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
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# Wave Scattering by Circular Arc Shaped Plates

by

**Umma Urka**

M.Phil Thesis

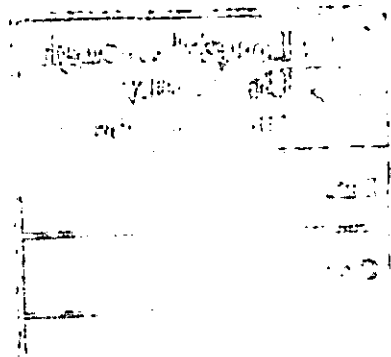
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For the Award of Master of Philosophy  
of Loughborough University of Technology

September 1995.

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Department of Mathematical Sciences.



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**To : My Daughter.**

## ABSTRACT

This work investigates the reflection and transmission properties of a circular arc plate which is submerged in deep water. The purpose is to compare the reflective properties of a circular arc plate with those for a submerged, circular cylinder in order to assess the suitability of using circular arc plates when constructing a water wave lens. Linear theory is assumed and two separate techniques are used to determine the wave field. The first involves expanding the potential as a series of multipole potentials outside a circular region and a series of nonsingular solutions of Laplace's equation within the region and matching the expansions on the boundary. The second technique is based on a variational procedure and is used to derive an explicit, approximate expression for the reflection coefficient, under the assumption that the plate is short compared with the other length scales in the problem. Results are presented which compare the approximate solution with the full numerical method for a variety of plates. Finally, the full numerical calculations of the reflection and transmission coefficients for a plate are compared with those for a submerged, circular cylinder.

# Declaration

I declare that I am responsible for the work submitted in this thesis, that the original work is my own except as specified in acknowledgements, and that neither the thesis nor the original work contained therein has been submitted to this or any other institution for a higher degree.

Umna Urka.

# Acknowledgements

I would like to express my very sincere thanks to my supervisor Dr. Maureen McIver for her continual encouragement, guidance, comments and valuable supervision for the research in this thesis. I benefited greatly from his mathematical intuition and insight. It was a privilege to work with her. I would also like to thank my husband Dr. Montaz Ali for his unbounded enthusiasm. Our countless discussions were invaluable to me. I would like to thank our secretaries Louise Howard and Helen Sherwood for helping me on the use of  $T_E X$ .



# Table of Contents

<b>Chapter 1</b>	1
1.1 Introduction	1
1.2 Equations of Motion	8
<b>Chapter 2</b>	12
2.1 Introduction	12
2.2 Problem Formulation	12
2.3 Mathematical Model	15
<b>Chapter 3</b>	23
3.1 A Variational Approximation	23
<b>Chapter 4</b>	31
4.1 Numerical Results and Discussion	31
<b>Chapter 5</b>	42
5.1 Conclusion	42
<b>References</b>	44

# Chapter 1

## 1.1 Introduction

Increasing demand for energy has encouraged scientists to devote themselves to various research projects on how to extract energy from renewable energy sources. Ocean waves represent an enormous energy resource and a great deal of effort has been made by the scientists around the world to economically extract energy from water waves. The theoretical aspects of the hydrodynamics of wave energy devices are discussed in Evans (1981). A Norwegian research group has also investigated the feasibility of constructing a water wave lens (Helstad, 1980) for focusing waves. A water wave lens consists of a system of submerged bodies, each of which is capable of retarding a wave by a different amount and the overall effect of the underwater system is to focus waves prior to harnessing their energy (Mehlum and Stamnes, 1978). The idea of constructing a water wave lens which would focus waves prior to extracting energy from them has been developed by Mehlum and Stamnes (1978) and more recently by Kinoshita and Murashige (1991).

Each lens element must clearly possess the property that it reflects very little of the incident wave, over a wide range of frequencies and directions. Such a lens system operates under the same principles that govern the focusing of light waves. As a wave enters the

shallower region over a submerged body, the wavelength is decreased and, as is well-known, the wave speed is reduced and a phase lag is introduced in the transmitted wave on the far side of the body. After passing over the lens the waves converge towards a focal point where a wave energy device may be installed. Mehlum (1980) considered the use of a submerged, circular cylinder as a lens element as Dean (1948) and Ursell (1950a) had showed that this body has the ideal property that it does not reflect normally incident waves of any frequency when placed in deep water. Total transmission of normally incident waves past other bodies does also occur but only at isolated frequencies (Newman, 1965 and Mei and Black, 1969). McIver (1985) also studied the use of a horizontal flat plate which is moored to the sea bed as a lens element as it may be easier to construct a sufficiently long plate than a sufficiently large cylinder to give the phase lag required in the transmitted wave.

The reflective properties of the bodies referred to in the previous paragraph have been determined using linear wave diffraction theory. This theory relies on the assumption that the water is inviscid and incompressible and that the motion is irrotational. Under these assumptions the wave motion is described by a velocity potential which satisfies Laplace's equation. In addition, the wave amplitudes are assumed to be small compared to the wavelength and the amplitude of any body motion is assumed to be correspondingly small and so the free surface and body boundary conditions may be linearised. Under linear theory the velocity potential is conventionally split up into the scattering potential, which is due to the wave diffraction by a fixed body and the radiation potential which is due to the radiation of waves by the moving body into otherwise calm water.

In two dimensions, the properties of the wave reflected or transmitted by a fixed body are measured by the reflection and transmission coefficients respectively. These coefficients are ratios of the amplitudes of the reflected and transmitted waves to the amplitude of

the incident wave. In general exact analytic solutions for the reflection and transmission coefficients are very hard to find. However Ursell (1947) and Dean (1945) obtained analytic solutions for the reflection and transmission coefficients associated with a finite vertical plate and a semi-infinite vertical plate respectively in infinite depth water. The work of Ursell is based on the Havelock wavemaker theory (Havelock, 1929). Extensions to the work of Ursell (1947) were made by John (1948) who considered barriers inclined at angles  $\pi/2n$  to the horizontal, although the solution rapidly becomes more complicated as  $n$  increases. Further extensions to submerged plates, obliquely incident waves and more than one barrier have been made by Evans (1970), Evans and Morris (1972) and Porter (1974). Evans (1970) considered scattering of surface waves by a fixed vertical plate, submerged beneath the free surface. His method of solution was a complex function technique. Shaw (1985) considered the problem of scattering by a surface-piercing plate, whose shape is slightly altered from being flat. Using perturbation techniques, Shaw found that to the first order, the problem is the same as that solved by Ursell (1947). However he found second order corrections to the reflection and transmission coefficients. Similarly, the horizontal plate has been the subject of many investigations. Although there does not exist an explicit solution for a finite, horizontal plate, numerical methods such as the finite element method (Patarapanch, 1984) and a matched eigenfunction expansion (McIver, 1985) have been used to obtain the values for the reflection and transmission coefficients. McIver (1985) employed the method of matched eigenfunction expansions to obtain the scattering and radiation potential due to diffraction of water waves by a moored, horizontal flat plate. In addition, the Wiener-Hopf technique has been used by Burke (1964) and Heins (1950) to generate explicit forms for the reflection coefficient associated with a semi-infinite plates, both submerged and in the free surface. More recently, Parsons and Martin (1992) have

developed a method based on hypersingular integral equations to calculate the reflection from a plate of arbitrary inclination. Their method may be generalised to plates which are not flat and Parsons and Martin (1994) presents some results for a submerged, circular arc plate.

In addition to the work done on plates there have been many investigations into the properties of submerged cylinders. The radiation and scattering of waves by a single cylinder have been investigated by Dean (1948) and Ursell (1950a). Dean employed the conformal mapping technique and showed that waves normally incident on a submerged circular cylinder suffer no reflection. The only effect the obstacle has at a great distance is that it produces a phase-difference between the incident and transmitted waves, while the amplitude of the waves is the same. Ursell (1950a) verified the results by using multipole potentials. Multipole potentials are singular solutions of Laplace's equation which satisfy the free surface condition, behave like waves radiating outwards at large distances from the singular point and, in infinite depth water, decay with depth. Ursell placed sets of these multipoles at the centre of the cylinder, choosing their strengths to satisfy the body boundary condition. Mathematically he derived the multipole potentials by repeated differentiation of the complex source potential with respect to the source point. Ursell (1950b) also established the uniqueness of the wave potential when a normal velocity distribution was applied on the cylinder surface. Ogilvie (1963) used Ursell's method to calculate the first-order and second-order forces on a cylinder. In fact, he extended Ursell's method for the specific problem, in which a cylinder is made to move along a circular orbit around its axis, and showed that there is wave radiation in one direction only. He also restablished the remarkable conclusion arrived at originally by Dean (1948) and subsequently confirmed by Ursell (1950a). The single circular cylinder was also studied

by Levine (1965) who used a Green's function technique to investigate waves incident on a cylinder at arbitrary angles and showed that zero reflection occurred at normal incidence. He developed a mode of analysis to extend the results previously obtained in the case of normal incidence. The Green's function technique is a boundary element method which was developed by John (1950). The advantage of this method is that it can be used for arbitrary shaped bodies and the disadvantage of the method is that for multiple bodies it becomes expensive both in terms of cpu time and computer storage, to model each body accurately. Davis (1974) developed a short wave approximation technique for wave scattering by a submerged circular cylinder. Evans et al (1979) further investigated the possibility of the use of a submerged circular cylinder as a wave energy device. Leppington and Siew (1980) extended the work of Davis (1974) for cylinders of elliptic cross-section. Mehlum (1980) derived a formula for the transmission coefficient using a conformal mapping technique for the problem of waves normally incident on a submerged cylinder. The method is an attractive one in terms of numerical calculation. A more general method for a submerged elliptic cylinder was derived by Grue and Palm (1984) using integral equations.

The work described so far has all been for a single cylinder. Wang (1970, 1981) extended Ursell's multipole method to two cylinders. Schmute (1971) used the integral equation method of Levine (1965) to investigate the radiation and scattering of waves by two submerged circular cylinders of arbitrary radii and arbitrary depth. The integral equation technique was further used by Schmute (1967) who looked at the scattering of waves by an infinite array of submerged circular cylinders each of which had the same size and depth but no numerical results were given. Chakrabarti (1979) also applied the boundary element method of John (1950) to a group of submerged cylinders and the numerical results were discussed for two parallel cylinders. In O'leary (1985) an extended

multipole potential technique has been used to investigate the radiation and scattering of waves by a group of any number of submerged, horizontal circular cylinders with arbitrary positions and radii.

