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
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**The Existence of Bistable Stationary Solutions of
Random Dynamical Systems Generated by
Stochastic Differential Equations and Random
Difference Equations**

by

Bo ZHOU

Doctoral Thesis

**Submitted in partial fulfillment of the requirements for
the award of**

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Supervisor: Prof. Huaizhong ZHAO





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Abstract

In this thesis, we study the existence of stationary solutions for two cases. One is for random difference equations. For this, we prove the existence and uniqueness of the stationary solutions in a finite-dimensional Euclidean space \mathbb{R}^d by applying the coupling method. The other one is for semilinear stochastic evolution equations. For this case, we follow Mohammed, Zhang and Zhao [25]'s work. In an infinite-dimensional Hilbert space \mathbb{H} , we release the Lipschitz constant restriction by using Arzela-Ascoli compactness argument. And we also weaken the globally bounded condition for F by applying forward and backward Gronwall inequality and coupling method.

Keywords: stationary solution, random dynamical system, random difference equation, semilinear stochastic evolution equation, coupling method, Gronwall inequality.

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Chapter 1

Introduction

Random dynamical systems arise in the modeling of many phenomena in physics, biology, climatology, economics, etc., when uncertainties or random influences, called noises, are taken into account. The need for studying random dynamical systems was presented by Ulam and von Neumann [37] in 1945. It has been pushed since the 1980s due to the discovery through the efforts that stochastic ordinary differential equations generate random dynamical systems, we refer the reader to [1], [17], [21], [22] and the references therein. In deterministic and random dynamical systems, to find the existence of stationary solutions and to construct local stable and unstable manifolds near a hyperbolic stationary point is a fundamental problem. In recent years, Mohammed and Scheutzow [24] has established that local stable and unstable manifolds exist for finite-dimensional stochastic ordinary differential equations. For semilinear stochastic evolution equations (see's) and stochastic partial differential equations (spde's), Mohammed, Zhang and Zhao [25] proved the existence of flows and cocycles and establish the existence of local stable and unstable manifolds near stationary solutions. However, in contrast to the deterministic dynamical systems, the existence of stationary solutions of stochastic dynamical systems generated e.g. by stochastic differential equations or stochastic partial differential equations, is a difficult and subtle problem. Actually, researchers usually assume there is an invariant set or a stationary solution or a fixed point, often assumed to be 0, then prove invariant manifolds and stability results at a point of the invariant set ([1], [14], [15], [20], [33]). In particular, for the existence of stationary solutions, results are only known in very few cases ([5], [10], [25], [34], [35], [39]). It is far from clear, in general.

The main objective of this thesis is to find the stationary solution in two different

type equations.

In chapter 2, we consider the following random difference equation in the finite-dimensional Euclidean space \mathbb{R}^d .

$$x_{n+1} = A(\theta^n \omega)x_n + F(\theta^n \omega, x_n),$$

where $n \in \mathbb{Z}$ and

$$F(\omega, 0) = 0.$$

Here A is a random $d \times d$ invertible matrix with entry elements from $\Omega \rightarrow \mathbb{R}$ and F is a function which satisfies the Lipschitz condition. We introduce some basic concepts on random dynamical systems, stationary solution, invariant manifold and multiplicative ergodic theorem. In Sections 2.4 to 2.6, we establish the structure of stable and unstable manifold theorem for random dynamical systems. Coupling method is introduced in order to find the corresponding stationary solution. Section 2.7 gives the main theorem (Theorem 2.7.1) and the related proof in details follows. Two key lemmas (Lemmas 2.6.1, 2.6.2) act as an important role in the proof. In Section 2.8, a gap condition problem will be mentioned and gives a possible method to solve it. Finally, we give some unsolved possible improvements and problems for the future research for this chapter in Section 2.9.

In chapter 3, our problems are studied in an infinite-dimensional Hilbert space \mathbb{H} . Under this space, we consider a semilinear stochastic evolution equation (semilinear see) with the additive noise of the form

$$\begin{aligned} du(t) &= [-Au(t) + F(u(t))]dt + B_0 dW(t), \\ u(0) &= x \in \mathbb{H}, \end{aligned}$$

where A is a closed linear operator from

$$D(A) \subset \mathbb{H} \rightarrow \mathbb{H},$$

B_0 is a bounded linear operator from

$$\mathbb{H} \rightarrow L_2(K, \mathbb{H}).$$

The function F is a nonlinear perturbation which satisfies the Lipschitz condition. On the Wiener space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$, $W(t)$, $t \geq 0$ is a Brownian motion. In the background section, we introduce the basic structures of semilinear stochastic evolution

equations, Oseledec-Ruelle version multiplicative ergodic theorem and local invariant manifold theorem in infinite-dimensional space. Mohammed, Zhang and Zhao's existence results (Propositions 3.1.5, 3.1.6) for stationary solution are introduced and two problems about the results are mentioned. In Section 3.2, we release the Lipschitz constant restriction in Propositions 3.1.5 and 3.1.6 by using Arzela-Ascoli compactness argument. They are presented as Propositions 3.2.1 and 3.2.2. The cost for this is that we lost the uniqueness property. In Section 3.3, it is a complicated work to weaken the globally boundedness condition for F . We try to find a better condition for F to replace the previous one. For the key equation

$$z(t) = \int_{-\infty}^t T_{t-s}P^+ F_n(z(s) + Y_1(s))ds - \int_t^{\infty} T_{t-s}P^- F_n(z(s) + Y_1(s))ds,$$

where

$$Y_1(t) = (\omega) \int_{-\infty}^t T_{t-s}P^+ B_0 dW(s) - (\omega) \int_t^{\infty} T_{t-s}P^- B_0 dW(s)$$

for all $z(\theta, \omega) \in C_B(T, \mathbb{H})$, all $\omega \in \Omega$ and

$$F_n := \begin{cases} F, & \text{if } |F| \leq n, \\ 0, & \text{otherwise,} \end{cases}$$

is a cut off function, we consider this equation as two parts corresponding to positive and negative eigenvalues of the operator A . We firstly have

$$\begin{aligned} z(t) &:= (z^+(t), z^-(t)), \\ Y_1(t) &:= (Y_1^+(t), Y_1^-(t)). \end{aligned}$$

Then

$$z^+(t) = \int_{-\infty}^t T_{t-s}P^+ F_n(z^+(s) + Y_1^+(s), z^-(s) + Y_1^-(s))ds$$

and

$$z^-(t) = - \int_t^{\infty} T_{t-s}P^- F_n(z^+(s) + Y_1^+(s), z^-(s) + Y_1^-(s))ds.$$

After a complicated form change, we have

$$(z^+)^2(t) \leq 2 \int_{-\infty}^t e^{-(t-s)2\mu_{m+1}} (z^+(s), P^+ F_n(z^+(s) + Y_1^+(s), z^-(s)))$$

$$+Y_1^-(s))ds$$

and

$$(z^-)^2(t) \leq -2 \int_t^\infty e^{-(t-s)2\mu_m} (z^-(s), P^- F_n(z^+(s) + Y_1^+(s), z^-(s) + Y_1^-(s))) ds.$$

Moreover, we follow the coupling method to consider the possible improvement. Forward and backward Gronwall inequalities are technically used in the proof in this section. Two propositions (Propositions 3.3.1, 3.3.2) conclude this section. As a result, a new version local invariant manifold theorem (Theorem 3.4.1) will be presented in Section 3.4.

Chapter 2

The Discrete Time RDS in a Finite-Dimensional Euclidean Space

\mathbb{R}^d

§2.1 Basic Concepts

In this section, we introduce some main basic concepts including random dynamical system, invariant measure, stationary solution and manifold before developing them in further research.

Random Dynamical System (RDS)

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. For a topological space E , $\mathcal{B}(E)$ denotes its Borel σ -algebra. We begin by giving the definition of dynamical systems, then extend to the random case.

Definition 2.1.1 *In general, a dynamical system (DS) is a tuple (T, M, ϕ) where T is a time set, M is a state space, ϕ is a function*

$$\begin{aligned}\phi : T \times M &\rightarrow M, \\ (t, x) &\mapsto \phi(t, x),\end{aligned}$$

with the following properties

1. $\phi(0, x) = x$,

$$2. \phi(t + s, x) = \phi(t, \phi(s, x)),$$

for all $t, s, t + s \in T$, and the variable $x \in M$ is the initial starting point of the dynamical system.

Remark 2.1.2

1. Dynamical systems are usually generated from differential equations or difference equations. When differential equations are employed, it is called continuous dynamical systems. When difference equations are employed, it is called discrete dynamical systems.
2. There are several choices for the time set T . When T is taken to be the reals $T = \mathbb{R}$, the dynamical system is called a flow. When T is restricted to the non-negative reals $T = \mathbb{R}^+$, it is called a semi-flow. When T is taken to be the integers $T = \mathbb{Z}$, it is a cascade or a map. When T is restricted to be the non-negative integers $T = \mathbb{Z}^+$, it is a semi-cascade. The set T is called two-sided time when it is taken \mathbb{R} or \mathbb{Z} , and one-sided time for \mathbb{R}^+ or \mathbb{Z}^+ .

From the Remark 2.1.2 (1), we can replace differential equations by stochastic differential equations. This process generates a flow from the solution to a stochastic differential equation. These flows are called random dynamical systems. We need a well-defined definition on their owns.

Definition 2.1.3 Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, the noise space. Let

$$\theta : T \times \Omega \rightarrow \Omega$$

be a measure preserving measurable dynamical system, i.e. we fix a time $s \in T$, the function θ_s :

$$\Omega \rightarrow \Omega$$

is a measure-preserving measurable function which means

$$\mathbb{P}(E) = \mathbb{P}(\theta_s^{-1}(E))$$

for all $E \in \mathcal{F}$ and $s \in T$ and θ also satisfies:

1. $\theta_0 = id_\Omega$ the identity function on Ω .

2. $\theta_t \circ \theta_s = \theta_{t+s}$ for all $s, t, t+s \in T$.

Let (X, d) be a complete separable metric space, the phase space. A measurable random dynamical system φ over θ is a function

$$\begin{aligned} \varphi : T \times \Omega \times X &\rightarrow X \\ (t, \omega, x) &\mapsto \varphi(t, \omega, x) \end{aligned}$$

with the following properties:

1. φ is a $(\mathcal{B}(T) \otimes \mathcal{F} \otimes \mathcal{B}(X), \mathcal{B}(X))$ measurable function.
2. φ satisfies the cocycle property:

$$\begin{aligned} \varphi(0, \omega) &= id_X, \\ \varphi(t+s, \omega) &= \varphi(t, \theta_s(\omega)) \circ \varphi(s, \omega) \end{aligned}$$

for almost all $\omega \in \Omega$.

Note \circ means composition, i.e. $(f \circ g)(x) = f(g(x))$.

Remark 2.1.4

1. A measurable RDS φ over θ is said to be continuous if the function for each $\omega \in \Omega$

$$\begin{aligned} \varphi(\cdot, \omega, \cdot) : T \times X &\rightarrow X \\ (t, x) &\mapsto \varphi(t, \omega, x) \end{aligned}$$

is continuous on $t \in T$ and $x \in X$.

2. When RDS φ is driven by a Wiener process $W : T \times \Omega \rightarrow X$, the function $\theta_s : \Omega \rightarrow \Omega$ given by

$$\theta_t(\omega)W(s, \omega) = W(t+s, \omega) - W(s, \omega)$$

is a measure preserving dynamical system.

3. For a given measurable RDS φ over θ , we consider a new map which is defined for all $t \in T$

$$\begin{aligned} \Theta(t) : \Omega \times X &\rightarrow \Omega \times X \\ (\omega, x) &\mapsto (\theta(t)\omega, \varphi(t, \omega)x) \end{aligned}$$

We call this map the skew product of the metric DS $(\Omega, \mathcal{F}, \mathbb{P}, (\theta(t))_{t \in T})$ and the cocycle $\varphi(t, \omega)$ on X . It is easy to see Θ is a measurable DS from $(T \times \Omega \times X)$ to $(\Omega \times X)$. Hence, all the RDS φ can be consequently regarded as a DS on a higher dimensional state space.

4. We present this definition as a figure.

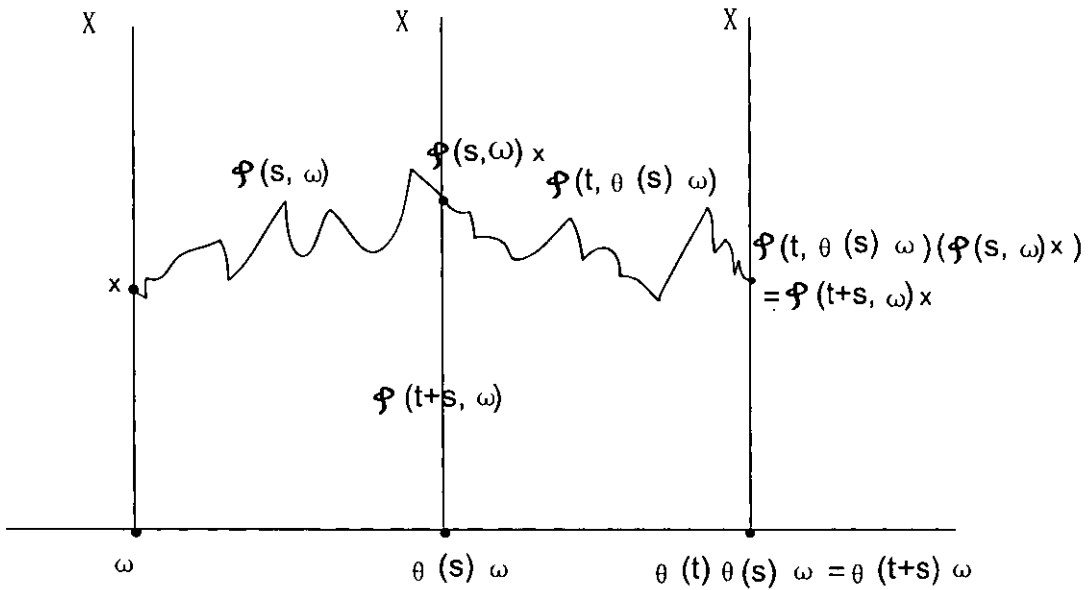


Figure 2.1: A random dynamical system

Invariant Measure

In the theory of dynamical systems, invariant measure is an important concept. The existence of an invariant measure under some conditions is always a central problem in dynamical systems. We firstly give a concise introduction of invariant measure in dynamical systems, then extend it to random dynamical systems.

Definition 2.1.5 In a DS (X, T, ϕ) , X is a state space, T is a time set, the function $\phi : T \times X \rightarrow X$ is DS map, a measure μ on X is said to be an invariant measure if and only if for each $t \in T$

$$\phi_t : X \rightarrow X,$$

we have

$$\mu(\phi_t^{-1}(A)) = \mu(A)$$

for all $A \in \mathcal{B}(X)$.

Example 2.1.6

In the one-dimensional real line \mathbb{R} , equipped with its Borel σ -algebra, for a fixed constant $a \in \mathbb{R}$, we consider the map

$$\begin{aligned} M_a : \mathbb{R} &\rightarrow \mathbb{R} \\ x &\mapsto x + a. \end{aligned}$$

Then it is easy to see the one-dimensional Lebesgue measure λ is an invariant measure for map M_a .

For RDS, we need to consider the random elements into this definition. In Remark 2.1.4 (3), we notice any RDS φ can be equivalent to consider as a DS, represented as a skew product Θ . Hence, we define the invariant measure for RDS by applying this application here.

