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Synchronization of a large number of continuous one-dimensional stochastic elements with time delayed mean field coupling

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Abstract

We study synchronization as a means of control of collective behavior of an ensemble of coupled stochastic units in which oscillations are induced merely by external noise. We determine the boundary of the synchronization domain of a large number of one-dimensional continuous stochastic elements with time delayed non-homogeneous mean-field coupling. Exact location of the synchronization threshold is shown to be a solution of the boundary value problem (BVP) which was derived from the linearized Fokker-Planck equation. Here the synchronization threshold is found by solving this BVP numerically. Approximate analytics is obtained by expanding the solution of the linearized Fokker-Planck equation into a series of eigenfunctions of the stationary Fokker-Planck operator. Bistable systems with a polynomial and piece-wise linear potential are considered as examples. Multistability and hysteresis is observed in the Langevin equations for finite noise intensity. In the limit of small noise intensities the critical coupling strength was shown to remain finite.

Key words:

synchronization, stochastic oscillations, delay, control

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1 Introduction

Collective behavior in a network of globally coupled elements [1], including the stochastic ones, has recently attracted much interest due to its relevance in physics, biology, medicine, economics and social sciences. Depending on the nature of the elements which compose the network, the whole variety of these networks can be roughly divided into two classes. The first class is

formed by networks of coupled oscillators or coupled excitable units which find applications in neuroscience [2,3], biology [4–8], chemistry [1,9], physics [10] and medicine [11,12]. The second class consists of ensembles, formed by the (non-excitable) units which cannot oscillate on their own. Prominent example of the latter class of networks is an ensemble of bistable stochastic elements which was previously used in physics, to describe critical phenomena [13] and phase transitions in spin systems [14], in biology, to study neural networks [15–17] and in social sciences to model decision making processes [18].

When decoupled, the elements in the network evolve independently of each other. In networks of stochastic elements the information about the initial distribution of their states is lost after a certain relaxation period, and the mean field approaches some constant value in the long time limit. The system is then known to be in the disordered state.

When coupling between the elements is introduced, two different phenomena can occur depending on the type of the elements the network is made of and on the type and the strength of the coupling. These phenomena are the first- and second-order phase transitions and synchronization. Some authors identify synchronization with the phase transition, implying that the latter occurs from the trivial non-oscillating to the oscillating state. However, this notation may lead to confusion, in case when a network of oscillators exhibits both kinds of transitions: from the non-oscillating to the oscillating state, and between two different non-oscillating states. To avoid a possible jumble we distinguish here between synchronization and phase transitions.

In the case of the phase transitions, the disordered state becomes unstable and the system undergoes a transition to a new (ordered) state with the new constant value of the mean field which is different from its value in the disordered state. If at the transition point the absolute value of the mean field changes continuously (discontinuously), the transition is regarded as second (first) order. Theory of the phase transition in a network of stochastic elements described by one-dimensional continuous equations with linear coupling was developed in Ref. [19].

Phenomenon of synchronization differs from the phase transition qualitatively. Suppose that each single element in the network exhibits noise-induced oscillations whose spectra do not contain peaks at non-zero frequencies. At some parameters of the network the individual elements are not synchronized, and the mean field is constant. Under certain conditions however, the motion of the elements can become synchronized resulting in an *oscillating* mean field. From the point of view of the bifurcation theory, the phase transition is associated with the (sub- or supercritical) pitchfork, while synchronization with the Andronov-Hopf bifurcation. Synchronization can be regarded as the mechanism for the control of the collective behavior of an ensemble of units. How-

ever, synchronization is not guaranteed to occur in a network with arbitrary structure of individual elements or arbitrary coupling between them.

Namely, synchronization was recently observed in a network of one-dimensional elements coupled through the time delayed mean field, where each single element represents a particle moving in a symmetric polynomial bistable potential [20,21], in a network of one-dimensional phase oscillators with non-linear coupling [22], and in a network of two-dimensional excitable elements with global linear coupling [23]. The simplest non-invasive way to induce synchronization in a network without interfering with its structure seems to be the use of delay in the coupling terms.

Here we consider a network of N one-dimensional elements, with i -th element described by

$$\dot{x}_i = -\frac{dU(x_i)}{dx_i} + \alpha h(x_i) \frac{1}{N} \sum_{j=1}^N f[x_j(t - \tau)] + \sqrt{2D} \xi_i(t), \quad (1)$$

where $U(x)$ is the potential each of the N particles is moving in, α is the coupling strength, $h(x)$ is the inhomogeneity of the coupling, D is the noise intensity and $\xi_i(t)$ is the white Gaussian noise with the correlation function given by $\langle \xi_i(t) \xi_k(t') \rangle = \delta_{ik} \delta(t - t')$, where δ_{ik} is the Kronecker delta and $\delta(x)$ is the Dirac delta function.

The elements in Eqs.(1) are coupled globally through the delayed average $\bar{f}(t - \tau) = (1/N) \sum_{i=1, N} f[x_i(t - \tau)]$. For large N we make an assumption that the individual elements of the network are statistically independent and therefore their joint probability density distribution is a product of the respective one-dimensional densities of individual elements. The latter is sometimes called molecular chaos approximation. Since all elements are identical, their probability densities are also identical. This allows us to describe the network of interacting elements Eqs.(1) using the one-particle distribution function $P(x, t)$. Previous studies have shown that the above approximation gives quite accurate results, provided that the function f is either a growing function [24,25], or a periodic one [22,23]. The Fokker-Planck equation for the one-particle distribution function is essentially nonlinear

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial x} \left[\frac{\partial U}{\partial x} P - \alpha h(x) \bar{f}(t - \tau) P \right] + D \frac{\partial^2 P}{\partial x^2}, \quad (2)$$

where the delayed mean field is calculated self-consistently $\bar{f}(t - \tau) = \int_{-\infty}^{\infty} f(x) P(x, t - \tau) dx$.

General question about whether or not synchronization can occur in the sys-

tem Eqs. (1) in the case of $\tau = 0$ and arbitrary f is still open. The answer is known only for the very special case of linear homogeneous coupling, i.e. for $f(x) = x$ and $h = \text{const}$. In Ref. [19] the so-called H -theorem was proven for Eqs. (1) with $f(x) = x$, $h = \text{const}$ and $\tau = 0$. According to the H -theorem, there exists a Lyapunov functional $H[P]$, such that $dH[P]/dt \leq 0$ if P is the solution of the Fokker-Planck Eq. (2). This excludes the possibility of any oscillating behavior in the system (1) without the delay. As the consequence of the H -theorem, synchronization of the large number of one-dimensional elements with *linear* mean-field coupling is not possible.

However, it should be emphasized that the H -theorem in Ref. [19] was proven under the assumption of periodic boundary conditions. For general boundary conditions the H -theorem is *not valid* and synchronization can occur even in the case of linear coupling without delay, as it was demonstrated for the ensemble of leaky integrate-and-fire neurons in Refs. [26,27].

Moreover, one-dimensional systems without delay can be synchronized if the coupling contains more than one term of the mean-field type, i.e. if it can be represented in the form $h_1(x_i)\bar{f}_1 + h_2(x_i)\bar{f}_2$, with $\bar{f}_{1,2} = (1/N) \sum_{j=1,N} f_{1,2}(x_j)$. This was shown in Ref. [22] for the system of non-linearly coupled phase oscillators. The oscillators are coupled via $\sin(x_i)c - \cos(x_i)s$, where $c = (1/N) \sum_{j=1,N} \cos(x_j)$ and $s = (1/N) \sum_{j=1,N} \sin(x_j)$. At the onset of the synchronization, c and s oscillate with identical frequency and certain phase shift.

