. In order to state his remarkable result, we need the following definition. Two metrics g and $\tilde{g}$ on M are called *strictly non-proportional* at the point $x \in \mathrm{M}$ if the $(1,1)$-tensor $\tilde{g}^{-1}g$ has n different eigenvalues at x. We then have the following theorem.

Theorem 3.4.1 (Levi-Civita) *Suppose that there are two metrics g and $\tilde{g}$ on* M *which are strictly non-proportional at the point p. Then there exists a sufficiently small local coordinate system $U \in M$ such that the two metrics are projectively equivalent on U if and only if they are given by*

$$\mathrm{d}s_g^2 = \sum_{i=1}^{n} \left[\left| \prod_{\substack{j=1 \\ j \neq i}}^{n} \Big(F_i(u^i) - F_j(u^j) \Big) \right| \mathrm{d}(u^i)^2 \right] \tag{3.4.2}$$

and

$$\mathrm{d}s_{\tilde{g}}^2 = \sum_{i=1}^{n} \left[\frac{1}{F_i(u^i) \prod_{\alpha=1}^{n} F_\alpha(u^\alpha)} \left| \prod_{\substack{j=1 \\ j \neq i}}^{n} \Big(F_i(u^i) - F_j(u^j) \Big) \right| \mathrm{d}(u^i)^2 \right] \tag{3.4.3}$$

where F_i is a positive function only of the variable u^i for all i.

For visual simplicity, let us consider the following two dimensional example. Write u^1, u^2 for the local coordinates and let $F_1(u^1)$ and $F_2(u^2)$ be positive functions. Then the metrics

$$\mathrm{d}s_g^2 = \Big(F_1(u^1) - F_2(u^2) \Big) \Big(\mathrm{d}(u^1)^2 + \mathrm{d}(u^2)^2 \Big) \tag{3.4.4}$$

and

$$\mathrm{d}s_{\tilde{g}}^2 = \left(\frac{1}{F_2(u^2)} - \frac{1}{F_1(u^1)} \right) \left(\frac{\mathrm{d}(u^1)^2}{F_1(u^1)} + \frac{\mathrm{d}(u^2)^2}{F_2(u^2)} \right) \tag{3.4.5}$$

are projectively equivalent. It is a straightforward verification that for $n = 2$ the metrics (3.4.2) and (3.4.3) respectively reduce to (3.4.4) and (3.4.5). After re-expressing them in their respective matrix forms

$$g = \begin{pmatrix} F_1(u^1) - F_2(u^2) & 0 \\ 0 & F_1(u^1) - F_2(u^2) \end{pmatrix} \text{ and } \tilde{g} = \begin{pmatrix} \frac{F_1(u^1)-F_2(u^2)}{F_1^2(u^1)F_2(u^2)} & 0 \\ 0 & \frac{F_1(u^1)-F_2(u^2)}{F_1(u^1)F_2^2(u^2)} \end{pmatrix},$$

we effortlessly compute

$$\tilde{g}^{-1}g = F_1(u^1)F_2(u^2) \begin{pmatrix} F_1(u^1) & 0 \\ 0 & F_2(u^2) \end{pmatrix},$$

which guarantees that our metrics are strictly non-proportional. It is here that we must give a deserved credit to Dini, who actually first proved the two dimensional version of Theorem 3.4.1 in his work [Din] dating back to 1869. An analogue of Dini's theorem for pseudo-Riemannian metrics can be found in [BMP]. We learn from this paper that the pseudo-Riemannian metrics of the form

$$\mathrm{d}s_g^2 = \Big(F_1(u^1) - F_2(u^2)\Big)\Big(\mathrm{d}(u^1)^2 - \mathrm{d}(u^2)^2\Big) \tag{3.4.6}$$

and

$$\mathrm{d}s_{\tilde{g}}^2 = \left(\frac{1}{F_2(u^2)} - \frac{1}{F_1(u^1)}\right)\left(\frac{\mathrm{d}(u^1)^2}{F_1(u^1)} - \frac{\mathrm{d}(u^2)^2}{F_2(u^2)}\right) \tag{3.4.7}$$

are projectively equivalent[7]. Unfortunately, apart from this particular case, Theorem 3.4.1 does not have a higher dimensional analogue on pseudo-Riemannian manifolds.

Looking more generally on this matter, one is naturally bound for seeking suitable transformations on arbitrary (pseudo)-Riemannian manifold (M, g) sending a metric to its equivalent. This necessitates the following general framework. A diffeomorphism $F : \mathrm{M} \longrightarrow \mathrm{M}$ is called a *projective (an affine) transformation* on (M, g) if the pull-

[7] Compare with the Riemannian metrics (3.4.4) and (3.4.5).

back metric F^*g is projectively (affinely) equivalent to g. It is not difficult to convince ourselves that the set of all projective transformations is a group. Then immediately arises the question "*how does this group differ to the isometry group of* (M, g)?" It can be shown that the group of projective transformations of the standard sphere S^n is bigger than its isometry group. Whence we arrive at the second most important problem in this area.

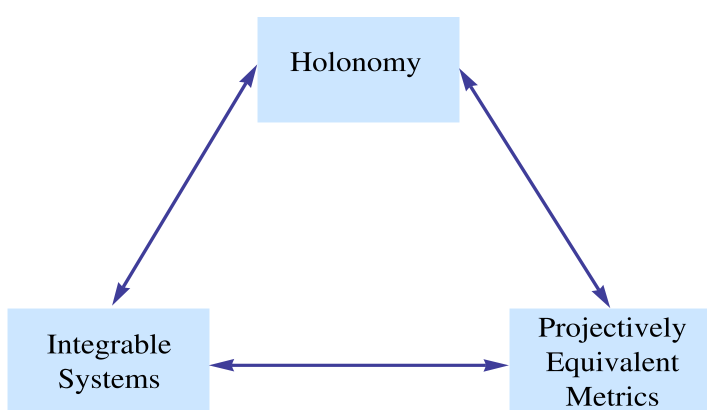
Problem 2 *Which (pseudo)-Riemannian manifolds* (M, g) *admit a group of projective transformations bigger than their isometry group?*

At this moment we shall leave the world of projectively equivalent metrics. For further details on Problems 1 and 2 the reader is referred to the survey papers [Ami, Mat] as well as the recent preprint [BM2].

3.5 The Motivation of this thesis

The motivation of a mathematical work of any kind is usually delivered at its beginning. However, we deliberately violate this custom and proclaim the motivation of this thesis at a rather later stage. By the end of this section, we hope, the reader would agree with the author's opinion that this is indeed the right place for such a discussion.

Recall that in the Preface, we visualised our work with the following diagram



In the previous three sections, each of the blocks of this diagram was discussed to an extent, sufficient for our inquiry. In this section, our goal is to throw some light on the relationships

Holonomy ⟷ Projectively equivalent metrics

and

Integrable Systems ⟷ Projectively equivalent metrics[8].

To begin with, assume that g and $\tilde{g}$ are two pseudo-Riemannian metrics on a pseudo-Riemannian manifold M. We write ∇ and $\tilde{\nabla}$ for their corresponding Levi-Civita connections. Recall that the geodesics for g and $\tilde{g}$ are respectively given as the solutions to the the following second order ordinary differential equations

$$\frac{\mathrm{d}^2 u^\lambda(t)}{\mathrm{d}t^2} + \Gamma^\lambda_{\mu\nu}\frac{\mathrm{d}u^\mu(t)}{\mathrm{d}t}\frac{\mathrm{d}u^\nu(t)}{\mathrm{d}t} = 0,$$
$$\frac{\mathrm{d}^2 \tilde{u}^\lambda(t)}{\mathrm{d}t^2} + \tilde{\Gamma}^\lambda_{\mu\nu}\frac{\mathrm{d}\tilde{u}^\mu(t)}{\mathrm{d}t}\frac{\mathrm{d}\tilde{u}^\nu(t)}{\mathrm{d}t} = 0. \tag{3.5.1}$$

If we now insist that our metrics are affinely equivalent, then equations (3.5.1) immediately affirm $\Gamma^\lambda_{\mu\nu} = \tilde{\Gamma}^\lambda_{\mu\nu}$. This, in turn, justifies

$$\nabla\tilde{g} = 0. \tag{3.5.2}$$

Furthermore, due to the standard one-to-one correspondence between symmetric bilinear forms and g-symmetric operators, the metric $\tilde{g}$ can be substituted with a suitable $(1,1)$-tensor field on M in the following sense. For any pair of tangent vectors ξ and η, we define

$$\tilde{g}(\xi,\eta) = g(L\xi,\eta). \tag{3.5.3}$$

[8]The relationship **Holonomy ⟷ Integrable systems** will be unveiled in Chapters 4 and 5.

Now, bearing this identity in mind, it is readily seen that (3.5.2) implies $\nabla L = 0$. Thus, by virtue of (3.5.2) and (3.5.3), we apprehend that the classification of the affinely equivalent pairs g and $\tilde{g}$ is tantamount to the classification of the pairs g and L, provided $\nabla L = 0$ and L is g-symmetric. This latter problem was partially solved by Kručkovič and Solodovnikov [KS]. In their approach, however, no use of holonomy was made. It turns out that the existence of a covariantly constant $(1,1)$-tensor field on M is the key, enabling us to look at the classification of the pairs g and L from the perspective of holonomy. Indeed, one can easily perceive the truth of the following proposition.

Proposition 3.5.4 *Consider the group* $G_L^0 = \{X \in \mathrm{SO}^0(g) \mid XLX^{-1} = L\}$*, where* $\mathrm{SO}^0(g)$ *is the connected component of the identity of the group* $\mathrm{SO}(g)$*. Then, the connection* ∇ *admits a covariantly constant* $(1,1)$*-tensor field if and only if* $\mathrm{Hol}(\nabla) \subset G_L^0$.

Proof. Let γ be an arbitrary smooth curve on M. There exists an appropriate parallel transport P_γ along γ such that the condition $\nabla_\gamma L = 0$ is equivalent to $P_\gamma L P_\gamma^{-1} = L$. This clearly implies $P_\gamma L = L P_\gamma$. Now, as some of these parallel transports will be along closed loops, we readily conclude that $\mathrm{Hol}(\nabla) \subset G_L^0$. □

At this juncture, the following remarks worth mentioning. Firstly, by L we understand the value of the $(1,1)$-tensor field at any fixed point $x_0 \in$ M. Secondly, the choice of $x_0 \in$ M does not play any important role, as L is covariantly constant. Thus, Proposition 3.5.4 naturally raises the question.

Are there any pairs g and L on a pseudo-Riemannian manifold, with $\nabla L = 0$ such that $\mathrm{Hol}(\nabla) \equiv G_L^0$*?*

In this thesis we shall give an affirmative answer to this question. Our approach will be primarily inspired by a remarkable relationship between the areas of integrable systems and projectively equivalent metrics. This rather unexpected relationship was recently noticed by A. Bolsinov, V. Kiosak and V. Matveev [BKM]. Incredibly, the main theorem of the latter paper is given two alternative proofs. Although this is a result generically concerned with projective equivalence of pseudo-Riemannian metrics, one of the proofs offered in [BKM] uses ideas from the theory of integrable systems on semisimple Lie algebras. Thus, in order to achieve a greater clarity, it is necessary to briefly outline the aforementioned paper. Its main result is the following theorem.

Theorem 3.5.5 (Bolsinov, Kiosak & Matveev) *Let g, $\tilde{g}$ and $\widehat{g}$ be three projectively equivalent metrics on a connected manifold M^n of dimension $n \geqslant 3$. Suppose there exists a point at which g, $\tilde{g}$ and $\widehat{g}$ are linearly independent. Then, the metrics g, $\tilde{g}$ and $\widehat{g}$ have constant curvature.*

It is worth noting that a local version of this theorem in the case of Riemannian metrics was known to Fubini [Fub1, Fub2]. His proof, however, was based on the Levi-Civita theorem (see Theorem 3.4.1) and hence is not applicable in the pseudo-Riemannian case. We must also note that in the two dimensional case as well as the case of metrics which are not strictly non-proportional, counterexamples of Theorem 3.5.5 are known (see for instance [BMM, Koe, Sha, Sol1, Sol2, Sol3]). We also note that, despite the global nature of Theorem 3.5.5, it is sufficient to give a local proof. This is secured by the following two facts. Firstly, if the metrics g, $\tilde{g}$ and $\widehat{g}$ are linearly dependent at every point of some neighbourhood of M, then they are linearly dependent at every point of the manifold. Secondly, if two projectively equivalent metrics are strictly non-proportional at least at one point, then they are strictly non-proportional at almost every point. For proofs the reader may consult [BKM].

A key point for both of the proofs of Theorem 3.5.5 is the following tensor reformulation of the projective equivalence property of two metrics. Given the metrics g and $\tilde{g}$ we consider the $(0,2)$-tensor

$$L_{ij} = \left|\frac{\det(\tilde{g})}{\det(g)}\right|^{\frac{1}{n+1}} \cdot g_{i\alpha}\tilde{g}^{\alpha\beta}g_{j\beta}$$

and the function

$$\lambda = \frac{1}{2}L_{\alpha\beta}g^{\alpha\beta}.$$

Then, under these assumptions, we have the following criterion for projective equivalence of two metrics.

Theorem 3.5.6 *The metrics g and $\tilde{g}$ are projectively equivalent if and only if*

$$\nabla_k L_{ij} = \frac{\partial\lambda}{\partial u^i}g_{jk} + \frac{\partial\lambda}{\partial u^j}g_{ik}, \tag{3.5.7}$$

where the covariant derivative is taken with respect to the metric g.

This reformulation was suggested by Sinjukov [Sin], but the reader may also wish to refer to [BM1, EM]. Now, the proof of Theorem 3.5.5 mostly constitutes an analysis of the integrability (compatibility) conditions of (3.5.7). These, being of indirect interest for our work, are omitted. However, bearing in mind the fact that the Riemann curvature operator can be thought of as the map $R : \mathfrak{so}(g) \longrightarrow \mathfrak{so}(g)$, a tedious computation establishes that the compatibility conditions of (3.5.7) can be rewritten in the form

$$[R(X), L] = [X, M].$$

Thus, we arrive at the following result.

Theorem 3.5.8 (Bolsinov, Kiosak & Matveev) *If g and $\tilde{g}$ are projectively equivalent, then the curvature tensor of g considered as a linear map*

$$R : \mathfrak{so}(g) \longrightarrow \mathfrak{so}(g)$$

is a Manakov operator, i.e., it satisfies the identity

$$[R(X), L] = [X, M] \quad \textit{for all } X \in \mathfrak{so}(g)$$

with L defined by $\tilde{g}^{-1}g = \det L \cdot L$ and M being the Hessian of $2\mathrm{tr}\, L$, i.e. $M^i_j = 2\nabla^i\nabla_j \mathrm{tr}\, L$.

The moral of this theorem is twofold. Firstly, it throws some light onto how the areas of Integrable systems and Projectively equivalent metrics are interrelated. Secondly, if g and $\tilde{g}$ are affinely equivalent, then L is automatically covariantly constant and therefore $M = 0$. Thus, the curvature tensor R satisfies a simpler equation, namely

$$[R(X), L] = 0.$$

Notice that formula (3.3.8) still defines a non-trivial operator, if $p(t)$ is a non-trivial polynomial satisfying $p(L) = M = 0$, for example, the minimal polynomial for L. Thus, we may well think of $R(X)$ as an element of the Lie algebra $\mathfrak{g}_L$, which will be shown to be the case in the next chapter. To put it another way, Theorem 3.5.8 motivates us to look for a Manakov operator which can also be thought of as a formal curvature operator. We shall construct such an example in the next chapter and by exploiting the power of Manakov operators we shall conclude that $\mathfrak{g}_L$ is a Berger algebra.

Chapter 4

Berger algebras related to g-symmetric operators

It is our primary concern in this chapter to prove that for a given g-symmetric operator L, the Lie algebra $\mathfrak{g}_L$ is a Berger algebra. We begin our exposition with a detailed analysis of the $(2;2)$-case and discuss how Manakov operators miraculously emerge on the horizon of our quest. We next show that, without loss of generality, we may confine ourselves to nilpotent g-symmetric operators. Finally, we give a proof of Theorem B.

4.1 The beginning: analysis of the (2;2)-case

Without any doubt, the abstractness and rigour of mathematics is always motivated by simple examples illustrating its main ideas. We thus initiate this chapter by considering a basic example first. Computational and straightforward in nature, it will both aid the reader's understanding of the problem and enable us to naturally conjecture the foremost result of this chapter.

For the purposes of the present section it suffices to consider an n-dimensional pseudo-Euclidean vector space V with standard basis $\{e_i\}_{i=1}^n$. We know from our discussion in the background Chapter 3 that to show that $\mathfrak{g}_L$ is Berger algebra it is sufficient to find a suitable formal curvature operator with image coinciding with $\mathfrak{g}_L$. Thus, we would naturally like to be able to resolve the following problem.

Problem 3 *Find all maps $R : \Lambda^2 V \longrightarrow \mathfrak{g}_L$ such that*

$$\begin{cases} (\heartsuit) \quad R(e_i \wedge e_j)e_k + R(e_j \wedge e_k)e_i + R(e_k \wedge e_i)e_j = 0, \\ (\clubsuit) \quad \mathrm{Im} R \equiv \mathfrak{g}_L. \end{cases} \tag{4.1.1}$$

Certainly, the solution of this problem splits into solving the system of equations ($\heartsuit$) and ensuring that the solutions indeed satisfy ($\clubsuit$). It deserves to be noticed that solving this problem in its full generality is rather difficult and no general solution is known to the author. The sheer difficulty, particularly lies in finding all formal curvature operators satisfying the property ($\clubsuit$). For good or ill, we shall not be able to derive a solution of (4.1.1) with the aid of some standard or algorithmic procedures. Instead, we shall define a map and check that it is a solution of (4.1.1). Fortunately, to the extent which this thesis requires, it will suffice to construct just one suitable solution. Approaching this problem, we observe that $R(e_i \wedge e_j) = -R(e_j \wedge e_i)$ holds true for any two basis vectors. Thus, without loss of generality we may assume $i \leqslant j \leqslant k$. Consequently, the system of equations ($\heartsuit$) in (4.1.1) reduces to a smaller one. Evidently, this is a system of equations consisting of $\binom{n}{3}$ equations.