Definition 2.1.7 For a measurable RDS φ over a dynamical system θ of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, define

$$P_\Omega : \Omega \times X \rightarrow \Omega$$

to be the projection onto Ω . We say a probability measure μ on $(\Omega \times X, \mathcal{F} \otimes \mathcal{B}(X))$ to be an invariant measure if for all $t \in T$

1. $\Theta(t)\mu = \mu$,
2. $P_\Omega\mu = \mathbb{P}$.

Here Θ is the skew product corresponding to φ .

An invariant measure is a measure which is preserved by some functions. For a RDS, to find an invariant measure is not obvious. The difficulty is to lift the invariant property from an θ -invariant \mathbb{P} on (Ω, \mathcal{F}) to an $\Theta(t)$ -invariant μ on $(\Omega \times X, \mathcal{F} \otimes \mathcal{B}(X))$. Normally, the invariant measure only comes with a DS measure \mathbb{P} on (Ω, \mathcal{F}) . Fortunately, this part has been carefully introduced by Arnold [1]. We will not give a

presentation for this part.

Stationary Solution

We now introduce the concept of stationary points or stationary solutions in dynamical systems and random dynamical systems, respectively. How to find stationary solutions in different situations is the main task in this thesis. We will discuss the cases due to finite-dimensional space and infinite-dimensional space.

Definition 2.1.8 For a deterministic DS (T, X, ϕ) ,

$$\phi : T \times X \rightarrow X,$$

a stationary solution is a fixed point $a \in X$, such that

$$\phi_t(a) = a$$

for all $t \in T$.

Example 2.1.9

We consider a simple case. For a DS $\phi_t : \mathbb{R} \rightarrow \mathbb{R}$ which is generated by a linear differential equation with initial starting point $x \in \mathbb{R}$

$$\begin{aligned} \frac{dy}{dt} &= -y, \\ y_0 &= x. \end{aligned}$$

It is easy to know that the solution is given by

$$\phi_t x = x e^{-t}.$$

Obviously, zero is the stationary solution for ϕ since

$$\phi_t 0 = 0.$$

Definition 2.1.10 For a measurable RDS φ on a state space $(X, \mathcal{B}(X))$ over a metric DS $(\Omega, \mathcal{F}, \mathbb{P}, \theta(t)_{t \in T})$:

$$\varphi : T \times \Omega \times X \rightarrow X,$$

a stationary solution is an \mathcal{F} -measurable random variable $Y : \Omega \rightarrow X$ such that

$$\varphi(t, \omega, Y(\omega)) = Y(\theta_t \omega)$$

for all $t \in T$ a.s.

Example 2.1.11

We consider a Ornstein-Uhlenbeck (OU) process y_t by the following simple stochastic differential equation

$$\begin{aligned} dy_t &= -y_t dt + dW_t(\omega), \\ y_0 &= x, \end{aligned}$$

where $W_t(\omega)$ denotes the Wiener process. This can be considered as a random perturbation in the dynamical system discussed in Example 2.1.9. Applying Itô formula to $e^t y_t$, we admit that the solution is given by

$$y_t = x e^{-t} + \int_0^t e^{-(t-s)} dW_s(\omega).$$

Assume φ is the RDS generated by this stochastic differential equation, thus y_t is replaced by

$$\varphi(t, \omega)x = x e^{-t} + \int_0^t e^{-(t-s)} dW_s(\omega).$$

Now, we consider the random variable with

$$Y(\omega) = \int_{-\infty}^0 e^s dW_s(\omega).$$

We are going to see this $Y(\omega)$ is a stationary solution. Hence, we need to check $\varphi(t, \omega)Y(\omega)$ and $Y(\theta_t \omega)$.

$$\begin{aligned} \varphi(t, \omega)Y(\omega) &= e^{-t} \int_{-\infty}^0 e^s dW_s(\omega) + \int_0^t e^{-(t-s)} dW_s(\omega) \\ &= \int_{-\infty}^t e^{-(t-s)} dW_s(\omega). \end{aligned}$$

By applying Remark 2.1.4 (2), we have

$$W(s, \theta_t(\omega)) = W(s + t, \omega) - W(t, \omega).$$

Then

$$\begin{aligned} Y(\theta_t \omega) &= \int_{-\infty}^0 e^s dW_s(\theta_t \omega) \\ &= \int_{-\infty}^0 e^s dW_{s+t}(\omega) \end{aligned}$$

$$= \int_{-\infty}^t e^{-(t-s)} dW_s(\omega).$$

The last step is obtained by applying the change of variables $s' = t + s$. Thus, we finally have that $Y(\omega)$ satisfies

$$\varphi(t, \omega)Y(\omega) = Y(\theta_t \omega).$$

It is a stationary point.

Invariant Manifolds

Manifold is an important mathematical space. It describes a space model on a Euclidean space. Each point on a manifold has a neighborhood which resembles Euclidean space, but the global structure is more complicated. Some simple examples include lines, two-dimensional planes, the surface of a sphere and so on. We here introduce some basic concepts of related deterministic manifolds. Now consider a DS (X, T, ϕ) , we have that an invariant manifold M is a manifold with the property that it is invariant under the flow such that for all $t \in T$

$$\phi(t)M = M.$$

If a is a stationary point of this DS, such that

$$\phi_t(a) = a,$$

the stable manifold of a is defined by

$$M^S(\phi, a) = \{x \in X : \phi_t(x) \rightarrow a \text{ as } t \rightarrow \infty\}$$

and the unstable manifold of a is defined by

$$M^U(\phi, a) = \{x \in X : \phi_{-t}(x) \rightarrow a \text{ as } t \rightarrow \infty\}.$$

For the random case, researchers usually assume the fixed point to be zero. we consider a measurable RDS φ on a state space $(X, \mathcal{B}(X))$ over a metric DS $(\Omega, \mathcal{F}, \mathbb{P}, \theta(t)_{t \in T})$. We call the set

$$M^+(\omega) = \{x \in X : \varphi(-t, \omega, x) \rightarrow 0 \text{ as } t \rightarrow \infty\}$$

unstable manifold. Similarly, we call the set

$$M^-(\omega) = \{x \in X : \varphi(t, \omega, x) \rightarrow 0 \text{ as } t \rightarrow \infty\}$$

stable manifold. In fact, in random dynamical systems, the existence of the fixed point is a more difficult problem than in dynamical systems.

§2.2 Stationary Solution and Invariant Measure

This section is devoted to discuss the relationship between invariant measures and stationary solutions. Both of them are the basic important concepts in DS and RDS. We try to prove a stationary solution can give an invariant measure here. For a RDS φ over θ , let μ denote an invariant probability measure on $(\Omega \times X, \mathcal{F} \otimes \mathcal{B}(X))$. Consider the function

$$\mu(\cdot) : \Omega \times \mathcal{B}(X) \rightarrow [0, 1],$$

if for all $A \in \mathcal{F} \otimes \mathcal{B}(X)$, we have

$$\mu(A) = \int_{\Omega} \int_X 1_A(\omega, x) \mu_{\omega}(dx) \mathbb{P}(d\omega),$$

or for all $f \in L^1(\mu)$

$$\int_{\Omega \times X} f d\mu = \int_{\Omega} \left(\int_X f(\omega, x) \mu_{\omega}(dx) \right) \mathbb{P}(d\omega).$$

We call such a function $\mu(\cdot)$ a factorization of the invariant measure μ . For simplicity, we write

$$\mu(d\omega, dx) = \mu_{\omega}(dx) \mathbb{P}(d\omega). \quad (2.1)$$

In Arnold [1], the existence and uniqueness for this kind of factorization of μ have been presented. Since μ is an invariant measure for the RDS φ over θ , we have by the definition for all $t \in T$

$$\Theta(t)\mu(F \times B) = \mu(F \times B) \quad (2.2)$$

for any $F \times B \in \mathcal{F} \otimes \mathcal{B}(X)$. By applying the factorization $\mu(\cdot)$ of μ for both sides of the equation

$$\begin{aligned} (\Theta(t)\mu)(F \times B) &= \mu(\Theta(t)^{-1}(F \times B)) \\ &= \int_{\theta^{-1}(t)F} \mu_{\omega}(\varphi^{-1}(t, \omega)B) \mathbb{P}(d\omega) \\ &= \int_{\theta^{-1}(t)F} (\varphi(t, \omega)\mu_{\omega})(B) \mathbb{P}(d\omega) \end{aligned}$$

and

$$\mu(F \times B) = \int_F \mu_{\omega}(B) \mathbb{P}(d\omega)$$

$$\begin{aligned}
 &= \int_F \mu_\omega(B)(\theta(t)\mathbb{P})(d\omega) \\
 &= \int_{\theta^{-1}(t)F} \mu_{\theta(t)\omega}(B)\mathbb{P}(d\omega).
 \end{aligned}$$

Hence, for any $F \in \mathcal{F}$ and $B \in \mathcal{B}(X)$

$$\int_{\theta^{-1}(t)F} (\varphi(t, \omega)\mu_\omega)(B)\mathbb{P}(d\omega) = \int_{\theta^{-1}(t)F} \mu_{\theta(t)\omega}(B)\mathbb{P}(d\omega).$$

This leads

$$(\varphi(t, \omega)\mu_\omega)(B) = \mu_{\theta(t)\omega}(B) \quad \mathbb{P} - a.s. \quad (2.3)$$

for all $B \in \mathcal{B}(X)$. To reverse the above process, one can see if (2.3) holds for any $B \in \mathcal{B}(X)$, then $\mu(d\omega, dx)$ defined by (2.1) satisfied (2.2). That is to say μ is an invariant measure. Here comes the idea. We now consider a special case, when the factorization of μ is a random Dirac measure i.e. for a random variable $Y : \Omega \rightarrow X$, it is defined by

$$\mu_\omega(B) = \delta_{Y(\omega)}(B) = \begin{cases} 1 & \text{if } Y(\omega) \in B \\ 0 & \text{if } Y(\omega) \notin B \end{cases}$$

for $B \in \mathcal{B}(X)$. Then the above equation reads as

$$\varphi(t, \omega)\delta_{Y(\omega)} = \delta_{Y(\theta(t)\omega)} \quad \mathbb{P} - a.s.$$

However, for $B \in \mathcal{B}(X)$

$$\begin{aligned}
 \varphi(t, \omega)\delta_{Y(\omega)}(B) &= \delta_{Y(\omega)}(\varphi^{-1}(t, \omega)B) \\
 &= \begin{cases} 1 & \text{if } Y(\omega) \in \varphi^{-1}(t, \omega)B \\ 0 & \text{if } Y(\omega) \notin \varphi^{-1}(t, \omega)B \end{cases} \\
 &= \begin{cases} 1 & \text{if } \varphi(t, \omega)Y(\omega) \in B \\ 0 & \text{if } \varphi(t, \omega)Y(\omega) \notin B \end{cases} \\
 &= \delta_{\varphi(t, \omega)Y(\omega)}(B) \quad \mathbb{P} - a.s.
 \end{aligned}$$

Thus,

$$\delta_{Y(\theta(t)\omega)} = \delta_{\varphi(t, \omega)Y(\omega)} \quad \mathbb{P} - a.s.$$

This leads

$$Y(\theta(t)\omega) = \varphi(t, \omega)Y(\omega) \quad \mathbb{P} - a.s.$$

Therefore, we can conclude that there exists a stationary point $Y(\omega)$ if and only if we can construct an invariant measure as a Dirac measure of $Y(\omega)$ as a factorization of the

invariant measure according to the stationary point and the measure can be expressed by

$$\mu(dx, d\omega) = \delta_{Y(\omega)}(dx)\mathbb{P}(d\omega).$$

Normally, to find an invariant measure, we apply the Krylov-Bogolyubov procedure. We can find related introductions in the DS or RDS books. In this thesis, we would like to point out that there have been extensive works on stability and invariant manifolds of random dynamical systems: Researchers usually assume there is an invariant set or a single point, a stationary solution or a fixed point, often assumed to be 0, then prove invariant manifolds and stability results at a point of the invariant set, see Arnold [1] and references therein, Ruelle [32], [33], Duan, Lu and Schaumluffuss [14], [15], Li and Lu [20], Mohammed, Zhang and Zhao [25]. But the invariant manifolds theory gives neither the existence results of the invariant set and the stationary solution nor a way to find them. In particular, for the existence of stationary solutions, results are only known in very few cases, see [5], [10], [25], [34], [35] and [39]. To find the stationary point, it is a different problem. Basically, the invariant measure does not necessarily give the stationary solution. In this thesis, we are concentrating on the existence of stationary points in different situations.

§2.3 Multiplicative Ergodic Theorem (MET)

This section is devoted to the presentation and discussion of Oseledets Multiplicative Ergodic Theorem in a random manner, basically following Goldsheid and Margulis [12]. MET is the theoretical background to compute Lyapunov exponents and it is a key theorem to study the different type DS. Here, we present a deterministic definition for Lyapunov exponent firstly.

Definition 2.3.1 *In the d -dimensional Euclidean phase space \mathbb{R}^d , $\Phi(t)$ is a linear DS generated by a linear differential (or difference) equation*

$$\dot{x}_t = A(t)x_t \quad (\text{or } x_{n+1} = A_n x_n).$$

For $x \in \mathbb{R}^d$, the Lyapunov exponent, for two-sided time $t \in T$, is defined by

$$\lambda(x) := \limsup_{t \rightarrow \pm\infty} \frac{1}{|t|} \log \|\Phi(t)x\|.$$

Here $\|\cdot\|$ defines the Euclidean norm.

In the following MET, we will see a random Lyapunov exponent. Since this chapter is for finite-dimensional space \mathbb{R}^d , we are going to see the MET version adapted in a finite-dimensional space. For the infinite-dimensional Hilbert space H in the next chapter, we will introduce Oseledec-Ruelle version of MET.

Theorem 2.3.2 (Multiplicative Ergodic Theorem)

From now on, we define Φ to be a linear RDS over θ , and

$$A : \Omega \rightarrow \mathbb{R}^{d \times d}$$

is a $d \times d$ invertible random matrix with elements in \mathbb{R} . Suppose that, the following integrable conditions are satisfied with

$$\begin{aligned} \log^+ \| A(\cdot) \| &\in L^1(\Omega, \mathcal{F}, \mathbb{P}), \\ \log^+ \| A^{-1}(\cdot) \| &\in L^1(\Omega, \mathcal{F}, \mathbb{P}), \end{aligned}$$

where \log is a logarithm function.

For a one-sided time $T = \mathbb{N}$, and a linear random dynamical system

$$\Phi(n, \omega) = A_{n-1}(\omega) \cdots A_0(\omega),$$

there exists an invariant set $\Omega_0 \in \mathcal{F}$ with full measure such that

$$\theta(n, \cdot)(\Omega_0) = \Omega_0 \quad \text{and} \quad \mathbb{P}(\Omega_0) = 1$$

for all $n \in \mathbb{N}$ and for each $\omega \in \Omega_0$ the limit

$$\Psi(\omega) := \lim_{n \rightarrow \infty} (\Phi(n, \omega)^* \Phi(n, \omega))^{\frac{1}{2n}}$$

exists. This $\Psi(\omega)$ is self-adjoint with a discrete spectrum

$$e^{\lambda_{p(\omega)}(\omega)} < \dots < e^{\lambda_1(\omega)}.$$

Let $U_{p(\omega)}(\omega), \dots, U_1(\omega)$ be the corresponding eigenvectors. Then we denote for $i = 1, \dots, p(\omega)$,

$$V_i(\omega) := U_{p(\omega)}(\omega) \oplus \dots \oplus U_i(\omega).$$

These $(V_i(\omega))_{i=1, \dots, p(\omega)}$ form a filtration for \mathbb{R}^d such that

$$V_{p(\omega)}(\omega) \subset \dots \subset V_i(\omega) \subset \dots \subset V_1(\omega) = \mathbb{R}^d,$$

and $d_i(\omega) = \dim U_i(\omega)$. Moreover, the random Lyapunov exponent

$$\lambda_i(\omega) = \lim_{n \rightarrow \infty} \log \|\Phi(n, \omega)x\|$$

for $x \in V_i(\omega) \setminus V_{i+1}(\omega)$, and

$$A(\omega)V_i(\omega) = V_i(\theta\omega)$$

for all $i \in \{1, \dots, p(\omega)\}$.