For non-zero delay time $\tau \neq 0$, synchronization in Eqs. (1) was observed in the case of the bistable potential U [20,21]. The boundary of the synchronization domain was calculated using a dichotomous approximation, which is based on the assumption of a small noise intensity D .

Apart from the general question of whether or not coupled stochastic elements can actually be synchronized, a more detailed information about the location of the synchronization threshold in the parameter space is often required. Depending on the nature of the elements in the network and on the coupling between them a number of methods exist which can be used to determine the location of the point of transition from the asynchronous to the synchronous regimes approximately. Among such methods are the above mentioned dichotomous approximation [20,21], Gaussian approximation [22] and mode expansion [28].

However all the above approximations work sufficiently well only under specific conditions. Thus, the dichotomous approximation is valid only for bistable potentials and small noise intensities. Gaussian approximation can be used in case when the distribution possesses a single Gauss-like peak, i.e. the shape of the distribution function is close to Gaussian function. Expansion of the probability density $P(x, t)$ into a series of orthonormal polynomials, e.g. Her-

mite polynomials [28], does not work for non-smooth functions and also for very small noise intensities, as will be discussed in detail below. Alternatively, the synchronization threshold can be computed straightforwardly by solving the Langevin equations for different system parameters. This method, however, requires significant computational power and suffers from the lack of precision.

Here we formulate a boundary value problem (BVP), whose solution gives the *exact* location of the boundary of the synchronization domain in the network of stochastic elements Eqs. (1) with *arbitrary* potential U , coupling function f and heterogeneity function h . We solve this boundary value problem numerically using continuation technique of [29] for linear homogeneous coupling $f(x) = x$, $h = \text{const}$, and two types of the symmetric bistable potential U : polynomial and piece-wise linear.

Following Ref. [30] we expand the solution of the linearized Fokker-Planck equation Eq. (2) into a series of eigenfunctions of the stationary Fokker-Planck operator and reduce the formulated BVP to a system of algebraic equations which determine the synchronization threshold as a codimension-one line in the parameter space together with the frequency of the synchronized oscillations at the onset of synchronization. For brevity, throughout the paper we will call the latter the onset frequency.

For a bistable polynomial potential U we demonstrate the existence of multistability and hysteresis at finite noise intensities. For both the polynomial and the piece-wise linear potential U we show that in the limit of small noise intensities D the critical coupling strength α which determines the point of transition from the asynchronous to the synchronous behavior remains finite.

2 Self-consistent approach: synchronization threshold

2.1 Boundary value problem

Stationary one-particle distribution $P_0(x)$ obtained from Eq. (2) reads

$$\begin{aligned} P_0(x) &= \frac{1}{C} \exp \left[\frac{1}{D} \left(-U(x) + \alpha \tilde{h}(x) \bar{f}_0 \right) \right], \\ C &= \int_{-\infty}^{\infty} \exp \left[\frac{1}{D} \left(-U(x) + \alpha \tilde{h}(x) \bar{f}_0 \right) \right] dx, \end{aligned} \quad (3)$$

where $\tilde{h}(x) = \int h(x) dx$ and stationary mean field \bar{f}_0 is computed self-consistently $\bar{f}_0 = \int_{-\infty}^{\infty} P_0(x) f(x) dx$.

Note that since synchronization arises as a result of sub- or supercritical Andronov-Hopf bifurcation of the mean field, at the instant of bifurcation the latter oscillates harmonically. Then assume that near, but slightly above synchronization threshold the mean field $\bar{f}(t - \tau)$ also oscillates harmonically with a small amplitude ϵ and some (unknown) frequency ω around its stationary value, i.e. we can put $\bar{f}(t - \tau) = \bar{f}_0 + \epsilon \cos \omega t$. We have chosen the initial phase of these oscillations to be zero at the initial time moment. With the ansatz $P(x, t) = P_0(x) + \epsilon \tilde{P}(x, t)$, the linearized Fokker-Planck equation reads

$$\frac{\partial \tilde{P}}{\partial t} = L_0[\tilde{P}] - \frac{\partial}{\partial x}[\alpha h(x)(\bar{f}_0 \tilde{P} + \cos \omega t P_0)], \quad (4)$$

where the operator L_0 is given by $L_0 = \partial^2 U / \partial x^2 + (\partial U / \partial x) \partial / \partial x + D \partial^2 / \partial x^2$. We look for the solution of Eq. (4) in the form

$$\tilde{P}(x, t) = u(x) \sin \omega t + v(x) \cos \omega t, \quad (5)$$

where $u(x)$ and $v(x)$ are often called response functions. Consequently, the mean field $\bar{f}(t)$ becomes

$$\begin{aligned} \bar{f}(t) &= \int P(x, t) f(x) dx \\ &= \bar{f}_0 + \epsilon [\langle u f \rangle \sin \omega t + \langle v f \rangle \cos \omega t], \end{aligned} \quad (6)$$

and $\bar{f}(t - \tau)$ is given by

$$\begin{aligned} \bar{f}(t - \tau) &= \bar{f}_0 + \epsilon [\langle u f \rangle \cos \omega \tau + \langle v f \rangle \sin \omega \tau] \sin \omega t \\ &\quad + \epsilon [\langle v f \rangle \cos \omega \tau - \langle u f \rangle \sin \omega \tau] \cos \omega t \\ &= \bar{f}_0 + \epsilon \cos \omega t, \end{aligned} \quad (7)$$

where $\langle u f \rangle = \int_{-\infty}^{\infty} u(x) f(x) dx$ and $\langle v f \rangle = \int_{-\infty}^{\infty} v(x) f(x) dx$.

Substituting Eq. (5) into the Eq. (4) we arrive at the following coupled equations for the unknown functions u , v and the onset frequency ω

$$\begin{aligned} \omega u &= L_0[v] - \alpha \frac{\partial}{\partial x} [h(x)(\bar{f}_0 v + P_0)], \\ -\omega v &= L_0[u] - \alpha \frac{\partial}{\partial x} [h(x) \bar{f}_0 u]. \end{aligned} \quad (8)$$

Additional coupling between the functions u and v is introduced from the self-consistent Eq. (7)

$$\begin{aligned}\langle uf \rangle \cos \omega \tau + \langle vf \rangle \sin \omega \tau &= 0, \\ \langle vf \rangle \cos \omega \tau - \langle uf \rangle \sin \omega \tau &= 1.\end{aligned}\tag{9}$$

In order to solve Eqs. (8,9) one should set the appropriate boundary conditions for u and v . Here we employ natural boundary conditions, i.e. we assume that the probability density vanishes at the infinity together with its first derivative. Therefore, the same applies to the functions $u(x)$ and $v(x)$.

Similar approach was used in Ref. [27] to compute the location of the synchronization threshold for the ensemble of coupled leaky integrate-and-fire neurons with different boundary conditions. Here we solve this problem for a general network of stochastic units with delay and with arbitrary boundary conditions. The boundary value problem Eqs. (8,9) determines the frequency ω and the codimension-one line in the parameter space where the transition from the asynchronous to the synchronous motion occurs.