Now, in order to achieve greater clarity, we shall discuss in full detail the simplest possible non-trivial example, namely the $(2;2)$-case. Recall that $L^{(2;2)}$ is a linear operator acting on a four dimensional space. Then by means of elementary combinatorics, the system of equations (♡) reduces to the following four equations

$$\begin{aligned} R(e_1\wedge e_2)e_3 + R(e_2\wedge e_3)e_1 + R(e_3\wedge e_1)e_2 = 0,\\ R(e_1\wedge e_2)e_4 + R(e_2\wedge e_4)e_1 + R(e_4\wedge e_1)e_2 = 0,\\ R(e_1\wedge e_3)e_4 + R(e_3\wedge e_4)e_1 + R(e_4\wedge e_1)e_3 = 0,\\ R(e_2\wedge e_3)e_4 + R(e_3\wedge e_4)e_2 + R(e_4\wedge e_2)e_3 = 0. \end{aligned} \tag{4.1.2}$$

Bearing in mind the standard identification of the space of skew-symmetric matrices with $\Lambda^2 V$, along with the trivial fact that the $e_i \wedge e_j$'s form a basis for the latter, we can justifiably rewrite the expression

$$x_1 e_1\wedge e_2 + x_2 e_1\wedge e_3 + x_3 e_1\wedge e_4 + x_4 e_2\wedge e_3 + x_5 e_2\wedge e_4 + x_6 e_3\wedge e_4$$

as the matrix

$$\begin{pmatrix} 0 & x_1 & x_2 & x_3\\ -x_1 & 0 & x_4 & x_5\\ -x_2 & -x_4 & 0 & x_6\\ -x_3 & -x_5 & -x_6 & 0 \end{pmatrix}.$$

Now, with the aid of the coefficients x_i we prescribe further the quantities $a(x) = \sum_{i=1}^{6} a_i x_i$ and $b(x) = \sum_{i=1}^{6} b_i x_i$, where $x_i, a_i, b_i \in \mathbb{R}$. By means of these last expressions we finally define the map $R : \Lambda^2 V \longrightarrow \mathfrak{g}_L^{(2;2)}$ so that each element of $\Lambda^2 V$ is sent to

$$\left(\begin{array}{cc|cc} 0 & 0 & a(x) & b(x)\\ 0 & 0 & 0 & a(x)\\ \hline -a(x) & -b(x) & 0 & 0\\ 0 & -a(x) & 0 & 0 \end{array}\right). \tag{4.1.3}$$

Notice that $\dim \mathfrak{g}_L^{(2;2)} = 2$ and $a(x)$ and $b(x)$ are mutually independent quantities - a fact which will be important for the conclusion of this example.

We observe that, substituting (4.1.3) in the first equation of (4.1.2) we read off

$$\left(\begin{array}{cc|cc} 0 & 0 & a_1 & b_1 \\ 0 & 0 & 0 & a_1 \\ \hline -a_1 & -b_1 & 0 & 0 \\ 0 & -a_1 & 0 & 0 \end{array}\right)\begin{pmatrix}0\\0\\1\\0\end{pmatrix} + \left(\begin{array}{cc|cc} 0 & 0 & a_4 & b_4 \\ 0 & 0 & 0 & a_4 \\ \hline -a_4 & -b_4 & 0 & 0 \\ 0 & -a_4 & 0 & 0 \end{array}\right)\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + \left(\begin{array}{cc|cc} 0 & 0 & -a_2 & -b_2 \\ 0 & 0 & 0 & -a_2 \\ \hline a_2 & b_2 & 0 & 0 \\ 0 & a_2 & 0 & 0 \end{array}\right)\begin{pmatrix}0\\1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

only to determine $a_1 = 0$, $a_2 = 0$ and $a_4 = b_2$. Treating the rest of the equations in (4.1.2) in the same manner, we readily conclude that the only non-zero coefficients are $a_5 = b_3 = b_4 \neq 0$. Furthermore, we observe that the coefficient b_5 does not affect our calculations. It is essentially an arbitrary coefficient. Write for brevity $\alpha = a_5 = b_3 = b_4$ and $b_5 = \beta$. Thus, we have shown that there exists a non-trivial linear map, such that it solves the system of equations (4.1.2). More succinctly, the map $R : \Lambda^2 V \longrightarrow \mathfrak{g}_L^{(2;2)}$ defined by

$$\begin{pmatrix} 0 & x_1 & x_2 & x_3 \\ -x_1 & 0 & x_4 & x_5 \\ -x_2 & -x_4 & 0 & x_6 \\ -x_3 & -x_5 & -x_6 & 0 \end{pmatrix} \longmapsto \left(\begin{array}{cc|cc} 0 & 0 & \alpha x_5 & \alpha(x_3+x_4)+\beta x_5 \\ 0 & 0 & 0 & \alpha x_5 \\ \hline -\alpha x_5 & -\alpha(x_3+x_4)-\beta x_5 & 0 & 0 \\ 0 & -\alpha x_5 & 0 & 0 \end{array}\right)$$

is a formal curvature operator for the Lie algebra $\mathfrak{g}_L^{(2;2)}$. The careful reader may have already noticed that in fact αx_5 and $\alpha(x_3 + x_5) + \beta x_5$ are indeed mutually independent and therefore drawn the conclusion that $\mathfrak{g}_L^{(2;2)} \equiv \mathrm{Im} R$. In other words, we have proven that $\mathfrak{g}_L^{(2;2)}$ is Berger algebra.

Following the same idea we compute further a various of higher dimensional cases only to establish that the corresponding centralisers are indeed Berger algebras. We have decided to append some of our results, should the reader require more worked examples. It is by virtue of these computations that we are naturally in a position to propose the following conjecture.

Conjecture 1 *For any g-symmetric operator L acting on some pseudo-Euclidean space (V, g) we have that $\mathfrak{g}_L$ is a Berger algebra.*

Incredibly, we shall end up with a generic proof of the fact that $\mathfrak{g}_L^{(k;n)}$ is a Berger algebra for any nilpotent g-symmetric operator $L^{(k;n)}$, which will be naturally implemented into the general proof. Beforehand, however, we shall need to take a look at the main tool which will enable us to give an affirmative answer to our conjecture. We have arrived at the point where we shall discuss the intimate relationship between Manakov operators and formal curvature operators.

4.2 The magic formula

This section aims at acquainting the reader with one simple but fundamentally important formula. Henceforth, it will be assumed that g is a pseudo-Riemannian metric on V and $L : V \longrightarrow V$ is a g-symmetric operator with minimal polynomial $p_{\min}(L)$. Moreover, the identification

$$\mathfrak{so}(g) \cong \Lambda^2 V \quad \text{given by} \qquad u \wedge v = v \otimes g(u) - u \otimes g(v) \ \text{ for any } \ u, v \in V, \tag{4.2.1}$$

will finally be exploited as was promised in the background chapter. It is this identification which will help us to perceive the earlier mentioned relationship between Manakov operators and formal curvature operators. Let us define the linear map $R : \mathfrak{so}(g) \longrightarrow \mathfrak{so}(g)$ by means of the formula

$$R(X) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} p_{\min}(L + tX), \tag{4.2.2}$$

where $p_{\min}$ is the minimal polynomial for the operator L. It has been shown in Section 3.3 that this map is a Manakov operator. We now show that it can be viewed as a formal curvature operator as well. For this purpose, let us investigate its basic properties. Firstly, we observe that the image of R is indeed contained in $\mathfrak{so}(g)$, which is $R(X) \in \mathfrak{so}(g)$ for any $X \in \mathfrak{so}(g)$. To see this, it suffices to show that $R(X)^* = -R(X)$. By assumption, we have that $L^* = L$ and $X^* = -X$, which imply that

$$\Big(p_{\min}(L + tX)\Big)^* = p_{\min}(L^* + tX^*) = p_{\min}(L - tX).$$

Now, using this last line along with the fact that the operations "$\frac{\mathrm{d}}{\mathrm{d}t}$" and "$*$" commute, we compute

$$\begin{aligned} R(X)^* &= \left(\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} p_{\min}(L + tX)\right)^* = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} p_{\min}(L - tX) \\ &= -\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} p_{\min}(L + tX) = -R(X). \end{aligned}$$

We have thus shown that the map (4.2.2) is well-defined. Note that this argument applies to any polynomial, not necessarily minimal. However, the minimality condition will shortly be exploited. Secondly, we perceive the truth of the fact that $R(X)$ commutes with L. In other words, we have that $R(X) \in \mathfrak{g}_L$ for all $X \in \mathfrak{so}(g)$. To prove this, we consider yet again the following identity

$$[p_{\min}(L + tX), L + tX] = 0.$$

We now only differentiate this last expression and evaluate it at $t = 0$ to obtain

$$\left[\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} p_{\min}(L + tX), L\right] + [p_{\min}(L), X] = 0.$$

Since $p_{\min}(L) = 0$ we conclude that $[R(X), L] = 0$. It is this conclusion along with $\Lambda^2 V \cong \mathfrak{so}(g)$, which motivates us viewing (4.2.2) as the map $R : \Lambda^2 V \longrightarrow \mathfrak{g}_L$. Having this in mind, it is a natural question to ask whether or not R satisfies the Bianchi identity. Before proving that it indeed does, we shall derive one useful formula. Assume that L is a nilpotent operator[1] of order k. Clearly, its minimal polynomial is $p_{\min}(L) = L^k = 0$. Bearing this in mind we write $p_{\min}(L + tX) = (L + tX)^k$ and see that

$$p_{\min}(L + tX) = L^k + t \sum_{p=0}^{k-1} L^{k-p-1} X L^p + O(t^2).$$

As $L^k = 0$ we rewrite (4.2.2) as

$$R(X) = \sum_{p=0}^{k-1} L^{k-p-1} X L^p = L^{k-1} X + L^{k-2} X L + \cdots + L X L^{k-2} + X L^{k-1}. \tag{4.2.3}$$

This formula will be very helpful in achieving our final goal. Generally speaking, by virtue of this formula we may think of the operator (4.2.3) as $R(X) = \sum_k C_k X D_k$, where C_k and D_k are powers of L and certainly g-symmetric operators themselves. To prove that (4.2.2) is a formal curvature operator it suffices to verify that the Bianchi identity holds true for operators mapping X to CXD for some g-symmetric operators C and D. To put it in another way, we only need to show that for any vectors u, v and w we have

$$C(u \wedge v)Dw + C(v \wedge w)Du + C(w \wedge u)Dv = 0. \tag{4.2.4}$$

[1] We shall see in Section 4.3 that for the purposes of the present inquiry it is sufficient to consider only nilpotent operators.

Note that $v \otimes g(u)w = g(u,w)v$ holds true for any three vectors u, v and w. Then using the identification (4.2.1) we compute

$$\begin{aligned} C(u \wedge v)Dw &= C\Big(v \otimes g(u) - u \otimes g(v)\Big)Dw = C\Big(v \otimes g(u)Dw - u \otimes g(v)Dw\Big) = \\ &= C\Big(g(u,Dw)\cdot v - g(v,Dw)\cdot u\Big) = g(u,Dw)\cdot Cv - g(v,Dw)\cdot Cu \end{aligned}$$

Similarly, we obtain the following two relations.

$$C(v \wedge w)Du = g(w,Du)\cdot Cv - g(v,Du)\cdot Cw$$

$$C(w \wedge u)Dv = g(u,Dv)\cdot Cw - g(w,Dv)\cdot Cu.$$

As both C and D are g-symmetric operators we conclude that (4.2.4) is indeed satisfied by any triple u, v and w. In summary, we have proven the following.

Proposition 4.2.5 *Let $L : V \to V$ be a nilpotent g-symmetric operator. Then* (4.2.2) *defines a formal curvature operator $R : \Lambda^2 V \simeq \mathfrak{so}(g) \to \mathfrak{g}_L$ for the Lie algebra $\mathfrak{g}_L$.*

This proposition tells us that R satisfies the Bianchi identity and its image is contained in $\mathfrak{g}_L$. But this means that the centraliser of our g-symmetric operator L is already a good candidate for being a Berger algebra. Yet again we have arrived at Conjecture 1. Luckily, by the end of this chapter we shall confirm that this conjecture is true. The key point of our proof will be exactly the use of formula (4.2.2). It is kind of magic that a formula from integrable systems could do so much work in the realm of holonomy, is it not? For this reason, we shall henceforth refer to (4.2.2) as well as its reincarnation (4.2.3) simply as *the magic formula*.

4.3 Reduction to nilpotent g-symmetric operators

Before embarking onto proving the main result of this chapter, which has been stated as **Theorem B** in the introductory chapter, the following point needs to be brought into prominence. Without loss of generality, it suffices to prove this result only for nilpotent g-symmetric operators[2]. The reduction of an arbitrary operator to a nilpotent one is natural and is an immediate consequence of two well-known facts to be mentioned below.

Firstly, assume that g is a non-degenerate bilinear form on a vector space V. Consider an arbitrary g-symmetric operator $L : V \longrightarrow V$. Then it is a matter of standard linear algebra that V decomposes into its L-invariant eigensubspaces, that is

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_s.$$

Each V_i is either a generalised eigensubspace corresponding to a real eigenvalue λ_i, or one corresponding to a pair of complex conjugate eigenvalues λ_j and $\overline{\lambda}_j$. Moreover, this decomposition is g-orthogonal, that is for any $i \neq j$ we have

$$g(V_i, V_j) = 0.$$

To see this, take the eigensubspace V_i of the eigenvalue λ_i and consider the operator $(L - \lambda_i \cdot \mathrm{Id})^k$ for some $k \in \mathbb{N}$. Clearly, $(L - \lambda_i \cdot \mathrm{Id})^k V_i = 0$. Now, this operator is g-symmetric and the eigensubspace V_j is invariant under its action, that is $(L - \lambda_i \cdot \mathrm{Id})^k V_j = V_j$ for $i \neq j$. We then readily compute

$$0 = g\Big((L - \lambda_i \cdot \mathrm{Id})^k V_i, V_j\Big) = g\Big(V_i, (L - \lambda_i \cdot \mathrm{Id})^k V_j\Big) = g(V_i, V_j).$$

[2]We remind the reader that in this thesis only singular operators are of interest.

If V_i is a generalised eigensubspace corresponding to a pair of complex conjugate eigenvalues λ_i and $\overline{\lambda}_i$, the same argument remains valid if applied to the operator

$$\left(\left(L - \lambda_i \cdot \mathrm{Id}\right)\left(L - \overline{\lambda}_i \cdot \mathrm{Id}\right)\right)^k.$$

Secondly and more importantly, this decomposition naturally yields a similar decomposition for $\mathfrak{g}_L$. To see this, we first observe that any generalised eigensubpace V_j is invariant under the action of an operator X commuting with L. This immediately justifies

$$G_L^0 = G_1^0 \times \cdots \times G_s^0, \tag{4.3.1}$$

where the Lie subgroups G_i^0 are naturally associated with their corresponding generalised eigensubspace V_i. Moreover, G_i^0 is the connected component of the centraliser of $L_i = L|_{V_i}$ in $\mathrm{O}(g|_{V_i})$. Thus, G_L^0 is reducible and by virtue of the de Rham - Wu splitting theorem it is a holonomy group if and only if each G_i^0 is. For our purposes, however, we shall only need a weaker version. Namely, if each G_i^0 is a holonomy group then so is G_L^0. Geometrically speaking, our aim will be to realise each G_i^0 as a holonomy group for some pseudo-Riemannian manifold $(M_i, g|_{V_i})$. Then the holonomy group for the pseudo-Riemannian direct product $M = M_1 \times \cdots \times M_s$ will be exactly $G_L^0 = G_1^0 \times \cdots \times G_s^0$. In addition, let us suppose that $\mathfrak{g}_L$ and $\mathfrak{g}_i$ are the Lie algebras for G_L^0 and G_i^0 respectively. Consequently, the decomposition (4.3.1) induces

$$\mathfrak{g}_L = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_s.$$

Now, it can be shown that if $\mathfrak{g}_L$ is a Berger algebra then all the $\mathfrak{g}_i$s are. Notice that, bearing Proposition 3.1.1 in mind we actually have $\mathfrak{g}_i = \mathfrak{g}_{L_i}$.

The moral of the present discussion is now evident. It suffices to prove **Theorem B** for only two special cases:

$$\begin{cases} (\spadesuit)\ L \text{ has a single real eigenvalue,} \\ (\diamondsuit)\ L \text{ has a pair of complex conjugate eigenvalues.} \end{cases}$$

Luckily, the proof of case ($\diamondsuit$) is not substantially different to the proof of case ($\spadesuit$). For this reason we shall devote ourselves to elaborating a complete proof of **Theorem B** for the case ($\spadesuit$). Thereafter, we shall be in a position to easily adapt it to the complex case ($\diamondsuit$).

It is obvious that g-symmetric operators with a single real eigenvalue are immediately nilpotent, provided $\lambda = 0$. Furthermore, without loss of generality, for $\lambda \neq 0$ we may consider the operator $(L - \lambda \cdot \mathrm{Id})$ instead, which is clearly nilpotent. This latter fact will be modified at the end of the chapter in order to adapt the techniques used in ($\spadesuit$) to ($\diamondsuit$). Thus, we shall henceforth consider g-symmetric operators with a single real eigenvalue, unless otherwise stated.

4.4 The $(k; n)$ - case

In this section we completely exhaust the $(k; n)$-case. As a result, not only shall we have exemplified the principal theorem of this chapter, but also and more importantly, we shall have laid a firm ground for its general proof.

Lemma 4.4.1 *Let L be a nilpotent g-symmetric operator of the type $L^{(k;n)}$, i.e., it consists of two Jordan blocks. Then its centraliser $\mathfrak{g}_L^{(k;n)}$ is a Berger algebra.*

For the sake of brevity we shall henceforth write L instead of $L^{(k;n)}$ and $\mathfrak{g}_L$ instead of $\mathfrak{g}_L^{(k;n)}$ throughout this section. Evidently, to prove this lemma it suffices to check that the image of the formal curvature operator defined by means of the magic formula coincides with $\mathfrak{g}_L$. At this juncture, it will also be worth reminding the reader that by virtue of Section

3.1 the $(k;n)$-case is determined as follows. With respect to the canonical basis, we shall write for convenience $L = \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix}$ with $L_1^k = 0$ and $L_2^n = 0$ for $2 \leqslant k \leqslant n$. Clearly, the minimal polynomial for L in this case is $p_{\min}(t) = t^n$. Furthermore, the metric g and the elements of $\mathfrak{so}(g)$ and $\mathfrak{g}_L$ are given by block matrices of the form

$$g = \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix}, \quad X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & A \\ \widetilde{A} & 0 \end{pmatrix},$$

where $g_1, X_{11} \in \mathcal{M}_{k\times k}$ and $g_2, X_{22} \in \mathcal{M}_{n\times n}$. More importantly, we also need to bear in mind that $X_{ji} = -g_j X_{ij}^\top g_i$ and $\widetilde{A} = -g_2 A^\top g_1$ hold true. Notice that $i, j = 1, 2$ label the blocks in the matrix X and are not indices in the usual sense.