For a two-sided time $T = \mathbb{Z}$, and a linear random dynamical system Φ defined by

$$\Phi(n, \omega) = \begin{cases} A(\theta^{n-1}\omega) \cdots A(\omega), & n > 0, \\ I, & n = 0, \\ A^{-1}(\theta^n\omega) \cdots A^{-1}(\theta^{-1}\omega), & n < 0, \end{cases}$$

all the statements in the one-sided time $T = \mathbb{N}$ still hold. Moreover, there exists another spectrum

$$e^{\lambda_{p^-}^-(\omega)} > \dots > e^{\lambda_1^-(\omega)}$$

and a backward filtration with

$$\psi(n, \omega) := \Phi(-n, \omega)$$

over θ^{-1} and

$$V_{p^-}^-(\omega) \subset \dots \subset V_1^-(\omega) = \mathbb{R}^d.$$

Here the following relationship with the forward filtration holds

$$\begin{cases} p(\omega) & = & p^-(\omega) \\ d_i(\omega) & = & d_{p(\omega)+1-i}^-(\omega) \\ \lambda_i(\omega) & = & -\lambda_{p(\omega)+1-i}^-(\omega) \end{cases}$$

for all $i = 1, \dots, p(\omega)$. Denote

$$E_i(\omega) := V_i(\omega) \cap V_{p(\omega)+1-i}^-(\omega)$$

for $i = 1, \dots, p(\omega)$, then we have the Oseledets splitting

$$\mathbb{R}^d = E_1(\omega) \oplus \dots \oplus E_{p(\omega)}(\omega).$$

Finally, the Lyapunov exponent is

$$\lambda_i(\omega) = \lim_{n \rightarrow \pm\infty} \frac{1}{n} \log \|\Phi(n, \omega)x\|$$

for $x \in E_i(\omega) \setminus \{0\}$. And the subspace $E_i(\omega)$ is invariant under A , this is to say for all $i \in \{1, \dots, p(\omega)\}$

$$A(\omega)E_i(\omega) = E_i(\theta\omega).$$

Remark 2.3.3

1. For the continuous time case $T = \mathbb{R}^+$ or \mathbb{R} , the MET still holds only with the two conditions $\log^+ \|A(\cdot)\| \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ and $\log^+ \|A^{-1}(\cdot)\| \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ changing to

$$\sup_{0 \leq t \leq 1} \log^+ \|\Phi(t, \omega)^{\pm 1}\| \in L^1(\Omega, \mathcal{F}, \mathbb{P}).$$

2. The functions $\lambda_i(\omega)$, $d_i(\omega)$, $p(\omega)$, $U_i(\omega)$, $V_i(\omega)$ and $E_i(\omega)$ mentioned in the theorem are all measurable.
3. The ergodic case in our theorem refers to the DS θ on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. If the measurable set $A \in \mathcal{F}$ is invariant i.e. $\theta(A) = A$, then

$$\mathbb{P}(A) = 0 \text{ or } 1.$$

In this case, we have the functions $p(\cdot)$, $d_i(\cdot)$ and $\lambda_i(\cdot)$ are constants on Ω_0 . The proof of this result was given by Krengel [19] and Steele [36].

4. From the theorem, we have the Oseledets splitting

$$\mathbb{R}^d = E_1(\omega) \oplus \dots \oplus E_{p(\omega)}(\omega).$$

We call the set of the different λ in each subspace such that

$$\{(\lambda_1(\omega), d_1(\omega)), (\lambda_2(\omega), d_2(\omega)), \dots, (\lambda_p(\omega), d_p(\omega))\}, \quad 1 \leq p \leq d$$

the spectrum of Lyapunov exponents.

§2.4 Preparations

In this section, we are going to do some preparations for the structure of the invariant manifold. From the last section MET, for a linear cocycle Φ with two-sided time over a DS $(\Omega, \mathcal{F}, \mathbb{P}, (\theta(t))_{t \in T})$, we have

$$\lim_{n \rightarrow \pm\infty} \frac{1}{n} \log \|\Phi(n, \omega)x\| = \lambda_i(\omega)$$

for $x \in E_i(\omega) \setminus \{0\}$, $i = 1, \dots, p(\omega)$. We change another form for this limit. For arbitrary $\epsilon > 0$,

$$\left| \frac{1}{n} \log \|\Phi(n, \omega)x\| - \lambda_i(\omega) \right| < \epsilon$$

which means for a given positive $\lambda_i(\omega)$ as $n \rightarrow \infty$, $\Phi(n, \omega)$ is increasing exponentially fast. For a given negative $\lambda_i(\omega)$ as $n \rightarrow -\infty$, $\Phi(n, \omega)$ is increasing exponentially fast. This is not a concise view for further development since Φ under this norm may go to infinity. In order to control the non-uniformity in $\Phi(t, \omega)$ for the construction of invariant manifolds, we need change the standard Euclidean norm in \mathbb{R}^d to a new one which does not change the Lyapunov spectrum. We consider this change by applying the above estimation.

Definition 2.4.1 *In the Euclidean space with Oseledets splitting*

$$\mathbb{R}^d = \bigoplus_{i=1}^p E_i(\omega),$$

Φ is a linear cocycle which satisfies the MET with two-sided time T , Ω_0 is the invariant set in the MET. For a fixed constant $\kappa > 0$ and all $\omega \in \Omega_0$, we define the random scalar product in \mathbb{R}^d by

$$\langle x, y \rangle_{\kappa, \omega} := \sum_{i=1}^p \langle x_i, y_i \rangle_{\kappa, \omega}$$

where $x, y \in \mathbb{R}^d$, $x_i, y_i \in E_i(\omega)$ and when $T = \mathbb{R}$,

$$\langle x_i, y_i \rangle_{\kappa, \omega} := \int_{-\infty}^{\infty} \frac{\langle \Phi(t, \omega)x_i, \Phi(t, \omega)y_i \rangle}{e^{2(\lambda_i t + \kappa|t|)}} dt,$$

when $T = \mathbb{Z}$,

$$\langle x_i, y_i \rangle_{\kappa, \omega} = \sum_{n \in \mathbb{Z}} \frac{\langle \Phi(n, \omega)x_i, \Phi(n, \omega)y_i \rangle}{e^{2(\lambda_i n + \kappa|n|)}}.$$

Then

$$\|x\|_{\kappa, \omega} = \langle x, x \rangle_{\kappa, \omega}^{\frac{1}{2}} = \left(\sum_{i=1}^p \|x_i\|_{\kappa, \omega}^2 \right)^{\frac{1}{2}},$$

and when $T = \mathbb{R}$,

$$\|x_i\|_{\kappa,\omega}^2 = \int_{-\infty}^{\infty} \frac{\|\Phi(t,\omega)x_i\|^2}{e^{2(\lambda_i t + \kappa|t|)}} dt,$$

when $T = \mathbb{Z}$,

$$\|x_i\|_{\kappa,\omega}^2 = \sum_{n \in \mathbb{Z}} \frac{\|\Phi(n,\omega)x_i\|^2}{e^{2(\lambda_i n + \kappa|n|)}}.$$

This defines the random norm corresponding to $\langle \cdot, \cdot \rangle_{\kappa,\omega}$.

Remark 2.4.2

1. The constant κ in the definition is chosen in an arbitrary manner, then fixed. The subscript ω of the norm describes a situation of the RDS. For example, the subscript $\theta\omega$ under this norm $\|\Phi(1,\omega)x\|_{\kappa,\theta\omega}$ means that we move the point from x to $\Phi(1,\omega)x$, in the same time, ω changes to $\theta\omega$.
2. To see the control of non-uniformity, we can easily prove under this norm for $x \in E_i(\omega)$

$$e^{\lambda_i t - \kappa|t|} \|x\|_{\kappa,\omega} \leq \|\Phi(t,\omega)x\|_{\kappa,\theta(t)\omega} \leq e^{\lambda_i t + \kappa|t|} \|x\|_{\kappa,\omega}.$$

Next, by using the above well defined random norm, we construct some Banach spaces, which allow for the exponential growth rate of their elements, following Wanner [38].

Definition 2.4.3 Under the space $(\mathbb{R}^d, \|\cdot\|_{\kappa,\omega})$, for $\alpha, \beta > 0$, $\omega \in \Omega$ and $T^\pm = T \cap \mathbb{R}^\pm$, set

$$X_{\beta^+,\omega} := \{h : h \text{ is measurable from } T^+ \rightarrow \mathbb{R}^d \text{ and} \\ \|h\|_{\beta^+,\omega} := \sup_{t \geq 0} \beta^{-t} \|h(t)\|_{\kappa,\theta(t)\omega} < \infty\},$$

$$X_{\alpha^-,\omega} := \{h : h \text{ is measurable from } T^- \rightarrow \mathbb{R}^d \text{ and} \\ \|h\|_{\alpha^-,\omega} := \sup_{t \leq 0} \alpha^{-t} \|h(t)\|_{\kappa,\theta(t)\omega} < \infty\},$$

$$X_{\alpha^-,\beta^+,\omega} := \{h : h \text{ is measurable from } T \rightarrow \mathbb{R}^d \text{ and} \\ \|h\|_{\alpha^-,\beta^+,\omega} := \sup(\|h(t)\|_{\alpha^-,\omega}, \|h(t)\|_{\beta^+,\omega}) < \infty\},$$

and

$$X_{\alpha,\omega} := X_{\alpha^-,\alpha^+,\omega}.$$

Remark 2.4.4

1. In the space $X_{\beta^+, \omega}$, the norm $\|h\|_{\beta^+, \omega}$ describes the function h grows at most like β^t forward in time. Similarly, in space $X_{\alpha^-, \omega}$, the norm $\|h\|_{\alpha^-, \omega}$ describes the function h grows at most like α^t backward in time. These spaces provide us a kind of functions with a growth speed.
2. The space $X_{\alpha, \omega}$ is also nonempty. Obviously, the zero function is in it. To see the other non-zero elements, we can consider the following method. We firstly pick up the non-zero elements from the space $h \in X_{\alpha^+, \omega}$ and $g \in X_{\alpha^-, \omega}$, then we can form the new function as the following

$$f(t) = \begin{cases} h(t) & t \geq 0, \\ g(t) & t \leq 0, \end{cases}$$

It is trivial to check this function f is in $X_{\alpha, \omega}$. In the later research, we are going to seek a stationary point in this space.

3. It is easy to see the following facts: if $\bar{\alpha} \leq \alpha$,

$$\|\cdot\|_{\bar{\alpha}^-, \omega} \leq \|\cdot\|_{\alpha^-, \omega};$$

and if $\beta \leq \bar{\beta}$,

$$\|\cdot\|_{\bar{\beta}^+, \omega} \leq \|\cdot\|_{\beta^+, \omega}.$$

As we mentioned, the new Banach space will allow for the exponential growth of their elements, we can see this by setting

$$\alpha_i = e^{\lambda_i - \kappa},$$

$$\beta_i = e^{\lambda_i + \kappa},$$

for $i = 1, \dots, p$. Here κ is a positive constant, which is chosen to be sufficiently small such that the intervals $[\lambda_i - \kappa, \lambda_i + \kappa]$, $i = 1, 2, \dots, p$, do not overlap. Assume that the linear cocycle $\Phi = \text{diag}(\Phi_1, \dots, \Phi_p)$ is block-diagonal and has a spectrum, then applying Remark 2.4.2 (2) for the block Φ_i of the linear cocycle Φ ,

$$\|\Phi_i(t, \omega)\|_{\kappa, \theta(t)\omega} \leq \beta_i^t, \quad t \geq 0,$$

$$\|\Phi_i(t, \omega)\|_{\kappa, \theta(t)\omega} \leq \alpha_i^t, \quad t \leq 0.$$

We next introduce a constant δ which plays an important role in the later parts of this chapter. We choose this constant δ by considering

$$0 < \delta < \min\left(\frac{\alpha_1 - \beta_2}{2}, \dots, \frac{\alpha_{p-1} - \beta_p}{2}, \frac{\alpha_p - 0}{2}\right).$$

By using this constant δ , we define the interval

$$\Gamma_i = [\beta_{i+1} + \delta, \alpha_i - \delta], \quad i = 1, \dots, p$$

the spectral gap between λ_{i+1} and λ_i . To have a visual feeling for this part, we describe these by Figure 2.2.

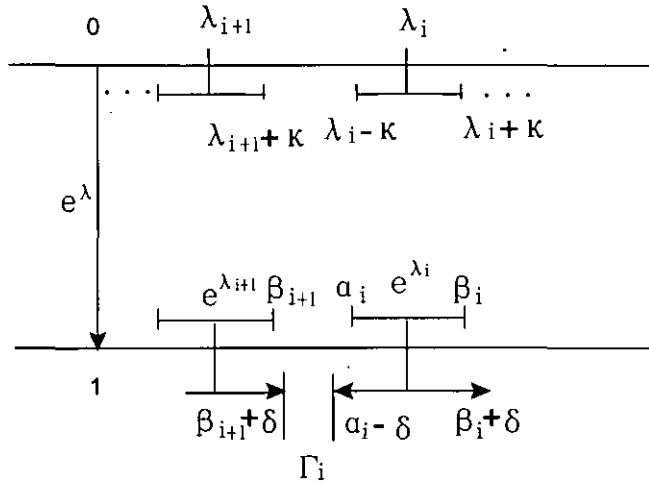


Figure 2.2: Spectral Gap

From this figure, we see, the constant κ and δ need to be chosen small enough so that the intervals do not overlap. Normally, we select a small κ , then fix it. However, δ will need to fit some other conditions.