In this chapter we have already discussed the wave scattering in two dimensions by vertical and horizontal, flat and non-flat plates. Following the plates, we discussed the reflective properties of a single and multiple circular cylinders. We also discussed the different techniques that have been used to tackle the wave scattering problems by plates, circular cylinders and even the cylinders of elliptic cross-section. Although a submerged horizontal plate reflects more energy than a submerged circular cylinder, it may be easier to construct plates of sufficient size to obtain the phase lags required in the transmitted wave when the bodies is acting as a lens element. Thus, the main objective of this thesis is to study the scattering of waves by a fixed submerged, circular arc plate. The reason this body is chosen is that it might be expected to reflect very little of the waves because in shape, it resembles the top part of a circular cylinder.

This thesis is organized as follows. In the next section of this chapter we review the derivation of the linearised equations of motion of water waves. In chapter 2 we determine the reflection and transmission coefficients for a submerged circular arc plate using the multipole potential technique of Ursell (1950a). In chapter 3 we derive approximate solutions for the reflection and transmission coefficients by using a variational approximation technique. In chapter 4 we illustrate our numerical results for long and short plates and compare the full theory with the approximate solution. Finally, some concluding remarks based on the numerical results are also given in chapter 5. We also make comments and give further research direction by weighing these results against the ideal properties of a lens element in the same chapter.

(The work in this thesis has already been accepted for publication in McIver and Urka  
(1995).)



## 1.2 Equations of Motion

The theory is concerned with an incompressible Newtonian fluid so that any motion is governed by the Navier-Stokes equations. These equations govern the conservation of mass and momentum and for an incompressible fluid they are as follows:

$$\nabla \cdot \underline{u} = 0 \quad (1.1)$$

and

$$\frac{\partial \underline{u}}{\partial t} + \underline{u} \cdot \nabla \underline{u} = -\nabla \left( \frac{p}{\rho} \right) + \underline{F} + \nu \nabla^2 \underline{u} \quad , \quad (1.2)$$

where  $\underline{u}$  is the velocity,  $p$ , the pressure,  $\rho$ , the density,  $\underline{F}$ , the body force and  $\nu$ , the kinematic viscosity. The body force is the gravitational force on the body and is given by

$$\underline{F} = (0, g, 0) = \nabla(gy) \quad , \quad (1.3)$$

where  $y$  is the  $y$ -coordinate of a coordinate system and is measured vertically downwards from the mean free surface.

The water is assumed to be inviscid, incompressible and of constant density. Therefore, in the presence of a conservative body force if a portion of the water is initially in irrotational motion then it can be seen from Kelvin's circulation theorem (Acheson, 1990) that the portion will always be in irrotational motion. However, it is assumed that the water is initially at rest and therefore, it follows that

$$\nabla \wedge \underline{u} = 0 \quad . \quad (1.4)$$

The equation (1.6) implies that for an inviscid irrotational flow, there is a velocity potential  $\Phi$  such that

$$\underline{u} = \nabla\Phi \quad . \quad (1.5)$$

Substitution of (1.5) into (1.1) gives the Laplace's equation for the potential

$$\nabla^2\Phi = 0 \quad . \quad (1.6)$$

Therefore if the velocity potential is known, then the pressure can be found from the momentum equation (1.2). By using the vector identity,

$$\underline{u} \cdot \nabla\underline{u} = \nabla(u^2/2) - \underline{u} \wedge (\nabla \wedge \underline{u}) \quad (1.7)$$

and irrotationality, we may rewrite equation (1.2), with  $\nu = 0$ , as

$$\nabla \left( \frac{\partial\Phi}{\partial t} + \frac{(\nabla\Phi)^2}{2} + \frac{p}{\rho} - gy \right) = 0 \quad . \quad (1.8)$$

The last equation is integrated to give,

$$p - p_o = -\rho \left( \frac{\partial\Phi}{\partial t} + \frac{(\nabla\Phi)^2}{2} - gy \right) \quad , \quad (1.9)$$

where  $p_o$  is a constant, usually chosen to equal atmospheric pressure which is assumed to be constant.

The boundary conditions at the free surface, at solid boundaries and at large distances from any bodies, must be added to the equations discussed so far. First the free surface boundary conditions will be considered. In two dimensions the vertical elevation of any point on the free surface is defined by,

$$y = \zeta(x, t) \quad . \quad (1.10)$$

Surface tension is considered negligible so the pressure just below the free surface equals atmospheric pressure just above the free surface, i.e.,  $p = p_o$  on  $y = \zeta(x, t)$ . Thus, from (1.9),

$$\frac{\partial \Phi}{\partial t} + \frac{(\nabla \Phi)^2}{2} - g\zeta = 0, \quad \text{on } y = \zeta \quad . \quad (1.11)$$

This is known as the dynamic free surface boundary condition. The kinematic free surface condition is derived by requiring a fluid particle on the free surface to remain on the free surface, i.e.,

$$\frac{D(y - \zeta)}{Dt} = 0 \quad \text{on } y = \zeta \quad , \quad (1.12)$$

where  $D/Dt$  is the Lagrangian derivative ( $\partial/\partial t + \underline{u} \cdot \nabla$ ). Equation (1.12) is expanded to give

$$\frac{\partial \Phi}{\partial t} - \frac{\partial \zeta}{\partial t} - \frac{\partial \Phi}{\partial x} \frac{\partial \zeta}{\partial x} = 0 \quad \text{on } y = \zeta \quad . \quad (1.13)$$

Equations (1.11) and (1.13) are now linearised about the mean free surface,  $y = 0$  to obtain

$$\frac{\partial \Phi}{\partial t} = g\zeta \quad \text{on } y = 0 \quad (1.14)$$

and

$$\frac{\partial \Phi}{\partial y} = \frac{\partial \zeta}{\partial t} \quad \text{on } y = 0 \quad (1.15)$$

which may be combined to give a boundary condition for the potential,

$$\frac{\partial^2 \Phi}{\partial t^2} - g \frac{\partial \Phi}{\partial y} = 0 \quad \text{on } y = 0 \quad . \quad (1.16)$$

The other boundary conditions which are needed are that of no flow through any solid body and some specification of the form of the wavefield at large distances.

It is now assumed that the motion is simple harmonic. Under the assumption of linear wave theory we can consider a plane wave of small amplitude of particular frequency,  $\omega$  and so it is convenient to represent the velocity potential as,

$$\Phi = \text{Re} \left[ -\frac{ig}{\omega} A\phi e^{-i\omega t} \right] , \quad (1.17)$$

where  $\phi$  is now a complex quantity and  $A$  is the amplitude of the incident wave. The free surface condition now becomes

$$\frac{\omega^2 \phi}{g} + \frac{\partial \phi}{\partial y} = 0 \quad \text{on} \quad y = 0 \quad . \quad (1.18)$$

# Chapter 2

## 2.1 Introduction

We consider the problem of wave scattering by a submerged circular arc plate and a mathematical model for the interaction of surface water waves with the arc plate is sought. Two-dimensional motion is considered and the solution to the scattering potential is obtained by using Ursell's multipole potential technique (Ursell, 1950a). The problem is to obtain an expression for the velocity potential and in particular the reflection and transmission coefficients.

## 2.2 Problem formulation

A two-dimensional cross-section through a circular arc plate, which is submerged at a fixed position in infinite deep water, is illustrated in figure 2.1.

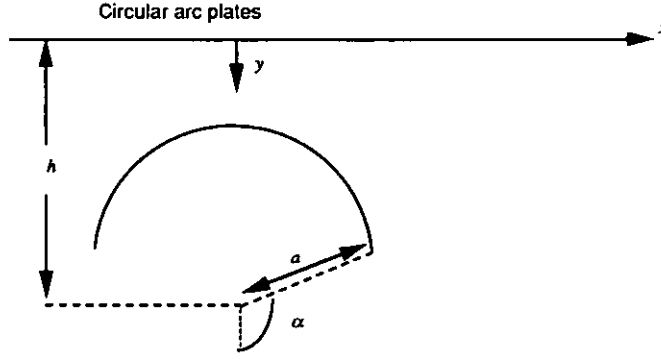


Figure 2.1

Cartesian axes are chosen with the  $x$ -axis along the free surface and  $y$ -axis pointing vertically downwards. The plate is assumed to occupy an arc of a circle. The centre of the circle is at the point  $(0, h)$  and the curve has radius  $a$ . Local polar coordinates  $(r, \theta)$  are defined at the centre of the circle and are related to the original, Cartesian coordinates by,

$$\begin{aligned} x &= r \sin \theta \quad , \\ y - h &= r \cos \theta \quad . \end{aligned} \tag{2.1}$$

The water is assumed to be inviscid and incompressible and the flow is irrotational and so from the theory in section 1.2 it may be described by a velocity potential

$$\Phi = \text{Re} \left[ -igA\phi(x, y)e^{-i\omega t} / \omega \right] \tag{2.2}$$

where  $\omega$  is the frequency of the incoming wave,  $A$  is its amplitude and  $g$  is the acceleration due to gravity. The complex potential  $\phi$  satisfies

$$\nabla^2 \phi = 0 \quad \text{in the fluid} \quad , \tag{2.3}$$

with the free surface boundary condition

$$K\phi + \frac{\partial \phi}{\partial y} = 0 \quad \text{on} \quad y = 0, \tag{2.4}$$

where  $K = \omega^2/g$ . The arc is assumed to be fixed and so

$$\frac{\partial \phi}{\partial r} = 0 \quad \text{on } r = a, \quad \alpha < \theta < -\alpha + 2\pi \quad . \quad (2.5)$$

In addition we assume that the fluid is at rest at large depths, thus

$$\nabla \phi \rightarrow 0 \quad \text{as } y \rightarrow \infty \quad . \quad (2.6)$$

A wave is incident on the arc from the left and so  $\phi$  satisfies the radiation condition

$$\phi \sim \begin{cases} e^{iKx-Ky} + Re^{-iKx-Ky} & \text{as } x \rightarrow -\infty, \\ Te^{iKx-Ky} & \text{as } x \rightarrow +\infty, \end{cases} \quad (2.7)$$

where  $T$  and  $R$  are the transmission and reflection coefficients respectively.