Once any particular solution of the Eqs. (8,9) is known for a given set of control parameters α , D and those defining the shape of the potential $U(x)$, it can be continued in the parameter space using the continuation technique [29]. We specify the functions $h(x)$, $f(x)$ and $U(x)$ and find the starting point for the continuation employing the following strategy. First we discretize the differential operators L_0 and $\partial/\partial x$ on the interval of $x \in [-L; L]$, where L is some large number, and write the system Eqs. (8) as a system of linear algebraic equations with respect to the values of u and v at the nodes of discretization. We solve this system for arbitrary frequency ω and arbitrary system parameters. Then we continue this solution of Eqs. (8) *without* the conditions Eqs. (9) in parameter ω until the two average values $\langle uf \rangle$ and $\langle vf \rangle$ of the coupling function f satisfy Eqs. (9), or, equivalently, fall on the unit circle, i.e. until $\langle uf \rangle^2 + \langle vf \rangle^2 = 1$. Then we find the smallest τ which solves the Eqs. (9). This is the starting point for the continuation of the solution of the boundary value problem Eqs. (8) with the conditions Eqs. (9).

2.2 Eigenfunction expansion: analytic results

The boundary value problem Eqs. (8,9) can be formally solved by expanding the response functions u and v into a series of eigenfunctions of the stationary Fokker-Planck operator $L_{\text{st}} = L_0 - \alpha \bar{f}_0 h(x)(\partial/\partial x) - \alpha \bar{f}_0 (\partial h/\partial x)$, as it was done in Ref. [30] in the case of $h = \text{const}$.

Denote by ψ_n the n -th right-eigenfunctions of the operator L_{st} , i.e.

$$L_{\text{st}} \psi_n = \lambda_n \psi_n,\tag{10}$$

where λ_n is the corresponding eigenvalue. Because the Fokker-Planck operator L_{st} is non-Hermitian, the eigenfunctions ψ_n are not orthonormal, i.e. $\int_{-\infty}^{\infty} \psi_n(x) \psi_m(x) dx \neq \delta_{nm}$. In order to be able to find the expansion coefficients for u and v , we need to know the left eigenfunctions $\tilde{\psi}_n(x)$ (eigenfunctions of the conjugate operator) of L_{st} that would satisfy the required orthonormality condition $\int_{-\infty}^{\infty} \tilde{\psi}_n(x) \psi_m(x) dx = \delta_{nm}$.

Standard technique is to transform the right-eigenfunctions ψ_n as follows [31]

$$\psi_n(x) = P_0(x)^{1/2} \Phi_n(x), \quad (11)$$

where $P_0(x)$ is given by Eq. (3). Functions $\Phi_n(x)$ then satisfy the stationary Schrödinger equation

$$H \Phi_n(x) = \lambda_n \Phi_n(x), \quad (12)$$

where the Hamiltonian H is given by

$$H = \left[\frac{1}{2} \left(\frac{\partial^2 U}{\partial x^2} - \alpha \frac{\partial h}{\partial x} \bar{f}_0 \right) - \frac{1}{4D} \left(\frac{\partial U}{\partial x} - \alpha h \bar{f}_0 \right)^2 \right] + D \frac{\partial^2}{\partial x^2}. \quad (13)$$

Eigenfunctions Φ_n of the Hermitian operator Eq. (13) are orthonormal, i.e.

$$\int_{-\infty}^{\infty} \Phi_n(x) \Phi_k(x) dx = \delta_{nk}. \quad (14)$$

From Eqs. (11,14) we deduce that the left-eigenfunctions $\tilde{\psi}_n$ of the operator L_{st} satisfying the condition

$$\int_{-\infty}^{\infty} \tilde{\psi}_n(x) \psi_k(x) dx = \delta_{nk}, \quad (15)$$

are related to ψ_n via

$$\begin{aligned} \tilde{\psi}_0(x) &= 1, \quad \psi_0(x) = P_0, \\ \tilde{\psi}_n(x) &= P_0^{-1} \psi_n(x), \quad n = 1, 2, \dots \end{aligned} \quad (16)$$

For further calculations it is important to notice that the eigenvalues λ_n are non-positive, i.e. $\lambda_0 = 0$, $0 > \lambda_1 > \lambda_2 > \dots > \lambda_n$ [31].

Now we expand the response functions u and v into series of the right-eigenfunctions ψ_n ,

$$u = \sum_{n=0}^{\infty} u_n \psi_n, \quad v = \sum_{n=0}^{\infty} v_n \psi_n. \quad (17)$$

The coefficients u_n and v_n of the expansions Eqs. (17) are determined from Eqs. (8) by multiplying both sides with $\tilde{\psi}_n$ and integrating over x from $-\infty$ to ∞ , namely,

$$\begin{aligned} u_n &= -\frac{\alpha\omega}{\lambda_n^2 + \omega^2} \left(\int_{-\infty}^{\infty} \tilde{\psi}_n \frac{\partial h}{\partial x} \psi_0 dx + \int_{-\infty}^{\infty} \tilde{\psi}_n h \frac{\partial \psi_0}{\partial x} dx \right), \\ v_n &= -\frac{\lambda_n u_n}{\omega}. \end{aligned} \quad (18)$$

From Eqs. (18) the mean field \bar{f} is given by

$$\begin{aligned} \bar{f}(t) &= \bar{f}_0 - \alpha\epsilon \sum_{i=1}^{\infty} \frac{g_n \omega}{\lambda_n^2 + \omega^2} \sin \omega t \\ &\quad + \alpha\epsilon \sum_{i=1}^{\infty} \frac{g_n \lambda_n}{\lambda_n^2 + \omega^2} \cos \omega t, \end{aligned} \quad (19)$$

where the coefficients g_n read

$$\begin{aligned} g_n &= \left(\int_{-\infty}^{\infty} f \psi_n dx \right) \left[\int_{-\infty}^{\infty} \tilde{\psi}_n \frac{\partial h}{\partial x} \psi_0 dx \right. \\ &\quad \left. + \int_{-\infty}^{\infty} \tilde{\psi}_n h \frac{\partial \psi_0}{\partial x} dx \right]. \end{aligned} \quad (20)$$

From Eq. (19) and Eqs. (9) we obtain the following system of algebraic equations which determines the synchronization threshold together with the unknown onset frequency ω

$$-\alpha \sum_{i=1}^{\infty} \frac{g_n \omega}{\lambda_n^2 + \omega^2} = -\sin \omega \tau, \quad \alpha \sum_{i=1}^{\infty} \frac{g_n \lambda_n}{\lambda_n^2 + \omega^2} = \cos \omega \tau. \quad (21)$$

If the coefficients g_n decrease rapidly with n , one can truncate the sums in Eqs. (21) after a few first terms. Truncation after the first term yields

$$\begin{aligned}\cos \omega \tau &= \text{sign}(\alpha g_1) \frac{\lambda_1}{\sqrt{\lambda_1^2 + \omega^2}}, \quad 1 = \frac{|\alpha g_1|}{\sqrt{\lambda_1^2 + \omega^2}}, \\ \sin \omega \tau &= \text{sign}(\alpha g_1) \frac{\omega}{\sqrt{\lambda_1^2 + \omega^2}}.\end{aligned}\tag{22}$$

As it will be shown below on the example of a symmetric bistable potential, the coefficients g_n are negative. With this assumption we analyze qualitatively the solution of the Eqs. (22).