We first observe that with the aid of formula (4.2.3) we can write

$$R(X) = \begin{pmatrix} R_{11}(X_{11}) & R_{12}(X_{12}) \\ R_{21}(X_{21}) & R_{22}(X_{22}) \end{pmatrix}, \tag{4.4.2}$$

where the following relations are satisfied

$$R_{11}(X_{11}) = L_1^{n-1} X_{11} + L_1^{n-2} X_{11} L_1 + \cdots + X_{11} L_1^{n-1}, \tag{4.4.3}$$

$$R_{12}(X_{12}) = L_1^{n-1} X_{12} + L_1^{n-2} X_{12} L_2 + \cdots + X_{12} L_2^{n-1}, \tag{4.4.4}$$

$$R_{21}(X_{21}) = L_2^{n-1} X_{21} + L_2^{n-2} X_{21} L_1 + \cdots + X_{21} L_1^{n-1}, \tag{4.4.5}$$

$$R_{22}(X_{22}) = L_2^{n-1} X_{22} + L_2^{n-2} X_{22} L_2 + \cdots + X_{22} L_2^{n-1}. \tag{4.4.6}$$

Now, these relations essentially tell us that the map $R : \mathfrak{so}(g) \longrightarrow \mathfrak{g}_L$ preserves the block-matrix structure, which means that every block $R_{ij}(X_{ij})$ depends only on its corresponding preimage block X_{ij}. Thereby, the proof of Lemma 4.4.1 reduces to showing that the following conditions hold true.

$$\begin{cases} R_{11}(X_{11}) = 0,\ R_{22}(X_{22}) = 0 \text{ and } R_{21}(X_{21}) = -g_2^\top R_{12}(X_{12}) g_1, \\ \text{the parameters } \alpha_1, \dots, \alpha_k \text{ defining } \mathfrak{g}_L \text{ are mutually independent.} \end{cases} \tag{4.4.7}$$

We have shown in Section 4.2 that R is the formal curvature operator for the Lie algebra $\mathfrak{g}_L$. This indeed implies that $\mathrm{Im}R \subset \mathfrak{g}_L$, whence $R_{11}(X_{11}) = R_{22}(X_{22}) = 0$ immediately holds true. We establish the rest of (4.4.7) by straightforward computation. Working out (4.4.4) we easily obtain the matrix

$$R_{12}(X_{12}) = \begin{pmatrix} 0 & \cdots & 0 & \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_k \\ 0 & \cdots & \cdots & 0 & \alpha_1 & \alpha_2 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \alpha_3 \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \alpha_1 & \alpha_2 \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \alpha_1 \end{pmatrix}, \tag{4.4.8}$$

with entries satisfying the relations

$$\begin{aligned} &\alpha_1 = x_{k1}, \\ &\alpha_2 = x_{k-1,1} + x_{k2}, \\ &\alpha_3 = x_{k-2,1} + x_{k-1,2} + x_{k3}, \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ &\alpha_k = x_{11} + x_{22} + x_{33} + \cdots + x_{kk}. \end{aligned}$$

Evidently, the α_is are mutually independent. Similarly, from equation (4.4.5) we derive

$$R_{21}(X_{21}) = \begin{pmatrix} -\alpha_1 & -\alpha_2 & -\alpha_3 & \ddots & -\alpha_k \\ 0 & -\alpha_1 & -\alpha_2 & -\alpha_3 & \ddots \\ 0 & \cdots & -\alpha_1 & -\alpha_2 & -\alpha_3 \\ 0 & \cdots & \cdots & -\alpha_1 & -\alpha_2 \\ 0 & \cdots & \cdots & \cdots & -\alpha_1 \\ 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix}. \tag{4.4.9}$$

We now observe that (4.4.8) and (4.4.9) relate precisely as $R_{21}(X_{21}) = -g_2^\top R_{12}(X_{12})g_1$, which settles (4.4.7). This completes the proof of Lemma 4.4.1 and completely exhausts the $(k;n)$-case.

4.5 The proof of the general case

Let us begin this section with a brief discussion of the special case when L consists of k blocks all of the same size. Remarkably, Lemma 4.4.1 can be naturally generalised. Strictly speaking, we can still define R exactly as in the previous section and no further modification will be needed. This is due to the fact that we do not observe any "interaction" between the different blocks as a result of the action of R on the elements of $\mathfrak{so}(g)$.

Let us illustrate this situation with the following example. Assume that L consists of k blocks each of which is of the form

$$L_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{4.5.1}$$

Since $L^2 = 0$, by the magic formula we have $R(X) = LX + XL$. We know from Proposition 3.1.6 that in this case the elements of $\mathfrak{so}(g)$ have the following block matrix form

$$X = \begin{pmatrix} B_1 & X_1 & X_2 & \cdots & \cdots & X_{k-1} \\ \tilde{X}_1 & B_2 & \cdots & \cdots & \cdots & \cdots \\ \tilde{X}_2 & \cdots & B_3 & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \cdots & \cdots & \cdots & \cdots & B_{k-1} & X_{\frac{k(k-1)}{2}} \\ \tilde{X}_{k-1} & \cdots & \cdots & \cdots & \tilde{X}_{\frac{k(k-1)}{2}} & B_k \end{pmatrix} \qquad (4.5.2)$$

with

$$B_j = \begin{pmatrix} \beta_j & 0 \\ 0 & -\beta_j \end{pmatrix}, \quad X_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \quad \text{and} \quad \tilde{X}_i = -\begin{pmatrix} d_i & b_i \\ c_i & a_i \end{pmatrix}, \qquad (4.5.3)$$

for all $1 \leqslant j \leqslant k$ and all $1 \leqslant i \leqslant \frac{k(k-1)}{2}$. Now, X and L are $k \times k$ block matrices. Thus we have that the $(ij)^{th}$entry of $R(X)$ is given by

$$R(X)_{ij} = (LX)_{ij} + (XL)_{ij} = L_0 X_{ij} + X_{ij} L_0.$$

Now, a straightforward computation shows that we have the following three possibilities for the images of X_{ij}.

• For $i = j$ we compute $R(X)_{ij} = 0$.

• For $i < j$ we have that $R(X)_{ij} = L_0 X_p + X_p L_0$ for some $1 \leqslant p \leqslant \frac{k(k-1)}{2}$, which yields

$$R(X)_{ij} = \begin{pmatrix} c_p & a_p + d_p \\ 0 & c_p \end{pmatrix}.$$

- For $i > j$ we similarly have $R(X)_{ij} = L_0\tilde{X}_p + \tilde{X}_p L_0$ for the same p as above and hence

$$R(X)_{ij} = \begin{pmatrix} -c_p & -a_p - d_p \\ 0 & -c_p \end{pmatrix}.$$

Apparently, the conclusion that $\mathfrak{g}_L$ is a Berger algebra is now immediate. Unfortunately, the most general situation is not quite as simple. It turns out that the case of many blocks of different sizes is different to the one just described. The principle difference lies in the fact that in this situation we have "interaction" between the different blocks upon the action of R. This difference is illustrated in two examples in Appendix A.4.

Let us now concentrate on the modification of the definition of R which is necessary for avoiding the aforesaid "interaction". Suppose that L and g are of the form (3.1.2) and that not all the blocks are of the same size. Note, that some blocks may still have the same size though. Let (V, g) and (V', g') be two pseudo-Euclidean vector spaces such that V' is a subspace of V and $g' = g|_{V'}$ is non-degenerate. Assume that $\mathfrak{h} \subset \mathfrak{so}(g')$ is a Berger subalgebra. Then if we consider the standard embedding $\mathfrak{so}(g') \longrightarrow \mathfrak{so}(g)$ induced by the inclusion $V' \longrightarrow V$, $\mathfrak{h}$ will also be a Berger subalgebra of $\mathfrak{so}(g)$. Moreover, if the map $R' : \mathfrak{so}(g') \longrightarrow \mathfrak{so}(g')$ is a formal curvature tensor, then its trivial extension $R : \mathfrak{so}(g) \to \mathfrak{so}(g)$ defined by

$$R \begin{pmatrix} X & Y \\ Z & W \end{pmatrix} = \begin{pmatrix} R'(X) & 0 \\ 0 & 0 \end{pmatrix} \tag{4.5.4}$$

is a formal curvature tensor too. More importantly, notice that this trivial extension of the curvature tensor still obeys Lemma 4.4.1. Thus, it is this observation that is the quintessence of what we are about to explain. Our goal must already be clear - we wish to build up a bigger curvature tensor out of (4.5.4) so that we could use Lemma 4.4.1

in a blockwise manner. This will result in circumventing the “interaction” of the blocks mentioned earlier. In practice, we introduce the operator $\widehat{R}_{12} : \mathfrak{so}(g) \longrightarrow \mathfrak{so}(g)$ defined by

$$\widehat{R}_{12}\begin{pmatrix} X_{11} & X_{12} & \cdots & X_{1k} \\ X_{21} & X_{22} & \cdots & X_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ X_{k1} & X_{k2} & \cdots & X_{kk} \end{pmatrix} = \begin{pmatrix} 0 & R_{12}(X_{12}) & \cdots & 0 \\ R_{21}(X_{21}) & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \tag{4.5.5}$$

where the only non zero blocks in the right hand side $R_{12}(X_{12})$ and $R_{21}(X_{21})$ are defined in the very same fashion as in (4.4.4) and (4.4.5). Then $\widehat{R}_{12}$ is a formal curvature tensor by construction. In addition, by dint of Corollary 3.1.9, its image coincides with the abelian subalgebra $\mathfrak{m}_{12} \subset \mathfrak{g}_L$. Then Lemma 4.4.1 asserts that $\mathfrak{m}_{12}$ is a Berger algebra. Similarly, we introduce the operators $\widehat{R}_{ij} : \mathfrak{so}(g) \to \mathfrak{so}(g)$ for arbitrary labels $i < j$. We then naturally define the operator

$$R = \sum_{i<j} \widehat{R}_{ij}. \tag{4.5.6}$$

To add rigour, we define an operator $R : \mathfrak{so}(g) \longrightarrow \mathfrak{g}_L$ via (4.5.6) such that

$$R\begin{pmatrix} X_{11} & X_{12} & \cdots & X_{1k} \\ X_{21} & X_{22} & \cdots & X_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ X_{k1} & X_{k2} & \cdots & X_{kk} \end{pmatrix} = \begin{pmatrix} 0 & R_{12}(X_{12}) & \cdots & R_{1k}(X_{1k}) \\ R_{21}(X_{21}) & 0 & \cdots & R_{2k}(X_{2k}) \\ \vdots & \vdots & \ddots & \vdots \\ R_{k1}(X_{k1}) & R_{k2}(X_{k2}) & \cdots & 0 \end{pmatrix} \tag{4.5.7}$$

and with the requirement that R acts on each block X_{ij} independently (compare with the proof of Lemma 4.4.1). Strictly speaking, each of its components $R_{ij} : X_{ij} \mapsto R_{ij}(X_{ij})$ is defined by

$$R_{ij}(X_{ij}) = L_i^{n_{ij}-1} X_{ij} + L_i^{n_{ij}-2} X_{ij} L_j + \cdots + X_{ij} L_j^{n_{ij}-1}, \tag{4.5.8}$$

where $n_{ij} = \max\{n_i, n_j\}$, and n_i, n_j are sizes of the nilpotent Jordan blocks L_i and L_j. Thus, we have finally arrived at the following proposition.

Proposition 4.5.9 *The operator R defined by* (4.5.7) *and* (4.5.8) *is a formal curvature tensor. Moreover,* $\operatorname{Im} R = \mathfrak{g}_L$ *and, therefore,* $\mathfrak{g}_L$ *is a Berger algebra.*

Proof. Firstly, by virtue of our construction we have that each $\widehat{R}_{ij}$ acts only on the blocks X_{ij} and X_{ji} and does not interfere with other blocks at all. Secondly, since each $\widehat{R}_{ij}$ is a formal curvature tensor so is R by linearity. Lastly, we have already explained that the image of $\widehat{R}_{ij}$ is the subalgebra $\mathfrak{m}_{ij}$. Then, by Corollary 3.1.9 we immediately have

$$\operatorname{Im} R = \sum_{i<j} \operatorname{Im} \widehat{R}_{ij} = \sum_{i<j} \mathfrak{m}_{ij} = \mathfrak{g}_L,$$

as required. □

Evidently, the truth of this proposition manifests into the case (♠) of **Theorem B**. In other words, the following theorem has been proven so far.

Theorem 4.5.10 *Let g be a pseudo-Riemannian metric (not necessarily Lorentzian) and L be a singular g-symmetric operator with a single real eigenvalue. Then its centraliser $\mathfrak{g}_L$ in $\mathfrak{so}(g)$ is a Berger algebra.*

Having established this result it now only remains to show that the complex case (♢) does not constitute any difficulty. Suppose that $L : V \longrightarrow V$ is a real g-symmetric operator with a pair of complex conjugate eigenvalues λ and $\overline{\lambda}$. Since Proposition 3.1.1 is valid for both real and complex vector spaces, we are naturally motivated to consider the complex canonical matrix representations for the operator L and the metric g. Thus, we first need to complexify L and g. We do this in the following manner. It is well-known from linear algebra that on any real vector space V there exists a canonical complex structure J, which by definition is an automorphism of V so that $J^2 = -\mathrm{Id}|_{\mathrm{V}}$. Now, the idea is to define a complex structure on V such that its i and $-i$ eigenspaces in $V^{\mathbb{C}}$ respectively coincide with the generalised eigenspaces V_λ and $V_{\overline{\lambda}}$ of L in V. This can be easily done by virtue of Proposition 3.1.1 which is still valid for the case of complex conjugate eigenvalues. Indeed, suppose that L has two complex conjugate eigenvalues with $\lambda = a + ib$ and $b > 0$. Then with respect to the canonical basis from Proposition 3.1.1 we have that L and J are given by the matrices

$$L = \begin{pmatrix} a & -b & 1 & 0 \\ b & a & 0 & 1 \\ 0 & 0 & a & b \\ 0 & 0 & -b & 0 \end{pmatrix} \qquad \text{and} \qquad J = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

This immediately allows us to think of L as a complex operator. Furthermore, as the complex structure J commutes with the operator L, it is g-symmetric. It is this property of J that enables us to complexify the metric g. We do this by introducing the bilinear form

$$g^{\mathbb{C}} : V \times V \longrightarrow \mathbb{C}$$

defined by

$$g^{\mathbb{C}}(u, v) = g(Ju, v) + ig(u, v). \tag{4.5.11}$$

We next observe that L viewed as a complex operator is $g^{\mathbb{C}}$-symmetric. Indeed, for any vectors $u, v \in V$ we certainly have

$$g^{\mathbb{C}}(Lu, v) = g(JLu, v) + ig(Lu, v) = g(LJu, v) + ig(u, Lv) = g(Ju, Lv) + ig(u, Lv) = g^{\mathbb{C}}(u, Lv).$$

To put it another way, we consider a complex coordinate system in V with respect to which the complex operator L and the complex metric $g^{\mathbb{C}}$ have matrix representations[3] given by Proposition 3.1.1. Moreover, in this complex coordinate system Propositions 3.1.6 and 3.1.8 remain valid for the complex Lie algebras $\mathfrak{so}(g^{\mathbb{C}})$ and $\mathfrak{g}_L^{\mathbb{C}}$. Notice also the obvious inclusions $\mathfrak{g}_L^{\mathbb{C}} \subset \mathfrak{so}(g^{\mathbb{C}}) \subset \mathfrak{so}(g)$. Thus, to show that $\mathfrak{g}_L^{\mathbb{C}}$ is a Berger algebra we first need to establish the $(k;n)$-case. In fact, Lemma 4.4.1 easily generalises to the complex case. Being a purely algebraic statement, this lemma does not cease to be valid for a complex operator L and a complex bilinear form $g^{\mathbb{C}}$. It only necessitates defining $R : \mathfrak{so}(g^{\mathbb{C}}) \longrightarrow \mathfrak{g}_L$ by means of the magic formula for a minimal polynomial $p_{min}(t) = (t-\lambda)^n$. However, in order to complete the proof in the general case, we need to take care of two things. Firstly, we have to prove the theorem on the larger Lie algebra $\mathfrak{so}(g)$. This is not a big issue since we can always define $R : \mathfrak{so}(g) \longrightarrow \mathfrak{g}_L$ and consider its restriction to the subalgebra $\mathfrak{so}(g^{\mathbb{C}})$. If the image of the restriction coincides with $\mathfrak{g}_L$, so will R. Secondly, and more importantly, the operator R must be real. For this reason we need to consider the real minimal polynomial $p_{min}(t) = (t-\lambda)^n(t-\overline{\lambda})^n$ instead. Then, thinking of L and $X \in \mathfrak{so}(g^{\mathbb{C}})$ as complex operators and using $(L - \lambda \cdot Id)^n = 0$, we compute the following.

$$
\begin{aligned}
R(X) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\Big((L-\lambda\cdot Id+tX)^n(L-\overline{\lambda}\cdot Id+tX)^n\Big) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\Big((L-\lambda\cdot Id)+tX\Big)^n\cdot(L-\overline{\lambda}\cdot Id)^n + \\
&+ (L-\lambda\cdot Id)^n\cdot\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\Big((L-\overline{\lambda}\cdot Id)+tX\Big)^n \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\Big((L-\lambda\cdot Id)+tX\Big)^n\cdot(L-\overline{\lambda}\cdot Id)^n.
\end{aligned}
$$

[3]Notice that the block-matrix structure for both L and $g^{\mathbb{C}}$ remains as in the real case. The only difference lies in the complex entries.

We thus reach the following conclusion. The operator in the first bracket is clearly the same as the one in Lemma 4.4.1 and therefore its image coincides with $\mathfrak{g}_L$. Furthermore, multiplying by the non-degenerate matrix $(L - \overline{\lambda} \cdot Id)^n$ does not change the dimension of the image of R and thus $\mathrm{Im} R = \mathfrak{g}_L$.