## 2.3 Mathematical Model

The potential  $\phi$  is constructed by splitting the fluid into two regions, namely region *I* (outside the arc) and region *II* (inside the arc). In region *I* the potential may be considered to arise from a normal velocity distribution around the circle surface which is precisely the problem studied by Ursell (1950a). He expressed the scattering potential for a submerged circular cylinder as a series of multipole potentials. These are the singular solution of Laplace's equation which satisfies the linear free surface condition, decay with depth and behave like waves radiating outwards as  $|x| \rightarrow \infty$ . Thus in region *I* the velocity potential  $\phi$  may be written as

$$\phi = e^{iKx - Ky} + Ka \sum_{n=1}^{\infty} \frac{a^n}{n} [P_n^s \phi_n^s + P_n^a \phi_n^a] \quad ; \quad (2.8)$$

where  $P_n^s$  and  $P_n^a$ ,  $\phi_n^s$  and  $\phi_n^a$  are the symmetric and anti-symmetric multipole strengths and multipole potential respectively. Thorne (1953) showed that the time-independent, symmetric,  $n$ -th order multipole potential which has a singularity at the point  $(0, h)$  is given by

$$\begin{aligned} \phi_n^s = & \frac{\cos n\theta}{r^n} + \frac{(-1)^{n-1}}{(n-1)!} \int_0^\infty \frac{(K+\ell)}{(K-\ell)} \ell^{n-1} e^{-\ell(y+h)} \cos \ell x \, d\ell \\ & + \frac{(-1)^n}{(n-1)!} 2\pi i K^n e^{-K(y+h)} \cos Kx \end{aligned} \quad (2.9)$$

and the corresponding anti-symmetric multipole potential has the representation

$$\begin{aligned} \phi_n^a = & \frac{\sin n\theta}{r^n} + \frac{(-1)^n}{(n-1)!} \int_0^\infty \frac{(K+\ell)}{(K-\ell)} \ell^{n-1} e^{-\ell(y+h)} \sin \ell x \, d\ell \\ & + \frac{(-1)^{n-1}}{(n-1)!} 2\pi i K^n e^{-K(y+h)} \sin Kx \end{aligned} \quad (2.10)$$

where  $\int$  denotes the principal value integral. By deforming the contour of integration in equations (2.9) and (2.10) it may be shown that

$$\phi_n^s \rightarrow \frac{(-1)^n}{(n-1)!} 2\pi i K^n e^{-K(y+h)} e^{iK|x|} \quad , \quad \text{as } |x| \rightarrow +\infty \quad (2.11)$$



and

$$\phi_n^a \rightarrow \text{Sgn}(x) \frac{(-1)^{n-1}}{(n-1)!} 2\pi K^n e^{-K(y+h)} e^{iK|x|} \quad , \quad \text{as } |x| \rightarrow +\infty \quad . \quad (2.12)$$

Substituting values for  $\phi_n^a$  and  $\phi_n^s$  from (2.12) and (2.11) into (2.8) we have

$$\begin{aligned} \phi \rightarrow e^{iKx-Ky} + Ka \sum_{n=1}^{\infty} \frac{a^n}{n} \left[ P_n^a \text{Sgn}(x) \frac{(-1)^{n-1}}{(n-1)!} 2\pi K^n e^{-K(y+h)} e^{iK|x|} \right. \\ \left. + P_n^s \frac{(-1)^n}{(n-1)!} 2\pi i K^n e^{-K(y+h)} e^{iK|x|} \right] \quad . \end{aligned} \quad (2.13)$$

Thus when  $x \rightarrow -\infty$  we have

$$\begin{aligned} \phi \sim e^{iKx-Ky} + Ka \sum_{n=1}^{\infty} \left[ P_n^a \frac{(-1)^n}{n!} 2\pi (Ka)^n e^{-K(y+h)} e^{-iKx} \right. \\ \left. + P_n^s \frac{(-1)^n}{n!} 2\pi i (Ka)^n e^{-K(y+h)} e^{-iKx} \right] \\ = e^{iKx-Ky} + 2\pi Ka e^{-Kh} \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} (Ka)^n (P_n^a + iP_n^s) e^{-iKx-Ky} \quad . \end{aligned} \quad (2.14)$$

Therefore, comparing (2.14) with (2.7) the reflection coefficient is given by

$$R = 2\pi Ka e^{-Kh} \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} (Ka)^n (P_n^a + iP_n^s) \quad . \quad (2.15)$$

Similarly when  $x \rightarrow +\infty$  we have

$$\begin{aligned} \phi \sim e^{iKx-Ky} + Ka \sum_{n=1}^{\infty} P_n^a \left[ \frac{(-1)^{n-1}}{n!} 2\pi (Ka)^n e^{-K(y+h)} e^{iKx} \right. \\ \left. + P_n^s \frac{(-1)^n}{n!} 2\pi i (Ka)^n e^{-K(y+h)} e^{iKx} \right] \\ = e^{iKx-Ky} \left[ 1 + 2\pi Ka e^{-Kh} \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} (Ka)^n (-P_n^a + iP_n^s) \right] \quad . \end{aligned} \quad (2.16)$$

Thus comparing (2.16) with (2.7) gives the transmission coefficient as

$$T = 1 + 2\pi Ka e^{-Kh} \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} (Ka)^n (-P_n^a + iP_n^s) \quad . \quad (2.17)$$

In region II,  $r < a$ , so the potential may be expressed as,

$$\phi = e^{iKx-Ky} + \sum_{n=0}^{\infty} \frac{r^n}{a^n} (A_n \cos n\theta + B_n \sin n\theta) \quad , \quad (2.18)$$

where  $A_n$  and  $B_n$  are further coefficients to be determined. (The incident wave term is included in (2.18) for later convenience. In principle, it could be expanded in a series of  $r^n \cos n\theta$  and  $r^n \sin n\theta$  and incorporated into the existing series.) The unknown coefficients in (2.8) and (2.18) are determined by requiring the potential in each region to satisfy the body boundary condition (2.5) and also by requiring  $\phi$  and  $\partial\phi/\partial r$  to be continuous on  $r = a$ ,  $-\alpha < \theta < \alpha$  ensuring continuity of pressure and velocity in the fluid. A combination of these conditions shows that  $\partial\phi/\partial r$  is continuous everywhere on  $r = a$  (except at the actual plate tips where there is a square root singularity in the velocity).

In order to satisfy the body boundary condition it is necessary to obtain series expansion of  $\phi_n^s$  and  $\phi_n^u$  in terms of  $r^m \cos m\theta$  and  $r^m \sin m\theta$  respectively (Evans et al., 1979).

Consider

$$I = \int_0^\infty b_n(\ell) e^{-\ell y - i\ell x} d\ell, \quad (2.19)$$

where

$$b_n(\ell) = \frac{K + \ell}{K - \ell} \ell^{n-1} e^{-\ell h} \quad (2.20)$$

then

$$I = \int_0^\infty b_n(\ell) e^{-\ell h} e^{-\ell r e^{i\theta}} d\ell = \sum_{m=0}^\infty \frac{(-1)^m r^m e^{im\theta}}{m!} \int_0^\infty b_n(\ell) e^{-\ell h} \ell^m d\ell, \quad 0 < r < 2h \quad (2.21)$$

Now it follows from (2.21) that

$$\phi_n^s = \frac{\cos n\theta}{r^n} + \sum_{m=0}^\infty A_{mn} r^m \cos m\theta \quad (2.22)$$

and

$$\phi_n^u = \frac{\sin n\theta}{r^n} + \sum_{m=0}^\infty A_{mn} r^m \sin m\theta \quad (2.23)$$

which is valid for  $0 < r < 2h$ , where

$$A_{mn} = \frac{(-1)^{m+n-1}}{m!(n-1)!} \int_0^\infty \frac{(K+\ell)}{(K-\ell)} \ell^{n+n-1} e^{-2\ell h} d\ell$$

$$+ \frac{(-1)^{m+n}}{m!(n-1)!} 2\pi i K^{m+n} e^{-2Kh} . \quad (2.24)$$

It is noticeable that the matrix  $A_{mn}$  appears in the expansion of both  $\phi_n^a$  and  $\phi_n^s$ . The incoming wave  $\phi_I$  can be written as follows

$$\phi_I = e^{iKx-Ky} = e^{iKr \sin \theta - K(h+r \cos \theta)}$$

$$= e^{-Kh} e^{-Krc^{i\theta}} = e^{-Kh} \sum_{n=0}^{\infty} \frac{(-1)^n K^n r^n}{n!} e^{-in\theta} . \quad (2.25)$$

The partial derivatives of  $\phi_n^s$ ,  $\phi_n^a$  and  $\phi_I$  with respect to  $r$  are as follows

$$\frac{\partial \phi_n^s}{\partial r} = \frac{-n \cos n\theta}{r^{n+1}} + \sum_{m=1}^{\infty} m r^{m-1} A_{mn} \cos m\theta , \quad (2.26)$$

$$\frac{\partial \phi_n^a}{\partial r} = \frac{-n \sin n\theta}{r^{n+1}} + \sum_{m=1}^{\infty} m r^{m-1} A_{mn} \sin m\theta \quad (2.27)$$

and

$$\frac{\partial \phi_I}{\partial r} = e^{-Kh} \sum_{n=1}^{\infty} \frac{(-1)^n K^n r^{n-1}}{(n-1)!} e^{-in\theta} \quad (2.28)$$

The expression for  $\phi_n^s$ ,  $\phi_n^a$  and  $\phi_I$  and their corresponding partial derivatives will be used in later calculations.