For a given RDS φ , we are always required to standardize it for deeper research. We call the RDS φ after this procedure be prepared. This means a measurable φ would be expressed by

$$\varphi(t, \omega, x) = \Phi(t, \omega)x + \psi(t, \omega, x).$$

In this form, Φ denote the linear part of RDS with

$$\Phi(t, \omega) := D\varphi(t, \omega, 0)$$

where D means the derivative of φ at point 0 and Φ is assumed to be block diagonal with a spectral theory, ψ denotes the nonlinear part with

$$\psi(t, \omega, x) := \dot{\varphi}(t, \omega, x) - \Phi(t, \omega)x.$$

In this chapter, we admit there is a one-to-one correspondence between the random dynamical system φ and a random differential equation. This means for a given random differential equation

$$\dot{x}_t = f(\theta_t \omega, x_t),$$

there is a unique RDS

$$t \mapsto \varphi(t, \omega)x$$

which solves the random differential equation. This has been discussed in Chapter 2 in Arnold [1] for local cases and global cases. In this part, we would like to consider a discrete time case, a random difference equation corresponding to a random dynamical system φ

$$x_{n+1} = \varphi(\theta^n \omega, x_n) = A(\theta^n \omega)x_n + F(\theta^n \omega, x_n), \quad (2.4)$$

where $n \in \mathbb{Z}$ and

$$F(\omega, 0) = 0.$$

A is a random $d \times d$ invertible matrix with elements from $\Omega \rightarrow \mathbb{R}$, φ is a measurable RDS which is assumed to be prepared. We denote $\varphi(\omega) := \varphi(1, \omega)$. Then the random difference equation (2.4) is equivalent to

$$\begin{aligned} \varphi(n, \omega, x) &= \Phi(n, \omega)x + \psi(n, \omega, x), \\ \psi(n, \omega, 0) &= 0, \end{aligned}$$

where

$$\Phi(n, \omega) := D\varphi(n, \omega, 0)$$

denotes a measurable linear RDS which satisfies the MET and $A(\omega) := \Phi(1, \omega)$. We define ψ the nonlinear part by

$$\psi(n, \omega, x) := \varphi(n, \omega, x) - \Phi(n, \omega)x$$

and $F(\omega, x) := \psi(1, \omega, x)$. From MET, we know there exists a finite spectrum $\lambda_1 > \dots > \lambda_p$ for this linear cocycle Φ . We pick up a j with $1 \leq j \leq p$ and consider the linear part A to be block-diagonal with $A = \text{diag}(A_1, \dots, A_p)$ written as

$$A = \begin{pmatrix} A^+ & 0 \\ 0 & A^- \end{pmatrix},$$

where $A^+ = \text{diag}(A_1, \dots, A_j)$ and $A^- = \text{diag}(A_{j+1}, \dots, A_p)$, the blocks $A_i(n, \omega)$ are linear cocycles of one-point spectrum $\{(\lambda_i, d_i)\}$. The nonlinear part is also considered as two parts

$$F = \begin{pmatrix} F^+ \\ F^- \end{pmatrix},$$

where

$$F^+ = \begin{pmatrix} F_1 \\ \vdots \\ F_j \end{pmatrix}, F^- = \begin{pmatrix} F_{j+1} \\ \vdots \\ F_p \end{pmatrix}.$$

Thus, the random difference equation (2.4), by the above partition due to j , will also be decomposed to two parts respectively:

$$\begin{cases} x_{n+1}^+ = A^+(\theta^n \omega)x_n^+ + F^+(\theta^n \omega, x_n^+, x_n^-), \\ x_{n+1}^- = A^-(\theta^n \omega)x_n^- + F^-(\theta^n \omega, x_n^+, x_n^-), \end{cases} \quad (2.5)$$

We call them unstable equation and stable equation. And the initial condition also changes to

$$\begin{cases} F^+(\omega, 0, 0) = 0 \\ F^-(\omega, 0, 0) = 0. \end{cases}$$

Since we choose a j and fix it, we will also assume

$$\begin{aligned} \alpha &:= \alpha_j = e^{\lambda_j - \kappa} \\ \beta &:= \beta_{j+1} = e^{\lambda_{j+1} + \kappa}. \end{aligned}$$

This leads

$$\begin{aligned} \|\Phi^+(n, \omega)\|_{\kappa, \theta^n \omega} &\leq \alpha^n, \quad n \leq 0 \\ \|\Phi^-(n, \omega)\|_{\kappa, \theta^n \omega} &\leq \beta^n, \quad n \geq 0. \end{aligned}$$

Take $\delta < \frac{\alpha - \beta}{2}$. We say F satisfies the Lipschitz condition if

$$\|F^\pm(\omega, x) - F^\pm(\omega, y)\|_{\kappa, \theta \omega} \leq L \|x - y\|_{\kappa, \omega}.$$

§2.5 Invariant Manifold and Coupling Method

In this section, we will introduce the invariant manifold theorem which is generated by the random difference equation (2.4)

$$x_{n+1} = A(\theta^n \omega)x_n + F(\theta^n \omega, x_n).$$

The coupling method is the main technical tool to prove this theorem. We consider the RDS φ with two-sided time on a d -dimensional Riemannian manifold $M = \mathbb{R}^d$, being differentiable from point to point. Define $T_x M$ to be the tangent space of M at the point x and

$$T_x M = \bigoplus_{i=1}^p E_i(\omega, x)$$

for $\omega \in \Omega$. From the MET, we already have had an invariant property for a linear RDS Φ generated by a linear random difference equation

$$x_{n+1} = A(\theta^n \omega) x_n$$

such that

$$A(\omega) E_i(\omega) = E_i(\theta \omega)$$

for all $i = 1, \dots, p(\omega)$. For simplicity, in this section, we denote by Ω the invariant set generated from MET with a full measure such that $\mathbb{P}(\Omega) = 1$. Our purpose here is to bend this invariant property from each Oseledets splitting subspace E_i to the submanifold M_i . Define

$$\Lambda := \{\lambda_1 > \dots > \lambda_p\}$$

where $\lambda_1, \dots, \lambda_p$ are the corresponding Lyapunov exponents of the linear RDS Φ . Choose any j with $1 \leq j \leq p$, then we define

$$\Lambda^+ := \{\lambda_1 > \dots > \lambda_j\},$$

$$\Lambda^- := \{\lambda_{j+1} > \dots > \lambda_p\}.$$

In our system with $M = \mathbb{R}^d$, we call the set

$$M^+(\omega) = M_{1j}(\omega) = \{x \in \mathbb{R}^d, \varphi(\cdot, \omega, x) \in X_{a^-, \omega}\}$$

unstable manifold corresponding to Λ^+ and the tangent space is

$$TM_{1j}(\omega) = \bigoplus_{k:1 \leq k \leq j} E_k(\omega).$$

Similarly, we call the set

$$M^-(\omega) = M_{jp}(\omega) = \{x \in \mathbb{R}^d, \varphi(\cdot, \omega, x) \in X_{b^-, \omega}\}$$

stable manifold corresponding to Λ^- and the tangent space is

$$TM_{jp}(\omega) = \bigoplus_{k:j+1 \leq k \leq p} E_k(\omega).$$

Here the constants a and b are from the spectral gap according to λ_j by taking from the different intervals

$$a \in [\beta_{j+1} + \delta, \alpha_j - \delta] =: \Gamma_{left}$$

$$b \in [\beta_j + \delta, \alpha_{j-1} - \delta] =: \Gamma_{right}$$

where α, β, δ and Γ are defined in the section of preparations. Then, for the convenience of the future discussion, we put

$$E^+ := E_{1j} = \bigoplus_{k:1 \leq k \leq j} E_k,$$

$$E^- := E_{jp} = \bigoplus_{k:j+1 \leq k \leq p} E_k,$$

and

$$\mathbb{R}^d = E^+ \oplus E^-.$$

We are now going to give the Global Invariant Manifold Theorem. For the deterministic case, Pesin [28] and [29] started this pioneering work. We can find the random case proof in Ruelle [32] and [33]. We will only emphasize on how the coupling method works on it. This theorem will be presented by two parts, unstable manifold and stable manifold, respectively.

Theorem 2.5.1 (Global Invariant Manifold Theorem)

For a two-sided discrete time case $T = \mathbb{Z}$, the RDS φ is prepared which is generated by the random difference equation (2.4), where the function F satisfies the Lipschitz condition with

$$\| F(\omega, x) - F(\omega, y) \|_{\kappa, \theta\omega} \leq L \| x - y \|_{\kappa, \omega}$$

for all $x, y \in \mathbb{R}^d$, and the Lipschitz constant L -satisfies

$$0 \leq L < \frac{\delta}{2}.$$

Then, the unstable manifold $M^+(\omega)$ according to Λ^+ can be expressed by a graph in $\mathbb{R}^d = E^+ \oplus E^-$

$$M^+(\omega) = \{x^+ \oplus m^+(\omega, x^+) : x^+ \in E^+\}$$

where $m^+(\omega, x^+)$ is uniquely determined by the given initial value x^+ , and $M^+(\omega)$ is φ -invariant such that

$$\varphi(n, \omega)M^+(\omega) = M^+(\theta^n \omega)$$

for all $n \in \mathbb{Z}$.

The stable manifold $M^-(\omega)$ according to Λ^- can be expressed by a graph in $\mathbb{R}^d = E^+ \oplus E^-$

$$M^-(\omega) = \{m^-(\omega, x^-) \oplus x^- : x^- \in E^-\}$$

where $m^-(\omega, x^-)$ is uniquely determined by the given initial value x^- , and $M^-(\omega)$ is φ -invariant such that

$$\varphi(n, \omega)M^-(\omega) = M^-(\theta^n \omega)$$

for all $n \in \mathbb{Z}$. We also have $M^+(\omega)$ and $M^-(\omega)$ do not intersect except at zero, such that

$$M^+(\omega) \cap M^-(\omega) = \{0\}.$$

Remark 2.5.2

1. The key functions

$$m^+(\omega, x^+) \in E^-$$

$$m^-(\omega, x^-) \in E^+$$

mentioned in the theorem are both measurable.

2. For the continuous time case $T = \mathbb{R}$, all the results still hold such that for given initial values x^+ and x^- , there exist uniquely determined graphs M^+ and M^- which satisfy

$$\varphi(t, \omega)M^+(\omega) = M^+(\theta^t \omega)$$

$$\varphi(t, \omega)M^-(\omega) = M^-(\theta^t \omega)$$

for all $t \in \mathbb{R}$.

3. In the higher regularity case, we consider the C^k ($k \geq 1$) RDS φ which means φ is k times differentiable with respect to x and the derivatives are continuous with respect to (t, x) . In this situation, the theorem still holds with M^+ and M^- are C^k manifold. However, this requires to fit the additional Gap conditions. We will discuss this in a separate section.

4. The Lipschitz constant L , satisfies the following restriction

$$0 \leq L < \frac{\delta}{2}.$$

This L may be taken small since this δ is required to be small enough. But this is necessary for the proof of the contraction. In fact, under the Lipschitz condition, this random norm $\|\cdot\|_{\kappa,\omega}$ is equivalent to the Euclidean norm. We can deduce this by the definition of the norm $\|\cdot\|_{\kappa,\omega}$.

5. To see the last assertion, we present Figure 2.3.

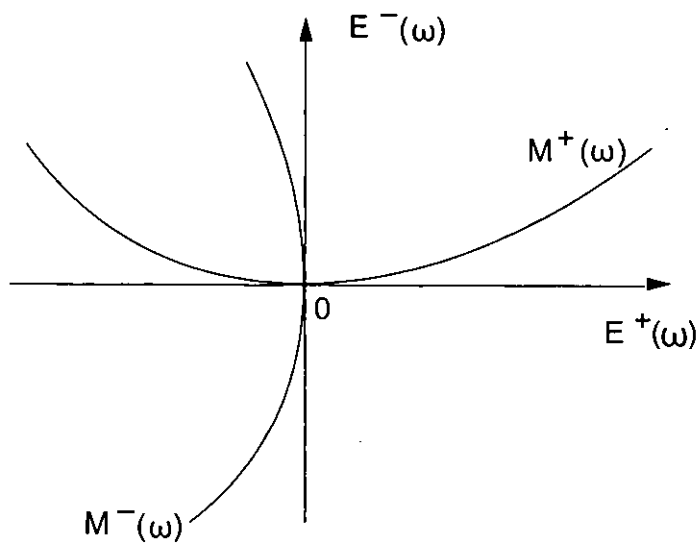


Figure 2.3: Zero is the stationary point of this system

To understand this theorem, we study from the definition of unstable manifold, we recall a unstable manifold M^+ is

$$M^+(\omega) = \{x \in \mathbb{R}^d, \varphi(\cdot, \omega, x) \in X_{a^-, \omega}\}$$

according to the spectrum interval $\Lambda^+ = \{\lambda_1 > \dots > \lambda_j\}$. By the definition of the space $X_{a^-, \omega}$, this is to say, there exists a random variable $V(\omega) \geq 0$ such that for $t \geq 0$

$$\|\varphi(t, \omega, x)\|_{\kappa, \theta(t)\omega} \leq V(\omega)a^{-t}.$$

The purpose of invariant manifold theorem is to find such a function $x \mapsto \varphi(t, \omega, x)$ to fit this inequality. We now decompose this RDS φ by two parts according to j which is taken from $1 \leq j \leq p$, then we have equation (2.5). We call the first equation of (2.5) the unstable equation and the second equation of (2.5) the stable equation. And the

finite-dimensional Euclidean space is also decomposed to two parts

$$\mathbb{R}^d = E^+ \oplus E^-.$$

To deal with these two equations, the coupling method plays an important role. The coupling method generally works for a group of two different dual equations. In our unstable manifold case, we pick up an initial value point $x^+ \in E^+$ and an arbitrary $\xi^-(\omega) \in X_{a^-, \omega}(E^-)$. Put $(x^+, \xi^-(\omega))$ into the unstable equation. For the initial value problem, the unstable equation has exactly one solution

$$\xi^+(\omega, x^+) \in X_{a^-, \omega}(E^+).$$

After inserting the couple $(\xi^+(\omega, x^+), \xi^-(\omega))$ into the stable equation, we obtain another unique solution by the iterations

$$\eta^-(\omega, x^+) \in X_{a^-, \omega}(E^-).$$

The final technique is to prove the mapping

$$\begin{aligned} X_{a^-, \omega}(E^-) &\rightarrow X_{a^-, \omega}(E^-) \\ \xi^-(\omega) &\mapsto \eta^-(\omega, x^+) \end{aligned}$$

is contracting. This will lead to a fixed point, denote it as $m^+(\omega, x^+) \in X_{a^-, \omega}(E^-)$. Then the graph

$$\{x^+ \oplus m^+(\omega, x^+) : x^+ \in E^+\}$$

gives the unstable manifold. Similar technique can apply to the stable manifold case. This coupling method is extremely useful. It is also widely used to solve many infinite-dimensional problems as well, see Bricmont, Kupiainen and Lefevere [4], Li and Lu [20] and references therein.

This theorem is based on an assumption that the RDS φ has a fixed point $x = 0$ i.e. the random dynamical system is prepared. Of course, if one knows there is a stationary solution (random fixed point) for the random dynamical system, one can always change the random dynamical system to a prepared one. The point here we mentioned is how to find the stationary solution. Without knowing the stationary solution, one cannot define the prepared random dynamical system. To improve this, we are going to consider the non-linear part function of the r.d.e (2.4) with

$$F(\omega, 0) = c(1, \omega)$$

where c is a random variable. In this situation, zero is obviously not the stationary point of the system. The theorem does not work any more. Our problem is to find such a stationary solution to fit this condition. The coupling method provides us a possibility to solve the intriguing problem.

§2.6 Two Key Lemmas

In the end of last section, we mentioned the problem we were going to devote. In this section, we present two key technical lemmas to help us reach our target. We borrow them from Arnold [1] and pick up some parts of the conclusions for the future use. We do a little form change so that we can understand them better. We will not give the proof for them.

Lemma 2.6.1 *Consider the random difference equation for a two-sided discrete time case $T = \mathbb{Z}$*

$$x_{n+1} = A(\theta^n \omega)x_n + f(n, \omega, x_n) + f_0(n+1, \omega), \quad (2.6)$$

where A is a $d \times d$ invertible random matrix and is measurable, f and f_0 are measurable functions. Assume that there exist constants $\beta > 0$ and $L \geq 0$ such that for each fixed ω , we have the following conditions

$$\begin{aligned} \|A(\omega)\|_{\kappa, \theta \omega} &\leq \beta, \\ f(n, \omega, 0) &= 0, \\ \|f(n, \omega, x) - f(n, \omega, y)\|_{\kappa, \theta^{n+1} \omega} &\leq L\|x - y\|_{\kappa, \omega}. \end{aligned}$$

Now let $\gamma > \beta + L$, suppose $f_0(\cdot, \omega) \in X_{\gamma, \omega}$, then there exists exactly one solution of (2.6) which $\xi(\cdot, \omega) \in X_{\gamma, \omega}$ and $\xi(\cdot, \omega)$ satisfies

$$\xi(n, \omega) = \sum_{i=-\infty}^n \Phi(n-i, \theta^i \omega)(f(i-1, \omega, \xi(i-1, \omega)) + f_0(i, \omega)), \quad (2.7)$$

and

$$\|\xi(\cdot, \omega)\|_{\gamma, \omega} \leq \frac{\gamma}{\gamma - (\beta + L)} \|f_0(\cdot, \omega)\|_{\gamma, \omega}, \quad (2.8)$$

where Φ is the linear cocycle generated by A such that

$$\Phi(n, \omega) = A(\theta^{n-1} \omega) \circ \cdots \circ A(\omega)$$

for $n \geq 1$ and \circ means composition.