Note that the proof of Proposition 4.5.9 was merely a block-wise generalisation of the $(k; n)$-case. Therefore, the truth of its complex counterpart is immediately guaranteed by the complex version of Lemma 4.4.1 that we have already mentioned above. This clarifies the complex case ($\diamondsuit$) and we close this chapter as it deserves. Namely, we proved the following theorem.

Theorem B *Let g be a pseudo-Riemannian metric (not necessarily Lorentzian) and L be a singular g-symmetric operator with centraliser $\mathfrak{g}_L$ in $\mathfrak{so}(g)$. Then $\mathfrak{g}_L$ is a Berger algebra.*

Chapter 5

Pseudo-Riemannian metrics realising $\mathfrak{g}_L$ as a holonomy algebra

In the present chapter we shall add some geometrical flavour into our work. This endeavour of ours will ultimately affirm that $\mathfrak{g}_L$ is indeed a holonomy algebra. We begin with the recollection of some properties of covariantly constant $(1,1)$-tensor fields. In the second section we give a concise description of the main problem of this chapter and discuss the strategy for its resolution. We thenceforward elaborate the construction of the pseudo-Riemannian metrics realising $\mathfrak{g}_L$ as a holonomy algebra.

5.1 Covariantly constant $(1,1)$-tensor fields

The covariantly constant $(1,1)$-tensor fields will be of paramount importance and of constant use in this chapter. For these reasons, we briefly introduce a few of their properties relevant to our purposes. At this juncture it is worth reminding the reader that the covariant derivative ∇_ξ is linear. Furthermore, it satisfies the Leibniz rule as well. This means that for any two tensor fields T_1 and T_2 we have the identity

$$\nabla_\xi\big(T_1T_2\big) = \big(\nabla_\xi T_1\big)T_2 + T_1\big(\nabla_\xi T_2\big),$$

for any tangent vector ξ. It must be noticed that T_1 and T_2 can be tensor fields of any type including vectors (covectors) in particular. To make some good use of this property, recall that for a given $(1,1)$-tensor field L one can define the map

$$\mathscr{N}_L : \Gamma(\mathrm{TM}) \times \Gamma(\mathrm{TM}) \longrightarrow \Gamma(\mathrm{TM})$$

by

$$\mathscr{N}_L(\xi, \eta) = [L\xi, L\eta] - L[L\xi, \eta] - L[\xi, L\eta] + L^2[\xi, \eta] \tag{5.1.1}$$

for any $\xi, \eta \in \Gamma(TM)$. This map is de facto a $(1,2)$-tensor and is known in the literature as the *Nijenhuis tensor*. In order to prove that $\mathscr{N}_L$ is indeed a tensor we first observe that the bilinearity of the commutator $[\cdot, \cdot]$ immediately implies

$$\mathscr{N}_L(\xi_1 + \xi_2, \eta) = \mathscr{N}_L(\xi_1, \eta) + \mathscr{N}_L(\xi_2, \eta)$$

for any $\xi, \eta \in \Gamma(\mathrm{TM})$. Our goal, however, is to show that for any two smooth functions f_1 and f_2 we have that

$$\mathscr{N}_L(f_1\xi_1 + f_2\xi_2, \eta) = f_1\mathscr{N}_L(\xi_1, \eta) + f_2\mathscr{N}_L(\xi_2, \eta).$$

Clearly, it is sufficient to establish the truth of

$$\mathscr{N}_L(f\xi, \eta) = f\mathscr{N}_L(\xi, \eta)$$

for any smooth function f. It is a straightforward computation to see that for any smooth function f and any $\xi, \eta \in \Gamma(\mathrm{TM})$ we have the following identity

$$[f\xi, \eta] = f[\xi, \eta] - (\eta f)\xi.$$

It is by means of this latter that we easily compute

$$[L(f\xi), L\eta] = f[\xi,\eta] - L\eta(f)L\xi,$$

$$L[L(f\xi),\eta] = fL[L\xi,\eta] - \eta(f)L^2\xi,$$

$$L[f\xi, L\eta] = fL[\xi, L\eta] - L\eta(f)L\xi,$$

$$L^2[f\xi, L\eta] = fL^2[\xi, L\eta] - \eta(f)L^2\xi.$$

It is now obvious that $\mathscr{N}_L(f\xi,\eta) = f\mathscr{N}_L(\xi,\eta)$, hence $\mathscr{N}_L$ is a tensor. We shall next see that in our case $\mathscr{N}_L$ actually vanishes. Since ∇ is torsion-free we have that $[\xi,\eta] = \nabla_\xi\eta - \nabla_\eta\xi$ and thus naturally rewrite (5.1.1) as

$$\begin{aligned}\mathscr{N}_L(\xi,\eta) &= \nabla_{L\xi}(L\eta) - \nabla_{L\eta}(L\xi) - L\Big(\nabla_{L\xi}\eta - \nabla_\eta(L\xi)\Big) - L\Big(\nabla_\xi(L\eta) - \nabla_{L\eta}\xi\Big) + \\ &+ L^2\Big(\nabla_\xi\eta - \nabla_\eta\xi\Big).\end{aligned}$$

Now, using the Leibniz rule for ∇_ξ we easily obtain

$$\begin{aligned}\mathscr{N}_L(\xi,\eta) &= L\nabla_{L\xi}\eta - \Big(\nabla_{L\xi}L\Big)\eta - L\nabla_{L\eta}\xi - \Big(\nabla_{L\eta}L\Big)\xi - L\nabla_{L\xi}\eta + L^2\nabla_\eta\xi + \\ &+ L\Big(\nabla_\eta L\Big)\xi - L^2\nabla_\xi\eta - L\Big(\nabla_\xi L\Big)\eta + L\nabla_{L\eta}\xi + L^2\nabla_\xi\eta - L^2\nabla_\eta\xi \\ &= -\Big(\nabla_{L\xi}L\Big)\eta - \Big(\nabla_{L\eta}L\Big)\xi + L\Big(\nabla_\eta L\Big)\xi - L\Big(\nabla_\xi L\Big)\eta.\end{aligned}$$

It is obvious that $\nabla L = 0$ implies $\nabla_\xi L = 0$ for any ξ. But then $\nabla_{L\xi}L = 0$ as well, hence $\mathscr{N}_L(\xi,\eta) = 0$.

It is well-known that every operator naturally splits into the sum of its symmetric and skew-symmetric parts. Write $L = L^s + L^a$ where L^s and L^a are the symmetric and skew-symmetric parts respectively. Now naturally arises the question whether or not both L^s and L^a are covariantly constant provided that L is? By assumption we have that $\nabla_\xi L = 0$. Then, it is not difficult to see that $(\nabla_\xi L)^* = \nabla_\xi(L^*)$. On the one hand, we have that

$$0 = (\nabla_\xi L)^* = (\nabla_\xi L^s)^* + (\nabla_\xi L^a)^* = \nabla_\xi(L^s)^* + \nabla_\xi(L^a)^* = \nabla_\xi L^s - \nabla_\xi L^a,$$

and therefore $\nabla_\xi L^s = \nabla_\xi L^a$. On the other hand, $0 = \nabla_\xi L = \nabla_\xi(L^s + L^a)$ implies that $\nabla_\xi L^s = -\nabla_\xi L^a$ and thus $\nabla_\xi L^s = \nabla_\xi L^a = 0$.

We next focus our attention on the eigenvalues of L. Are they constant provided $\nabla L = 0$? The answer is that they are. To see this, let us choose a curve $\gamma : [0,1] \longrightarrow M$ with ends $\gamma(0) = p$ and $\gamma(1) = q$. Consider $\mathrm{T}_p\mathrm{M}$ and suppose its basis is $e_1(0), ..., e_n(0)$. Now, let us parallelly transport it along the curve γ to the point q. Then, as parallel transport is an isomorphism of tangent spaces, we only end up with a basis for $\mathrm{T}_q\mathrm{M}$, say $e_1(1), ..., e_n(1)$. Further, as $\nabla_\gamma L = 0$ is tantamount to $P_\gamma L(p) P_\gamma^{-1} = L(q)$, where P_γ is precisely the parallel transport just mentioned, we conclude that the matrix of L remains the same with respect to both the bases $e_1(0), ..., e_n(0)$ and $e_1(1), ..., e_n(1)$. Thus, the eigenvalues of any covariantly constant $(1,1)$-tensor field are necessarily constant. In summary, we have proven the following theorem.

Theorem 5.1.2 *Let* M *be a manifold with Levi-Civita connection* ∇ *and suppose that* $L : \mathrm{TM} \longrightarrow \mathrm{TM}$ *is a covariantly constant* $(1,1)$*-tensor field, i.e.* $\nabla L = 0$. *Then*

(i) the eigenvalues of L *are constant,*

(ii) both the symmetric and skew-symmetric parts of L *are covariantly constant,*

(iii) the Nijenhuis tensor for L *vanishes.*

It must be noticed that part (i) of this last theorem will be of particular importance for our approach.

We conclude this section with the following discussion. For the purposes of this investigation we need to understand how the curvature operator $R(\xi,\eta)$ acts upon a given $(1,1)$-tensor field. In order to distinguish this case to the usual one when the curvature acts on tangent vector fields, we shall write

$$R_{\xi,\eta}(L) = \nabla_\xi \nabla_\eta(L) - \nabla_\eta \nabla_\xi(L) - \nabla_{[\xi,\eta]}(L),$$

for some arbitrary $(1,1)$-tensor field L. Certainly, for the sake of consistency, we shall write

$$R_{\xi,\eta}(\zeta) = R(\xi,\eta)(\zeta),$$

for any tangent vector field ζ. Now, a back of the envelope computation yields

$$R_{\xi,\eta}(T_1T_2) = \big(R_{\xi,\eta}(T_1)\big)T_2 + T_1\big(R_{\xi,\eta}(T_2)\big), \tag{5.1.3}$$

for any two $(1,1)$-tensor fields T_1 and T_2. Since (5.1.3) is nothing but a reincarnation of the Lebniz rule for the covariant derivative, it holds true for any types of tensor fields. We make immediate use of this property by proving the following handy proposition.

Proposition 5.1.4 *The action of the curvature operator upon a* $(1,1)$*-tensor field* L *is determined by* $R_{\xi,\eta}(L) = [R(\xi,\eta), L]$.

Proof. Let ζ be arbitrary tangent vector field. Then, by notation and virtue of (5.1.3) we easily have

$$R_{\xi,\eta}(L)\zeta = R_{\xi,\eta}(L\zeta) - LR_{\xi,\eta}(\zeta) = \Big(R(\xi,\eta)L - LR(\xi,\eta)\Big)\zeta = [R(\xi,\eta), L]\zeta,$$

as required. □

5.2 Description of the problem

In order to achieve the final goal of this dissertation, we need to explicitly construct pseudo-Riemannian metrics which realise $\mathfrak{g}_L$ as its holonomy algebra. To put it another way, we shall settle by the end of this chapter the following geometric problem.

Problem 4 *Let us consider the linear operator* $L : \mathrm{T}_{u_0}\mathrm{M} \to \mathrm{T}_{u_0}\mathrm{M}$ *for a given manifold* M*. We then wish to find (locally!) a pseudo-Riemannian metric* g *on* M *and a* $(1,1)$*-tensor field* $L(u)$ *such that*

1. $\nabla L(u) = 0$ *with the initial condition* $L(u_0) = L$*,*

2. $\mathfrak{hol}\,(\nabla) = \mathfrak{g}_L$.

The following three remarks must be brought into prominence. Firstly, the condition $\nabla L(u) = 0$ guarantees that $\mathfrak{hol}\,(\nabla) \subset \mathfrak{g}_L$ (see Proposition 3.5.4). Secondly, we have the inclusion $\operatorname{Im} R(u_0) \subset \mathfrak{hol}\,(\nabla)$ where $u_0 \in \mathrm{M}$ is a fixed point and R is now the Riemann curvature tensor of g[1]. Lastly, by virtue of the Ambrose-Singer holonomy theorem, to show that the second condition holds true it suffices to show that $R(u_0)$ coincides with the formal curvature tensor defined previously by means of the magic formula, i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} p_{\min}(L + tX).$$

Notice that this last remark naturally requires L to be g-symmetric and we shall assume it as such throughout the entire chapter. Now, having stated the problem let us briefly describe our strategy towards its solution. We shall rely upon two important geometric facts.

[1]Opposed to the previous chapter, by the end of the present chapter R will stand for the Riemann curvature tensor .

Proposition 5.2.1 *For every metric g there exists a local coordinate system such that $\frac{\partial g_{ij}}{\partial u^\alpha}(0) = 0$ for all i, j, α. In particular, in this coordinate system we have $\Gamma^k_{ij}(0) = 0$ and the components of the curvature tensor at $u_0 = 0$ are defined as some combinations of the second derivatives of g.*

This is a standard result and the reader may wish to refer to [CCL] for a proof. Its usefulness lies in the fact that it enables us to simplify our computations at the point $u_0 = 0$. This will essentially result in ignoring the linear terms of the metric g. Due to A. P. Shirokov [Shi], the second geometric result suggests a remarkable property of covariantly constant $(1,1)$-tensor fields.

Theorem 5.2.2 (Shirokov) *If L satisfies $\nabla L = 0$ for a symmetric connection ∇, then there exists a local coordinate system $u^1, \ldots, u^n$ in which L is constant.*

This theorem is essentially our starting point. Suppose that $u^1, ..., u^n$ is the coordinate system from the Shirokov theorem. We shall soon see that in such a coordinate system, the equation $\nabla L = 0$ can be conveniently rewritten in the following form:

$$\left(\frac{\partial g_{ip}}{\partial u^\beta} - \frac{\partial g_{i\beta}}{\partial u^p}\right) L^\beta_k = \left(\frac{\partial g_{i\beta}}{\partial u^k} - \frac{\partial g_{ik}}{\partial u^\beta}\right) L^\beta_p. \tag{5.2.3}$$

This equation is obviously linear and therefore if we represent g as a power series in u, then (5.2.3) must hold true for each term of this expansion. This motivates us to set $L(u) = const$ and then to look for metrics $g(u)$ in the "constant + quadratic" form, which in other words is

$$g_{ij}(u) = g^0_{ij} + \sum_{p,q} \mathcal{B}_{ij,pq} u^p u^q. \tag{5.2.4}$$

Clearly, $\mathcal{B}$ satisfies the obvious symmetry relations $\mathcal{B}_{ij,pq} = \mathcal{B}_{ji,pq}$ and $\mathcal{B}_{ij,pq} = \mathcal{B}_{ij,qp}$. It is this choice of "quadratic" metrics (5.2.4) that enables us to attack Problem 4 by algebraic means. In other words we are able to translate the original geometric problem into a purely algebraic one. This will be done in the following three steps. Firstly, we shall show the condition that L is g-symmetric reads

$$\mathcal{B}_{ij,pq}L^i_l = \mathcal{B}_{il,pq}L^i_j. \tag{5.2.5}$$

This simple property will be of constant use in most of our computations. Secondly, we shall derive one very useful formula for the curvature tensor in terms of $\mathcal{B}$. More precisely, we shall prove that the curvature tensor of g at the origin $u_0 = 0$ takes the following form

$$R^i_{\alpha\beta,k} = g^{0is}\Big(\mathcal{B}_{\beta s,\alpha k} + \mathcal{B}_{\alpha k,\beta s} - \mathcal{B}_{\beta k,\alpha s} - \mathcal{B}_{\alpha s,\beta k}\Big). \tag{5.2.6}$$

In particular, R (at the origin) depends linearly on $\mathcal{B}$, which is

$$R_{\lambda_1\mathcal{B}_1+\lambda_2\mathcal{B}_2} = \lambda_1 R_{\mathcal{B}_1} + \lambda_2 R_{\mathcal{B}_2}.$$

Thirdly, and more importantly, we shall show that the condition $\nabla L = 0$ amounts to the following equation for $\mathcal{B}$

$$\Big(\mathcal{B}_{ip,\beta q} - \mathcal{B}_{i\beta,pq}\Big)L^\beta_k = \Big(\mathcal{B}_{\beta i,kq} - \mathcal{B}_{ik,\beta q}\Big)L^\beta_p. \tag{5.2.7}$$

Thus, by virtue of these three facts, the realisation problem reduces to finding a $\mathcal{B}$ satisfying (5.2.5), (5.2.7) and such that (5.2.6) coincides with the formal curvature tensor as defined in Theorem 4.5.10. From the formal viewpoint, this is a system of linear equations on $\mathcal{B}$ which needs to be solved.

5.3 Prerequisites and lemmata

In this section, we discuss the technical results needed for our construction. More precisely, this section explicitly justifies the algebraic reformulation of Problem 4. We shall first quickly establish (5.2.5).

Lemma 5.3.1 *Let L be a constant g-symmetric operator. Then $g_{\alpha s}L^{\alpha}_{k} = g_{\alpha k}L^{\alpha}_{s}$. In particular, $\mathcal{B}_{\alpha s,pq}L^{\alpha}_{k} = \mathcal{B}_{\alpha k,pq}L^{\alpha}_{s}$.*

Proof. By definition for any two vectors ξ and η, we have

$$g(L\xi, \eta) = g(\xi, L\eta).$$

Writing both sides of this equation in coordinates we have the following consecutive implications

$$g_{\alpha\beta}(L\xi)^{\alpha}\eta^{\beta} = g_{\alpha\beta}\xi^{\alpha}(L\eta)^{\beta}$$
$$\Downarrow$$
$$g_{\alpha\beta}L^{\alpha}_{k}\xi^{k}\eta^{\beta} = g_{\alpha\beta}L^{\beta}_{s}\xi^{\alpha}\eta^{\beta}$$
$$\Downarrow$$
$$g_{\alpha\beta}L^{\alpha}_{k}\delta^{\beta}_{s}\xi^{k}\eta^{s} = g_{\alpha\beta}L^{\beta}_{s}\delta^{\alpha}_{k}\xi^{k}\eta^{s}$$
$$\Downarrow$$
$$g_{\alpha s}L^{\alpha}_{k} = g_{k\beta}L^{\beta}_{s}.$$

Since α and β are summation indices, the first relation holds true. Moreover, it immediately implies the second, as $g^{0}_{\alpha s}L^{\alpha}_{k} = g^{0}_{\alpha k}L^{\alpha}_{s}$ holds true as well. ◻

To continue, let us recall that the Christoffel symbols for a metric g are given by

$$\Gamma^{l}_{ij} = \frac{1}{2}g^{lp}\left(\frac{\partial g_{pj}}{\partial u^{i}} + \frac{\partial g_{ip}}{\partial u^{j}} - \frac{\partial g_{ij}}{\partial u^{p}}\right)$$

and that the components of the curvature tensor for g are computed via

$$R^{l}_{ijk} = \frac{\partial \Gamma^{l}_{jk}}{\partial u^{i}} - \frac{\partial \Gamma^{l}_{ik}}{\partial u^{j}} + \Gamma^{r}_{jk}\Gamma^{l}_{ir} - \Gamma^{r}_{ik}\Gamma^{l}_{jr}.$$

Then using the former we easily settle the following lemma.