Thus in regions *I* and *II* expressions for  $\phi$  in (2.8) and (2.18) are differentiated with respect to  $r$  and equated on  $r = a$ . This gives

$$\frac{\partial}{\partial r} (e^{iKx-Ky}) + Ka \sum_{n=1}^{\infty} \frac{a^n}{n} (P_n^a \frac{\partial \phi_n^a}{\partial r} + P_n^s \frac{\partial \phi_n^s}{\partial r})$$

$$= \frac{\partial}{\partial r} (e^{iKx-Ky}) + \sum_{n=1}^{\infty} \frac{n}{a} (A_n \cos n\theta + B_n \sin n\theta) . \quad (2.29)$$

Substituting the values of  $\partial\phi_n^a/\partial r$  and  $\partial\phi_n^s/\partial r$  from (2.27) and (2.26) into (2.29) we get

$$\begin{aligned} & Ka \sum_{n=1}^{\infty} \left[ P_n^s \left( -\cos n\theta + \sum_{m=1}^{\infty} m M_{mn} \cos m\theta \right) + P_n^a \left( -\sin n\theta + \sum_{m=1}^{\infty} m M_{mn} \sin m\theta \right) \right] \\ &= \sum_{n=1}^{\infty} n (A_n \cos n\theta + B_n \sin n\theta), \end{aligned} \quad (2.30)$$

where

$$M_{mn} = A_{mn} \frac{a^{m+n}}{n}. \quad (2.31)$$

Multiplying (2.30) by  $\cos k\theta$  and integrating from  $-\pi$  to  $+\pi$  we have

$$Ka \left[ -\frac{P_k^s}{k} + \sum_{n=1}^{\infty} P_n^s M_{kn} \right] = A_k \quad k = 1, 2, \dots \quad (2.32)$$

Similarly multiplying (2.30) by  $\sin k\theta$  and integrating from  $-\pi$  to  $+\pi$  we have

$$Ka \left[ -\frac{P_k^a}{k} + \sum_{n=1}^{\infty} P_n^a M_{kn} \right] = B_k \quad k = 1, 2, \dots \quad (2.33)$$

Equations (2.32) and (2.33) represent expressions for the unknown coefficients in the region *II* expansion of the potential in terms of the coefficients in the region *I* expansion. Further relations between the sets of coefficients, in regions *I* and *II* are obtained by requiring continuity of pressure on the fluid interface section of  $r = a$  and zero normal velocity on the arc section of  $r = a$ . From (2.8) and (2.18), continuity of  $\phi$  on  $r = a$ ,  $-\alpha < \theta < \alpha$  gives

$$Ka \sum_{n=1}^{\infty} \frac{a^n}{n} [P_n^s \phi_n^s + P_n^a \phi_n^a] = \sum_{n=0}^{\infty} \frac{r^n}{a^n} (A_n \cos n\theta + B_n \sin n\theta), \quad -\alpha < \theta < \alpha. \quad (2.34)$$

Substituting the expansions of  $\phi_n^s$  and  $\phi_n^a$  from (2.22) and (2.23) into (2.34) we have

$$\begin{aligned} & Ka \sum_{n=1}^{\infty} \left[ P_n^s \left\{ \frac{\cos n\theta}{n} + \sum_{m=0}^{\infty} M_{mn} \cos m\theta \right\} + P_n^a \left\{ \frac{\sin n\theta}{n} + \sum_{m=1}^{\infty} M_{mn} \sin m\theta \right\} \right] \\ &= \sum_{n=0}^{\infty} (A_n \cos n\theta + B_n \sin n\theta), \quad -\alpha < \theta < \alpha. \end{aligned} \quad (2.35)$$

Multiplying (2.35) by  $\cos s\theta$  and integrating from  $-\alpha$  to  $+\alpha$  we have

$$\sum_{n=1}^{\infty} \left[ \frac{P_n^s}{n} G_{ns} + P_n^s \sum_{m=0}^{\infty} M_{mn} G_{ms} \right] = \sum_{n=0}^{\infty} \frac{A_n G_{ns}}{Ka}, \quad s = 0, 1, \dots, \quad (2.36)$$

where

$$G_{ns} = \int_{-\alpha}^{+\alpha} \cos n\theta \cos s\theta \, d\theta \quad (2.37)$$

Similarly multiplying (2.35) by  $\sin s\theta$  and integrating from  $-\alpha$  to  $+\alpha$  we have

$$\sum_{n=1}^{\infty} \left[ \frac{P_n^a}{n} H_{ns} + P_n^a \sum_{m=1}^{\infty} M_{mn} H_{ms} \right] = \sum_{n=1}^{\infty} \frac{B_n H_{ns}}{Ka}, \quad s = 1, 2, \dots, \quad (2.38)$$

where

$$H_{ns} = \int_{-\alpha}^{+\alpha} \sin n\theta \sin s\theta \, d\theta \quad (2.39)$$

By eliminating  $A_n$  and  $B_n$  from (2.36) and (2.38) using (2.32) and (2.33) we get two sets of equations for the coefficients in the region  $I$  expansion, namely,

$$\sum_{n=1}^{\infty} P_n^s \left[ \frac{2G_{ns}}{n} + M_{0n} G_{0s} \right] = \frac{A_0 G_{0s}}{Ka}, \quad s = 0, 1, 2, \dots, \quad (2.40)$$

and

$$\sum_{n=1}^{\infty} P_n^a \left( \frac{2H_{ns}}{n} \right) = 0 \quad s = 1, 2, \dots \quad (2.41)$$

Clearly these equations cannot produce a unique solution for the multipole coefficient unless all information in the problem, especially the body boundary condition (2.5), is specified. After the expansion for  $\phi$  in region  $I$  in (2.8) is differentiated with respect to  $r$ , the body boundary condition,  $\partial\phi/\partial r = 0$  on  $r = a$  is applied and yields

$$\frac{\partial\phi}{\partial r} = \frac{\partial}{\partial r} (e^{iKx-Ky}) + Ka \sum_{n=1}^{\infty} \frac{a^n}{n} \left( P_n^a \frac{\partial\phi_n^a}{\partial r} + P_n^s \frac{\partial\phi_n^s}{\partial r} \right) \quad (2.42)$$

Substituting the expansions for  $\partial\phi_n^s/\partial r$  and  $\partial\phi_n^a/\partial r$  and the derivative of the incident wave from (2.26) and (2.27) and (2.28) into (2.42) we have

$$\begin{aligned} & Ka \sum_{n=1}^{\infty} \left[ P_n^s \left( -\cos n\theta + \sum_{m=1}^{\infty} m M_{mn} \cos m\theta \right) \right. \\ & \left. + P_n^a \left( -\sin n\theta + \sum_{m=1}^{\infty} m M_{mn} \sin m\theta \right) \right] \\ & = -e^{-Kh} \sum_{n=1}^{\infty} \frac{(-1)^n (Ka)^n}{(n-1)!} (\cos n\theta - i \sin n\theta), \quad \alpha < \theta < 2\pi - \alpha. \end{aligned} \quad (2.43)$$

Multiplying (2.43) by  $\cos s\theta$  and integrating from  $\alpha$  to  $-\alpha + 2\pi$  we obtain

$$Ka \sum_{n=1}^{\infty} P_n^s \left( -E_{ns} + \sum_{m=1}^{\infty} m M_{mn} E_{ms} \right) = -e^{-Kh} \sum_{n=1}^{\infty} \frac{(-1)^n (Ka)^n}{(n-1)!} E_{ns} \quad (2.44)$$

where

$$E_{ns} = \int_{\alpha}^{2\pi-\alpha} \cos n\theta \cos s\theta \, d\theta \quad s = 1, 2, \dots \quad (2.45)$$

Similarly multiplying (2.43) by  $\sin s\theta$  and integrating from  $\alpha$  to  $-\alpha + 2\pi$  we obtain

$$Ka \sum_{n=1}^{\infty} P_n^a \left( -F_{ns} + \sum_{m=1}^{\infty} m M_{mn} F_{ms} \right) = ie^{-Kh} \sum_{n=1}^{\infty} \frac{(-1)^n (Ka)^n}{(n-1)!} F_{ns} \quad (2.46)$$

where

$$F_{ns} = \int_{\alpha}^{2\pi-\alpha} \sin n\theta \sin s\theta \, d\theta \quad s = 1, 2, \dots \quad (2.47)$$

Clearly equations (2.44) and (2.46) do not include any information about the potential or its derivative on the part of the circle  $-\alpha < \theta < \alpha$  and therefore they cannot be sufficient to determine the coefficients  $P_n^s$  and  $P_n^a$ . In order to use all the information on the line  $r = a$  we solve two particular combinations of the systems of equations, namely the combination  $\lambda \times (2.44) - \mu \times s(2.40)/2$  and  $\lambda \times (2.46) - \mu \times s(2.41)/2$ , where  $\lambda$  and  $\mu$  ( $0 \leq \lambda, \mu \leq 1$ ) are user supplied parameters, chosen to obtain the numerical convergence. After numerical experimentation we found that  $\mu = 1$  and  $\lambda = 1$  gave the best results. Of

course other combinations of equations may be taken but the motivation for these choice is that they were found to produce the best numerical convergence. The only exceptions to the above choice were when we considered the limiting case of a circular cylinder, for which we took  $\mu = 0$ ,  $\lambda = 1$  and the limit in which the plate shrinks to zero when we took  $\mu = 1$ ,  $\lambda = 0$ . After the coefficients  $A_n, n = 1, 2, \dots$  are eliminated from (2.36) by using (2.32), the combination  $\lambda \times (2.44) - \mu \times s(2.40)/2$  gives

$$\begin{aligned} & \lambda \sum_{n=1}^{\infty} P_n^s \left[ -E_{ns} + \sum_{m=1}^{\infty} m M_{mn} E_{ms} \right] - \mu \sum_{n=1}^{\infty} P_n^s \left( \frac{s G_{ns}}{n} + \frac{s}{2} M_{0n} G_{0s} \right) + \frac{s}{2} \frac{A_0 \mu G_{0s}}{K a} \\ & = -\lambda e^{-K h} \sum_{n=1}^{\infty} \frac{(-1)^n (K a)^{n-1}}{(n-1)!} E_{ns} \quad s = 1, 2, \dots \end{aligned} \quad (2.48)$$

In addition there is one extra equation which comes from (2.40) for  $s = 0$ , namely

$$\sum_{n=1}^{\infty} P_n^s \left[ \frac{2 G_{n0}}{n} + M_{0n} G_{00} \right] = \frac{A_0 G_{00}}{K a} \quad (2.49)$$

Substituting the value of  $A_0$  from (2.49) in (2.48) we have

$$\begin{aligned} & \lambda \sum_{n=1}^{\infty} P_n^s \left[ -E_{ns} + \sum_{m=1}^{\infty} m M_{mn} E_{ms} \right] - \mu \sum_{n=1}^{\infty} P_n^s \left( \frac{s G_{ns}}{n} - \frac{s}{2} \frac{G_{n0} G_{0s}}{G_{00}} \right) \\ & = -\lambda e^{-K h} \sum_{n=1}^{\infty} \frac{(-1)^n (K a)^{n-1}}{(n-1)!} E_{ns} \quad s = 1, 2, \dots \end{aligned} \quad (2.50)$$

Similarly by taking  $\lambda \times (2.46) - \mu \times s(2.41)/2$  we have

$$\begin{aligned} & \lambda \sum_{n=1}^{\infty} P_n^a \left[ -F_{ns} + \sum_{m=1}^{\infty} m M_{mn} F_{ms} \right] - \mu \sum_{n=1}^{\infty} P_n^a \left( \frac{s H_{ns}}{n} \right) \\ & = \lambda i e^{-K h} \sum_{n=1}^{\infty} \frac{(-1)^n (K a)^{(n-1)}}{(n-1)!} F_{ns} \quad s = 1, 2, \dots \end{aligned} \quad (2.51)$$

The systems of equations for the coefficients  $P_n^s$  and  $P_n^a$  are truncated and solved numerically using the NAG library routine f04adf. The values of the reflection and transmission coefficients,  $R$  and  $T$  are then found from (2.15) and (2.17) respectively. The full set of results for the reflection and transmission coefficients for various frequencies are discussed in chapter 4.