For $\alpha > 0$ it follows from the second equation in Eqs. (22) that the smallest critical α corresponds to vanishing onset frequency $\omega = 0$, i.e. $\alpha = \lambda_1/g_1$. This is the point of the phase transition which is associated with the pitch-fork bifurcation of the stationary state. Interestingly, for a bistable symmetric polynomial potential U and homogeneous coupling $h = \text{const}$, α given by the approximate Eqs. (22) coincides with its exact value, as shown in Ref. [19]. The latter is given by $\alpha_p = D/\langle x^2 \rangle_0$, where $\langle x^2 \rangle_0$ is the variance in the stationary state P_0 [19].

The next solution of Eqs. (22) for $\alpha > 0$ corresponds to some finite frequency ω_1 and critical coupling $\alpha_1 = -\sqrt{\lambda_1^2 + \omega_1^2}/g_1 > \alpha_p$. This means that the Andronov-Hopf bifurcation of the stationary state occurs within the region where the stationary state is already unstable as a result of primary pitch-fork bifurcation at $\alpha = \alpha_p$. Consequently, synchronization can not occur for positive coupling strength $\alpha > 0$.

For negative coupling $\alpha < 0$, the vanishing frequency $\omega = 0$ is no longer a solution of Eqs. (22), because $\lambda_1 < 0$. The smallest critical frequency belongs to the interval $\omega \in [\pi/(2\tau), 3\pi/(2\tau)]$, as it follows from the first and the third Eqs. (22). Therefore, the primary bifurcation of the stationary state is the Andronov-Hopf bifurcation which corresponds to the synchronization threshold. For the special case of bistable symmetric polynomial potential U and homogeneous coupling $h = \text{const}$ the above findings were confirmed in Refs. [20,21].

3 Bistable polynomial potential

To solve the boundary value problem Eqs. (8) with the integral conditions Eqs. (9) and also to test the validity of the truncated eigenmode expansion Eqs. (22) we revisit the case of symmetric polynomial potential and homogeneous linear coupling considered in Refs. [20,21]

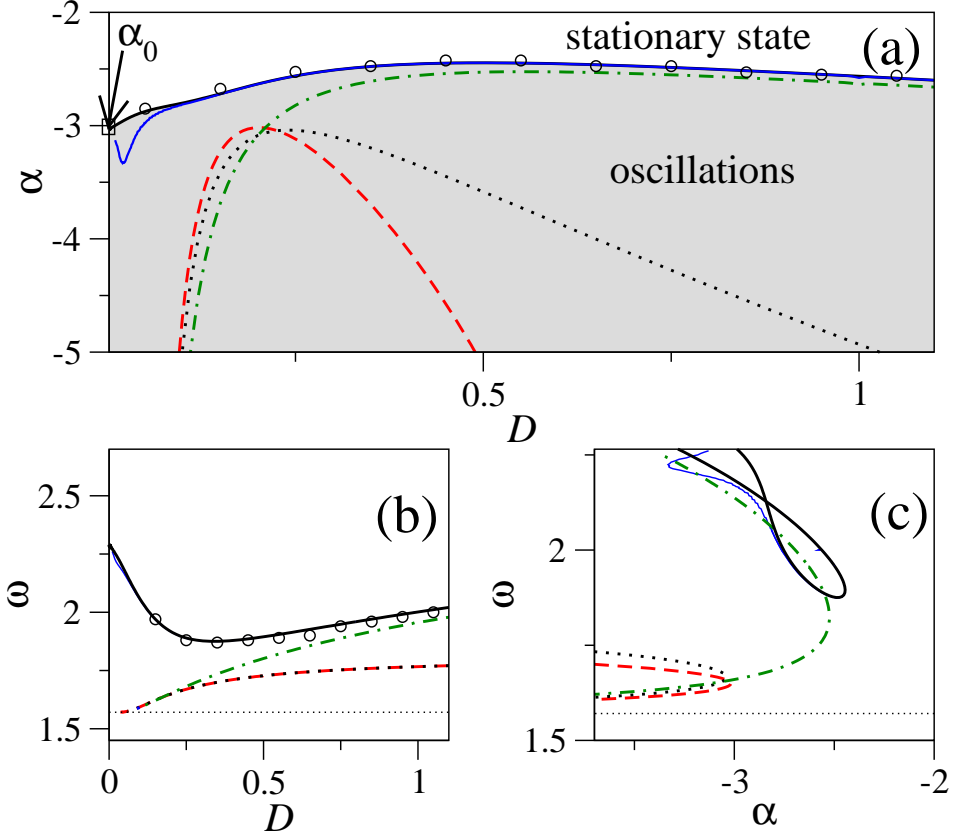


Fig. 1. (color online) (a) Synchronization in a network of stochastic bistable elements described by Eqs. (1),(23). Synchronization threshold in the parameter space (α, D) at fixed delay time $\tau = 1$. (b) and (c) Onset frequency ω as a function of D and α , respectively. Line coding as in the main text.

$$U = -\frac{x^2}{2} + \frac{x^4}{4}, \quad f(x) = x, \quad h(x) = 1. \quad (23)$$

Transition to synchronization occurs from the trivial solution $\bar{f}_0 = \bar{x}_0 = 0$ at negative coupling strength $\alpha < 0$.

The first coefficient g_1 as well as the first eigenvalue λ_1 in Eqs.(22) can be expressed as functions of the noise strength D using the well-known asymptotic results [31,30,32]. In the small D limit one obtains $\lambda_1 = 2q_k$ and $g_1 = \langle x^2 \rangle_0 \lambda_1 / D$, where $q_k = (1/2\pi) \sqrt{U''(0)U''(c)} \exp[-(U(0) - U(c))/D]$ stands for the Kramer's transition rate, c is the location of the minimum of the potential U and $\langle x^2 \rangle_0$ is the variance of x in the stationary (non-synchronous) state. This simplifies Eqs. (22) to

$$\begin{aligned}\cos(\omega\tau) &= \pm \frac{2q_k}{\sqrt{4q_k^2 + \omega^2}}, \quad 1 = \pm \alpha \frac{\langle x^2 \rangle_0}{D} \frac{2q_k}{\sqrt{4q_k^2 + \omega^2}}, \\ \sin(\omega\tau) &= \mp \frac{\omega}{\sqrt{4q_k^2 + \omega^2}},\end{aligned}\tag{24}$$

where the upper (lower) sign corresponds to positive (negative) coupling α .

In Fig. 1 we compare synchronization threshold obtained by solving numerically the boundary value problem Eqs. (8,9), with the one obtained by truncating the eigenmode expansion Eqs. (21) after the first, or the first three non-vanishing terms. Note that for even functions $U(x)$ and $h(x)$ and for even n , the eigenfunctions ψ_n are symmetric yielding $g_n = 0$. The coefficients g_n as well as the eigenvalues λ_n are computed by solving the eigenvalue problem Eq. (10) with the continuation technique [29]. This data is then used to solve the algebraic Eqs. (21) and Eqs. (22) to determine the location of the synchronization threshold in the parameter space (α, D) together with the onset frequency ω . We also compare these results with the dichotomous approximation [20,21] and with the threshold obtained by solving Eqs. (24).