Lemma 5.3.2 *The Christoffel's symbols for the metric $g_{ij}(u) = g^0_{ij} + \mathcal{B}_{ij,pq}u^p u^q$ are given by*

$$\Gamma^i_{\beta k} = g^{0is}\Big(\mathcal{B}_{sk,\beta t} + \mathcal{B}_{\beta s,kt} - \mathcal{B}_{\beta k,st}\Big)u^t.$$

Proof. The following straightforward computation affirms the claim.

$$\begin{aligned}
\Gamma^i_{\beta k} &= \frac{1}{2}g^{0is}\Big(\frac{\partial g_{sk}}{\partial u^\beta} + \frac{\partial g_{\beta s}}{\partial u^k} - \frac{\partial g_{\beta k}}{\partial u^s}\Big) \\
&= \frac{1}{2}g^{0is}\Big(\mathcal{B}_{sk,rt}\frac{\partial(u^r u^t)}{\partial u^\beta} + \mathcal{B}_{\beta s,rt}\frac{\partial(u^r u^t)}{\partial u^k} - \mathcal{B}_{\beta k,rt}\frac{\partial(u^r u^t)}{\partial u^s}\Big) \\
&= \frac{1}{2}g^{0is}\Big(\mathcal{B}_{sk,rt}(\delta^r_\beta u^t + u^r\delta^t_\beta) + \mathcal{B}_{\beta s,rt}(\delta^r_k u^t + u^r\delta^t_k) - \mathcal{B}_{\beta k,rt}(\delta^r_s u^t + u^r\delta^t_s)\Big) \\
&= g^{0is}\Big(\mathcal{B}_{sk,\beta t} + \mathcal{B}_{\beta s,kt} - \mathcal{B}_{\beta k,st}\Big)u^t.
\end{aligned}$$

□

Now, by means of this lemma, we easily prove further

Lemma 5.3.3 *Let $g_{ij}(u) = g^0_{ij} + \mathcal{B}_{ij,pq}u^p u^q$. Then the components of its curvature tensor at the origin are given by*

$$R^i_{\alpha\beta,k} = g^{0is}\Big(\mathcal{B}_{\beta s,\alpha k} + \mathcal{B}_{\alpha k,\beta s} - \mathcal{B}_{\beta k,\alpha s} - \mathcal{B}_{\alpha s,\beta k}\Big). \tag{5.3.4}$$

Proof. By virtue of Proposition 5.2.1 we have that $\Gamma^\alpha_{\beta\gamma}(0) = 0$ for all α, β, γ, and so $R^i_{\alpha\beta,k} = \frac{\partial \Gamma^i_{\beta k}}{\partial u^\alpha} - \frac{\partial \Gamma^i_{\alpha k}}{\partial u^\beta}$. Then using Lemma 5.3.2 we immediately obtain

$$\begin{aligned}
R^i_{\alpha\beta,k} &= g^{0is}\bigg(\Big(\mathcal{B}_{sk,\beta t} + \mathcal{B}_{\beta s,kt} - \mathcal{B}_{\beta k,st}\Big)\delta^t_\alpha - \Big(\mathcal{B}_{sk,\alpha t} + \mathcal{B}_{\alpha s,kt} - \mathcal{B}_{\alpha k,st}\Big)\delta^t_\beta\bigg) \\
&= g^{0is}\Big(\mathcal{B}_{\beta s,\alpha k} + \mathcal{B}_{\alpha k,\beta s} - \mathcal{B}_{\beta k,\alpha s} - \mathcal{B}_{\alpha s,\beta k}\Big).
\end{aligned}$$

□

This Lemma clearly justifies the formula (5.2.6). However, it remains to show that at the origin R depends linearly on $\mathcal{B}$. We establish this fact with the following proposition.

Proposition 5.3.5 *Let g and $\tilde{g}$ be two metrics of the above "quadratic" form with equal constant terms. Then, at the origin, the curvature tensor for $g+\tilde{g}$ is proportional to the sum of the curvature tensors for g and $\tilde{g}$ respectively.*

Proof. Write $R^i_{\alpha\beta,k}$ and $\tilde{R}^i_{\alpha\beta,k}$ for the curvature tensors of g and $\tilde{g}$ respectively. Write the two metrics in coordinates as

$$g_{ij} = g^0_{ij} + \mathcal{B}_{ij,pg} u^p u^q$$

and

$$\tilde{g}_{ij} = g^0_{ij} + \tilde{\mathcal{B}}_{ij,pg} u^p u^q.$$

We then have that the components of the metric $g+\tilde{g}$ are given by

$$(g+\tilde{g})_{ij} = 2g^0_{ij} + \Big(\mathcal{B}_{ij,pq} + \tilde{\mathcal{B}}_{ij,pg}\Big) u^p u^q.$$

Thus we compute the curvature tensor for the metric $g+\tilde{g}$, denoted by $\widehat{R}^i_{\alpha\beta,k}$.

$$\begin{aligned}
\widehat{R}^i_{\alpha\beta,k} &= 2g^{0^{is}}\Big(\widehat{\mathcal{B}}_{\beta s,\alpha k} + \widehat{\mathcal{B}}_{\alpha k,\beta s} - \widehat{\mathcal{B}}_{\beta k,\alpha s} - \widehat{\mathcal{B}}_{\alpha s,\beta k}\Big) \\
&= 2g^{0^{is}}\Big(\mathcal{B}_{\beta s,\alpha k} + \mathcal{B}_{\alpha k,\beta s} - \mathcal{B}_{\beta k,\alpha s} - \mathcal{B}_{\alpha s,\beta k}\Big) + 2g^{0^{is}}\Big(\tilde{\mathcal{B}}_{\beta s,\alpha k} + \tilde{\mathcal{B}}_{\alpha k,\beta s} - \tilde{\mathcal{B}}_{\beta k,\alpha s} - \tilde{\mathcal{B}}_{\alpha s,\beta k}\Big) \\
&= 2\Big(R^i_{\alpha\beta,k} + \tilde{R}^i_{\alpha\beta,k}\Big).
\end{aligned}$$

□

We finally need to settle (5.2.7). Let L be an operator such that in some coordinate system it is independent of the local coordinates, i.e., $\frac{\partial L}{\partial u^\alpha} = 0$ for all u^α. We are interested in solving the following system of equations

$$\nabla L = 0. \tag{5.3.6}$$

By solving these equations we mean that L is known and the metric is unknown. Recall that in components (5.3.6) is written as

$$\nabla_k L^i_j = \frac{\partial L^i_j}{\partial u^k} + \mathit{\Gamma}^i_{lk} L^l_j - \mathit{\Gamma}^l_{jk} L^i_l = 0. \tag{5.3.7}$$

Then, as L is constant, we can rewrite our original equation (5.3.6) in the following simpler form

$$\mathit{\Gamma}^i_{lk} L^l_j = \mathit{\Gamma}^l_{jk} L^i_l. \tag{5.3.8}$$

Now, working simultaneously on both sides of (5.3.8), we have that

$$g^{is}\left(\tfrac{\partial g_{ks}}{\partial u^l} + \tfrac{\partial g_{sl}}{\partial u^k} - \tfrac{\partial g_{lk}}{\partial u^s}\right) L^l_j = g^{lp}\left(\tfrac{\partial g_{kp}}{\partial u^j} + \tfrac{\partial g_{jp}}{\partial u^k} - \tfrac{\partial g_{jk}}{\partial u^p}\right) L^i_l$$

$$\Downarrow$$

$$g_{\alpha i} \cdot g^{is}\left(\tfrac{\partial g_{ks}}{\partial u^l} + \tfrac{\partial g_{sl}}{\partial u^k} - \tfrac{\partial g_{lk}}{\partial u^s}\right) L^l_j = g_{\alpha i} \cdot g^{lp}\left(\tfrac{\partial g_{kp}}{\partial u^j} + \tfrac{\partial g_{jp}}{\partial u^k} - \tfrac{\partial g_{jk}}{\partial u^p}\right) L^i_l$$

Since $g_{\alpha i}L^i_l = g_{li}L^i_\alpha$, we have further

$$\delta^s_\alpha \left(\frac{\partial g_{ks}}{\partial u^l} + \frac{\partial g_{sl}}{\partial u^k} - \frac{\partial g_{lk}}{\partial u^s}\right) L^l_j = \delta^p_i \left(\frac{\partial g_{kp}}{\partial u^j} + \frac{\partial g_{jp}}{\partial u^k} - \frac{\partial g_{jk}}{\partial u^p}\right) L^i_\alpha$$

$$\Downarrow$$

$$\left(\frac{\partial g_{k\alpha}}{\partial u^l} + \frac{\partial g_{\alpha l}}{\partial u^k} - \frac{\partial g_{lk}}{\partial u^\alpha}\right) L^l_j = \left(\frac{\partial g_{ki}}{\partial u^j} + \frac{\partial g_{ji}}{\partial u^k} - \frac{\partial g_{jk}}{\partial u^i}\right) L^i_\alpha$$

$$\Downarrow$$

$$\left(\frac{\partial g_{k\alpha}}{\partial u^l} + \frac{\partial g_{\alpha l}}{\partial u^k} - \frac{\partial g_{lk}}{\partial u^\alpha}\right) \delta^l_i L^i_j = \left(\frac{\partial g_{ki}}{\partial u^j} + \frac{\partial g_{ji}}{\partial u^k} - \frac{\partial g_{jk}}{\partial u^i}\right) L^i_\alpha$$

$$\Downarrow$$

$$\left(\frac{\partial g_{k\alpha}}{\partial u^i} + \frac{\partial g_{\alpha i}}{\partial u^k} - \frac{\partial g_{ik}}{\partial u^\alpha}\right) L^i_j = \left(\frac{\partial g_{ki}}{\partial u^j} + \frac{\partial g_{ji}}{\partial u^k} - \frac{\partial g_{jk}}{\partial u^i}\right) L^i_\alpha$$

$$\Downarrow$$

$$\left(\frac{\partial g_{k\alpha}}{\partial u^i} - \frac{\partial g_{ik}}{\partial u^\alpha}\right) L^i_j + \frac{\partial}{\partial u^k}(g_{\alpha i}L^i_j) = \left(\frac{\partial g_{ki}}{\partial u^j} - \frac{\partial g_{jk}}{\partial u^i}\right) L^i_\alpha + \frac{\partial}{\partial u^k}(g_{ji}L^i_\alpha)$$

Clearly, $\frac{\partial}{\partial u^k}(g_{\alpha i}L^i_j) = \frac{\partial}{\partial u^k}(g_{ji}L^i_\alpha)$, and so we obtain the following linear differential equation for the metric g_{ij}

$$\left(\frac{\partial g_{k\alpha}}{\partial u^i} - \frac{\partial g_{ik}}{\partial u^\alpha}\right) L^i_j = \left(\frac{\partial g_{ki}}{\partial u^j} - \frac{\partial g_{jk}}{\partial u^i}\right) L^i_\alpha.$$

Putting $\alpha = p, i = \beta, k = i$ and $j = k$ we have that (5.3.6) reduces to a system of first order linear differential equations for g_{ij}, i.e.,

$$\left(\frac{\partial g_{ip}}{\partial u^\beta} - \frac{\partial g_{i\beta}}{\partial u^p}\right) L^\beta_k = \left(\frac{\partial g_{i\beta}}{\partial u^k} - \frac{\partial g_{ik}}{\partial u^\beta}\right) L^\beta_p. \tag{5.3.9}$$

Since the metric is of the form $g_{ij}(u) = g^0_{ij} + \mathcal{B}_{ij,pq}u^p u^q$, we have the following

$$\begin{aligned}
\left(\frac{\partial g_{ip}}{\partial u^\beta} - \frac{\partial g_{i\beta}}{\partial u^p}\right) L^\beta_k &= \left(\mathcal{B}_{ip,st}\frac{\partial}{\partial u^\beta}(u^s u^t) - \mathcal{B}_{i\beta,st}\frac{\partial}{\partial u^p}(u^s u^t)\right) L^\beta_k \\
&= \Big(\mathcal{B}_{ip,st}(\delta^s_\beta u^t + u^s\delta^t_\beta) - \mathcal{B}_{i\beta,st}(\delta^s_p u^t + u^s\delta^t_p)\Big) L^\beta_k \\
&= 2\Big(\mathcal{B}_{ip,\beta t} - \mathcal{B}_{i\beta,pt}\Big) u^t L^\beta_k.
\end{aligned}$$

Similarly, we have

$$\left(\frac{\partial g_{i\beta}}{\partial u^k} - \frac{\partial g_{ik}}{\partial u^\beta}\right) L^\beta_p = 2\Big(\mathcal{B}_{i\beta,kt} - \mathcal{B}_{ik,\beta t}\Big) u^t L^\beta_p.$$

Now, as L does not depend on u^t, we easily conclude that (5.3.9) is equivalent to

$$\Big(\mathcal{B}_{ip,\beta t} - \mathcal{B}_{i\beta,pt}\Big) L^\beta_k = \Big(\mathcal{B}_{i\beta,kt} - \mathcal{B}_{ik,\beta t}\Big) L^\beta_p. \tag{5.3.10}$$

Conversely, suppose that there exist bilinear forms $\mathcal{B}_{ij,pq}$ such that (5.3.10) holds. Then, by virtue of Lemma 5.3.1 and the symmetry of $\mathcal{B}$'s we have that $\mathcal{B}_{p\beta,it}L^\beta_k = \mathcal{B}_{\beta k,it}L^\beta_p$. Thus, starting from (5.3.10) we have the following

$$\Big(\mathcal{B}_{p\beta,it} + \mathcal{B}_{ip,\beta t} - \mathcal{B}_{i\beta,pt}\Big) L^\beta_k = \Big(\mathcal{B}_{\beta k,it} + \mathcal{B}_{i\beta,kt} - \mathcal{B}_{ik,\beta t}\Big) L^\beta_p$$

$$g^{jp}\Big(\mathcal{B}_{p\beta,it} + \mathcal{B}_{ip,\beta t} - \mathcal{B}_{i\beta,pt}\Big) L^\beta_k u^t = g^{pj}\Big(\mathcal{B}_{\beta k,it} + \mathcal{B}_{i\beta,kt} - \mathcal{B}_{ik,\beta t}\Big)\delta^\beta_j L^j_p u^t$$

$$\Gamma^j_{i\beta} L^\beta_k = g^{pj}\Big(\mathcal{B}_{jk,it} + \mathcal{B}_{ij,kt} - \mathcal{B}_{ik,jt}\Big) L^j_p u^t$$

$$\Gamma^j_{i\beta} L^\beta_k = \Gamma^p_{ik} L^j_p.$$

Clearly, the last line is exactly (5.3.8). We have therefore just proven the following proposition.

Proposition 5.3.11 *Let g be a metric of the type $g_{ij}(u) = g^0_{ij} + \mathcal{B}_{ij,pq}u^p u^q$ with Levi-Civita connection ∇. Suppose that L is a g-symmetric operator which is constant with respect to our local coordinate system. Then, $\nabla L = 0$ if and only if the following identity holds true*

$$\left(\mathcal{B}_{ip,\beta t} - \mathcal{B}_{i\beta,pt}\right)L^\beta_k = \left(\mathcal{B}_{i\beta,kt} - \mathcal{B}_{ik,\beta t}\right)L^\beta_p.$$

5.4 One special case of pseudo-Riemannian metrics realising $\mathfrak{g}_L$ as a holonomy algebra

We now aim our attention at one very special case. Following the discussion in Section 5.2 we know that in order to complete our inquiry we have to find a suitable $\mathcal{B}_{ij,\alpha\beta}$ satisfying the algebraic conditions (5.2.5), (5.2.6) and (5.2.7). It is not difficult to conjecture at this point that $\mathcal{B}_{ij,\alpha\beta}$ should be constructed by means of L and g^0. *But how and where to start?* Considering possibly the simplest example, in the first half of this section we shall make an "intelligent guess" of what $\mathcal{B}_{ij,\alpha\beta}$ would be. It will not be very difficult thereafter to predict the general form for $\mathcal{B}_{ij,\alpha\beta}$.

For the purposes of the present section we shall confine ourselves to considering a linear operator of the type $L^{(2;2)}$. According to the algebraic discussion given in Chapter 4 we have that its formal curvature tensor is given by the formula

$$R(X) = LX + XL, \tag{5.4.1}$$

where $X = \xi \wedge \eta = \xi \otimes g(\eta) - \eta \otimes g(\xi)$. Under these suppositions we prove the following lemma.