# Chapter 3

## 3.1 A Variational Approximation

In this section, approximations to the reflection and transmission coefficients are obtained using the Schwinger variational procedure. This technique has been applied by many people. Miles (1967) considered the diffraction of gravity waves at a discontinuous change in depth between two horizontal bottoms. He developed and illustrated a scattering matrix formulation for the diffraction of gravity waves and associated Schwinger variational principal. The variational approximations for the elements of the scattering matrix were developed. Evans and Morris (1972) also used an approximation method associated with a fixed vertical barrier immersed to a given depth. They used this method to obtain accurate upper and lower bounds for the reflection and transmission coefficients for all angles of incidence and all wavelengths. The method involved the use of a one-term Galerkin approximation to the solution of an integral equation which is equivalent to Schwinger variational approach.

It is convenient to derive  $V(\theta)$  to be the jump potential across the circle  $r = a$  and



this is expanded on  $r = a$  using (2.18) and (2.8) to give

$$V(\theta) = Ka \sum_{n=1}^{\infty} \left[ P_n^s \frac{\cos n\theta}{n} + P_n^a \frac{\sin n\theta}{n} + \sum_{m=0}^{\infty} M_{mn} (P_n^a \sin m\theta + P_n^s \cos m\theta) \right] - \sum_{n=0}^{\infty} (B_n \sin n\theta + A_n \cos n\theta) \quad , \quad -\pi < \theta < \pi . \quad (3.1)$$

This equation is split into two parts, namely the symmetric and antisymmetric parts,

$$V(\theta) = V_s(\theta) + V_a(\theta) \quad . \quad (3.2)$$

From (3.1) the symmetric part is

$$V_s(\theta) = Ka \sum_{n=1}^{\infty} P_n^s \left[ \frac{\cos n\theta}{n} + \sum_{m=0}^{\infty} M_{mn} \cos m\theta \right] - \sum_{n=0}^{\infty} A_n \cos n\theta \quad , \quad -\pi < \theta < \pi \quad (3.3)$$

and the antisymmetric part is

$$V_a(\theta) = Ka \sum_{n=1}^{\infty} P_n^a \left[ \frac{\sin n\theta}{n} + \sum_{m=1}^{\infty} M_{mn} \sin m\theta \right] - \sum_{n=1}^{\infty} B_n \sin n\theta \quad , \quad -\pi < \theta < \pi . \quad (3.4)$$

The coefficient  $A_n$  and  $B_n$ ,  $n = 1, \dots$ , are eliminated from (3.3) and (3.4) using (2.32) and (2.33) to give

$$V_s(\theta) = 2Ka \sum_{n=1}^{\infty} P_n^s \frac{\cos n\theta}{n} - A_o + Ka \sum_{m=1}^{\infty} P_n^s M_{on} \quad , \quad -\pi < \theta < \pi \quad (3.5)$$

and

$$V_a(\theta) = 2Ka \sum_{n=1}^{\infty} P_n^a \frac{\sin n\theta}{n} \quad , \quad -\pi < \theta < \pi \quad . \quad (3.6)$$

Multiplication of (3.5) by  $\cos m\theta$  and integration between  $-\pi$  to  $\pi$  gives

$$\int_{-\pi}^{\pi} V_s(\theta) \cos m\theta d\theta = 2Ka \sum_{n=1}^{\infty} \frac{P_n^s}{n} \int_{-\pi}^{\pi} \cos n\theta \cos m\theta d\theta - \int_{-\pi}^{\pi} A_o \cos m\theta d\theta + Ka \sum_{m=1}^{\infty} P_n^s M_{on} \int_{-\pi}^{\pi} \cos m\theta d\theta .$$

As the trigonometric functions are orthogonal on  $[-\pi, \pi]$  this gives

$$2\pi Ka \frac{P_m^s}{m} = \int_{-\pi}^{\pi} V_s(\theta) \cos m\theta d\theta, \quad m = 1, \dots \quad (3.7)$$

However,

$$\int_{-\pi}^{\pi} V_s(\theta) \cos m\theta d\theta = \int_{-\alpha}^{\alpha} V_s(\theta) \cos m\theta d\theta + \int_{\alpha}^{2\pi-\alpha} V_s(\theta) \cos m\theta d\theta$$

and the potential jump  $V_s(\theta)$  is zero on  $[-\alpha, \alpha]$  and so (3.7) becomes

$$2\pi Ka \frac{P_m^s}{m} = \int_{\alpha}^{2\pi-\alpha} V_s(\theta) \cos m\theta d\theta, \quad m = 1, \dots \quad (3.8)$$

Similarly, multiplication of (3.6) by  $\sin m\theta$  and integration between  $-\pi$  to  $\pi$  gives

$$2\pi Ka \frac{P_m^u}{m} = \int_{\alpha}^{2\pi-\alpha} V_u(\theta) \sin m\theta d\theta, \quad m = 1, \dots \quad (3.9)$$

It is also convenient to define  $U(\theta)$  by

$$U(\theta) = a \frac{\partial}{\partial r} [\phi - e^{iKx - Ky}] \quad \text{on } r = a, \quad -\pi < \theta \leq \pi, \quad \theta \neq \pm\alpha. \quad (3.10)$$

From (2.8) we can write

$$U(\theta) = Ka \sum_{n=1}^{\infty} \left[ P_n^s \left\{ -\cos n\theta + \sum_{m=1}^{\infty} m M_{mn} \cos m\theta \right\} + P_n^u \left\{ -\sin n\theta + \sum_{m=1}^{\infty} m M_{mn} \sin m\theta \right\} \right] \quad (3.11)$$

This is split into two parts, a symmetric and an antisymmetric part,

$$U(\theta) = U_s(\theta) + U_a(\theta) \quad (3.12)$$

From (3.11) the symmetric part is given by

$$\begin{aligned} & U_s(\theta) - iKa \sum_{n=1}^{\infty} P_n^s \sum_{m=1}^{\infty} m M_{mn}^i \cos m\theta \\ & = Ka \sum_{n=1}^{\infty} P_n^s \left[ -\cos n\theta + \sum_{m=1}^{\infty} m M_{mn}^r \cos m\theta \right] \quad (3.13) \end{aligned}$$

and the antisymmetric part is given by

$$\begin{aligned}
U_a(\theta) - iKa \sum_{n=1}^{\infty} P_n^a \sum_{m=1}^{\infty} m M_{mn}^i \sin m\theta \\
= Ka \sum_{n=1}^{\infty} P_n^a \left[ -\sin n\theta + \sum_{m=1}^{\infty} m M_{mn}^r \sin m\theta \right] .
\end{aligned} \tag{3.14}$$

where  $M_{mn}^r$  and  $M_{mn}^i$  are the real and imaginary parts of  $M_{mn}$  respectively. Using the body boundary condition  $\partial\phi/\partial r = 0$ , on  $r = a$ ,  $\alpha < \theta < 2\pi - \alpha$  yields

$$\begin{aligned}
U_s(\theta) + U_a(\theta) &= \sum_{n=1}^{\infty} \left[ a \frac{\partial\phi}{\partial r} - a \frac{\partial}{\partial r} (e^{iKx - Ky}) \right] \\
&= -e^{Kh} \sum_{n=1}^{\infty} \frac{(-1)^n (Ka)^n}{(n-1)!} (\cos n\theta - i \sin n\theta) \quad \text{on } \alpha < \theta < 2\pi - \alpha .
\end{aligned} \tag{3.15}$$

From (3.15) we can write

$$U_s(\theta) = -e^{-Kh} \sum_{n=1}^{\infty} \frac{(-1)^n (Ka)^n}{(n-1)!} \cos n\theta \quad \text{on } \alpha < \theta < 2\pi - \alpha \tag{3.16}$$

and

$$U_a(\theta) = ie^{-Kh} \sum_{n=1}^{\infty} \frac{(-1)^n (Ka)^n}{(n-1)!} \sin n\theta \quad \text{on } \alpha < \theta < 2\pi - \alpha . \tag{3.17}$$

From (2.15) the reflection coefficient may be written as  $R = R_s + R_a$  where

$$R_s = 2\pi ie^{-Kh} \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} (Ka)^{n+1} P_n^s \tag{3.18}$$

and

$$R_a = 2\pi e^{-Kh} \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} (Ka)^{n+1} P_n^a \tag{3.19}$$

This may be used together with (2.31), (3.16) and (3.17) to rewrite the left-hand sides of (3.13) and (3.14) which gives

$$U_s(\theta)(1 + R_s) = Ka \sum_{n=1}^{\infty} P_n^s \left[ -\cos n\theta + \sum_{m=1}^{\infty} m M_{mn}^r \cos m\theta \right] \quad \text{on } \alpha < \theta < 2\pi - \alpha \tag{3.20}$$

and

$$U_a(\theta)(1-R_a) = Ka \sum_{n=1}^{\infty} P_n^a \left[ -\sin n\theta + \sum_{m=1}^{\infty} mM_{mn}^r \sin m\theta \right] \quad \text{on } \alpha < \theta < 2\pi - \alpha, \quad (3.21)$$

where  $U_s(\theta)$  and  $U_a(\theta)$  are given explicitly by (3.16) and (3.17). Thus, substitution of  $P_n^s$  and  $P_n^a$  in terms of  $V_s(\theta)$  and  $V_a(\theta)$  from (3.8) and (3.9) into the right-hand sides of (3.20) and (3.21) yields

$$\begin{aligned} U_s(\theta)(1+R_s) &= \frac{1}{2\pi} \sum_{n=1}^{\infty} n \left[ -\int_{\alpha}^{2\pi-\alpha} V_s(\theta') \cos n\theta \cos n\theta' d\theta' \right. \\ &\quad \left. + \sum_{m=1}^{\infty} mM_{mn}^r \int_{\alpha}^{2\pi-\alpha} V_s(\theta') \cos m\theta \cos n\theta' d\theta' \right] \end{aligned} \quad (3.22)$$

and

$$\begin{aligned} U_a(\theta)(1+R_a) &= \frac{1}{2\pi} \sum_{n=1}^{\infty} n \left[ -\int_{\alpha}^{2\pi-\alpha} V_a(\theta') \cos n\theta \cos n\theta' d\theta' \right. \\ &\quad \left. + \sum_{m=1}^{\infty} mM_{mn}^r \int_{\alpha}^{2\pi-\alpha} V_a(\theta') \cos m\theta \cos n\theta' d\theta' \right] \end{aligned} \quad (3.23)$$