Fig. 1(a) shows the boundary of the synchronization domain in the parameter space (α, D) at fixed delay time $\tau = 1$. The boundary, obtained by solving the boundary value problem Eqs. (8,9), is given by the thick solid line. In the shaded area below this line stable periodic solutions of the non-linear Fokker-Planck Eq. (2) exist, whereas above this line the trivial solution with $\bar{x}_0 = 0$ is stable. Dotted-dashed line was obtained by keeping only the first non-vanishing term g_1 in Eqs. (21). Thin solid line corresponds to the first three non-vanishing terms g_n , ($n = 1, 3, 5$) in Eqs. (21). Dotted line was obtained from the approximation Eqs. (24) and dashed line shows the prediction of the dichotomous model [20,21]. Circles correspond to a direct simulation of the Langevin Eqs. (1).

As we see, the roughest prediction gives the dichotomous approximation, followed by the asymptotic Eqs. (24). The best agreement with the threshold obtained by solving the BVP Eqs. (8,9) is given by Eqs. (21) truncated after three non-vanishing terms g_n .

Figs. 1(b,c) show the frequency ω at the synchronization threshold as a function of the noise strength D and of the coupling strength α , respectively. Horizontal dotted line corresponds to the level of $\pi/(2\tau)$.

From Fig. 1(a) we see that for large noise intensities D it is sufficient to keep only a few first non-vanishing terms in expansion Eq. (19). However, for small D the error produced by the truncation of Eqs. (19) becomes significant. In fact, in the limit $D \rightarrow 0$, the critical α remains finite, whereas the dichotomous approximation and the first term truncation of Eqs. (21) give the value of the

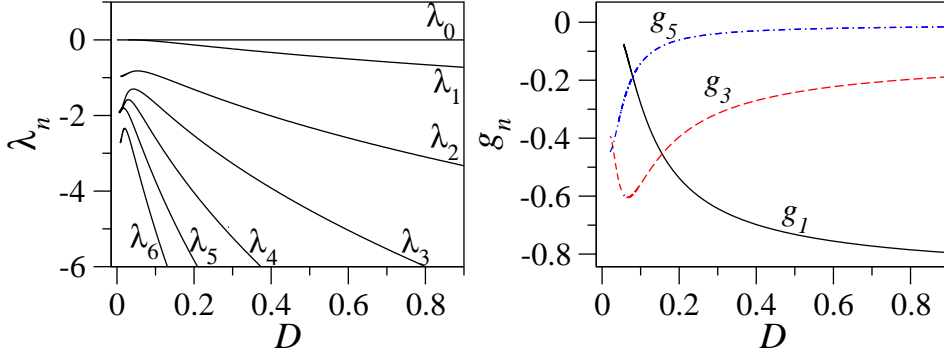


Fig. 2. (a) The first seven eigenvalues λ_n , including $\lambda_0 = 0$, of the Fokker-Planck operator Eq. (2) with U from Eq. (23) vs D . (b) The first three non-vanishing coefficients g_n .

critical coupling strength which diverges as D approaches zero. For instance, the asymptotic behavior of α in the approximation Eqs. (24) can be computed analytically. First, we notice that the onset frequency ω remains finite as D approaches zero. In fact, ω is close to $\pi/(2\tau)$ when $D \ll 1$. This allows us to neglect $4q_k^2$ as compared to ω^2 in the second equation in Eqs. (24) and obtain

$$\alpha \approx -\frac{\pi^2 D}{2\tau} \frac{1}{\sqrt{U''(0)U''(c)}} \exp \left[\frac{U(0) - U(c)}{D} \right], \quad (25)$$

where we put approximately $\langle x^2 \rangle_0 = 1$.

This mismatch can be understood from Fig. 2(a) and Fig. 2(b) where the first seven eigenvalues λ_n and the first three non-vanishing coefficients g_n are plotted as functions of the noise intensity D , respectively. The coefficient g_1 becomes comparable with all the other coefficients g_n , ($n = 3, 5, \dots$) for $D \leq 0.2$. Simultaneously, the eigenvalue λ_1 tends to zero as $D \rightarrow 0$, whereas λ_n , ($n = 3, 5, \dots$) and the threshold frequency ω remain finite. This shows that the first term in Eqs. (21) becomes negligibly small as compared to the next terms in the series. Therefore, more terms should be taken into account as $D \rightarrow 0$.

3.1 Multistability and hysteresis

The approach based on the linearization of the Fokker-Planck Eq. (2) determines the location of the Andronov-Hopf bifurcation of the mean field but does not indicate whether this bifurcation is super- or subcritical. If the bifurcation is supercritical, the transition from the asynchronous to the synchronous states occurs continuously. However, if the bifurcation is subcritical, a hysteretic behavior becomes possible implying multistability, i.e. coexistence of the stable

trivial (zero non-oscillating mean field) and stable periodic solutions.

To find the boundary of the hysteretic behavior, we follow [28] and expand the distribution density $P(x, t)$ in the series of Hermite polynomials

$$P(x, t) = \sum_{n=0}^{\infty} r_n(t) H_n(x) e^{-x^2}, \quad r_0 = \frac{1}{\sqrt{\pi}}. \quad (26)$$

As shown in Ref. [28], with the ansatz Eq. (26) one obtains from Eq. (2) and natural boundary conditions a system of infinitely many coupled nonlinear equations with time delay for the coefficients $r_n(t)$ [28]. We keep the first 18 equations in this hierarchy and analyze it for the bifurcations with DDE-BIFTOOL [33] and PDDE-CONT [34].

Note that the mode expansion Eq. (26) fails when the noise intensity D becomes small. As D approaches zero, the stationary distribution Eq. (3) becomes delta-like shaped at $x = \pm 1$ so that the coefficients r_n in Eq. (26) decrease only slowly with n . Therefore, the results obtained using Eq. (26) for small values of D are less accurate than for large D .

For the bistable potential Eqs. (23), we found hysteresis for negative coupling α and intermediate noise intensities D , as it is shown in Fig. 3(a). The boundary of the hysteresis domain is formed, from one side, by the sub-critical Andronov-Hopf bifurcation of the fixed point, and, from the other side, by the saddle-node bifurcation of the limit cycle which emerged at the Andronov-Hopf point. The locus of the saddle-node bifurcation was found using the expansion Eq. (26). Eqs. (1) was rewritten in terms of $r_n(t)$ and then analyzed for bifurcations using PDDE-CONT [34]. The existence of the hysteresis was confirmed by a direct numerical simulation of the Langevin equations Eqs. (1), as shown in Fig. 3(b).

3.2 Limit of small noise intensities

As we have seen in Fig. 1(a), as D vanishes the critical coupling α remains finite, i.e. $\lim_{D \rightarrow 0} \alpha = \alpha_0$. The physical implication of this result is as follows. Suppose that we start with a deterministic system of coupled equations Eqs. (1). Because each element in the network cannot oscillate without noise at least for the range of the values of τ considered, synchronization in the ensemble of the elements is not possible with $D = 0$. However, if arbitrarily small additive noise is present $D \neq 0$, there exists a finite coupling strength α_0 such that the dynamics of the elements becomes coherent resulting in the oscillating mean field.