Lemma 2.6.2 Consider the random difference equation (2.6). Suppose that there exist constants $\alpha > 0$ and $L \geq 0$ such that for each fixed ω ,

$$\begin{aligned} \|A(\omega)^{-1}\|_{\kappa, \theta\omega} &\leq \alpha^{-1}, \\ f(n, \omega, 0) &= 0, \\ \|f(n, \omega, x) - f(n, \omega, y)\|_{\kappa, \theta^{n+1}\omega} &\leq L\|x - y\|_{\kappa, \omega}. \end{aligned}$$

Let $0 < \gamma < \alpha - L$, assume $f_0(\cdot, \omega) \in X_{\gamma, \omega}$, then there exists exactly one solution of (2.6) which $\xi(\cdot, \omega) \in X_{\gamma, \omega}$ and $\xi(\cdot, \omega)$ satisfies

$$\xi(n, \omega) = - \sum_{i=n+1}^{\infty} \Phi(n-i, \theta^i \omega) (f(i-1, \omega, \xi(i-1, \omega)) + f_0(i, \omega)), \quad (2.9)$$

and

$$\|\xi(\cdot, \omega)\|_{\gamma, \omega} \leq \frac{\gamma}{\alpha - (L + \gamma)} \|f_0(\cdot, \omega)\|_{\gamma, \omega}, \quad (2.10)$$

where Φ is the linear cocycle generated by A^{-1} such that

$$\Phi(n, \omega) = A(\theta^n \omega)^{-1} \circ \dots \circ A(\theta^{-1} \omega)^{-1}$$

for $n \leq -1$.

Remark 2.6.3

1. For both lemmas, the existence of the solutions does not depend on the initial values. This is an extremely strongly result. This is made possible by working on the space $X_{\gamma, \omega}$.
2. We will apply these two lemmas in the coupling method to find a stationary solution. This requires the two conditions $\gamma > \beta + L$ and $0 < \gamma < \alpha - L$ are both satisfied. We will in the next section introduce a particularly choosing constant L to fit two conditions automatically.
3. The random difference equation (2.6) in two lemmas acts in a different manner. In Lemma 2.6.1, since $\|A(\omega)\|_{\kappa, \theta\omega} \leq \beta$, the equation works forward in time. In Lemma 2.6.2, since $\|A(\omega)^{-1}\|_{\kappa, \theta\omega} \leq \alpha^{-1}$, the equation follows the backward in time order.

§2.7 Main Results

In this section, we introduce the main result of this chapter about the stationary solution for a random difference equation in the finite-dimensional Euclidean space \mathbb{R}^d . We firstly recall the random difference equation (2.4) which is mentioned in the preparation section

$$x_{n+1} = \varphi(\theta^n \omega, x_n) = A(\theta^n \omega)x_n + F(\theta^n \omega, x_n),$$

where $n \in \mathbb{Z}$, φ is the RDS generated by this equation. In this section, we change the condition for F as

$$F(\omega, 0) = c(1, \omega)$$

where c is a random variable in \mathbb{R}^d . We then denote

$$\Phi(n, \omega) = \begin{cases} A(\theta^{n-1}\omega) \cdots A(\omega), & n > 0; \\ I, & n = 0; \\ A^{-1}(\theta^n \omega) \cdots A^{-1}(\theta^{-1}\omega), & n < 0, \end{cases}$$

which is a linear measurable RDS satisfying the MET. Comparing the definition of the prepared RDS, in our case, we notice that zero is not a stationary point. Other conditions in the prepared RDS are still needed. We hence also have the equivalent equation

$$\begin{aligned} \varphi(n, \omega, x) &= \Phi(n, \omega)x + \psi(n, \omega, x), \\ \psi(n, \omega, 0) &= c(n, \omega), \end{aligned}$$

where the nonlinear part is defined by

$$\psi(n, \omega, x) := \varphi(n, \omega, x) - \Phi(n, \omega)x.$$

According to the spectrum $\{\lambda_1 > \cdots > \lambda_p\}$, we pick up a j with $1 \leq j \leq p$. Then the space is splitting into two parts

$$\mathbb{R}^d = E^+(\omega) \oplus E^-(\omega)$$

which is corresponding to the spectrum interval

$$\Lambda^+ = \{\lambda_1 > \lambda_2 > \cdots > \lambda_j\}$$

and

$$\Lambda^- = \{\lambda_{j+1} > \cdots > \lambda_p\}.$$

It is easy to see that the random difference equation (2.4) can be changed to the coupling unstable and stable equations (2.5)

$$\begin{cases} x_{n+1}^+ = A^+(\theta^n \omega)x_n^+ + F^+(\theta^n \omega, x_n^+, x_n^-), \\ x_{n+1}^- = A^-(\theta^n \omega)x_n^- + F^-(\theta^n \omega, x_n^+, x_n^-), \end{cases}$$

and the initial condition is

$$\begin{cases} F^+(\omega, 0, 0) = c^+(1, \omega) \\ F^-(\omega, 0, 0) = c^-(1, \omega) \end{cases}$$

where $c^+ \in E^+$ and $c^- \in E^-$ are random variables. One can do some linear transformations if necessary to device the coupling equations. We are now ready to state the theorem.

Theorem 2.7.1

Consider the random difference equation (2.4) for a two-sided discrete time case $T = \mathbb{Z}$, the RDS φ satisfies the above conditions in this section. Assume the function F satisfies the Lipschitz condition

$$\|F(\omega, x) - F(\omega, y)\|_{\kappa, \theta\omega} \leq L\|x - y\|_{\kappa, \omega},$$

for all $x, y \in \mathbb{R}^d$. The constant L satisfies

$$0 \leq L < \frac{\delta}{2}.$$

Choose any j with $1 \leq j \leq p$, and also choose one constant a in the spectral gap Γ defined by

$$a := a_j \in \Gamma := [\beta + \delta, \alpha - \delta],$$

where

$$\begin{aligned} \alpha &:= \alpha_j = e^{\lambda_j - \kappa}, \\ \beta &:= \beta_{j+1} = e^{\lambda_{j+1} + \kappa}, \end{aligned}$$

and

$$\delta : 0 < \delta < \min\left(\frac{\alpha_1 - \beta_2}{2}, \dots, \frac{\alpha_{p-1} - \beta_p}{2}, \frac{\alpha_p - 0}{2}\right).$$

Then we have the expression (2.4) replaced by (2.5). Assume the random variables $c(\cdot, \omega) \in X_{a, \omega}$, then there exists exactly a pair of solutions

$$\xi_*^+(\cdot, \omega) \in X_{a, \omega}(E^+),$$

and

$$\xi_*^-(\cdot, \omega) \in X_{a,\omega}(E^-)$$

of a pair of unstable and stable equations (2.5) without any initial conditions and satisfying for all $n \in \mathbb{Z}$

$$\xi_*^+(n, \omega) = - \sum_{i=n+1}^{\infty} \Phi^+(n-i, \theta^i \omega)(F^+(i-1, \omega, \xi_*^+(i-1, \omega), \xi_*^-(i-1, \omega)))$$

and

$$\xi_*^-(n, \omega) = \sum_{i=-\infty}^n \Phi^-(n-i, \theta^i \omega)(F^-(i-1, \omega, \xi_*^+(i-1, \omega), \xi_*^-(i-1, \omega))).$$

Moreover, assume

$$Y(\omega) := \{(\xi_*^+(0, \omega), \xi_*^-(0, \omega))\}$$

in \mathbb{R}^d , then $Y(\omega)$ is the stationary point, such that

$$\varphi(n, \omega, Y(\omega)) = Y(\theta^n \omega)$$

for all $n \in \mathbb{Z}$.

Proof. The main idea of the proof is to apply the coupling method. We divide the proof into four steps.

Step 1. Stable Equation

Given an arbitrary $\xi^+(\cdot, \omega) \in X_{a,\omega}(E^+)$, we consider the stable equation

$$x_{n+1}^- = A^-(\theta^n \omega)x_n^- + F^-(\theta^n \omega, \xi^+(n, \omega), x_n^-).$$

In order to apply Lemma 2.6.1, we set

$$f(n, \omega, x^-) := F^-(\theta^n \omega, \xi^+(n, \omega), x^-) - F^-(\theta^n \omega, \xi^+(n, \omega), 0)$$

and

$$\begin{aligned} f_0(n+1, \omega) &:= F^-(\theta^n \omega, \xi^+(n, \omega), 0) \\ &= F^-(\theta^n \omega, \xi^+(n, \omega), 0) - F^-(\theta^n \omega, 0, 0) + c^-(n+1, \omega). \end{aligned} \quad (2.11)$$

Then, this stable equation is written as

$$x_{n+1}^- = A^-(\theta^n \omega)x_n^- + f(n, \omega, x_n^-) + f_0(n+1, \omega).$$

To check the conditions of Lemma 2.6.1, we have

$$\|A^-(\omega)\|_{\kappa, \theta\omega} \leq \beta$$

and

$$f(n, \omega, 0) = 0.$$

To check the Lipschitz condition of f , we have

$$\begin{aligned} & \|f(n, \omega, x^-) - f(n, \omega, y^-)\|_{\kappa, \theta^{n+1}\omega} \\ &= \|F^-(\theta^n\omega, \xi^+(n, \omega), x^-) - F^-(\theta^n\omega, \xi^+(n, \omega), y^-)\|_{\kappa, \theta^{n+1}\omega} \\ &\leq L\|x^- - y^-\|_{\kappa, \omega}. \end{aligned}$$

To check $f_0(\cdot, \omega) \in X_{a, \omega}(E^-)$, multiplying $a^{-(n+1)}$ on both sides of (2.11), then

$$\begin{aligned} a^{-(n+1)}\|f_0(n+1, \omega)\|_{\kappa, \theta^{n+1}\omega} &\leq a^{-(n+1)}[\|F^-(\theta^n\omega, \xi^+(n, \omega), 0) \\ &\quad - F^-(\theta^n\omega, 0, 0)\|_{\kappa, \theta^{n+1}\omega} + \|c^-(n+1, \omega)\|_{\kappa, \theta^{n+1}\omega}] \\ &\leq \frac{L}{a}a^{-n}(\|\xi^+(n, \omega)\|_{\kappa, \theta^n\omega} + \|c^-(n+1, \omega)\|_{\kappa, \theta^{n+1}\omega}). \end{aligned}$$

Then, by the definition of $\|\cdot\|_{a, \omega}$

$$\|f_0(\cdot, \omega)\|_{a, \omega} \leq \frac{L}{a}\|\xi^+(\cdot, \omega)\|_{a, \omega} + \|c^-(\cdot, \omega)\|_{a, \omega} < \infty, \quad (2.12)$$

which means $f_0(\cdot, \omega) \in X_{a, \omega}(E^-)$. Hence, by Lemma 2.6.1, for this stable equation, there exists exactly one solution which has the property $\xi^-(\cdot, \omega) \in X_{a, \omega}(E^-)$ and by (2.7) and (2.8) we have

$$\begin{aligned} \xi^-(n, \omega) &= \sum_{i=-\infty}^n \Phi^-(n-i, \theta^i\omega)(f(i-1, \omega, \xi^-(i-1, \omega)) + f_0(i, \omega)) \\ &= \sum_{i=-\infty}^n \Phi^-(n-i, \theta^i\omega)(F^-(i-1, \omega, \xi^+(i-1, \omega), \xi^-(i-1, \omega))), \end{aligned}$$

and

$$\|\xi^-(\cdot, \omega)\|_{a, \omega} \leq \frac{a}{a - (\beta + L)}\|f_0(\cdot, \omega)\|_{a, \omega}. \quad (2.13)$$

Substituting (2.12) into (2.13), we have

$$\|\xi^-(\cdot, \omega)\|_{a, \omega} \leq \frac{L}{a - (\beta + L)}\|\xi^+(\cdot, \omega)\|_{a, \omega} + \frac{a}{a - (\beta + L)}\|c^-(\cdot, \omega)\|_{a, \omega}. \quad (2.14)$$

Step 2. Unstable Equation

For an arbitrary $\xi^+(\cdot, \omega) \in X_{a,\omega}(E^+)$, we have a unique solution $\xi^-(\cdot, \omega) \in X_{a,\omega}(E^-)$ to fit the stable equation. Putting “solutions” (ξ^+, ξ^-) into the unstable equation, we now consider

$$x_{n+1}^+ = A^+(\theta^n \omega)x_n^+ + F^+(\theta^n \omega, \xi^+(n, \omega), \xi^-(n, \omega)).$$

In order to apply Lemma 2.6.2, we set

$$f(n, \omega, x^+) := 0$$

and

$$\begin{aligned} f_0(n+1, \omega) &:= F^+(\theta^n \omega, \xi^+(n, \omega), \xi^-(n, \omega)) \\ &= F^+(\theta^n \omega, \xi^+(n, \omega), \xi^-(n, \omega)) - F^+(\theta^n \omega, 0, 0) + c^+(n+1, \omega). \end{aligned}$$

Then, this unstable equation is written as the required form of Lemma 2.6.2 as follows:

$$x_{n+1}^+ = A^+(\theta^n \omega)x_n^+ + f(n, \omega, x_n^+) + f_0(n+1, \omega).$$

To check the conditions of Lemma 2.6.2, we have

$$\|A^+(\theta^{-1}\omega)\|_{\kappa, \theta\omega} \leq \alpha^{-1}$$

and

$$f(n, \omega, 0) = 0.$$

To check $f_0(\cdot, \omega) \in X_{a,\omega}(E^+)$, multiplying $a^{-(n+1)}$ both sides, then

$$\begin{aligned} a^{-(n+1)}\|f_0(n+1, \omega)\|_{\kappa, \theta^{n+1}\omega} &\leq a^{-(n+1)}(\|F^+(\theta^n \omega, \xi^+(n, \omega), \xi^-(n, \omega)) \\ &\quad - F^+(\theta^n \omega, 0, 0)\|_{\kappa, \theta^{n+1}\omega} + \|c^+(n+1, \omega)\|_{\kappa, \theta^{n+1}\omega}) \\ &\leq \frac{L}{a}a^{-n}(\|\xi^+(n, \omega)\|_{\kappa, \theta^n \omega} + \|\xi^-(n, \omega)\|_{\kappa, \theta^n \omega}) \\ &\quad + a^{-(n+1)}\|c^+(n+1, \omega)\|_{\kappa, \theta^{n+1}\omega}. \end{aligned}$$

Then, by the definition of $\|\cdot\|_{a,\omega}$

$$\|f_0(\cdot, \omega)\|_{a,\omega} \leq \frac{L}{a}[\|\xi^+(\cdot, \omega)\|_{a,\omega} + \|\xi^-(\cdot, \omega)\|_{a,\omega}] + \|c^+(\cdot, \omega)\|_{a,\omega} < \infty. \quad (2.15)$$