From (3.22) and (3.23) we get

$$U_s(\theta)(1+R_s) = \sum_{n=1}^{\infty} \int_{\alpha}^{2\pi-\alpha} k_{ns}(\theta, \theta') V_s(\theta') d\theta', \quad \alpha < \theta < 2\pi - \alpha \quad (3.24)$$

where

$$k_{ns}(\theta, \theta') = \frac{n}{2\pi} \left[ -\cos n\theta \cos n\theta' + \sum_{m=1}^{\infty} mM_{mn}^r \cos m\theta \cos n\theta' \right] \quad (3.25)$$

and

$$U_a(\theta)(1-R_a) = \sum_{n=1}^{\infty} \int_{\alpha}^{2\pi-\alpha} k_{na}(\theta, \theta') V_a(\theta') d\theta', \quad \alpha < \theta < 2\pi - \alpha \quad (3.26)$$

where

$$k_{na}(\theta, \theta') = \frac{n}{2\pi} \left[ -\sin n\theta \sin n\theta' + \sum_{m=1}^{\infty} mM_{mn}^r \sin m\theta \sin n\theta' \right] \quad (3.27)$$

It is not possible to interchange the order of summation and integration in (3.24) and (3.26) as the resulting series would not converge. (Evans and Morris (1972) overcame a

similar problem by introducing an artificial exponential decay factor in the kernel of the equation and taking the limit as the exponent tends to zero. However, it is not necessary that the order of integration and summation should be changed here.) By writing

$$V_s(\theta) = (1 + R_s)X_s(\theta) \quad (3.28)$$

and

$$V_a(\theta) = i(1 - R_a)X_a(\theta) \quad (3.29)$$

the following integral equations are obtained for  $X_s(\theta)$  and  $X_a(\theta)$

$$\sum_{n=1}^{\infty} \int_{\alpha}^{2\pi-\alpha} k_{ns}(\theta, \theta') X_s(\theta') d\theta' = U_s(\theta), \quad \alpha < \theta < 2\pi - \alpha \quad (3.30)$$

and

$$\sum_{n=1}^{\infty} \int_{\alpha}^{2\pi-\alpha} k_{na}(\theta, \theta') X_a(\theta') d\theta' = -iU_a(\theta), \quad \alpha < \theta < 2\pi - \alpha \quad (3.31)$$

The quantities  $k_{ns}(\theta, \theta')$ ,  $k_{na}(\theta, \theta')$ ,  $U_s(\theta)$  and  $-iU_a(\theta)$  are real so  $X_s$  and  $X_a$  must be real functions. From (3.18) using (3.16) and (3.8) it may be shown that

$$R_s = -i \int_{\alpha}^{2\pi-\alpha} V_s(\theta) U_s(\theta) d\theta \quad (3.32)$$

and so by rewriting  $V_s$  in terms  $X_s$  using (3.28),  $R_s$  may be written as

$$R_s = \frac{A_s}{i - A_s} \quad (3.33)$$

where

$$A_s = \int_{\alpha}^{2\pi-\alpha} X_s(\theta) U_s(\theta) d\theta \quad (3.34)$$

and  $A_s$  is real quantity. A similar analysis gives

$$R_a = \frac{A_a}{A_a - i} \quad (3.35)$$

where

$$A_a = \int_{\alpha}^{2\pi-\alpha} X_a(\theta)(-iU_a(\theta))d\theta \quad (3.36)$$

and  $A_a$  is also a real quantity.

A variational approximation to  $X_s$  is sought in the form  $X_s(\theta) = a_s W_s(\theta)$  where  $a_s$  is a constant and is chosen so that

$$\int_{\alpha}^{2\pi-\alpha} a_s W_s(\theta) U_s(\theta) d\theta = \int_{\alpha}^{2\pi-\alpha} a_s W_s(\theta) \sum_{n=1}^{\infty} \int_{\alpha}^{2\pi-\alpha} k_{ns}(\theta, \theta') a_s W_s(\theta') d\theta' d\theta . \quad (3.37)$$

Substitution of this approximation into (3.34) yields

$$A_s = \frac{\left[ \int_{\alpha}^{2\pi-\alpha} U_s(\theta) W_s(\theta) d\theta \right]^2}{\int_{\alpha}^{2\pi-\alpha} W_s(\theta) \sum_{n=1}^{\infty} \int_{\alpha}^{2\pi-\alpha} k_{ns}(\theta, \theta') W_s(\theta') d\theta' d\theta} \quad (3.38)$$

The success of the approximation depends on a suitable choice for the function  $W_s(\theta)$ . There are square root singularities in the velocity at the tips of the plates and the simplest way to model this is to choose

$$W_s(\theta) = (\theta - \alpha)^{1/2} (2\pi - \alpha - \theta)^{1/2}, \quad \alpha \leq \theta \leq 2\pi - \alpha . \quad (3.39)$$

Such an approximation allows only a very simple variation in potential along the length of the plate and so may be expected to give good results only when the plate is short compared to the wavelength and occupies a small part of the circle. An approximation similar to ours was also used by Mei and Petroni (1973) when considering the wave scattering by a vertical, circular harbour which contains a narrow gap. More Recently, Porter and Evans (1995) developed a variational method based on a Galerkin approximation for scattering problems involving vertical barriers with gaps in finite depth. Substitution of (3.39) into (3.38) yields

$$A_s = \frac{2\pi e^{-2Kh} \left[ \sum_{n=1}^{\infty} \frac{(Ka)^n}{n!} J_1(n(\pi - \alpha)) \right]^2}{\sum_{n=1}^{\infty} \left[ -\frac{J_1^2(n(\pi - \alpha))}{n} + \sum_{m=1}^{\infty} (-1)^{m+n} M_{mn}^r J_1(n(\pi - \alpha)) J_1(m(\pi - \alpha)) \right]} \quad (3.40)$$

where Gradshteyn and Ryzhik (1980, 3.752.2) has been used to write

$$\int_{\alpha}^{2\pi-\alpha} \cos n\theta (\theta - \alpha)^{1/2} (2\pi - \alpha - \theta)^{1/2} d\theta = \frac{(-1)^n}{n} \pi (\pi - \alpha) J_1(n(\pi - \alpha)), \quad (3.41)$$

where  $J_1$  is a Bessel function of the first kind.

The corresponding variational approximation to  $X_u(\theta)$  is given by  $X_u(\theta) = a_u W_u(\theta)$

where

$$W_u(\theta) = (\theta - \pi)(\theta - \alpha)^{1/2} (2\pi - \alpha - \theta)^{1/2}, \quad \alpha \leq \theta \leq 2\pi - \alpha \quad (3.42)$$

and the constant  $a_u$  satisfies

$$\int_{\alpha}^{2\pi-\alpha} a_u W_u(\theta) (-iU_s(\theta)) d\theta = \int_{\alpha}^{2\pi-\alpha} a_u W_u(\theta) \sum_{n=1}^{\infty} \int_{\alpha}^{2\pi-\alpha} k_{nu}(\theta, \theta') a_u W_u(\theta') d\theta' d\theta \quad (3.43)$$

A similar analysis yields

$$A_u = \frac{2\pi e^{-2Kh} \left[ \sum_{n=1}^{\infty} \frac{(K_u)^n}{n!} J_2(n(\pi - \alpha)) \right]^2}{\sum_{n=1}^{\infty} \left[ -\frac{J_2^2(n(\pi - \alpha))}{n} + \sum_{m=1}^{\infty} (-1)^{m+n} M_{mn}^r J_2(n(\pi - \alpha)) J_2(m(\pi - \alpha)) \right]} \quad (3.44)$$

where Gradshteyn and Ryzhik (1980, 3.771.10) has been used to write

$$\int_{\alpha}^{2\pi-\alpha} \sin n\theta (\theta - \pi)(\theta - \alpha)^{1/2} (2\pi - \alpha - \theta)^{1/2} d\theta = \frac{(-1)^n}{n} \pi (\pi - \alpha)^2 J_2(n(\pi - \alpha)), \quad (3.45)$$

where  $J_2$  is a Bessel function of the first kind. The expressions for  $A_s$  and  $A_u$  in (3.40) and (3.44) are substituted into (3.33) and (3.35) and the resulting approximation to the reflection coefficient is compared with the results from the full numerical solution in the next chapter.

# Chapter 4

## 4.1 Numerical Results and Discussion

In this chapter we carry out numerical studies and comparisons. We compare the numerical results for the reflection coefficient for a number of circular arc plates with those for a submerged circular cylinder in order to assess the suitability of using circular arc plates when constructing a water wave lens. Results obtained from both the variational approximation and the matched series expansion techniques are also compared graphically. For the implementation of our program both for the circular cylinder and for the arc plate we have used following input parameters: (i)  $a/h$ , where  $a$  is the radius and  $h$  is the depth of the centre of the cylinder respectively (see Figure 2.1 in chapter 2), (ii) the angle  $\alpha$  which fixes the length of the plate, ( $\alpha = 0$  corresponds to the circular cylinder) and the frequency parameter  $Ka$ . From numerical experiments it was found that 256 multipole potentials were usually sufficient to ensure that the numerical results were accurate to two decimal places and in several cases less terms were required. In Table 4.1 we represent how  $|R|$  changes with the number of multipoles for  $a/h = 0.9$  and  $\alpha = 0.7\pi$ .