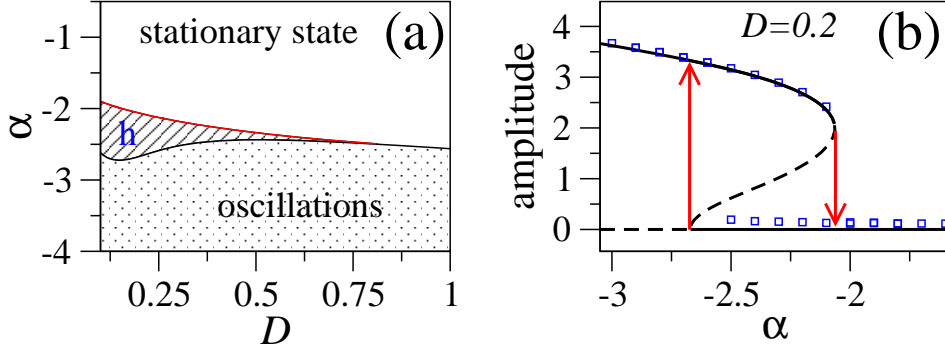


Fig. 3. (color online) (a) Region of hysteretic behavior h in the parameter plane (α, D) at $\tau = 1$. (b) Amplitude of the oscillations of the mean field as a function of the coupling strength α at fixed noise intensity $D = 0.2$. Solid (dashed) lines correspond to stable (unstable) solutions. Squares are the result of a direct numerical simulation of the Langevin equations Eqs. (1).

To show that the finiteness of α as D approaches zero is not a numerical artifact, we study analytically the asymptotic solution of the boundary value problem Eqs. (8) in this limit. To this end we notice that

$$L_{\text{st}} \left[\frac{\partial P_0}{\partial x} \right] = -U_{xxx} P_0 + U_{xx} \frac{U_x}{D} P_0. \quad (27)$$

For small D , the first term in Eq. (27) can be neglected as compared with the second one. The derivative of the stationary distribution $\partial P_0 / \partial x = -U_x P_0 / D$ differs significantly from zero only close to $x = \pm 1$, where the second derivative of the potential $U_{xx} = -(1 - 3x^2)$ is finite. By expanding U_{xx} around $x = \pm 1$ into Taylor series and keeping first two terms in this expansion, we approximately set $U_{xx} \approx 2 \pm 6(x \mp 1)$ and rewrite Eq. (27) in the form

$$L_{\text{st}} \left[\frac{\partial P_0}{\partial x} \right] \approx -2 \frac{\partial P_0}{\partial x} \mp 6(x \mp 1) \frac{\partial P_0}{\partial x}, \quad (28)$$

where the upper (lower) sign corresponds to positive (negative) x . Similarly we have

$$L_{\text{st}} \left[(x \mp 1) \frac{\partial P_0}{\partial x} \right] \approx -4(x \mp 1) \frac{\partial P_0}{\partial x}, \quad D \rightarrow 0. \quad (29)$$

This suggests the following substitution for the response functions u and v

$$\begin{aligned} u &= A \frac{\partial P_0}{\partial x} \pm A'(x \mp 1) \frac{\partial P_0}{\partial x}, \\ v &= B \frac{\partial P_0}{\partial x} \pm B'(x \mp 1) \frac{\partial P_0}{\partial x}, \end{aligned} \quad (30)$$

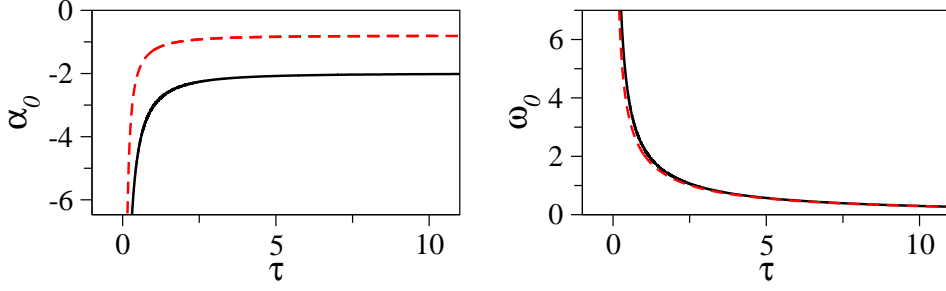


Fig. 4. (color online) (a) The limiting value of coupling strength $\alpha = \alpha_0$ as a function of the delay time τ computed (solid line) from the BVP Eqs. (8,9) and its approximate value (dashed line) obtained from Eqs. (33). (b) Limiting value of the onset frequency $\lim_{D \rightarrow 0} \omega = \omega_0$ vs delay time.

where A , A' , B and B' are some unknown constants. Substituting Eqs. (30) into Eqs. (8) with Eqs. (28,29) and comparing the coefficients before $\partial P_0 / \partial x$ and $x \partial P_0 / \partial x$, we determine the constants A , A' , B and B'

$$\begin{aligned} A' &= \frac{36\omega_0\alpha_0}{(\omega_0^2 + 4)(\omega_0^2 + 16)}, \quad B' = -\frac{6\alpha_0(\omega_0^2 - 8)}{(\omega_0^2 + 4)(\omega_0^2 + 16)}, \\ A &= -\frac{\omega_0\alpha_0}{\omega_0^2 + 4}, \quad B = -\frac{2\alpha_0}{\omega_0^2 + 4}, \end{aligned} \quad (31)$$

where α and ω were replaced by their limiting values α_0 and ω_0 , respectively.

Now notice that the integrals $\langle ux \rangle$ and $\langle vx \rangle$ can be calculated analytically. For instance $\langle ux \rangle$ is given by

$$\int_{-\infty}^{\infty} xu \, dx = \int_{-\infty}^{\infty} x(A - A') \frac{\partial P_0}{\partial x} \, dx = -(A - A'). \quad (32)$$

From the integral conditions Eqs. (9), Eqs. (31) and Eqs. (32) we then obtain the following two coupled equations for the critical α_0 and ω_0

$$\begin{aligned} \frac{4\alpha_0(\omega_0^2 - 20)}{(\omega_0^2 + 4)(\omega_0^2 + 16)} + \cos \omega_0 \tau &= 0, \\ \frac{\alpha_0\omega_0(\omega_0^2 + 52)}{(\omega_0^2 + 4)(\omega_0^2 + 16)} + \sin \omega_0 \tau &= 0. \end{aligned} \quad (33)$$

Fig. 4(a) and Fig. 4(b) show how α_0 and ω_0 depend on the delay time τ . Solid lines correspond to α_0 computed from the BVP Eqs. (8,9), dashed lines obtained by solving Eqs. (33). As we see, with growing τ the limiting critical coupling α_0 increases from $\alpha_0 = -\infty$ to some finite negative value, whereas ω_0 decreases monotonically to zero. The significance of this observation is as

follows: without delay in coupling ($\tau = 0$), no coupling was strong enough to induce synchronization in the network. However, an arbitrary small but finite τ grants the network the capacity to synchronize at finite noise intensity.

4 Piece-wise linear symmetric bistable potential

As the next example, we consider a symmetric bistable piece-wise linear potential

$$U = \begin{cases} -x - 1, & x < -1 \\ x + 1, & -1 < x < 0 \\ -x + 1, & 0 < x < 1 \\ x - 1, & x > 1. \end{cases}, \quad h(x) = 1, \quad (34)$$

which is shown by the solid line in Fig. 5(a).

This case is qualitatively different from the polynomial potential Eq. (23). Neither dichotomous approximation, nor the truncated eigenmode expansion can be used for Eq. (34) for the following reasons.

Dichotomous approximation fails because the second derivative $\partial^2 U / \partial x^2$ of the potential Eq. (34) is given by the delta-function at $x = \pm 1$ and $x = 0$, i.e.

$$\frac{\partial^2 U}{\partial x^2} = 2\delta(x + 1) - 2\delta(x) + 2\delta(x - 1). \quad (35)$$

Therefore, the Kramers rate q_k in Eq. (24) is ill-defined.