It follows that $f_0(\cdot, \omega) \in X_{a,\omega}(E^+)$. Hence, by Lemma 2.6.2, there is a unique solution $\eta^+(\cdot, \omega) \in X_{a,\omega}(E^+)$ to the unstable equation. By (2.9), (2.10) and $f(n, \omega, 0) = 0$, then

$$\eta^+(n, \omega) = - \sum_{i=n+1}^{\infty} \Phi^+(n-i, \theta^i \omega)(f(i-1, \omega, \eta^+(i-1, \omega)) + f_0(i, \omega))$$

$$= - \sum_{i=n+1}^{\infty} \Phi^+(n-i, \theta^i \omega)(F^+(i-1, \omega, \xi^+(i-1, \omega), \xi^-(i-1, \omega))),$$

and

$$\|\eta^+(\cdot, \omega)\|_{a, \omega} \leq \frac{a}{\alpha - a} \|f_0(\cdot, \omega)\|_{a, \omega}. \quad (2.16)$$

Substituting (2.15) into (2.16)

$$\begin{aligned} \|\eta^+(\cdot, \omega)\|_{a, \omega} &\leq \frac{L}{\alpha - a} (\|\xi^+(\cdot, \omega)\|_{a, \omega} + \|\xi^-(\cdot, \omega)\|_{a, \omega}) \\ &\quad + \frac{a}{\alpha - a} \|c(\cdot, \omega)\|_{a, \omega}. \end{aligned} \quad (2.17)$$

Now, we can replace the term $\|\xi^-(\cdot, \omega)\|_{a, \omega}$ by (2.14) in (2.17) to lead to the following inequality

$$\begin{aligned} \|\eta^+(\cdot, \omega)\|_{a, \omega} &\leq \frac{L(a - \beta)}{(\alpha - a)(a - \beta - L)} \|\xi^+(\cdot, \omega)\|_{a, \omega} \\ &\quad + \frac{a(a - \beta)}{(\alpha - a)(a - \beta - L)} \|c(\cdot, \omega)\|_{a, \omega}. \end{aligned} \quad (2.18)$$

Step 3. Contraction

From step 2, one can define an operator for each fixed $\omega \in \Omega$

$$\begin{aligned} T_\omega : X_{a, \omega}(E^+) &\rightarrow X_{a, \omega}(E^+), \\ T_\omega \xi^+(\cdot, \omega) &= \eta^+(\cdot, \omega). \end{aligned}$$

To see this operator is contracting, we consider ξ_1^+ and ξ_2^+ in $X_{a, \omega}(E^+)$. Then from the stable equation, we can determine the unique solutions ξ_1^- and ξ_2^- in $X_{a, \omega}(E^-)$ respectively, by the arguments in step 1. Let $\zeta := \xi_2^- - \xi_1^-$. Then this ζ satisfies a stable equation

$$y_{n+1}^- = A^-(\theta^n \omega) y_n^- + F^-(\theta^n \omega, \xi_2^+, y_n^- + \xi_1^-) - F^-(\theta^n \omega, \xi_1^+, \xi_1^-).$$

In order to apply Lemma 2.6.1, we set

$$f(n, \omega, y^-) := F^-(\theta^n \omega, \xi_2^+, y^- + \xi_1^-) - F^-(\theta^n \omega, \xi_1^+, \xi_1^-)$$

and

$$f_0(n+1, \omega) := -F^-(\theta^n \omega, \xi_1^+, \xi_1^-) + F^-(\theta^n \omega, \xi_2^+, \xi_1^-).$$

Then, this stable equation is written in the following form

$$y_{n+1}^- = A^-(\theta^n \omega) y_n^- + f(n, \omega, y_n^-) + f_0(n+1, \omega).$$

We now can apply the procedure in Step 1. In this case, we notice the random variables $c(n, \omega) = 0$. By using Lemma 2.6.1 again,

$$\|\zeta\|_{a,\omega} = \|\xi_2^- - \xi_1^-\|_{a,\omega} \leq \frac{L}{a - (\beta + L)} \|\xi_2^+ - \xi_1^+\|_{a,\omega}. \quad (2.19)$$

Now, denote $\eta_1^+ = T_\omega \xi_1^+$, $\eta_2^+ = T_\omega \xi_2^+$. The difference $\eta_2^+ - \eta_1^+ \in X_{a,\omega}(E^+)$ which solves the unstable equation

$$x_{n+1}^+ = A^+(\theta^n \omega) x_n^+ + F^+(\theta^n \omega, \xi_2^+, \xi_2^-) - F^+(\theta^n \omega, \xi_1^+, \xi_1^-).$$

In order to apply Lemma 2.6.2, we set

$$f(n, \omega, x^+) := 0$$

and

$$f_0(n+1, \omega) := F^+(\theta^n \omega, \xi_2^+, \xi_2^-) - F^+(\theta^n \omega, \xi_1^+, \xi_1^-).$$

Then, this unstable equation changes to

$$x_{n+1}^+ = A^+(\theta^n \omega) x_n^+ + f(n, \omega, x_n^+) + f_0(n+1, \omega).$$

We can apply the procedure in Step 2. Here also comes the random variable $c(n, \omega) = 0$. According to Lemma 2.6.2,

$$\|\eta_2^+ - \eta_1^+\|_{a,\omega} \leq \frac{L}{\alpha - a} (\|\xi_2^+ - \xi_1^+\|_{a,\omega} + \|\xi_2^- - \xi_1^-\|_{a,\omega}). \quad (2.20)$$

Hence, we substitute $\|\xi_2^- - \xi_1^-\|_{a,\omega}$ by (2.19) into (2.20)

$$\|\eta_2^+ - \eta_1^+\|_{a,\omega} \leq \frac{L(a - \beta)}{(\alpha - a)(a - \beta - L)} \|\xi_2^+ - \xi_1^+\|_{a,\omega}.$$

To see T_ω is contracting, we require the constant

$$\frac{L(a - \beta)}{(\alpha - a)(a - \beta - L)} < 1.$$

Since $a \in [\beta + \delta, \alpha - \delta]$, we have

$$\frac{L(a - \beta)}{(\alpha - a)(a - \beta - L)} \leq \frac{L}{\delta} \frac{(a - \beta)}{(a - \beta - L)}$$

$$\begin{aligned} &\leq \frac{L}{\delta} \frac{\delta}{\delta - L} \\ &= \frac{L}{\delta - L}. \end{aligned}$$

Thus,

$$\|T_\omega \xi_2^+ - T_\omega \xi_1^+\|_{a,\omega} \leq \frac{L}{\delta - L} \|\xi_2^+ - \xi_1^+\|_{a,\omega}.$$

This means T_ω is contracting when the Lipschitz constant $L < \frac{\delta}{2}$. Therefore, by using the fixed point theorem for this contracting operator T_ω , there exists a unique fixed point $\xi_\star^+(\cdot, \omega) \in X_{a,\omega}(E^+)$. We replace this fixed point from Step 1, then there exists exactly one corresponding $\xi_\star^-(\cdot, \omega) \in X_{a,\omega}(E^-)$ and the pair $(\xi_\star^+, \xi_\star^-)$ solves the pair equations (2.5) and satisfies

$$\xi_\star^+(n, \omega) = - \sum_{i=n+1}^{\infty} \Phi^+(n-i, \theta^i \omega) (F^+(i-1, \omega, \xi_\star^+(i-1, \omega), \xi_\star^-(i-1, \omega))),$$

and

$$\xi_\star^-(n, \omega) = \sum_{i=-\infty}^n \Phi^-(n-i, \theta^i \omega) (F^-(i-1, \omega, \xi_\star^+(i-1, \omega), \xi_\star^-(i-1, \omega))).$$

Step 4. Stationary Solution

Denote $Y(\omega) := \{(\xi_\star^+(0, \omega), \xi_\star^-(0, \omega))\}$ for all $\omega \in \Omega$, to prove $Y(\omega)$ is the stationary point, it is equivalent to prove that

$$\varphi(n, \omega)Y(\omega) = Y(\theta^n \omega).$$

By the cocycle property, it is equivalent to prove

$$\varphi(1, \omega)Y(\omega) = Y(\theta \omega).$$

Now, by the uniqueness of the solution $Y(\omega)$ from the fixed point theorem, we have that the set

$$M^\star(\omega) = \{x \in \mathbb{R}^d : \varphi(\cdot, \omega, x) \in X_{a,\omega}\}$$

is a one point set and the unique point is

$$x = Y(\omega) = \{(\xi_\star^+(0, \omega), \xi_\star^-(0, \omega))\}.$$

According to $M^*(\omega)$, when the variable ω is changed to $\theta\omega$, we define

$$M^*(\theta\omega) = \{x \in \mathbb{R}^d : \varphi(\cdot, \theta\omega, x) \in X_{a,\theta\omega}\}.$$

This set is also a one point set with the only point $Y(\theta\omega)$. Now from the cocycle property, we have for $x \in M^*(\omega)$

$$\varphi(n+1, \omega, x) = \varphi(n, \theta\omega, \varphi(1, \omega, x)).$$

Since $\varphi(\cdot, \omega, x) \in X_{a,\omega}$, we have

$$\sup_{n+1 \geq 0} a^{-(n+1)} \|\varphi(n+1, \omega, x)\| < \infty.$$

It is easy to see

$$\sup_{n \geq 0} a^{-n} \|\varphi(n, \theta\omega, \varphi(1, \omega, x))\| < \infty.$$

By the definition of the space $X_{a,\theta\omega}$, this leads to $\varphi(\cdot, \theta\omega, \varphi(1, \omega, x)) \in X_{a,\theta\omega}$. Hence,

$$\varphi(1, \omega, x) \in M^*(\theta\omega).$$

Similar argument, we pick one element $x \in M^*(\theta\omega)$,

$$\varphi(1, \omega, x)^{-1} \in M^*(\omega).$$

This leads

$$\varphi(1, \omega)M^*(\omega) = M^*(\theta\omega).$$

Since $M^*(\omega)$ and $M^*(\theta\omega)$ are both one-point sets, we have that $Y(\omega)$ also satisfies the invariant property

$$\varphi(1, \omega)Y(\omega) = Y(\theta\omega).$$

So $Y(\omega)$ is the stationary point. This completes the proof of the theorem. \sharp

Remark 2.7.2

1. For the coupling method, it is trivial to change the coupling order. In our proof, we deal with the stable equation firstly, then the unstable one. Actually, we can also consider the unstable equation then the stable one. By the uniqueness of the solutions, we have the two conclusions are coincident since they are under the same RDS φ .

2. The restriction for the Lipschitz constant L is

$$0 \leq L < \frac{\delta}{2}.$$

With this condition, the requirements of the two key lemmas are easily met. However, it is quite strong condition since L is required to be small enough. This is needed to make the map T_ω a contraction.

3. In our case, we assume the random variable $c(\cdot, \omega) \in X_{a,\omega}$ and set the initial value $F(\omega, 0) = c(1, \omega)$. This is another restricted condition. Meanwhile, we can only choose to consider this situation, since the whole structure is under the space $X_{a,\omega}$. This provides us a restricted RDS φ .

4. The pair solutions $(\xi_*^+(n, \omega), \xi_*^-(n, \omega))$ will not be equal to zero if the random variable $c(n, \omega)$ is not identical to zero. This is easy to check from the construction of the solution in the proof of the theorem.

5. In the theorem, we notice the coupling stable and unstable equations are decomposed by j ,

$$1 \leq j \leq p.$$

For a special case when $j = p$, we will only consider the unstable equation

$$x_{n+1}^+ = A^+(\theta^n \omega)x_n^+ + F^+(\theta^n \omega, \xi^+(n, \omega)).$$

By applying Lemma 2.6.2, we can still obtain the stationary point. In this case, we can have restriction of the lipschitz condition reduced to

$$0 \leq L < \alpha - a.$$

This restriction is to fit Lemma 2.6.2.

It is interesting to observe the relationship between the stationary solution and the invariant manifolds. For the structure of the manifolds, we follow Arnold's work. We still keep our settings at the beginning of this section. But now, we can define the linear cocycle Φ by

$$\Phi(n, \omega) = D\varphi(n, \omega, Y(\omega)).$$

Different from the invariant manifold section, the fixed point now changes from zero to $Y(\omega)$. Firstly, for any j with $1 \leq j \leq p$, to construct the invariant unstable manifold $M^+(\omega)$ corresponding to the spectral interval

$$\Lambda^+ := \{\lambda_1 > \dots > \lambda_j\},$$

we solve the unstable equation for the initial value $x^+ \in E^+$ and given $\xi^-(\cdot, \omega) \in X_{a^-, \omega}(E^-)$ to have

$$\xi^+(\cdot, \omega, x^+) \in X_{a^-, \omega}(E^+).$$

Then replace the pair solution $(\xi^+(\cdot, \omega, x^+), \xi^-(\cdot, \omega))$ into the stable equation to generate a unique solution

$$\eta^-(\cdot, \omega, x^+) \in X_{a^-, \omega}(E^-).$$

After proving the operator

$$\begin{aligned} T_{\omega, x^+} : X_{a^-, \omega}(E^-) &\rightarrow X_{a^-, \omega}(E^-) \\ T_{\omega, x^+} \xi^-(\cdot, \omega) &= \eta^-(\cdot, \omega, x^+) \end{aligned}$$

is contracting, we have the unique point $\xi_*^-(\cdot, \omega, x^+) \in X_{a^-, \omega}(E^-)$. Therefore, for each given $x^+ \in E^+$,

$$x^- = m^+(\omega, x^+) := \xi_*^-(0, \omega, x^+) \in X_{a^-, \omega}(E^-)$$

is uniquely determined. Hence, M^+ is defined as a graph in $E^+ \oplus E^-$

$$M^+(\omega) = \{(x^+, m^+(\omega, x^+)) : x^+ \in E^+\}.$$

Similarly, we obtain the invariant stable manifold

$$M^-(\omega) = \{(m^-(\omega, x^-), x^-) : x^- \in E^-\}.$$

corresponding to the complementary spectral interval

$$\Lambda^- := \{\lambda_{j+1} > \dots > \lambda_p\}.$$

Comparing with the proof of our theorem, we find the key difference is the choice of the initial value. The manifolds depend on the initial point. Due to this difference, we claim one theorem which shows the connection of the manifolds and the stationary solution.

Theorem 2.7.3

Assume all the conditions in Theorem 2.7.1 are satisfied, $Y(\omega)$ is the stationary point of this system constructed in Theorem 2.7.1. For any j with $1 \leq j \leq p$, $M^+(\omega)$ and $M^-(\omega)$ are the corresponding unstable and the stable manifold with respect to the different spectral intervals Λ^+ and Λ^- . Then, we have

$$M^+(\omega) \cap M^-(\omega) = \{Y(\omega)\}$$

for all $\omega \in \Omega$.

Proof. The proof is simple. We only need to prove

1. $Y(\omega) \in M^+(\omega)$
2. $Y(\omega) \in M^-(\omega)$.

For (1), we can particular choose the initial value $x^+ = \xi_*^+(0, \omega)$ which is in E^+ part of $Y(\omega)$. Thus for the fixed ω , $Y(\omega) \in M^+(\omega)$ by the structure of the unstable manifold. The claim (2) can be obtained by using a similar argument. From the structure of stable and unstable manifold, we know they have the only one intersection point $Y(\omega)$. Hence the assertion of the theorem is satisfied. $\#$

Comparing with the invariant theorem, we have a familiar figure for this result. See Figure 2.4.