Lemma 5.4.2 *The components for the formal curvature tensor* (5.4.1) *are given by*

$$R^i_{\alpha\beta,k} = L^i_\alpha g_{\beta k} - L^i_\beta g_{\alpha k} + \delta^i_\alpha g_{\beta s} L^s_k - \delta^i_\beta g_{\alpha s} L^s_k. \tag{5.4.3}$$

Proof. For any three vectors ξ, η and ζ we have the following

$$\begin{aligned} R(\xi \wedge \eta)\zeta &= L\Big(\xi \otimes g(\eta) - \eta \otimes g(\xi)\Big)\zeta + \Big(\xi \otimes g(\eta) - \eta \otimes g(\xi)\Big)L\zeta \\ &= (L\xi) \cdot g(\eta, \zeta) - (L\eta) \cdot g(\xi, \zeta) + \xi \cdot g(\eta, L\zeta) - \eta \cdot g(\xi, L\zeta). \end{aligned}$$

Which, written in coordinates, is

$$\begin{aligned} \Big(R(\xi \wedge \eta)\zeta\Big)^i &= L^i_\alpha \xi^\alpha g_{\beta k} \eta^\beta \zeta^k - L^i_\beta \eta^\beta g_{\alpha k} \xi^\alpha \zeta^k + \xi^i g_{\beta k} \eta^\beta (L\zeta)^k - \eta^i g_{\alpha k} \xi^\alpha (L\zeta)^k \\ &= L^i_\alpha g_{\beta k} \xi^\alpha \eta^\beta \zeta^k - L^i_\beta g_{\alpha k} \xi^\alpha \eta^\beta \zeta^k + \delta^i_\alpha g_{\beta s} L^s_k \xi^\alpha \eta^\beta \zeta^k - \delta^i_\beta g_{\alpha s} L^s_k \xi^\alpha \eta^\beta \zeta^k \\ &= \Big(L^i_\alpha g_{\beta k} - L^i_\beta g_{\alpha k} + \delta^i_\alpha g_{\beta s} L^s_k - \delta^i_\beta g_{\alpha s} L^s_k\Big) \xi^\alpha \eta^\beta \zeta^k \\ &\equiv R^i_{\alpha\beta,k} \xi^\alpha \eta^\beta \zeta^k. \end{aligned}$$

□

Now, our intelligent guess is based upon the just proven lemma as well as formula (5.3.4). By construction $\mathcal{B}_{ij,\alpha\beta}$ is such that the Riemann curvature tensor for the quadratic metric $g_{ij}(u) = g^0_{ij} + \sum_{\alpha,\beta} \mathcal{B}_{ij,\alpha\beta} u^\alpha u^\beta$ coincides with the formal curvature tensor $R(X) = LX + XL$. To put it another way, $\mathcal{B}_{ij,\alpha\beta}$ must satisfy the following system of equations

$$\mathcal{B}_{\beta s,\alpha k} + \mathcal{B}_{\alpha k,\beta s} - \mathcal{B}_{\beta k,\alpha s} - \mathcal{B}_{\alpha s,\beta k} = L_{\alpha s} g_{\beta k} - L_{\beta s} g_{\alpha k} + L_{\beta k} g_{\alpha s} - L_{\alpha k} g_{\beta s}. \tag{5.4.4}$$

The right hand side of (5.4.4) is obtained from the right hand side of (5.4.9) using the identity $L_{ij} = g_{is} L^s_j$. Now, it is a matter of straightforward verification to see that

$$\mathcal{B}_{\beta s,\alpha k} = -L_{\beta s} g_{\alpha k} \quad \text{and} \quad \mathcal{B}_{\beta s,\alpha k} = -L_{\alpha k} g_{\beta s}$$

are two particular solutions of (5.4.4). Furthermore, since (5.4.4) is a system of linear equations then so is a linear combination of its solutions. This observation motivates us to look for $\mathcal{B}$'s which are linear combinations of $L_{\alpha k} g_{\beta s}$. Thus, our intelligent guess brought us to the following proposition.

Proposition 5.4.5 *Let g^0 be some constant metric and L be a g^0-symmetric operator which is constant in the local coordinate system $u^1, ..., u^n$ and is nilpotent of order 2. Consider the metric*

$$g_{ij}(u) = g^0_{ij} - \frac{1}{2} \sum_{\alpha,\beta} \Big(L_{ij} g^0_{\alpha\beta} + L_{\alpha\beta} g^0_{ij} \Big) u^\alpha u^\beta, \tag{5.4.6}$$

where $L_{ij} = g^0_{ik} L^k_j$. Then L is g-symmetric and $\nabla L = 0$, where ∇ is the Levi-Civita connection for g. Furthermore, we have that Riemann curvature tensor for the metric (5.4.6)*is given by*

$$R(X) = LX + XL.$$

Proof. Firstly, we observe that $L_{ij} L^i_t = g^0_{ik} L^k_j L^i_t = L_{kt} L^k_j \equiv L_{it} L^i_j$. Using further the fact that L is g^0-symmetric we easily compute

$$\begin{aligned}
\mathcal{B}_{ij,\alpha\beta} L^i_t &= -\frac{1}{2} \Big(L_{ij} g^0_{\alpha\beta} + L_{\alpha\beta} g^0_{ij} \Big) L^i_t = -\frac{1}{2} \Big(L_{ij} g^0_{\alpha\beta} L^i_t + L_{\alpha\beta} g^0_{ij} L^i_t \Big) \\
&= -\frac{1}{2} \Big(L_{it} g^0_{\alpha\beta} L^i_j + L_{\alpha\beta} g^0_{il} L^i_j \Big) = -\frac{1}{2} \Big(L_{it} g^0_{\alpha\beta} + L_{\alpha\beta} g^0_{il} \Big) L^i_j \\
&= \mathcal{B}_{it,\alpha\beta} L^i_j.
\end{aligned}$$

Now, by Lemma 5.3.1 we conclude that L is indeed g-symmetric. Secondly, we already know from Proposition 5.3.11 that to prove that $\nabla L = 0$ is to show that

$$(\mathcal{B}_{ip,\beta q} - \mathcal{B}_{i\beta,pq})L_k^\beta = (\mathcal{B}_{\beta i,kq} - B_{ik,\beta q})L_p^\beta. \tag{5.4.7}$$

Since $L^2 = 0$ is rewritten in coordinates as $L_k^i L_j^k = 0$, we first compute

$$\begin{aligned}
\Big(\mathcal{B}_{ip,\beta q} - \mathcal{B}_{i\beta,pq}\Big)L_k^\beta &= \Big(L_{ip}g_{\beta q}^0 + L_{\beta q}g_{ip}^0\Big)L_k^\beta - \Big(L_{i\beta}g_{pq}^0 + L_{pq}g_{i\beta}^0\Big)L_k^\beta \\
&= L_{ip}g_{\beta q}^0 L_k^\beta + g_{qs}L_\beta^s g_{ip}^0 L_k^\beta - g_{is}L_\beta^s g_{pq}^0 L_k^\beta - \\
&- L_{pq}g_{i\beta}^0 L_k^\beta = \Big(L_{ip}g_{\beta q}^0 - L_{pq}g_{i\beta}^0\Big)L_k^\beta \\
&= \Big(L_{ip}g_{\beta q}^0 - L_{pq}g_{i\beta}^0\Big)\delta_k^p L_p^\beta = \Big(L_{ik}g_{\beta q}^0 - L_{kq}g_{i\beta}^0\Big)L_p^\beta.
\end{aligned}$$

Similarly, we obtain

$$\Big(\mathcal{B}_{\beta i,kq} - \mathcal{B}_{ik,\beta q}\Big)L_p^\beta = \Big(L_{kq}g_{\beta i}^0 - L_{ik}g_{\beta q}^0\Big)L_p^\beta.$$

Now, as the indices q and i both run from 1 to n, nothing changes if we swap them and therefore $\nabla L = 0$. Finally, using formula (5.3.4), we compute the Riemann curvature tensor for the metric (5.4.6). Namely,

$$
\begin{aligned}
\mathcal{B}_{\beta s,\alpha k} \;&+\; \mathcal{B}_{\alpha k,\beta s} - \mathcal{B}_{\beta k,\alpha s} - \mathcal{B}_{\alpha s,\beta k} = -\frac{1}{2}\Big(L_{\beta s}g^0_{\alpha k} + L_{\alpha k}g^0_{\beta s}\Big) - \\
&- \;\frac{1}{2}\Big(L_{\alpha k}g^0_{\beta s} + L_{\beta s}g^0_{\alpha k}\Big) + \frac{1}{2}\Big(L_{\beta k}g^0_{\alpha s} + L_{\alpha s}g^0_{\beta k}\Big) + \frac{1}{2}\Big(L_{\alpha s}g^0_{\beta k} + L_{\beta k}g^0_{\alpha s}\Big) \\
&= \;L_{\alpha s}g^0_{\beta k} - L_{\beta s}g^0_{\alpha k} + L_{\beta k}g^0_{\alpha s} - L_{\alpha k}g^0_{\beta s}.
\end{aligned}
$$

This shows that the Riemann curvature tensor is indeed given by $R(X) = LX + XL$ and the proof is complete. □

At this juncture, the following important remark should be made. On the one hand, we have just brought into prominence the metrics of the form

$$g_{ij}(u) = g^0_{ij} - \frac{1}{2}\mathcal{B}_{ij,\alpha\beta}u^\alpha u^\beta. \tag{5.4.8}$$

On the other hand, the reader may recall the following well-known formula for the metric tensor in Riemann normal coordinates

$$g_{\mu\nu}(u) = g_{\mu\nu} - \frac{1}{3}R_{\mu\alpha\nu\beta}u^\alpha u^\beta + \cdots\,, \tag{5.4.9}$$

where $R_{\mu\alpha\nu\beta}$ is the Riemann tensor. Let us briefly outline the difference between the formulas (5.4.8) and (5.4.9). It is customary in Riemannian geometry to consider the Taylor expansion of the metric tensor. Indeed, since the metric components are smooth functions we can expand each component in a Taylor series about a given point p as

$$g_{\mu\nu}(p+x) = g_{\mu\nu} + g_{\mu\nu,\alpha}x^\alpha + \frac{1}{2!}g_{\mu\nu,\alpha\beta}x^\alpha x^\beta + \frac{1}{3!}g_{\mu\nu,\alpha\beta\gamma}x^\alpha x^\beta x^\gamma + \cdots\,, \tag{5.4.10}$$

where $g_{\mu\nu}$ is evaluated at the point p and as usual the symbol $g_{\mu\nu,\alpha\beta\gamma...}$ denotes the partial derivatives of $g_{\mu\nu}$ with respect to $x^\alpha, x^\beta, x^\gamma...$ at the point p. It is clear that the second order derivatives of the metric possess the following symmetries $g_{\mu\nu,\alpha\beta} = g_{\nu\mu,\alpha\beta} = g_{\mu\nu,\beta\alpha}$. Now, at the origin of coordinates such that the first order derivatives of the metric vanish one can compute for the components of the Riemann tensor

$$R_{\mu\nu\alpha\beta} = \frac{1}{2}\Big(g_{\mu\beta,\alpha\nu} - g_{\mu\alpha,\nu\beta} - g_{\nu\beta,\mu\alpha} + g_{\alpha\nu,\mu\beta}\Big), \tag{5.4.11}$$

which in our notation[2] is equivalent to formula 5.3.4. It is sometimes convenient to work with *Riemann normal coordinates* which by definition are such that $g_{\mu\nu,\alpha} = 0$ and with the following additional symmetries of the second order derivatives of the metric

$$g_{ab,cd} = g_{cd,ab} \text{ and } g_{ab,cd} + g_{ac,db} + g_{ad,bc} = 0. \tag{5.4.12}$$

It is due to this symmetries that the expression for the components of the Riemann tensor simplifies to

$$R_{\mu\nu\alpha\beta} = \frac{1}{2}\Big(g_{\mu\beta,\alpha\nu} - g_{\mu\alpha,\nu\beta}\Big). \tag{5.4.13}$$

Notice that (5.4.13) indeed implies (5.4.9) but this is only valid in the case of Riemann normal coordinates. In contrast, in this thesis we consider coordinates such that only the first order derivatives of the metric vanish and do not assume the special additional symmetry for the second order derivatives of the metric. For this reason our formula (5.4.8) has different second order terms than the second order terms in the well-known formula (5.4.9).

With Proposition 5.4.5 we have solved Problem 4 for one very special case. At the end of this section the following remark must be made. A nilpotent operator of degree $k > 2$

[2] In our notation $\mathcal{B}_{\mu\nu,\alpha\beta} \equiv \frac{1}{2} g_{\mu\nu,\alpha\beta}$

may be of many different types. For instance, the operators of types $(m;k)$, $(l;m;k)$ and $(l;m;k;k;k)$ are all nilpotent of degree k, provided that $k = \max(k,l,m)$. However, this will not be any obstacle, since we shall first settle Problem 4 for the important $(k;n)$-case, which is de facto the main issue (see Section 4.5).

5.5 The general construction

We are now just a step away from the climax of this thesis. In its final section, we shall discuss the general construction of the class of pseudo-Riemannian metrics realising the Lie algebra $\mathfrak{g}_L$ as a holonomy algebra. Since the main result herein is of a rather general nature, it is preferable to work with invariant notation. Thus, in order to avoid the clumsy coordinate computations, we shall be working within the following framework.

Starting from the metric $g_{ij} = g^0_{ij} + \mathcal{B}_{ij}(u,u)$, we wish to rewrite the bilinear form $\mathcal{B}_{ij}(u,u) = \mathcal{B}_{ij,pq}u^p u^q$ in invariant terms. Bearing in mind the discussion in the previous section we are motivated to write, up to a factor, $\mathcal{B} = \sum_\alpha \mathcal{C}_\alpha \otimes \mathcal{D}_\alpha$, where $\mathcal{C}$ and $\mathcal{D}$ are bilinear forms associated with some g^0-symmetric operators C and D. For greater clarity, let us first consider the bilinear form $\mathcal{B} = \mathcal{C} \otimes \mathcal{D}$. This expression for $\mathcal{B}$ clearly allows us to write $\mathcal{B}_{ij,pq} = \mathcal{C}_{ij} \cdot \mathcal{D}_{pq}$ with $\mathcal{C}_{ij} = (g^0)_{i\alpha} C^\alpha_j$, $\mathcal{D}_{pq} = (g^0)_{p\alpha} D^\alpha_q$. Henceforth, the "curly" capitals will denote forms whereas the usual ones their corresponding operators. In this new language we shall first prove the following proposition.

Proposition 5.5.1 *Assuming $\mathcal{B} = -\frac{1}{2}\mathcal{C} \otimes \mathcal{D}$, the algebraic identities* (5.2.5)*,* (5.2.6) *and* (5.2.7) *are respectively rewritten as*

$$CL = LC, \tag{5.2.5$'$}$$

$$R(X) = -CXD + (CXD)^*, \tag{5.2.6$'$}$$

$$[CXD, L] + [CXD, L]^* = 0. \tag{5.2.7$'$}$$

Remark. It must be noted that while (5.2.6′) necessitates $X \in \mathfrak{so}(g^0)$, formula (5.2.7′) holds true for the more general case $X \in \mathfrak{gl}(V)$.

Proof. Clearly, our goal is to rewrite the algebraic identities (5.2.5), (5.2.6) and (5.2.7) in an invariant form. Recall that the first one was $\mathcal{B}_{\alpha s,pq} L^\alpha_k = \mathcal{B}_{\alpha k,pq} L^\alpha_s$. We compute for the left hand side of this last expression

$$\mathcal{B}_{\alpha s,pq} L^\alpha_k = g^0_{\alpha a} C^a_s g^0_{pa} D^a_q L^\alpha_k = g^0_{\alpha a} C^a_s L^\alpha_k g^0_{pa} D^a_q = C^a_s L_{ak} g^0_{pa} D^a_q = g^0_{kt} C^a_s L^t_a g^0_{pa} D^a_q.$$

Similarly, the right hand side reduces to

$$\mathcal{B}_{\alpha k,pq} L^\alpha_s = g^0_{kt} C^t_\alpha L^\alpha_s g^0_{pa} D^a_q.$$

Thus, $\mathcal{B}_{\alpha s,pq} L^\alpha_k = \mathcal{B}_{\alpha k,pq} L^\alpha_s$ is tantamount to $C^a_s L^t_a = C^t_\alpha L^\alpha_s$, which is precisely $CL = LC$. We perceive the truth of (5.2.6′) by virtue of the following computation.

$$\begin{aligned}
R^i_{\alpha\beta,k} X^{\alpha\beta} &= (g^0)^{is}\Big(\mathcal{B}_{\beta s,\alpha k} + \mathcal{B}_{\alpha k,\beta s} - \mathcal{B}_{\beta k,\alpha s} - \mathcal{B}_{\alpha s,\beta k}\Big) X^{\alpha\beta} \\
&= -\frac{1}{2}(g^0)^{is}\Big(\mathcal{C}_{\beta s}\mathcal{D}_{\alpha k} X^{\alpha\beta} + \mathcal{C}_{\alpha k}\mathcal{D}_{\beta s} X^{\alpha\beta} - \mathcal{C}_{\beta k}\mathcal{D}_{\alpha s} X^{\alpha\beta} - \mathcal{C}_{\alpha s}\mathcal{D}_{\beta k} X^{\alpha\beta}\Big) \\
&= -\frac{1}{2}(g^0)^{is}\Big((DXC)_{ks} + (CXD)_{ks} - (DXC)_{sk} - (CXD)_{sk}\Big) \\
&= -(g^0)^{is}\Big((CXD)_{sk} + (DXC)_{sk}\Big) = -\Big((CXD)^i_k + (DXC)^i_k\Big).
\end{aligned}$$

Notice that we have used the obvious fact that both the forms $(CXD)_{ks}$ and $(DXC)_{ks}$

are g^0 skew-symmetric. We thus obtain, in invariant terms, the following identity

$$R(X) = -CXD - DXC.$$

Now, (5.2.6′) is immediately justified by virtue of $(CXD)^* = -DXC$.