Table 4.1

$N = \text{Number of multipoles}$				
Ka	64	128	256	512
.100000	.025887	.025588	.025420	.025323
.300000	.207476	.205654	.204626	.204034
.500000	.407376	.406672	.406272	.406014
.700000	.463471	.466631	.468451	.469436
.900000	.403314	.409886	.413702	.415818
1.100000	.296530	.304553	.309253	.311888
1.300000	.190695	.198247	.202706	.205223
1.500000	.106611	.112679	.116284	.118328
1.700000	.047565	.051912	.054509	.055987
1.900000	.009833	.012605	.014273	.015225
2.100000	.011754	.010276	.009375	.008858
2.300000	.021893	.021403	.021088	.020906
2.500000	.024396	.024605	.024705	.024761
2.700000	.022218	.022869	.023234	.023441
2.900000	.017565	.018446	.018949	.019236
3.000000	.014826	.015755	.016288	.016593

In particular, many fewer terms were needed to model the wave scattering by a circular cylinder than any one of the plates. This is thought to be because the velocity potential is modelled by a series of smooth functions and in order to produce a singularity in its derivative at the tips of the plate, the coefficients in the series have to decay at a certain rate which is not as fast as the coefficients in a series for which the velocity is bounded. Thus more terms were needed in the series expansion for a plate geometry to obtain the same level of accuracy in any calculation. In all our calculations 256 multipole potentials were used.

We first consider the matched eigenfunction expansion method and check our model

by reproducing the results of Ursell (1950a) for the submerged, circular cylinder in the limit as  $\alpha \rightarrow 0$ . Our numerical scheme also ensured that the energy was conserved (i.e.,  $|R|^2 + |T|^2 = 1$ ). In Ursell's formulation the transmission coefficient,  $T^u$ , is given by

$$T^u = 1 - 4\pi i(Ka)e^{-Kh}(\sigma b)^{-1} \sum_{n=1}^{\infty} \frac{p_n}{n!} (Ka)^n - 4\pi(Ka)e^{-Kh}(\sigma b)^{-1} \sum_{n=1}^{\infty} \frac{q_n}{n!} (Ka)^n, \quad (4.1)$$

where the period of simple harmonic motion is  $2\pi/\sigma$ . The unknown coefficients are  $(\sigma b)^{-1}p_n$  and  $(\sigma b)^{-1}q_n$  where  $b$  is the amplitude of the wave motion. We now compare this form with the representation for the transmission coefficient derived in (2.17) in the limit as  $\alpha \rightarrow 0$ . (In this limit, the two models should coincide.) Comparing these two transmission coefficients we have the following relationships between the unknown coefficients of two models,

$$-(\sigma b)^{-1}p_n = (-1)^{n+1} \frac{1}{2} [\Im(P_n^a) - \Re(P_n^s)] \quad (4.2)$$

and

$$-(\sigma b)^{-1}q_n = (-1)^{n+1} \frac{1}{2} [\Im(P_n^s) - \Re(P_n^a)] \quad , \quad (4.3)$$

where  $\Im$  and  $\Re$  are real and imaginary parts. Ursell (1950a) used five multipole potentials in his numerical work. Clearly it is enough to reproduce the values for the unknown coefficients in order to show that the reflection and transmission coefficients are the same in each case. The values of the unknown parameters  $(\sigma b)^{-1}p_n$  and  $(\sigma b)^{-1}q_n$  have been exactly reproduced upto four decimal places as shown in Table 4.2 below, where the column under U represents the values of Ursell (1950a) and the column under C represents our calculated values.

Table 4.2

i	$-(\sigma b)^{-1}p_i$		$-(\sigma b)^{-1}q_i$	
	U	C	U	C
1	0.07195	0.07199	-0.06771	-0.06776
2	0.16241	0.16246	-0.15284	-0.15290
3	0.15341	0.15343	-0.14437	-0.14440
4	0.09680	0.09679	-0.09109	-0.09110
5	0.04954	0.04953	-0.04662	-0.04661

Next we compare our results with those of Mehlum (1980) for the circular cylinder. In his graphical representation Mehlum plotted the phase shift in the transmission coefficient as a function of the radius and depth of the circular cylinder. He considered scaled values for radius and depth of the cylinder,  $A = a/\lambda$  and  $D = d/\lambda$ , where  $a$  is the radius and  $d$  is the depth of the cylinder and  $\lambda$  is the wavelength. We only plot the results for a particular value of  $D$ , namely  $D = 0.45$ . In our implementation appropriate rearrangements of the parameters have been carried out in order to provide a fair comparison. We took  $a/h = A/D$  and  $Ka = 2\pi A$ . We consider  $A$  in the range of  $[0.05, 0.45]$  inclusive. Since the values of  $A$  affect the values of  $a/h$  therefore for a particular set of values for  $Ka$  we had to calculate the values for  $a/h$ . The program was run for each value of  $Ka$  and its corresponding value for  $a/h$  and the phase shift,  $\delta$ , has been found for each run by using  $\delta = \tan^{-1}(T_i/T_r)$ , where  $T_i$  and  $T_r$  are the real and imaginary parts of the transmission coefficient. The graph is shown in Figure (4.1).

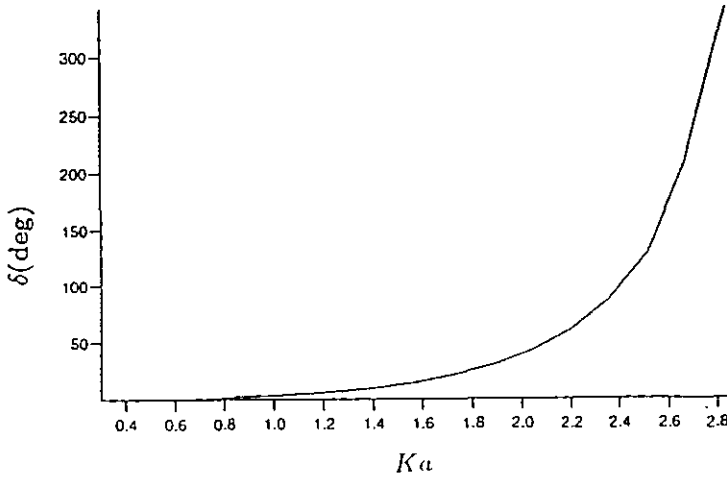


Fig. 4.1 : Phase shift against  $Ka$  for the corresponding input values of Mehlum;  $\alpha=0.0$

we compare the phase shift for a set of values for  $D$  at some particular frequencies and the results are shown in Table 4.3. In this Table the column under  $O$  represents our own calculated result and the column under  $M$  represents the approximate value for Mehlum. From this table it is clear that the comparison of our result Mehlum's shows good agreement.

Table 4.3 Comparison of phase shift ( $\delta$ )

$M$	$O$	$D$	$A$
44.00	44.10	0.30	0.20
129.00	128.34	0.45	0.40
80.00	80.51	0.70	0.60
94.00	93.55	0.90	0.80

Next we compare the values of the reflection and transmission coefficients with those given by Parsons and Martin (1994). Parsons and Martin formulated the scattering potential problem as a hypersingular integral equation for the unknown discontinuity in potential across the arc plate and they solved the integral equation numerically using collocation and Chebyshev expansions. We consider a number of different plates for comparison and as it

can be seen in Table 4.4 that two decimal place agreement was obtained in the majority of cases.

Table 4.4

Parameters	$Ka$	$ R ^\dagger$	$\arg(R^\dagger)$	$ R ^\ddagger$	$\arg(R^\ddagger)$
$a/h = 0.9139$	0.5305	0.7223	2.8094	0.7237	2.7998
$\alpha = 0.7\pi$	1.0610	0.2743	-2.3145	0.2852	-2.3264
	1.5915	0.0518	1.1029	0.0506	1.0992
	2.1221	0.0485	1.0874	0.0507	1.0845
$a/h = 0.8884$	0.3979	0.4820	2.5410	0.4841	2.5352
$\alpha = 0.6\pi$	0.7958	0.2586	-2.6607	0.2648	-2.6675
	1.1937	0.0083	-2.3713	0.0097	-2.3738
	1.5915	0.0277	0.7732	0.0283	0.7718
$a/h = 0.8642$	0.3183	0.2935	2.3490	0.2957	2.3460
$\alpha = 0.5\pi$	0.6366	0.2057	-2.9432	0.2095	-2.9466
	0.9549	0.0353	-2.6399	0.0364	-2.6408
	1.2732	0.0083	0.5278	0.0084	0.5278

<sup>†</sup> Matched series results; <sup>‡</sup> Parsons and Martin (1994)

We next compare both the full and approximate methods for a number of arc plates. We compare the values of  $|R|$  calculated from the full numerical solution with those obtained from the variational approximation for plates of different lengths but for which  $a/h = 0.8$ . Figures 4.2-4.4 show graphs of  $|R|$  plotted against  $Ka$ . First we consider the plate with the shortest length equal to  $0.2\pi a$ , which is equivalent to a value of  $\alpha = 0.9\pi$ .

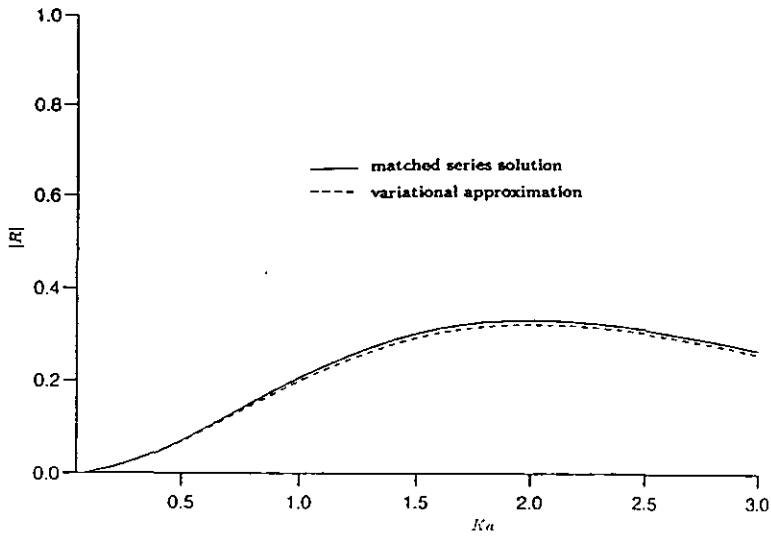


Fig. 4.2: Comparison of approximate and exact values of  $|R|$ ;  $a/h=0.8, \alpha=0.9\pi$