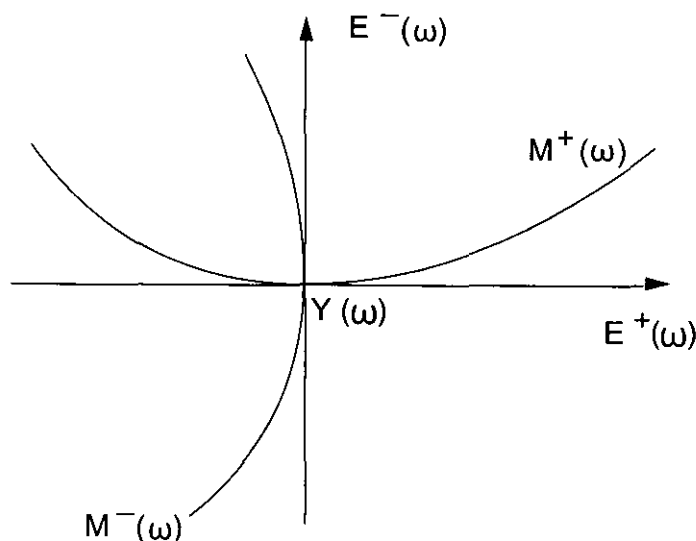


Figure 2.4: $Y(\omega)$ is the stationary point of our system

This theorem allows us to prove that $Y(\omega)$ is the stationary point by a simple argument. We have $M^+(\omega)$ and $M^-(\omega)$ are both invariant sets. Since the intersection of two invariant sets are invariant, $\{Y(\omega)\} = M^+(\omega) \cap M^-(\omega)$ automatically is the

invariant one point set. And it is the only one by the uniqueness of $Y(\omega)$. Recall the invariant manifold theorem, Theorem 2.5.1 and Theorem 2.7.3 describe the same situation. The intersection of stable and unstable manifolds is the stationary point. On the other hand, there is an interesting point as follows: For the stationary solution $\{Y(\omega)\} := \{(\xi_*^+(0, \omega), \xi_*^-(0, \omega))\}$, we can move points on $M^+(\omega)$ from time $+\infty$ back to time 0 by $\varphi(\cdot, \omega)$ with the exponential rate at most $-\lambda_j(\omega)$. This also implies the points on $M^-(\omega)$ can be moved to $Y(\omega)$ from time $-\infty$ forward to time 0 by $\varphi(\cdot, \omega)$ with the exponential rate at most $\lambda_{j+1}(\omega)$. This explains why the manifolds are called unstable and stable.

§2.8 Gap Conditions

We found this gap condition problem in the higher regularity of invariant manifold theorem. Higher regularity here means the invariant manifolds are C^k if the RDS φ is. This happens when we consider a C^k , ($k \geq 1$) RDS φ , some additional conditions are required to guarantee the spectral gap wide enough. However, the additional conditions are rigorous to satisfy in some ways. In this section, our purpose is to find a reasonable method to eliminate the influence of them.

To see this gap condition problem, we firstly recall the definition of spectral gap. For example, we have a center manifold M_{ij} according to the spectral interval for $1 \leq i, j \leq p$

$$\Lambda_{ij} = \lambda_i > \dots > \lambda_j.$$

Here the center manifold M_{ij} is a unstable manifold when $i = 1$ and is a stable manifold when $j = p$. Hence, this gap conditions will include the unstable case and the stable case. Assume

$$[\alpha_i, \beta_i] = [e^{\lambda_i - \kappa}, e^{\lambda_i + \kappa}],$$

where κ is sufficiently small to guarantee $\alpha_i > \beta_{i+1}$. And taking

$$0 < \delta < \min\left(\frac{\alpha_1 - \beta_2}{2}, \dots, \frac{\alpha_{p-1} - \beta_p}{2}, \frac{\alpha_p - 0}{2}\right).$$

Thus, define the spectral gap between λ_{i+1} and λ_i as

$$\Gamma := [\beta_{i+1} + \delta, \alpha_i - \delta], \quad i = 1, \dots, p-1.$$

We see Figure 2.5 which will help us to have a clear picture for these notation. The gap

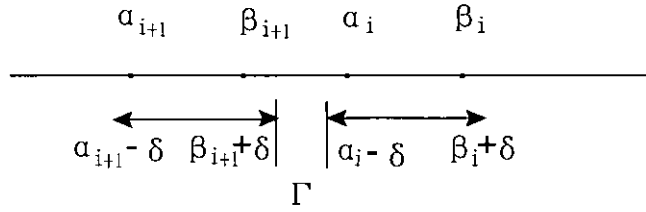


Figure 2.5: Spectral Gap

conditions are introduced in Arnold [1] to prove the higher regularity of the invariant manifolds. We specially pick up the conditions from the Theorem 7.3.19 in Arnold [1]. For $k \geq 2$, for the center manifold M_{ij} in \mathbb{R}^d , the gap conditions are presented as follows.

1. The spectral gap

$$\Gamma_{right} := [\beta_i + \delta, \alpha_{i-1} - \delta]$$

to the right of Λ_{ij} is wide enough such that we can choose two numbers $b, \bar{b} \in \Gamma_{right}$ with $b < \bar{b}$ for which, moreover, also $b^q < \bar{b}$ for every $q = 2, \dots, k$. There is no condition for $i = 1$, i.e. for the unstable manifolds.

2. The spectral gap

$$\Gamma_{left} := [\beta_{j+1} + \delta, \alpha_j - \delta]$$

to the left of Λ_{ij} is wide enough such that we can choose two numbers $a, \bar{a} \in \Gamma_{left}$ with $\bar{a} < a$ for which, moreover, also $\bar{a} < a^q$ for every $q = 2, \dots, k$. There is no condition for $j = p$, i.e. for the stable manifolds.

The gap conditions act an important role for the proof of this theorem in Arnold [1]. Actually they are not always be possible to be satisfied. For example, we consider a simple situation

$$\alpha_i < 1 < \beta_i$$

and

$$\begin{aligned} \Gamma_{right} &= [\beta_i + \delta, \alpha_{i-1} - \delta], \\ \Gamma_{left} &= [\beta_{i+1} + \delta, \alpha_i - \delta]. \end{aligned}$$

This will be impossible for either side. However, in some special cases, the gap conditions are automatically satisfied.

Case 1 If $\Gamma_{right} \cap (0, 1] \neq \emptyset$, we can choose $\bar{b} = 1$ and $b \in (0, 1]$. Then $b^q < \bar{b}$ for all $q \geq 2$.

Case 2 If $\Gamma_{left} \cap [1, \infty) \neq \emptyset$, we can choose $\bar{a} = 1$ and $a \in [1, \infty)$. Then $\bar{a} < a^q$ for all $q \geq 2$.

For a center manifold, we need to satisfy the two gap conditions in the same time. However, for the stable and unstable manifolds, we only need to satisfy Case 1 or Case 2. Hence, this gives us an idea to eliminate the gap conditions for the stable case or the unstable case.

We start with the beginning RDS φ structure in the preparation section. For a given prepared C^k RDS φ over θ in \mathbb{R}^d , we have

$$\varphi(n, \omega, x) = \Phi(n, \omega)x + \psi(n, \omega, x),$$

where

$$\Phi(t, \omega) := D\varphi(t, \omega, 0)$$

defines a linear cocycle over θ . And

$$\psi(t, \omega, x) := \varphi(t, \omega, x) - \Phi(t, \omega)x$$

is the nonlinear part with $\psi(t, \omega, 0) = 0$, $D\psi(t, \omega, 0) = 0$ and $\psi(0, \omega, x) = 0$. By the Theorem 7.3.19 in Arnold [1], we have if the corresponding manifold $M_{ij}(\omega)$ is a C^k manifold, it has to satisfy the gap conditions. Now, consider a unstable two-sided discrete time case, we need to consider the gap condition for Γ_{left} . For the linear part Φ of the RDS φ , we have the corresponding Lyapunov exponents

$$\lambda_1 > \lambda_2 > \dots > \lambda_j > \lambda_{j+1} > \dots > \lambda_p.$$

We can obtain the unstable manifold M_{1j} by considering the corresponding spectrum interval

$$\Lambda_{1j} = \{\lambda_1 > \lambda_2 > \dots > \lambda_j\}.$$

Assume

$$a = \frac{\lambda_j + \lambda_{j+1}}{2}.$$

We replace the Lyapunov exponents by putting

$$\lambda_1 - a > \lambda_2 - a > \dots > \lambda_j - a > 0 > \lambda_{j+1} - a > \dots > \lambda_p - a.$$

Next we let $\hat{\lambda}_1 = \lambda_1 - a, \dots, \hat{\lambda}_p = \lambda_p - a$. Then

$$\hat{\lambda}_1 > \hat{\lambda}_2 > \dots > \hat{\lambda}_j > 0 > \hat{\lambda}_{j+1} > \dots > \hat{\lambda}_p.$$

This structure is meaningful. We may consider it from the beginning by setting $\hat{\Phi}(n, \omega) = \Phi(n, \omega)e^{-an}$. It is easy to see that $\hat{\Phi}(n, \omega)$ is still a linear cocycle for x . By applying MET to $\hat{\Phi}(n, \omega)$, we have

$$\begin{aligned} \hat{\lambda}_i &= \lim_{n \rightarrow \infty} \frac{1}{n} \log \|\hat{\Phi}(n, \omega)x\| \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \log \|\Phi(n, \omega)e^{-an}x\| \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \log \|e^{-an}\| \|\Phi(n, \omega)x\| \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \log \|\Phi(n, \omega)x\| - a \\ &= \lambda_i - a \end{aligned}$$

Hence, we have the initial RDS change to

$$x_n = e^{-an}\Phi(n, \omega)x_0 + e^{-an}\psi(n, \omega, x_0).$$

Assume

$$\begin{aligned} \hat{\Phi}(n, \omega) &= e^{-an}\Phi(n, \omega), \\ \hat{\psi}(n, \omega, x) &= e^{-an}\psi(n, \omega, x). \end{aligned}$$

Thus, we can define a new RDS

$$\hat{\varphi}(n, \omega, x) = \hat{\Phi}(n, \omega)x + \hat{\psi}(n, \omega, x)$$

such that if φ is C^k , so is $\hat{\varphi}$. And the gap condition for this $\hat{\varphi}$ is exactly the same as the case 2

$$\Gamma_{left} \cap [1, \infty) \neq \emptyset,$$

which is satisfied automatically. Then the corresponding unstable manifold is C^k . The stable case is in a similar manner, and the gap condition changes to Γ_{right} . We can deal with it by using a similar argument.

§2.9 Further Research

In the last, let us consider some possible further directions of the research. Firstly, in this chapter, we consider a finite-dimensional space \mathbb{R}^d . Note $X_{a, \omega}$ is a Banach space

defined on \mathbb{R}^d . From the structure of $X_{a,\omega}$, we know there are very limited elements in $X_{a,\omega}$. Our stationary point is also generated in this space. It will be a challenge to build a new space which contains much more common elements and all results in Theorem 2.7.1 still hold.

Secondly, we recall the Lipschitz constant restrictions in Lemma 2.6.1, Lemma 2.6.2 and Theorem 2.7.1

$$\begin{aligned}\gamma &> \beta + L, \\ 0 &< \gamma < \alpha - L, \\ 0 &\leq L < \frac{\delta}{2}.\end{aligned}$$

These restrictions are used to prove the contraction, then apply the Banach fixed point theorem. Actually, to prove the existence, we can also apply other fixed point theorem such as Schauder fixed point theorem. Hence the Lipschitz constant restriction may have a chance to be omitted. But we lose the uniqueness of the stationary solution. There are some difficulties. It is not clear now the technique would work without the uniqueness of the solution.

Chapter 3

The Continuous Time RDS in an Infinite-Dimensional Hilbert Space

\mathbb{H}

§3.1 Background

In the first part of this chapter, we devote to introduce the background and to set up of the main problems which we are interested in.

§3.1.1 Semilinear Stochastic Evolution Equations

We start our work with a stochastic differential equation. Let \mathbb{H} be a separable real Hilbert space. We consider the semilinear stochastic evolution equations (semilinear see) with the additive noise of the form

$$\begin{aligned} du(t) &= [-Au(t) + F(u(t))]dt + B_0dW(t), \\ u(0) &= x \in \mathbb{H}, \end{aligned} \tag{3.1}$$

for $t \geq 0$. In the above semilinear see (3.1), we denote A to be as a closed linear operator from

$$D(A) \subset \mathbb{H} \rightarrow \mathbb{H}.$$

Suppose $-A$ generates a strongly continuous semigroup T_t of bounded linear operators from

$$T_t : \mathbb{H} \rightarrow \mathbb{H},$$

for $t \geq 0$. Let E be another separable real Hilbert space. Suppose $W(t)$, $t \geq 0$ is an E -valued Brownian motion which is defined on the canonical filtered Wiener space

$(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ and with a separable Hilbert space K , where $K \subset E$ is a Hilbert-Schmidt embedding. In this structure, Ω is the space of all continuous paths with the mapping

$$\omega : \mathbb{R} \rightarrow E$$

such that $\omega(0) = 0$, \mathcal{F} is its Borel σ -field, \mathcal{F}_t is the sub- σ -field of \mathcal{F} which is generated by all

$$\Omega \ni \omega \mapsto \omega(u) \in E$$

for $u \leq t$ and \mathbb{P} is Wiener measure on Ω . The Brownian motion is given by for all $\omega \in \Omega$ and $t \in \mathbb{R}$

$$W(t, \omega) := \omega(t).$$

Also it may be written as

$$W(t) = \sum_{k=1}^{\infty} W^k(t) f_k, \quad t \in \mathbb{R}$$

where $\{f_k, k \geq 1\}$ is a complete orthonormal basis of K and $(W^k, k \geq 1)$ are standard independent one-dimensional Wiener processes. In general, this series converges in E , not in K . We refer it to readers for Chapter 4 of Da Prato and Zabczyk [6] for details. Next, we denote by

$$L_2(K, \mathbb{H}) \subset L(K, \mathbb{H}),$$

be the Hilbert space of all Hilbert-Schmidt operators

$$S : K \rightarrow \mathbb{H}$$

with the norm

$$\|S\|_2 := \left[\sum_{k=1}^{\infty} |S(f_k)|^2 \right]^{\frac{1}{2}}$$

where $|\cdot|$ is the norm on \mathbb{H} , and $L(K, \mathbb{H})$ be the Banach space of all bounded linear operators from K to \mathbb{H} with the uniform norm such that for any $B \in L(K, \mathbb{H})$ and any $v \in K$

$$\|B\| := \sup_{|v| \leq 1} |B(v)|.$$

Suppose

$$B_0 \in L_2(K, \mathbb{H})$$

be a bounded linear operator. For this $L_2(K, \mathbb{H})$ space, see Mohammed, Zhang and Zhao [25] for some other related discussions. Assume that the operator A in (3.1) also

has a complete orthonormal system of eigenvectors $\{e_n : n \geq 1\}$ with corresponding the eigenvalues $\{\mu_n : n \geq 1\}$, such that

$$Ae_n = \mu_n e_n$$

for $n \geq 1$. The function

$$F : \mathbb{H} \rightarrow \mathbb{H}$$

defines a nonlinear perturbation which satisfies the Lipschitz condition with the constant L

$$|F(v_1) - F(v_2)| \leq L |v_1 - v_2|,$$

for $v_1, v_2 \in \mathbb{H}$.

§3.1.2 Oseledec-Ruelle version MET

From the finite-dimensional space \mathbb{R}^d , we have already introduced MET. In this section, we present an intensive infinite-dimensional version MET. This work has been done by Ruelle [33].