Our starting point in proving the last algebraic identity is the fact that the invariant formula $A + A^* = 0$ is rewritten in matrix terms as $gA + A^\top g = 0$. Then, by virtue of the latter, we write (5.2.7′) in coordinates as

$$g^0_{i\alpha}\Big(C^i_k X^k_l D^l_j L^j_s - L^i_j C^j_k X^k_l D^l_s\Big) + g^0_{is}\Big(C^i_k X^k_l D^l_j L^j_\alpha - L^i_j C^j_k X^k_l D^l_\alpha\Big) = 0.$$

Now, using the g^0-symmetry of L, which is $g^0_{i\alpha} L^i_j = g^0_{ij} L^i_\alpha$, as well as the obvious identity $g^0_{ij} L^i_\alpha C^j_k = L^i_\alpha \mathcal{C}_{ik} = L^j_\alpha \mathcal{C}_{jk}$, we reduce the last expression to

$$\Big(\mathcal{C}_{sk}\mathcal{D}_{j\beta} - \mathcal{C}_{jk}\mathcal{D}_{s\beta}\Big) L^j_\alpha = \Big(\mathcal{C}_{jk}\mathcal{D}_{\alpha\beta} - \mathcal{C}_{\alpha k}\mathcal{D}_{j\beta}\Big) L^j_s.$$

Clearly, this last expression is equivalent to

$$\Big(\mathcal{B}_{sk,j\beta} - \mathcal{B}_{jk,s\beta}\Big) L^j_\alpha = \Big(\mathcal{B}_{jk,\alpha\beta} - \mathcal{B}_{\alpha k,j\beta}\Big) L^j_s,$$

which is precisely (5.2.7) and therefore the proof is complete. □

More generally, if $\mathcal{B} = \sum_\alpha \mathcal{C}_\alpha \otimes \mathcal{D}_\alpha$, then the corresponding conditions on $\mathcal{B}$ are obtained from (5.2.5′), (5.2.6′) and (5.2.7′) simply by summing over α. This motivates us to consider the following general framework. Let $B = \sum_\alpha C_\alpha \otimes D_\alpha$ where C_α and D_α are g^0-symmetric operators. Consider B as the linear map

$$B : \mathfrak{gl}(V) \longrightarrow \mathfrak{gl}(V)$$

$$B(X) = \sum_{\alpha} C_\alpha X D_\alpha.$$

In other words, $B(X)$ is obtained from B by replacing $\otimes$ by X. Then the algebraic identities (5.2.5′), (5.2.6′) and (5.2.7′) can be conveniently rewritten as

$$[C_\alpha, L] = 0, \tag{5.2.5″}$$

$$R(X) = -B(X) + B(X)^*, \tag{5.2.6″}$$

$$[B(X), L] + [B(X), L]^* = 0. \tag{5.2.7″}$$

It is not difficult to observe that if $B(X) = CXD + DXC$ we are actually in the case of Proposition 5.4.5. Moreover, we have in this case that $B(X) = -B(X)^*$ and therefore formula (5.2.6″) shows how to reconstruct B from R. Since $R(X) = -2B(X)$ we can easily guess what is the general form of B. Indeed, replacing X by $\otimes$ yields $B = -\frac{1}{2}R(\otimes)$. Note that this last expression simply means that, up to a factor and some permutation of indices, R and B coincide as tensors of type $(2,2)$. This simple observation motivates us to consider

$$B = -\frac{1}{2} \cdot \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} p_{\min}(L + t \cdot \otimes), \tag{5.5.2}$$

where $p_{\min}(\lambda)$ is the minimal polynomial of L. This formula looks a bit strange, but, in fact, it defines a tensor B of type $(2,2)$ whose meaning is very simple. If the minimal polynomial of L is $p_{min}(t) = \sum_{m=0}^{n} a_m t^m$, then

$$B = -\frac{1}{2} \cdot \sum_{m=0}^{n} a_m \sum_{j=0}^{m-1} L^{m-1-j} \otimes L^j. \tag{5.5.3}$$

This formula is obtained from the right hand side of (4.2.2), i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\left(\sum_{m=0}^{n} a_m \Big(L + t\cdot X\Big)^m\right) = \sum_{m=0}^{n} a_m \sum_{j=0}^{m-1} L^{m-1-j} X L^j,$$

by substituting $\otimes$ instead of X. With this formula in mind, we are in a position to state and prove the foremost result of this chapter.

Theorem 5.5.4 *Assume that L is a g^0-symmetric operator which is constant in coordinates u. Define the quadratic metric $g(u) = g^0 + \mathcal{B}(u,u)$ with $\mathcal{B}_{ij,pq} = g^0_{i\alpha} g^0_{p\beta} B^{\alpha,\beta}_{j,q}$, where B is constructed from L by virtue of* (5.5.2) (*or, equivalently, by* (5.5.3)). *Then*

1) *L is g-symmetric,*

2) $\nabla L = 0$, *where ∇ is the Levi-Civita connection for g,*

3) *The Riemann curvature tensor for g at the origin is defined by* (4.2.2), *i.e.,*

$$R(X) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} p_{\min}(L + tX).$$

Proof. Since B is of the form $\sum_\alpha C_\alpha \otimes D_\alpha$, where C_α and D_α are some powers of L, we shall use the power of formulas (5.2.5″), (5.2.6″) and (5.2.7″). Firstly, statement 1) is equivalent to (5.2.5″) and is therefore obvious. Secondly, by virtue of (5.2.7″), to prove 2) is to show that

$$[B(X), L] = 0, \quad \text{where } B(X) = -\frac{1}{2}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} p_{\min}(L + t\cdot X).$$

But this has been already done for $-2B(X)$ in Section 4.2. Finally, to compute the Riemann curvature tensor R at the origin we make use of (5.2.6″). We have

$$R(X) = -B(X) + B(X)^* = -2B(X) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} p_{\min}(L + tX),$$

as stated. Notice that the discussion in Section 4.2 infers that $B(X)$ belongs to $\mathfrak{so}(g^0)$, which is $B(X) = -B(X)^*$. The proof is complete. $\square$

Theorem 5.5.4 along with Theorem B solves Problem 4 for the most important $(k;n)$-case. As for the general case, one proceeds in the exactly same fashion as we did in Section 4.5. Namely, one firstly splits L into Jordan blocks and defines for each pair of Jordan blocks L_i and L_j a formal curvature tensor $\widehat{R}_{ij}$ (see Section 4.5). Secondly, by virtue of (5.5.2) this formal curvature tensor can be realised by the appropriate quadratic metric $g(u) = g^0 + \widehat{B}_{ij}(u,u)$ satisfying $\nabla L^{(i;j)} = 0$. Finally, setting

$$g(x) = g^0 + \mathcal{B}(u,u) \quad \text{with} \quad B = \sum_{i,j} \widehat{B}_{ij},$$

we immediately observe that, by linearity, $\nabla L = 0$ holds true. Moreover, the Riemann curvature tensor for this metric coincides with $R_{\text{formal}} = \sum_{i<j} \widehat{R}_{ij}$ from Theorem 4.5.10. Thus, we arrive at the climax of this thesis. For a given smooth connected manifold M we consider the linear operator $L : T_p\mathrm{M} \longrightarrow T_p\mathrm{M}$. We then have the following theorem.

Theorem A *Let* M *be a smooth manifold,* $p \in \mathrm{M}$ *be a point and* g^0 *be a symmetric non-degenerate bilinear form on* $\mathrm{T}_p\mathrm{M}$ *and* $L_0 : \mathrm{T}_p\mathrm{M} \longrightarrow \mathrm{T}_p\mathrm{M}$ *be a* g^0*-symmetric operator. Then, in a local neighbourhood* U *of* p*, there exist a pseudo-Riemannian metric* g *and a* $(1,1)$*-tensor field* L *such that*

1)$g \mid_{\mathrm{T}_p\mathrm{M}} = g^0$,

2)$L \mid_{\mathrm{T}_p\mathrm{M}} = L_0$,

3)L *is* g*-symmetric,*

4) The centraliser $\mathfrak{g}_L$ *of* L *in the Lie algebra* $\mathfrak{so}(g)$ *is a holonomy algebra for the Levi-Civita connection of the metric* g.

Appendix A

A few worked examples of Berger algebras related to g-symmetric operators

In this addendum we briefly consider the Lie algebras $\mathfrak{g}_L^{(k_1;k_2)}$ and $\mathfrak{g}_L^{(k_1;k_2;k_3)}$ for a few different values of k_1, k_2 and k_3. We know from our discussion in Chapter 4 that these are all examples of Berger algebras related to the g-symmetric operators of the types $L^{(k_1;k_2)}$ and $L^{(k_1;k_2;k_3)}$, respectively. From the computational viewpoint, however, this conclusion is not always straightforward. While in the former case we can easily draw the conclusion that $\mathfrak{g}_L^{(k_1;k_2)}$ is indeed a Berger algebra, in the latter we perceive the necessity of a general formal proof. Following the idea of Section 4.1 we shall present in sections A.1 to A.3 several particular solutions of the following problem.

Find a map $R : \Lambda^2 V \longrightarrow \mathfrak{g}_L^{(k_1;k_2)}$ *such that* $R(e_i \wedge e_j)e_k + R(e_j \wedge e_k)e_i + R(e_k \wedge e_i)e_j = 0$ *and* $\mathrm{Im} R \equiv \mathfrak{g}_L^{(k_1;k_2)}$.

We already know that the solution of this problem asserts that $\mathfrak{g}_L^{(k_1;k_2)}$ is a Berger algebra. For the sake of brevity we shall only give the upshot of our computations in a tabular form. Recall that the above problem was stated as Problem 3 in Section 4.1 and we discussed in detail one special solution for the $(2;2)$-case. In brief, we used the standard identification of the space of skew-symmetric matrices with $\Lambda^2 V$ and considered the map $R : \Lambda^2 V \longrightarrow \mathfrak{g}_L^{(2;2)}$ defined by

$$\begin{pmatrix} 0 & x_1 & x_2 & x_3 \\ -x_1 & 0 & x_4 & x_5 \\ -x_2 & -x_4 & 0 & x_6 \\ -x_3 & -x_5 & -x_6 & 0 \end{pmatrix} \longmapsto \left(\begin{array}{cc|cc} 0 & 0 & a(x) & b(x) \\ 0 & 0 & 0 & a(x) \\ \hline -a(x) & -b(x) & 0 & 0 \\ 0 & -a(x) & 0 & 0 \end{array}\right),$$

where $a(x) = \sum_{i=1}^{6} a_i x_i$, $b(x) = \sum_{i=1}^{6} b_i x_i$, and $x_i, a_i, b_i \in \mathbb{R}$. Through straightforward computation we showed that the map

$$\begin{pmatrix} 0 & x_1 & x_2 & x_3 \\ -x_1 & 0 & x_4 & x_5 \\ -x_2 & -x_4 & 0 & x_6 \\ -x_3 & -x_5 & -x_6 & 0 \end{pmatrix} \longmapsto \left(\begin{array}{cc|cc} 0 & 0 & \alpha x_5 & \alpha(x_3 + x_4) + \beta x_5 \\ 0 & 0 & 0 & \alpha x_5 \\ \hline -\alpha x_5 & -\alpha(x_3 + x_4) - \beta x_5 & 0 & 0 \\ 0 & -\alpha x_5 & 0 & 0 \end{array}\right),$$

where $\alpha = a_5 = b_3 = b_4$ and $\beta = b_5$ were the only non-zero coefficients in $a(x)$ and $b(x)$, solves the problem above. We also remind the reader that β was an arbitrary coefficient which did not appear in our computations. We shall now demonstrate that the outcome of this computation can be neatly represented in a tabular form. The entry of the first row of our table will clearly indicate the case we are interested in. In this particular case we shall simply write $(2;2)$-case. The coefficients a_i form the second row, followed by the row of their corresponding values. Similarly, for the coefficients b_i and their values. We thus have the following table.

(2;2) - case					
a_1	a_2	a_3	a_4	a_5	a_6
0	0	0	0	α	0
b_1	b_2	b_3	b_4	b_5	b_6
0	0	α	α	β	0

Now, bearing in mind our definition of R as well as the matrix representation of $\mathfrak{g}_L^{(2;2)}$, we easily reach the same conclusion as before - $\mathfrak{g}_L^{(2;2)}$ is a Berger algebra. It turns out that similar conclusions could be swiftly drawn from the corresponding tables of the $(k_1; k_2)$-case for arbitrary values of k_1 and k_2. The map R will be defined in the same manner as above but the dimension and the matrix structure of $\mathfrak{g}_L^{(2;2)}$ will be different. For this reason we adopt the following simple convention. Write $a(x) = \sum_{i=1}^{\frac{n}{2}(n-1)} a_i x_i$ for the diagonal elements of the upper right block of the matrix of $\mathfrak{g}_L^{(2;2)}$, where n is the dimension of the underlying vector space V. We write further $b(x) = \sum_{i=1}^{\frac{n}{2}(n-1)} b_i x_i$, $c(x) = \sum_{i=1}^{\frac{n}{2}(n-1)} c_i x_i$, $d(x) = \sum_{i=1}^{\frac{n}{2}(n-1)} d_i x_i$ and so on for the consecutive respective upper diagonals. Suppose for instance that we are working in the $(4;4)$-case. Then the elements of $\mathfrak{g}_L^{(4;4)}$ are written as

$$\left(\begin{array}{cccc|cccc}
0 & 0 & 0 & 0 & a(x) & b(x) & c(x) & d(x) \\
0 & 0 & 0 & 0 & 0 & a(x) & b(x) & c(x) \\
0 & 0 & 0 & 0 & 0 & 0 & a(x) & b(x) \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & a(x) \\
\hline
-a(x) & -b(x) & -c(x) & -d(x) & 0 & 0 & 0 & 0 \\
0 & -a(x) & -b(x) & -c(x) & 0 & 0 & 0 & 0 \\
0 & 0 & -a(x) & -b(x) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -a(x) & 0 & 0 & 0 & 0
\end{array}\right).$$

Now, with this in mind, the reader must be able to read the tables in sections A.1 to A.3 and to see that they indeed imply that $\mathfrak{g}_L^{(2;k)}$, $\mathfrak{g}_L^{(k;2)}$ and $\mathfrak{g}_L^{(k_1;k_2)}$ are examples of Berger algebras related to the g-symmetric operators of the types $L^{(2;k)}$, $L^{(k;2)}$ and $L^{(k_1;k_2)}$, respectively. A certain pattern for every different case is clearly recognisable. This pattern allows us to quickly find more solutions of the aforementioned problem without doing all the calculations - we only need to follow the patterns. Moreover, the patterns appearing in sections A.1 and A.2 clearly display the isomorphism $\mathfrak{g}_L^{(2;k)} \cong \mathfrak{g}_L^{(k;2)}$. We remind the reader that we have already used this fact in Chapter 4. Alas, the general situation is far more complex. To demonstrate this we consider in section A.4 two examples for the $(k_1; k_2; k_3)$-case. We now define R as above but such that the elements of its image are of the form

$$\left(\begin{array}{cc|ccc|cccc}
0 & 0 & 0 & a(x) & b(x) & 0 & 0 & c(x) & d(x) \\
0 & 0 & 0 & 0 & a(x) & 0 & 0 & 0 & c(x) \\
\hline
-a(x) & -b(x) & 0 & 0 & 0 & 0 & e(x) & f(x) & g(x) \\
0 & -a(x) & 0 & 0 & 0 & 0 & 0 & e(x) & f(x) \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e(x) \\
\hline
-c(x) & -d(x) & -e(x) & -f(x) & -g(x) & 0 & 0 & 0 & 0 \\
0 & -c(x) & 0 & -e(x) & -f(x) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -e(x) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right).$$

The quantities $a(x)$ to $g(x)$ are defined just as before. Then the tables given in section A.4 represent formal curvature operators for the corresponding Lie algebras $\mathfrak{g}_L^{(k_1;k_2;k_3)}$. However, there is neither a pattern amongst them nor an easy way to see that the images of these formal curvature operators coincide with $\mathfrak{g}_L^{(k_1;k_2;k_3)}$.

A.1 First four examples of the (2;k)-case

(2;2) - case					
a_1	a_2	a_3	a_4	a_5	a_6
0	0	0	0	α	0
b_1	b_2	b_3	b_4	b_5	b_6
0	0	α	α	β	0

(2;3) - case									
a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}
0	0	0	0	0	0	α	0	0	0
b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}
0	0	0	α	0	α	β	0	0	0

(2;4) - case														
a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}	a_{13}	a_{14}	a_{15}
0	0	0	0	0	0	0	0	α	0	0	0	0	0	0
b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	b_{11}	b_{12}	b_{13}	b_{14}	b_{15}
0	0	0	0	α	0	0	α	β	0	0	0	0	0	0

(2;5) - case																				
a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}	a_{13}	a_{14}	a_{15}	a_{16}	a_{17}	a_{18}	a_{19}	a_{20}	a_{21}
0	0	0	0	0	0	0	0	0	0	α	0	0	0	0	0	0	0	0	0	0
b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	b_{11}	b_{12}	b_{13}	b_{14}	b_{15}	b_{16}	b_{17}	b_{18}	b_{19}	b_{20}	b_{21}
0	0	0	0	0	α	0	0	0	α	β	0	0	0	0	0	0	0	0	0	0

A.2 First four examples of the (k;2)-case

(2;2) - case					
a_1	a_2	a_3	a_4	a_5	a_6
0	0	0	0	α	0
b_1	b_2	b_3	b_4	b_5	b_6
0	0	α	α	β	0

(3;2) - case									
a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}
0	0	0	0	0	0	0	0	α	0
b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}
0	0	0	0	0	0	α	α	β	0

(4;2) - case														
a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}	a_{13}	a_{14}	a_{15}
0	0	0	0	0	0	0	0	0	0	0	0	0	α	0
b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	b_{11}	b_{12}	b_{13}	b_{14}	b_{15}
0	0	0	0	0	0	0	0	0	0	0	α	α	β	0

(5;2) - case																				
a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}	a_{13}	a_{14}	a_{15}	a_{16}	a_{17}	a_{18}	a_{19}	a_{20}	a_{21}
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	α	0
b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	b_{11}	b_{12}	b_{13}	b_{14}	b_{15}	b_{16}	b_{17}	b_{18}	b_{19}	b_{20}	b_{21}
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	α	α	β	0

A.3 First three examples of the (k;k)-case

(2;2) - case					
a_1	a_2	a_3	a_4	a_5	a_6
0	0	0	0	α	0
b_1	b_2	b_3	b_4	b_5	b_6
0	0	α	α	β	0

(3;3) - case														
a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}	a_{13}	a_{14}	a_{15}
0	0	0	0	0	0	0	0	0	0	0	α	0	0	0
b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	b_{11}	b_{12}	b_{13}	b_{14}	b_{15}
0	0	0	0	0	0	0	0	α	0	α	β	0	0	0
c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}	c_{15}
0	0	0	0	α	0	0	α	β	α	β	γ	0	0	0

(4;4) - case																											
a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}	a_{13}	a_{14}	a_{15}	a_{16}	a_{17}	a_{18}	a_{19}	a_{20}	a_{21}	a_{22}	a_{23}	a_{24}	a_{25}	a_{26}	a_{27}	a_{28}
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	α	0	0	0	0	0	0
b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	b_{11}	b_{12}	b_{13}	b_{14}	b_{15}	b_{16}	b_{17}	b_{18}	b_{19}	b_{20}	b_{21}	b_{22}	b_{23}	b_{24}	b_{25}	b_{26}	b_{27}	b_{28}
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	α	0	0	α	β	0	0	0	0	0	0
c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}	c_{15}	c_{16}	c_{17}	c_{18}	c_{19}	c_{20}	c_{21}	c_{22}	c_{23}	c_{24}	c_{25}	c_{26}	c_{27}	c_{28}
0	0	0	0	0	0	0	0	0	0	0	0	α	0	0	0	α	β	0	α	β	γ	0	0	0	0	0	0
d_1	d_2	d_3	d_4	d_5	d_6	d_7	d_8	d_9	d_{10}	d_{11}	d_{12}	d_{13}	d_{14}	d_{15}	d_{16}	d_{17}	d_{18}	d_{19}	d_{20}	d_{21}	d_{22}	d_{23}	d_{24}	d_{25}	d_{26}	d_{27}	d_{28}
0	0	0	0	0	0	α	0	0	0	0	α	β	0	0	α	β	γ	α	β	γ	δ	0	0	0	0	0	0

A.4 Two examples of the $(k_1; k_2; k_3)$-case.