It is clear from Figure 4.2 that the approximation to the reflection coefficient is very close to the exact solution over the whole range of frequencies considered. Therefore as expected the approximation has produced good results for the short plate. The disadvantage of the approximation is that it does not allow for large variations in the potential along the length of the plate and so is expected to perform well only when the plate is short compared to the wavelength and occupies a small fraction of a circle. Analogous results were obtained by Mei and Petroni (1973) who used a similar approximation when modelling the wave scattering by a vertical, circular harbour which contains a narrow gap. We next consider the second largest plate of length equal to  $0.6\pi a$ , (equivalent of a value of  $\alpha = 0.7\pi$ ). Figure 4.3 clearly shows that whilst the approximation is good at low frequencies, it starts to diverge from the true solution at  $Ka \approx 0.4$ . However better approximations could be obtained by employing a more accurate technique such as by choosing  $W_s(\theta)$  and  $W_a(\theta)$  in (3.39) and (3.42) in terms of Chebychev polynomials (Porter and Evans, 1995). Porter and Evans have recently produced extremely accurate results for the scattering of waves by

vertical barriers in finite depth. They have used a direct Galerkin method as an alternative variational approach. Their approach is based on deriving complementary bounds on quantities of interest. Evans and Morris (1972) have also proved that quantities calculated using a variational procedure provide bounds for the exact quantities. They obtained good complementary bounds for the reflection coefficient. In this case, the numerical evidence indicates that the values of  $A_s$  and  $A_a$  generated from the variational approximation are negative and, in magnitude, are lower bounds for the exact values. However, we were unable to prove this because we were unable to show that the operators in (3.30) and (3.31) are negative definite. Even if the variational approximation does yield bounds for  $A_s$  and  $A_a$  these do not translate a bound for the total reflection coefficient because of the way it is constructed from (3.33) and (3.35) by

$$R = \frac{A_s}{i - A_s} + \frac{A_a}{A_a - i} \quad . \quad (4.4)$$

This is apparent in Figure 4.3 where the approximate solution sometimes underestimates and sometimes overestimates the magnitude of the reflection coefficient. In Figure 4.4 we compare the approximate and full solution for a semi-circular plate. In this case, the approximation is not particularly good except at very low frequencies and for plates of this length or longer it would be desirable to seek other approximations such as those based on high or low frequency asymptotics or even small gap approximations.

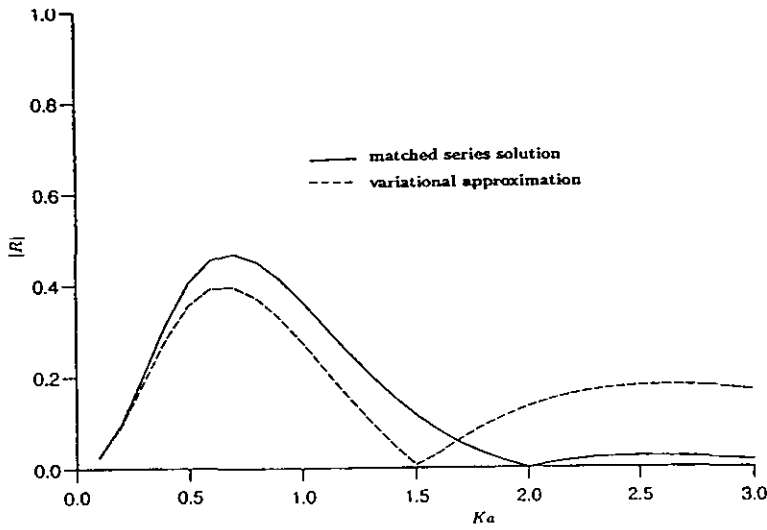


Fig. 4.3 : Comparison of approximate and exact values of  $|R|$ ;  $a/h=0.8, \alpha=0.7\pi$

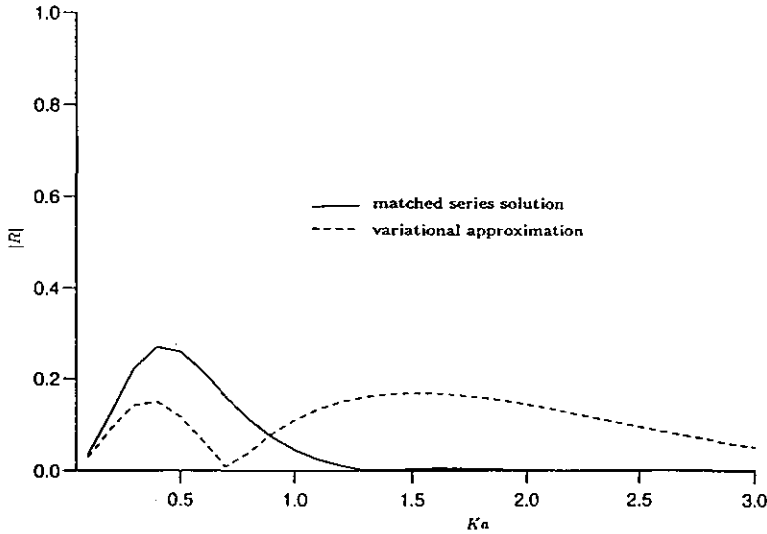


Fig. 4.4 : Comparison of approximate and exact values of  $|R|$ ;  $a/h=0.8, \alpha=0.5\pi$

In Figure 4.5 we compare the reflection coefficients for a set of plates of different lengths at a depth of  $a/h = 0.9$ . The matched series expansion method has been used for these calculations. The motivation behind this is to assess the suitability of the arc plate as a lens element. The corresponding reflection coefficient for the circular cylinder is of course zero for all frequencies. As expected, as  $\alpha$  decreases and plate occupies more of circle, so the reflection coefficient decreases on average. In particular, for plates which occupy half a circle or more there is very little reflection.



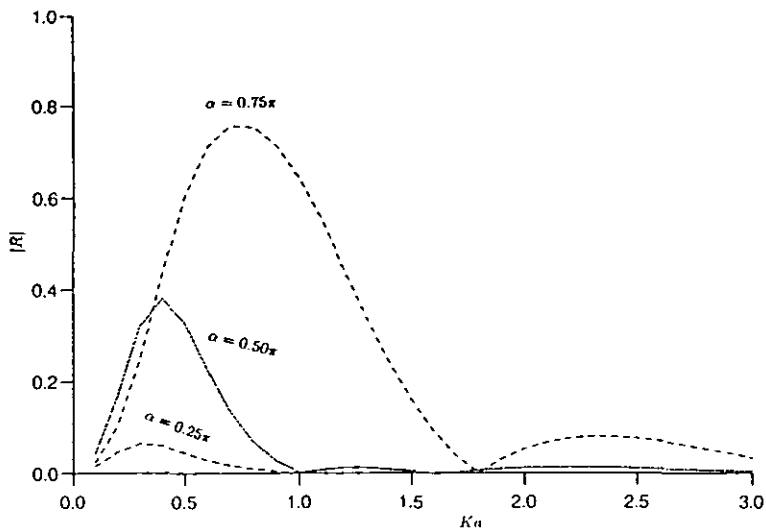


Fig. 4.5 : Variation of  $|R|$  with frequency;  $a/h=0.9$

However, for the shortest plate with a value of  $\alpha = 0.75\pi$  there can be substantial amounts of reflection at low frequencies. Finally in Figure 4.6 we compare the phase of the transmission coefficient for the same range of frequency and for the same set of plates as were considered in Figure 4.5.

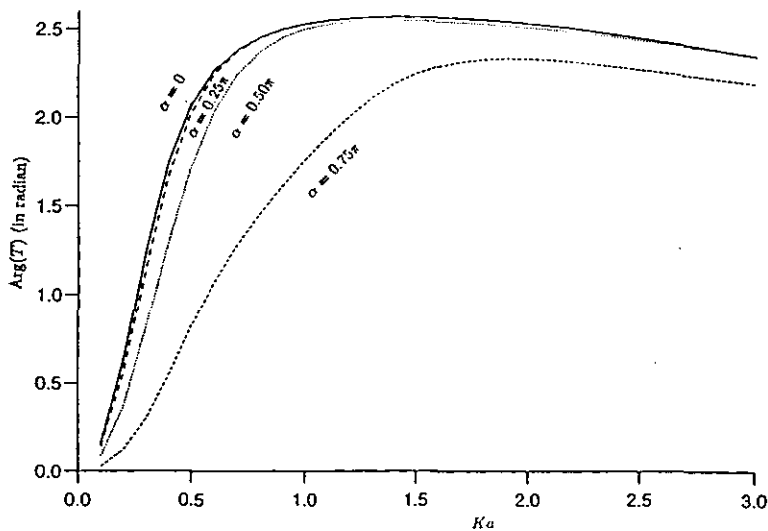


Fig. 4.6 : Variation of  $\text{Arg}(T)$  with frequency;  $a/h=0.9$

In addition, the results for the corresponding circular cylinder are also given. If we consider graphs for  $\alpha = 0.50\pi$  and  $\alpha = 0.25\pi$  in Figure 4.6 it can be seen that the variation in the

phase between these two cases of the transmission coefficients is very small. Therefore, in comparison with the circular cylinder one can infer that the variation in the phase of the transmission coefficient does not significantly depend on the position of the ends of the plates which occupy half a circle or more. It is as if the wave field only 'sees' the top part of the body and the fact that the lower part of the cylinder is missing has an insignificant effect. Thus in circumstances in which the circular cylinder produces suitable phase shifts to be used as a lens element, a circular arc plate occupying a least half of the same circle should also be a candidate.

# Chapter 5

## 5.1 Conclusion

A Norwegian research group has demonstrated that a water-wave lens may be constructed out of a system of under water structures. In this work we considered modelling one of these structures by a fixed submerged, circular arc plate. The wave scattering by the arc plate has been investigated using linear theory. Two methods, namely a matched series expansion and a variational approximation were used to investigate the reflective properties of the arc plate. The first method employed Ursell's multipole potentials technique (Ursell, 1950a) while an explicit approximate solution which modelled exactly the behaviour of the velocity potential at the tips of the plate was derived using the second method. The full numerical solution was used to compare the reflection and transmission coefficients associated with a number of plates with the corresponding coefficients for a submerged, circular cylinder and it was found that there was very little difference between the reflective properties of plates which occupy a half circle or more and those of a circular cylinder.

The approximation to the reflection coefficient proved to be accurate for plates which were short compared with the wavelength and occupied a small part of a circle. This was

expected because of the way the square roots singularities in the velocity at the tips of the plate were modelled. Better approximations could be obtained by modelling the velocity more accurately using a series of Chebychev polynomials multiplied by the square root singularity in a similar fashion to that done by Porter and Evans (1995) for a barrier with a single gap.

Further work needs to be done to obtain an approximation for longer plates. For the long plates the reflection coefficient for a wide range of frequencies should become very small. In these circumstances, it may be more appropriate to develop an approximate solution based on a cylinder with a small gap in it.

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