The truncated eigenmode expansion can not be used either, because the eigenvalue problem Eq. (10) has only two discrete eigenvalues λ , whereas the rest of the eigenvalues form a continuum. This can be seen by solving the stationary Schrödinger Eq. (12) with $\partial U / \partial x = \pm 1$ and $\partial^2 U / \partial x^2$ from Eq. (35)

$$D \frac{\partial^2 \Phi_n}{\partial x^2} - \frac{\Phi_n}{4D} + \frac{1}{2} \frac{\partial^2 U}{\partial x^2} \Phi_n = \lambda_n \Phi_n. \quad (36)$$

For $x \neq 0$ and $x \neq \pm 1$, the solution of the Eq. (36) is given by a linear combination of the exponents

$$\Phi_n = C_1 \exp(\gamma_n x) + C_2 \exp(-\gamma_n x), \quad (37)$$

where $\gamma_n = \sqrt{\lambda_n/D + 1/(4D^2)}$. Because the eigenvalues λ_n in Eq. (12) and Eq. (36) are real and non-positive, the exponent γ_n can be either real or purely imaginary.

Consider first the case when γ_n is real, i.e. $\lambda_n \geq -1/(4D)$. The eigenfunctions Φ_n can be divided into classes: symmetric (even) Φ_n^s or antisymmetric (odd) Φ_n^a . Obviously, antisymmetric eigenfunctions Φ_n^a take the form

$$\begin{aligned}\Phi_n^a &= C_1 \exp(\gamma_n x), \text{ if } x \leq -1 \\ \Phi_n^a &= C_2 [\exp(\gamma_n x) - \exp(-\gamma_n x)], \text{ if } -1 \leq x \leq 0\end{aligned}\tag{38}$$

The two constants C_1 and C_2 are determined from the following three conditions: the normalization condition Eq. (14), continuity of the eigenfunction Φ_n^a at $x = -1$ and one extra condition which is obtained by integrating Eq. (36) over x from $x = -1 - \Delta$ to $x = -1 + \Delta$, with some small Δ . The last condition yields

$$\lim_{\Delta \rightarrow 0} D \left[\frac{\partial \Phi_n^a(-1 + \Delta)}{\partial x} - \frac{\partial \Phi_n^a(-1 - \Delta)}{\partial x} \right] + \Phi_n^a(-1) = 0.\tag{39}$$

From the continuity condition at $x = -1$ and Eq. (39) we obtain the following two equations for C_1 and C_2

$$\begin{aligned}0 &= 2D\gamma_n C_2 \cosh \gamma_n - D\gamma_n C_1 e^{-\gamma_n} + C_1 e^{-\gamma_n}, \\ 0 &= C_1 e^{-\gamma_n} - 2C_2 \sinh \gamma_n.\end{aligned}\tag{40}$$

Eqs. (40) have a non-trivial solution only if γ_n satisfies the following transcendental equation

$$2D\gamma_n e^{\gamma_n} + e^{-\gamma_n} - e^{\gamma_n} = 0.\tag{41}$$

From Eq. (41) it is clear that a single non-zero γ_n exists if $D < 1$. Denote this value of γ_n by γ_1 . The corresponding eigenvalue λ_n then reads

$$\lambda_1 = D\gamma_1^2 - \frac{1}{4D}.\tag{42}$$

Now consider a symmetric eigenfunction Φ_n^s , which can be represented in the form

$$\begin{aligned}
\Phi_n^s &= C_1 \exp(\gamma_n x), \quad \text{if } x \leq -1 \\
\Phi_n^s &= C_2 \exp(\gamma_n x) + C_3 \exp(-\gamma_n x), \quad \text{if } -1 \leq x \leq 0
\end{aligned}
\tag{43}$$

There are four conditions for three unknown coefficients C_i in Eqs. (43). These are the continuity condition for Φ_n^s at $x = -1$, the normalization condition Eq. (14), the Eq. (39) and one extra condition obtained by integrating Eq. (36) near $x = 0$, i.e.

$$\lim_{\Delta \rightarrow 0} D \left[\frac{\partial \Phi_n^s(\Delta)}{\partial x} - \frac{\partial \Phi_n^s(-\Delta)}{\partial x} \right] - \Phi_n^s(0) = 0.
\tag{44}$$

From Eq. (39), Eq. (44) and the continuity condition at $x = -1$ we obtain three linear homogeneous equations for C_1 , C_2 and C_3

$$\begin{aligned}
0 &= C_1 e^{-\gamma_n} = C_2 e^{-\gamma_n} + C_3 e^{\gamma_n}, \\
0 &= \gamma_n D [C_2 e^{-\gamma_n} - C_3 e^{\gamma_n} - C_1 e^{-\gamma_n}] + C_1 e^{-\gamma_n}, \\
0 &= 2\gamma_n D (C_3 - C_2) - (C_2 + C_3).
\end{aligned}
\tag{45}$$

System Eqs. (45) has a non-trivial solution only if γ_n satisfies

$$(1 - 2\gamma_n D)(1 + 2\gamma_n D - e^{-2\gamma_n}) = 0.
\tag{46}$$

The only non-trivial solution of Eq. (46) is $\gamma_0 = 1/(2D)$, which corresponds to zero eigenvalue $\lambda_0 = D\gamma_0^2 - 1/(4D) = 0$.

If $\lambda_n < -1/(4D)$, the exponent γ_n is purely imaginary. In this case the spectrum $\lambda_n = \lambda(k)$ forms a continuum. It coincides with the spectrum of the Fokker-Planck operator with the V-shaped potential [31,35]

$$\lambda(k) = - \left(\frac{1}{4D} + Dk^2 \right).
\tag{47}$$

Summarizing, we see that the spectrum of the stationary Schrödinger Eq. (12) with the symmetric bistable piece-wise linear potential Eq. (34) has at most two discrete eigenvalues: zero eigenvalue $\lambda_0 = 0$ which corresponds to a symmetric eigenfunction Φ_0 and a non-zero one, given by Eq. (42) which exists only for $D < 1$ and corresponds to the antisymmetric eigenfunction Φ_1 . The rest of the spectrum forms a continuum Eq. (47).

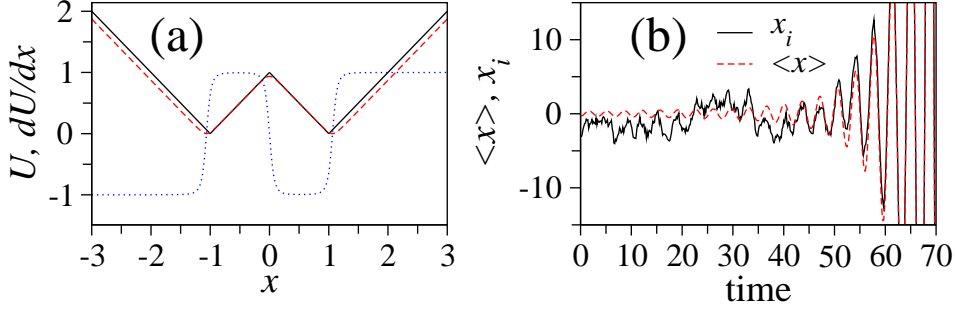


Fig. 5. (color online) (a) (solid line) Piece-wise linear potential Eq. (34), (dashed line) approximation by the differentiable potential Eq. (48) with $K = 10$, (dotted line) derivative dU/dx of the potential given by Eq. (48). (b) Entrainment of the unbounded oscillations of the mean field $\langle x \rangle$ at $D = 0.8$ and $\alpha = -2.1$: solid line – oscillation of a single element, dashed-line – mean field $\langle x \rangle$ vs. time.