(2;3;2) - case																				
a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}	a_{13}	a_{14}	a_{15}	a_{16}	a_{17}	a_{18}	a_{19}	a_{20}	a_{21}
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	α	0
b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	b_{11}	b_{12}	b_{13}	b_{14}	b_{15}	b_{16}	b_{17}	b_{18}	b_{19}	b_{20}	b_{21}
0	0	0	0	0	0	0	0	ρ	0	β	0	0	0	0	0	0	α	γ	φ	0
c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}	c_{15}	c_{16}	c_{17}	c_{18}	c_{19}	c_{20}	c_{21}
0	0	0	0	0	0	0	0	0	0	ψ	0	0	0	0	0	0	0	0	0	0
d_1	d_2	d_3	d_4	d_5	d_6	d_7	d_8	d_9	d_{10}	d_{11}	d_{12}	d_{13}	d_{14}	d_{15}	d_{16}	d_{17}	d_{18}	d_{19}	d_{20}	d_{21}
0	0	0	0	0	ψ	0	0	β	ψ	σ	0	0	0	0	0	0	0	0	ω	0
e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}	e_{16}	e_{17}	e_{18}	e_{19}	e_{20}	e_{21}
0	0	0	0	0	0	0	0	γ	0	0	0	0	0	0	0	0	0	0	χ	0
f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}	f_{11}	f_{12}	f_{13}	f_{14}	f_{15}	f_{16}	f_{17}	f_{18}	f_{19}	f_{20}	f_{21}
0	0	0	α	0	0	0	γ	φ	0	ω	0	0	0	0	0	0	χ	χ	τ	0

(2;3;4) - case																																			
a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}	a_{13}	a_{14}	a_{15}	a_{16}	a_{17}	a_{18}	a_{19}	a_{20}	a_{21}	a_{22}	a_{23}	a_{24}	a_{25}	a_{26}	a_{27}	a_{28}	a_{29}	a_{30}	a_{31}	a_{32}	a_{33}	a_{34}	a_{35}	a_{36}
0	0	0	0	0	0	0	0	0	0	α_7	0	0	0	α_6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	α_3	0	0	α_1	0	0	0
b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	b_{11}	b_{12}	b_{13}	b_{14}	b_{15}	b_{16}	b_{17}	b_{18}	b_{19}	b_{20}	b_{21}	b_{22}	b_{23}	b_{24}	b_{25}	b_{26}	b_{27}	b_{28}	b_{29}	b_{30}	b_{31}	b_{32}	b_{33}	b_{34}	b_{35}	b_{36}
0	0	0	α_7	0	0	0	α_6	0	α_7	α_{12}	0	0	α_6	α_9	0	0	0	0	0	0	0	0	0	0	α_3	0	0	α_3	α_{10}	0	α_1	α_1	0	0	0
c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}	c_{15}	c_{16}	c_{17}	c_{18}	c_{19}	c_{20}	c_{21}	c_{22}	c_{23}	c_{24}	c_{25}	c_{26}	c_{27}	c_{28}	c_{29}	c_{30}	c_{31}	c_{32}	c_{33}	c_{34}	c_{35}	c_{36}
0	0	0	0	0	0	0	0	0	0	α_6	0	0	0	α_8	0	0	0	0	0	0	0	0	0	0	0	α_1	0	0	α_5	0	0	0	0	0	0
d_1	d_2	d_3	d_4	d_5	d_6	d_7	d_8	d_9	d_{10}	d_{11}	d_{12}	d_{13}	d_{14}	d_{15}	d_{16}	d_{17}	d_{18}	d_{19}	d_{20}	d_{21}	d_{22}	d_{23}	d_{24}	d_{25}	d_{26}	d_{27}	d_{28}	d_{29}	d_{30}	d_{31}	d_{32}	d_{33}	d_{34}	d_{35}	d_{36}
0	0	0	α_6	0	0	0	α_8	0	α_6	α_9	0	0	α_8	α_{13}	0	0	0	0	0	0	0	α_1	0	0	α_5	α_1	0	α_5	α_{11}	0	0	0	0	0	0
e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}	e_{16}	e_{17}	e_{18}	e_{19}	e_{20}	e_{21}	e_{22}	e_{23}	e_{24}	e_{25}	e_{26}	e_{27}	e_{28}	e_{29}	e_{30}	e_{31}	e_{32}	e_{33}	e_{34}	e_{35}	e_{36}
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	α_2	0	0	0	0	0	0
f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}	f_{11}	f_{12}	f_{13}	f_{14}	f_{15}	f_{16}	f_{17}	f_{18}	f_{19}	f_{20}	f_{21}	f_{22}	f_{23}	f_{24}	f_{25}	f_{26}	f_{27}	f_{28}	f_{29}	f_{30}	f_{31}	f_{32}	f_{33}	f_{34}	f_{35}	f_{36}
0	0	0	0	0	0	0	0	0	0	α_3	α_1	0	0	α_5	0	0	0	0	0	0	0	0	0	0	α_2	0	0	α_2	α_4	0	0	0	0	0	0
g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8	g_9	g_{10}	g_{11}	g_{12}	g_{13}	g_{14}	g_{15}	g_{16}	g_{17}	g_{18}	g_{19}	g_{20}	g_{21}	g_{22}	g_{23}	g_{24}	g_{25}	g_{26}	g_{27}	g_{28}	g_{29}	g_{30}	g_{31}	g_{32}	g_{33}	g_{34}	g_{35}	g_{36}
0	0	0	α_3	α_1	0	0	α_3	0	α_3	α_{10}	α_1	0	α_5	α_{11}	0	0	0	0	0	α_2	0	0	0	α_2	α_4	0	α_2	α_4	α_{14}	0	0	0	0	0	0

Bibliography

[AS] W. Ambrose, I. M. Singer, *A theorem on holonomy*, Trans. Amer. Math. Soc. **75** (1953) 428–443.

[Ami] A. V. Aminova, *Projective transformations of pseudo-Riemannian manifolds*, Geometry, 9. J. Math. Sci. (N. Y.) **113** no. 3 (2003) 367–470.

[Bel] E. Beltrami, *Resoluzione del problema: riportari i punti di una superficie sopra un piano in modo che le linee geodetische vengano rappresentante da linee rette*, Ann. Mat., **1** no. 7 (1865) 185204.

[BI1] L. Bérard-Bergery, A. Ikemakhen, *On the Holonomy of Lorentzian Manifolds*, Proc. of Symposia in Pure Mathematics **54** (1993) Part 2: 27–39.

[BI2] L. Bérard-Bergery, A. Ikemakhen, *Sur l'holonomie des variétés pseudo-riemaniennes de signature* (n, n), Bull. Soc. Math. France **125** (1997) 93-114.

[Ber1] M. Berger, *Sur les groupes d'holonomie des variétés a connexion affine et des variétés Riemanniennes*, Bull. Soc. Math. France **83** (1955) 279–330.

[Ber2] M. Berger, *A panoramic view of Riemannian geometry*, Springer-Verlag Berlin Heidelberg (2003).

[Bes] A. Besse, *Einstein Manifolds*, Ergebnisse der Mathematik und ihner Grenzgebiete 3. Folge Band 10, Springer-Verlag, (1987).

[BKM] A. V. Bolsinov, V. Kiosak, V. S. Matveev, *A Fubini theorem for pseudo-Riemannian geodesically equivalent metrics*, J. London Math. Soc. (2) **80** (2009) 341–356.

[BM1] A. Bolsinov, V. Matveev, *Geometrical interpretation of Benenti systems*, J. Geom. Phys. **44** (2003) 489-506.

[BM2] A. Bolsinov, V. Matveev, *Local normal forms for geodesically equivalent pseudo-Riemannian metrics*, arXive: math.DG/1301.2492

[BMM] R. Bryant, G. Manno, V. Matveev, *A solution of a problem of Sophus Lie: Normal forms of 2-dim metrics admitting two projective vector fields*, Math. Ann. **340** no.2 (2008) 437-463, arXiv:0705.3592 .

[BMP] A. V. Bolsinov, V. S. Matveev, G. Pucacco, *Normal forms for pseudo-Riemannian 2-dimensional metrics whose geodesic flows admit integrals quadratic in momenta* // Journal of Geometry and Physics **59** (2009) 1048-1062.

[BL] A. Borel, A. Lichnerowicz, *Groupes d'holonomie des variétés riemanniennes*, C.R.Acad.Sci.Paris **234** (1952) 1835-1837.

[Bou] C. Boubel, *On the holonomy of Lorentzian metrics*, Annales de la faculté des sciences de Toulouse **16** no. 3 (2007) 427–475. Prépublication de l'ENS Lyon / ENS Lyon preprint no. 323 (2004).

[Bry1] R. Bryant, *Metrics with exceptional holonomy*, Ann. Math. **126** (1987) 525-576.

[Bry2] R. Bryant, *Classical, exceptional, and exotic holonomies: A status report*, Besse, A. L. (ed.), Actes de la table ronde de géométrie différentielle en lhonneur de Marcel Berger, Luminy, France, 12–18 juillet 1992. Soc. Math. France. Sémin. Congr. 1 (1996) 93-165.

[Bry3] R. Bryant, *Recent advances in the theory of holonomy*, Astérisque **266** (5) (2000) 351–374. (Expos No. 861).

[BS] R. Bryant, S. Salamon, *On the construction of some complete metrics with exceptional holonomy*, Duke. Math. Jour. **58** (1989) 829-850

[BT] A. Bolsinov, D. Tsonev, *On one class of holonomy groups in pseudo-Riemannian geometry*, arXive: 1107.2361 (2011).

[Car1] É. Cartan, *Sur les variétés à connexion affine et la théorie de la relativité généralisée I & II*, Ann.Sci.Ecol.Norm.Sup. **40** (1923) 325-412 et **41**, 1-25 (1924)ou Oeuvres complétes, tome III, 659-746 et 799-824.

[Car2] É. Cartan, *La géométrie des espaces de Riemann*, Mémorial des Sciences Mathématiques, Paris, Gauthier-Villars, vol. 5 (1925).

[Car3] É. Cartan, *Sur une classe remarquable d'espaces de Riemann*, Bull. Soc. Math. France **54** (1926) 214–264, **55**, (1927) 114–134 ou Oeuvres complétes, tome I, vol. 2 , 587-659.

[Car4] É. Cartan, *Les groupes d'holonomie des espaces généralisés*, Acta.Math. **48**, 1–42 (1926) ou Oeuvres complétes, tome III, vol. 2, 997–1038.

[Cha] I. Chavel, *Riemannian Geometry: A Modern Introduction*, Cambridge University Press(1993).

[CCL] S. S. Chern, W. H. Chen and K. S. Lam, *Lectures on Differential Geometry*, World Scientific Publishing Co. Pte. Ltd. (1999).

[deRha] G. de Rham, *Sur la réductibilité d'un espace de Riemann*, Comm.Math.Helv. **26** (1952) 328-344.

[Din] U. Dini, *Sopra un problema che si presenta nella teoria generale delle rappresentazioni geografice di una superficie su un'altra*, Ann. Mat., ser.2, **3** (1869) 269-293.

[DNF] B. A. Dubrovin, A. T. Fomenko, S. P. Novikov, *Modern geometry - methods and applications. Part I. The geometry of surfaces, transformation groups, and fields*, Sringer-Verlag New York 1984 (1992).

[EM] M. Eastwood, V. Matveev, *Metric connections in projective differential geometry*, Symmetries and Overdetermined Systems of Partial Differential Equations (Minneapolis, MN, 2006), 339-351, IMA Vol. Math. Appl., **144** (2007) Springer, New York.

[Fom] A. T. Fomenko, *Integrability and Nonintegrability in Geometry and Mechanics*, Kluwer Academic Publishers, Dordrecht/Boston/London (1988).

[FT] A. T. Fomenko, V. V. Trofimov, *Integrable systems on Lie Algebras and Symmetric Spaces* Gordon and Breach, London/New York (1988).

[Fub1] G. Fubini, *Sui gruppi transformazioni geodetiche*, Mem. Acc. Torino **53** (1903) 261-313.

[Fub2] G. Fubini, *Sulle copie di varieta geodeticamente applicabili*, Acc. Lincei **14** (1905) 678-683 (1 Sem.), 315-322 (2 Sem).

[Gal1] A. Galaev, *Remark on holonomy groups of pseudo-Riemannian manifolds of signature* $(2, n+2)$, arXive: math.DG/0406397 (2004).

[Gal2] A. Galaev, *The space of curvature tensors for holonomy algebras of Lorentzian manifolds*, Differential Geometry and its Applications, **22** (2005), no.1, 1-18.

[Gal3] A. Galaev, *Metrics that realize all Lorentzian holonomy algebras*, Int. J. Geom. Methods Mod. Phys., Italy. ISSN 0219-8878, 2006, **3** (2006), no. 5&6, 1025–1045.

[Gal4] A. Galaev, *Holonomy groups and special geometric structures on pseudo-Kählerian manifolds of index 2*, arXive: math.DG/0612392 (2006).

[GL] A. S. Galaev, T. Leistner, *Holonomy Groups of Lorentzian Manifolds: Classification, Examples, and Applications*, in Recent Developments in Pseudo-Riemannian Geometry // ESI Lect. Math. Phys., Eur., 53–96, Math. Soc., Zürich, (2008).

[Her] H. Hertz, *The principles of mechanics presented in a new form*, London, Macmillan (1899). (English translation of *Die prinzipien der mechanik in neuem zusammenhängen dargestellt*, Leipzig, posthumously published in 1894).

[HO] J. Hano, H. Ozeki,*On the holonomy group of linear connections*, Nagoya Math. Jour. **10** (1956) 97-100.

[Ike1] A. Ikemakhen, *Examples of indecomposable non-irreducible Lorentzian manifolds*, Ann. Sci. Math. Québec **20** no. 1 (1996) 53–66.

[Ike2] A. Ikemakhen, *Sur l'holonomie des variétés pseudo-riemaniennes de signature* $(2, 2+n)$, Publ.Math. **43** (1) (1999) 55-84.

[Joy1] D. Joyce, *Compact Riemannian 7-manifolds with holonomy G2. I*, J. Diff. Geom. **43** (1996) 291–328.

[Joy2] D. Joyce, *Compact Riemannian 7-manifolds with holonomy G2. II*, J. Diff. Geom. **43** (1996) 329–375.

[Joy3] D. Joyce, *A new construction of compact 8-manifolds with holonomy* Spin(7), J. Diff. Geom. **53** (1999) 89–130.

[Joy4] D. Joyce, *Compact manifolds with special holonomy*, Oxford University Press (2000).

[KN] S. Kobayashi, K. Nomizu, *Foundations of Differential Geometry*, Vol. I, Wiley Classics Library (1996).

[Koe] G. Koenigs, *Sur les géodesiques a intégrales quadratiques*, Note II from Darboux' "Leçns sur la théorie générale des surfaces", Vol. IV, Chelsea Publishing, (1896).

[KS] G. I. Kručkovič, A. S. Solodovnikov, *Constant symmetric tensors in Riemannian spaces*, Izv. Vysš. Učebn. Zaved. Matematika (1959) no. 3 (10), 147–158 (in Russian).

[LR] P. Lancaster, L. Rodman, *Canonical forms for hermitian matrix pairs under strict equivalence and congruence*, SIAM Review, **47** (2005) 407-443.

[Lei] T. Leistner, *On the classification of Lorentzian holonomy groups*, J. Diff. Geom., **76** (3) (2007) 423-484.

[Lev] T. Levi-Civita, *Sulle transformazioni delle equazioni dinamiche*, Ann. Mat. (2^a) **24** (1896) 255–300.

[Man] S. V. Manakov, *Note on the integration of Euler's equation of the dynamics of an N-dimensional rigid body*, Funct. Anal. Appl. **11** (1976) 328–329.

[Mat] V. S. Matveev, *Beltrami problem, Lichnerowicz-Obata conjecture and applications of integrable systems in differential geometry*, Tr. Semin. Vektorn. Tenzorn. Anal, **26** (2005) 214–238.

[MF] A. S. Mischenko, A. T. Fomenko , *Euler equations on finite dimensional Lie groups*, Izv. Akad. Nauk SSSR Ser. Mat. **42** (1978) 396–415 (in Russian); Math. USSR–Izv. **12** (1978) 371–389 (in English).

[MS] S. Merkulov, L. Schwachhöfer, *Classification of irreducible holonomies of torsion-free affine connections*, Ann. Math. **150** (1999) 77–149.

[Sal] S. Salamon, *Riemannian geometry and holonomy groups*, Pitman Research Notes in Mathematical Science, Longman Scientific & Technical, (1989).

[Sch] L. Schwachhöfer, *Connections with irreducible holonomy representations*, Adv. Math. **160** (2001) no.1, 1–80.

[Sha] I. G. Shandra, *On the geodesic mobility of Riemannian spaces*, Math. Notes **68** (2000), no.3-4 528-532.

[Shi] A. P. Shirokov, *On a property of covariantly constant affinors*, Dokl. Akad. Nauk SSSR (N.S.) **102** (1955) 461–464 (in Russian).

[Sin] N. S. Sinjukov, *Geodesic mappings of Riemannian spaces*, Nauka, Moscow, (1979), (in Russian).

[Sol1] A. S. Solodovnikov, *Projective transformations of Riemannian spaces*, Uspehi Mat. Nauk (N.S.) **11** (1956) no. 4 (70), 45-116 (in Russian).

[Sol2] A. S. Solodovnikov, *Spaces with common geodesics*, Trudy Sem. Vektor.Tenzor. Anal. **11** (1961), 43-102.

[Sol3] A. S. Solodovnikov, *Geometric description of all posssible representtions of a Riemannian metric in Levi-Civita form*, Trudy Sem. Vektor. Tenzor. Anal. **12** (1963) 131-173.

[Tho] G. Thompson, *The integrability of a field of endomorphisms*, Mathematica Bohemica, **127** (2002) No. 4, 605–611.

[Wu] H. Wu, *On the de Rham decomposition theorem*, Illinois.J.Math. **8** (1964) 291-311.