This shows that in case of the piece-wise linear potential Eq. (34) one should rely on the numerical solution of the BVP Eqs. (8,9). When using the continuation technique [29], all the functions involved, including their derivatives, must be continuous which does not allow us to use the potential Eq. (34) directly. To overcome this problem, we notice that with sufficiently large number K , the modulus $|x|$ can be replaced by $(2x/\pi) \arctan(Kx)$ [29]. With this approximation we rewrite the potential Eq. (34) in terms of continuously differentiable functions

$$U(x) = \frac{2(e(x) - 1)}{\pi} \arctan[K(e(x) - 1)],$$

$$e(x) = \frac{2x}{\pi} \arctan(Kx). \quad (48)$$

Fig. 6(a) shows the synchronization threshold for $K = 10$ and different delay times τ , as indicated in the legend. Fig. 6(b) represents the onset frequency ω as a function of the noise intensity D . Circles in Fig. 6(a) show the boundary of the synchronization domain and in Fig. 6(b) the onset frequency obtained by directly solving the Langevin Eqs. (1) with the original piece-wise linear potential Eq. (34).

As we see from Fig. 6(a), as τ decreases, the boundary of the synchronization domain moves towards larger (by the absolute value) coupling strength α and eventually disappears as τ approaches zero. The onset frequency ω increases with decreasing τ .

Another peculiarity of the piece-wise linear potential is a relatively weak stability of the two equilibria at $x = \pm 1$ as compared to the polynomial potential. The force F that returns each single oscillator to the equilibrium is constant for the piece-wise linear potential, $F = \pm 1$, while it increases with x as $F = \pm x^3$

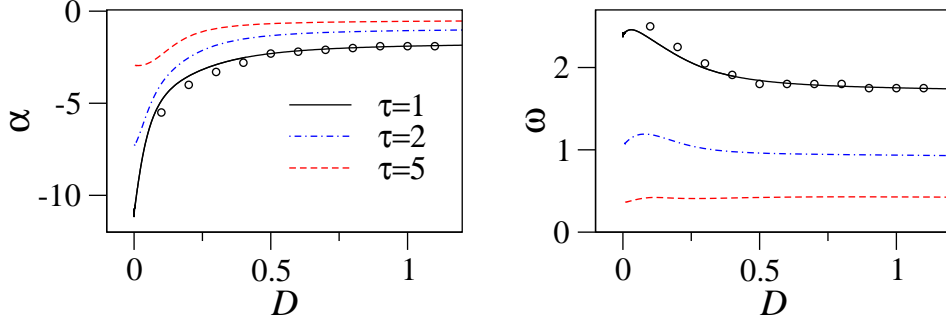


Fig. 6. (color online) (a) Synchronization threshold for the ensemble of bistable elements Eqs. (1) with the piece-wise linear potential Eq. (48) with $K = 10$. Circles represent the results of the direct simulation of the Langevin Eqs. (1) with the potential Eq. (34). (b) Onset frequency ω as a function of the noise intensity D for three different values of the delay time τ , as in the legend in (a).

for the polynomial potential.

This leads to an interesting effect in case of the piece-wise linear potential. There appears to be no stable limit cycle with finite amplitude in the region of synchronous behaviour. Single elements in the network oscillate in phase with one-another (and, therefore, with the mean field $\langle x \rangle$) with the exponentially increasing amplitude, as shown in Fig. 5(b). In our simulations with 10^4 oscillators we observed unlimited exponential growth of the amplitude of oscillations as soon as the parameters were changed to be within the region of synchronous behaviour.

5 Conclusion

To conclude, we formulated a boundary value problem Eqs. (8,9) which allows us to compute exactly the boundary of the synchronization domain of a system of stochastic one-dimensional elements with non-homogeneous time-delayed coupling. By expanding response functions u and v into a series of eigenfunctions of the stationary Fokker-Planck operator, we reduced the BVP Eqs. (8,9) to a system of two algebraic equations which determine the critical coupling strength α as well as the frequency ω at the onset of synchronization with the given noise intensity D and delay time τ . The validity of the truncated expansion Eqs. (21) was demonstrated on the example of the system of bistable stochastic oscillators in a symmetric polynomial potential. We show that the first term truncation Eqs. (22) as well as the previously used dichotomous approximation [20,21] fail to give an acceptable estimation for the synchronization threshold for small noise intensities D . The smaller the value of D is, the larger number of expansion terms should be taken into account. In the limit of $D \rightarrow 0$ the BVP Eqs. (8,9) was solved numerically using the

continuation technique [29].

For a symmetric bistable polynomial potential we demonstrated the existence of multistability and hysteresis in the system of Eqs. (1) for moderate values of the noise intensity D . To this end we compared in Fig. 3(b) the amplitude of the mean field $\bar{x}(t)$ computed by numerically solving the Langevin equations Eqs. (1) with that obtained using the mode expansion Eq. (26). We found that for moderate noise intensities D ($D \lesssim 0.75$ at $\tau = 1$), the time-periodic solution of Eqs. (1) loses its stability at a certain value of the coupling strength α via a saddle-node bifurcation. Hereby if α increases, the stable periodic solution disappears and the amplitude of the mean field changes abruptly to zero up to the fluctuations of order $1/\sqrt{N}$, where N is the number of elements in the network [36]. Conversely, if α decreases, the mean field remains constant until the point of the subcritical Andronov-Hopf bifurcation where the solution jumps to a stable periodic orbit with finite amplitude.

The case of a piece-wise linear bistable potential is considered as an example where the truncated eigenmode expansion Eqs. (21) as well as the dichotomous approximation fail by definition. Firstly, continuous spectrum of eigenvalues of the stationary Fokker-Planck operator does not allow us to use the eigenmode expansion. Secondly, piece-wise linearity leads to a divergent second derivative of the potential and, therefore, to the failure of the Kramers formula for the transition rates which is used in the dichotomous approximation.

For both the polynomial and the piece-wise linear potentials, we show that in the limit of vanishing noise intensities D the onset coupling strength α which determines the point of transition from the asynchronous to the synchronous regime, remains finite. For the polynomial potential this result was confirmed analytically by finding small- D asymptotic solution of the BVP Eqs. (8,9).

Finally, the practical implication of this study is the possibility to use delay in the coupling term in order to control the collective behavior of network elements. Namely, if each network element is one-dimensional, the removal of the delay guarantees that the system with periodic boundary conditions will never be synchronized at any coupling strength, whatever large. This can be important in applications related to neurobiology where synchronization of neurons in certain brain areas is associated with conditions like epilepsy and Parkinson's disease. On the other hand, by changing the positive value of the delay one can control the location of synchronization threshold and thus make the system more or less robust with respect to the possible fluctuations in coupling strength or noise intensity. In fact, we have shown that by making τ small, we force the system to enter the regime when the absolute value of the critical coupling strength α becomes very large implying that synchronization is suppressed disregarding the value of the noise intensity.

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