Theorem 3.1.1 (Oseledec-Ruelle MET)

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. We define (T, θ) be a $L(\mathbb{H})$ -valued measurable RDS. T is a function

$$T : \mathbb{R}^+ \times \Omega \rightarrow L(\mathbb{H}).$$

And θ is a group of \mathbb{P} -preserving ergodic transformations on $(\Omega, \mathcal{F}, \mathbb{P})$ from

$$\mathbb{R} \times \Omega \rightarrow \Omega.$$

Suppose that

$$E \sup_{0 \leq t \leq 1} \log^+ \|T(t, \cdot)\|_{L(\mathbb{H})} + E \sup_{0 \leq t \leq 1} \log^+ \|T(1-t, \theta(t, \cdot))\|_{L(\mathbb{H})} < \infty.$$

Then, there exists an invariant set $\Omega_0 \in \mathcal{F}$ with full measure such that for all $t \in \mathbb{R}^+$

$$\theta(t, \cdot)(\Omega_0) \subseteq \Omega_0 \text{ and } \mathbb{P}(\Omega_0) = 1,$$

and for each $\omega \in \Omega_0$, the limit

$$\Lambda(\omega) := \lim_{t \rightarrow \infty} (T(t, \omega)^* T(t, \omega))^{\frac{1}{2t}}$$

exists. This $\Lambda(\omega)$ is self-adjoint, non-negative with a discrete spectrum

$$e^{\lambda_1} > e^{\lambda_2} > e^{\lambda_3} > \dots$$

Let $F_1(\omega), F_2(\omega), F_3(\omega), \dots$ be the span of corresponding eigenvectors. Then denote that for $i = 1, 2, \dots$

$$\begin{aligned} E_1(\omega) &:= \mathbb{H} \\ E_i(\omega) &:= [\oplus_{j=1}^{i-1} F_j(\omega)]^\perp \end{aligned}$$

for $i > 1$. When $i = \infty$, set $\lambda_i = -\infty$ and

$$E_\infty := \ker \Lambda(\omega).$$

Then $(E_i(\omega))_{i=1,2,\dots}$ forms a filtration for \mathbb{H} , such that

$$E_\infty \subset \dots \subset E_{i+1}(\omega) \subset E_i(\omega) \subset \dots \subset E_1(\omega) = \mathbb{H},$$

and

$$m_i := \dim F_i(\omega).$$

Then the Lyapunov exponent will be expressed as

$$\lambda_i = \lim_{t \rightarrow \infty} \frac{1}{t} \log |T(t, \omega)x|$$

for $x \in E_i(\omega) \setminus E_{i+1}(\omega)$, and

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log |T(t, \omega)x| = -\infty$$

if $x \in E_\infty(\omega)$. The invariance property is

$$T(t, \omega)(E_i(\omega)) \subseteq E_i(\theta(t, \omega))$$

for all $t \geq 0, i \geq 1$.

Remark 3.1.2

1. $L(\mathbb{H})$ is the Banach space of bounded linear operator from

$$\mathbb{H} \rightarrow \mathbb{H}$$

with the uniform operator norm such that for $B \in L(\mathbb{H})$

$$\|B\|_{L(\mathbb{H})} := \sup_{|v| \leq 1} |B(v)|$$

for $v \in \mathbb{H}$.

2. Comparing with Theorem 2.3.2 and Remark 2.3.3, we have the two corresponding integrable conditions. They both require that the logarithm RDS is integrable in two-sided time.
3. From this theorem, we obtain an orthogonal splitting of the infinite-dimensional space \mathbb{H} by two parts. One is for the positive eigenvalues $\{\lambda_1, \lambda_2, \dots, \lambda_m\}$, the other one is for the negative eigenvalues $\{\lambda_n : n \geq m\}$. And the space \mathbb{H} can be written as

$$\mathbb{H} := \mathbb{H}^+ \oplus \mathbb{H}^-.$$

We see from this, \mathbb{H}^+ is a finite-dimensional subspace, and \mathbb{H}^- is an infinite-dimensional subspace.

4. To well understand this theorem, we have a figure below.

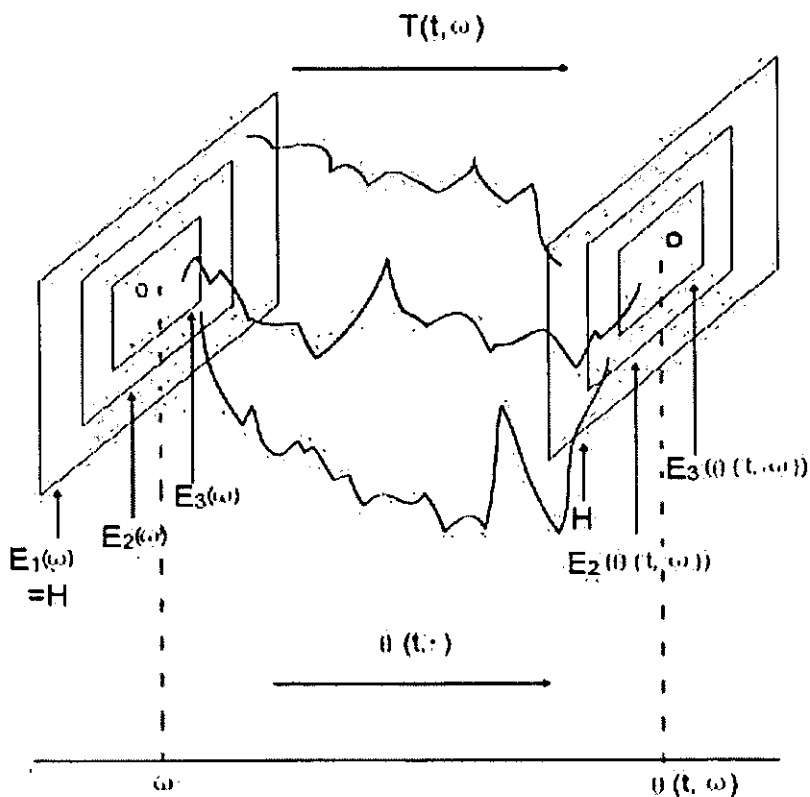


Figure 3.1: The Oseledec-Ruelle Theorem

§3.1.3 Infinite-Dimensional Local Invariant Manifold Theorem

In this part, we are going to introduce a beautiful work which will show us, in a random manner, the structure of the local invariant manifold in an infinite-dimensional space \mathbb{H} . This work has been done by Mohammed, Zhang and Zhao [25]. Firstly, we give some notations as before. \mathbb{H} is a separable Hilbert space. We denote by $B(x, \rho)$ the open ball in \mathbb{H} , with radius ρ and center point $x \in \mathbb{H}$, $\overline{B}(x, \rho)$ denotes the corresponding closed ball. Normally, we say a stationary point is hyperbolic if the eigenvalues of the linearized system have non-zero real part. In our RDS (U, θ) , we say a stationary point $Y(\omega)$ is hyperbolic if the corresponding linearized cocycle $(DU(t, \omega, Y(\omega)), \theta(\omega))$ has a non-zero Lyapunov spectrum

$$\dots < \lambda_{i+1} < \lambda_i < \dots < \lambda_2 < \lambda_1$$

such that $\lambda_i \neq 0$ for all $i \geq 1$. And the stationary point satisfies

$$\int_{\Omega} \log^+ \sup_{0 \leq t_1, t_2 \leq a} \| DU(t_2, \theta(t_1, \omega), Y(\theta(t_1, \omega))) \|_{L(\mathbb{H})} d\mathbb{P}(\omega) < \infty$$

for $a \in (0, \infty)$.

Theorem 3.1.3 (Local Invariant Manifold Theorem)

For a separable Hilbert space \mathbb{H} , let (U, θ) be a measurable RDS, where U is a measurable function defined from

$$\begin{aligned} (0, \infty) \times \Omega \times \overline{B}(0, \rho) &\rightarrow \mathbb{H}, \\ (t, \omega, x) &\mapsto U(t, \omega, x). \end{aligned}$$

Let Y be a hyperbolic stationary point of the RDS (U, θ) which satisfies the following condition

$$\int_{\Omega} \log^+ \sup_{0 \leq t_1, t_2 \leq a} \| U(t_2, \theta(t_1, \omega), Y(\theta(t_1, \omega))) \|_{L(\mathbb{H})} d\mathbb{P}(\omega) < \infty$$

for any fixed $0 < \rho, a < \infty$. Define the linearized RDS $(DU(t, \omega, Y(\omega)), \theta(t, \omega), t \geq 0)$ admits the discrete Lyapunov spectrum

$$\dots < \lambda_{i+1} < \lambda_i < \dots < \lambda_2 < \lambda_1.$$

We specially pick up

$$\lambda_{i_0} := \max\{\lambda_i : \lambda_i < 0\}.$$

If all finite λ_i are positive, set $\lambda_{i_0} := -\infty$. If all the λ_i are negative, set $\lambda_{i_0-1} := \infty$. We then choose and fix

$$\epsilon_1 \in (0, -\lambda_{i_0}) \text{ and } \epsilon_2 \in (0, \lambda_{i_0-1}).$$

Then, there exists an invariant set $\Omega^* \in \mathcal{F}$ with full measure such that

$$\theta(t, \cdot)(\Omega^*) = \Omega^* \text{ and } \mathbb{P}(\Omega^*) = 1$$

for all $t \in \mathbb{R}$. Assume the functions ρ_i, β_i are maps from

$$\Omega^* \rightarrow (0, 1),$$

where $\beta_i > 0, \rho_i > 0, i = 1, 2$ are random variables, such that for each $\omega \in \Omega^*$, we have the following:

Stable case

There is a submanifold $\mathcal{S}(\omega)$ of $\overline{B}(Y(\omega), \rho_1(\omega))$. For $\lambda_{i_0} > -\infty$, $\mathcal{S}(\omega)$ is the set of all $x \in \overline{B}(Y(\omega), \rho_1(\omega))$ such that

$$|U(n, \omega, x) - Y(\theta(n, \omega))| \leq \beta_1(\omega)e^{(\lambda_{i_0} + \epsilon_1)n}$$

for all $n \geq 0$. If $\lambda_{i_0} = -\infty$, $\mathcal{S}(\omega)$ is the set of all $x \in \overline{B}(Y(\omega), \rho_1(\omega))$ such that for all $n \geq 0$

$$|U(n, \omega, x) - Y(\theta(n, \omega))| \leq \beta_1(\omega)e^{\lambda n}$$

where $\lambda \in (-\infty, n)$. Furthermore, for all $x \in \mathcal{S}(\omega)$,

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log |U(t, \omega, x) - Y(\theta(t, \omega))| \leq \lambda_{i_0}.$$

For the linearized RDS $(DU(t, \omega, Y(\omega)), \theta(\omega))$, we define $\tilde{\mathcal{S}}(\omega)$ is the corresponding submanifold of it, then each $\tilde{\mathcal{S}}(\omega)$ is tangent to $\mathcal{S}(\omega)$ at the stationary point $Y(\omega)$, such that

$$T_{Y(\omega)}\mathcal{S}(\omega) = \tilde{\mathcal{S}}(\omega).$$

And $\mathcal{S}(\omega)$ is local invariant such that there exists $\tau_1(\omega) \geq 0$ with

$$U(t, \omega)(\mathcal{S}(\omega)) \subseteq \mathcal{S}(\theta(t, \omega))$$

and

$$DU(t, \omega)(\tilde{\mathcal{S}}(\omega)) \subseteq \tilde{\mathcal{S}}(\theta(t, \omega))$$

for all $t \geq \tau_1(\omega)$.

Unstable case

There is a submanifold $\mathcal{U}(\omega)$ of $\overline{B}(Y(\omega), \rho_2(\omega))$. For $\lambda_{i_0-1} < \infty$, $\mathcal{U}(\omega)$ is the set of all $x \in \overline{B}(Y(\omega), \rho_2(\omega))$ with the property

$$U(1, \theta(-n, \omega), y(-n, \omega)) = y(-(n-1), \omega)$$

where the process $y(\cdot, \omega)$ is defined from

$$\{-n : n \geq 0\} \rightarrow \mathbb{H}$$

and $y(0, \omega) = x$ and

$$|y(-n, \omega) - Y(\theta(-n, \omega))| \leq \beta_2(\omega) e^{-(\lambda_{i_0-1} - \epsilon_2)n}$$

for each $n \geq 1$. If $\lambda_{i_0-1} = \infty$, $\mathcal{U}(\omega)$ is the set of all $x \in \overline{B}(Y(\omega), \rho_2(\omega))$ such that for all $n \geq 0$

$$|y(-n, \omega) - Y(\theta(-n, \omega))| \leq \beta_2(\omega) e^{-\lambda n}$$

where $\lambda \in (0, \infty)$. Furthermore, we have

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log |y(-t, \omega) - Y(\theta(-t, \omega))| \leq -\lambda_{i_0-1},$$

for all $x \in \mathcal{U}(\omega)$. For the linearized RDS $(DU(t, \omega, Y(\omega)), \theta(\omega))$, we define $\tilde{\mathcal{U}}(\omega)$ to be the corresponding submanifold, then each $\tilde{\mathcal{U}}(\omega)$ is tangent to $\mathcal{U}(\omega)$ at the stationary point $Y(\omega)$, such that

$$T_{Y(\omega)}\mathcal{U}(\omega) = \tilde{\mathcal{U}}(\omega).$$

And $\mathcal{U}(\omega)$ is local invariant such that there exists $\tau_2(\omega) \geq 0$ with

$$\mathcal{U}(\omega) \subseteq U(t, \theta(-t, \omega))(\mathcal{U}(\theta(-t, \omega)))$$

and

$$DU(t, \theta(-t, \omega))(\tilde{\mathcal{U}}(\theta(-t, \omega))) = \tilde{\mathcal{U}}(\omega)$$

for all $t \geq \tau_2(\omega)$.

Remark 3.1.4

1. As a result, the Hilbert space \mathbb{H} can be splitting by two tangent spaces according to the stable manifold $\mathcal{S}(\omega)$ and the unstable manifold $\mathcal{U}(\omega)$. Hence, we naturally have

$$\mathbb{H} = T_{Y(\omega)}\mathcal{U}(\omega) \oplus T_{Y(\omega)}\mathcal{S}(\omega)$$

at the stationary point $Y(\omega)$, where $T_{Y(\omega)}\mathcal{U}(\omega)$, $T_{Y(\omega)}\mathcal{S}(\omega)$ are the tangent spaces of $\mathcal{U}(\omega)$ and $\mathcal{S}(\omega)$ at the point $Y(\omega)$ respectively.

2. The existence of the RDS (U, θ) in the infinite-dimensional Hilbert space is a difficult problem. Fortunately, Mohammed, Zhang and Zhao [25] has proved the existence of RDS corresponding a large class of the semilinear stochastic evolution equations and semilinear stochastic partial differential equations. This brings us a big convenience for further research.
3. In the higher regularity case, we consider a $C^{k,\epsilon}$ ($k \geq 1, \epsilon \in (0, 1]$) type RDS (U, θ) . Under this situation, all the assertions of Theorem 3.1.3 still hold, and $\mathcal{S}(\omega)$ and $\mathcal{U}(\omega)$ are the corresponding $C^{k,\epsilon}$ ($k \geq 1, \epsilon \in (0, 1]$) manifolds. We note here that $C^{k,\epsilon}$ describe a set of functions of f with the following properties. If E, N are real Banach space, we denote $L^{(k)}(E, N)$ be the Banach space of all k -multilinear maps such that

$$A : E^k \rightarrow N$$

with the uniform norm

$$\| A \| := \sup\{ | A(v_1, v_2, \dots, v_k) | : v_i \in E, | v_i | \leq 1, i = 1, \dots, k \}.$$

Suppose $U \subseteq E$ is an open set, the map

$$f : U \rightarrow N$$

is said to be of class $C^{k,\epsilon}$ if it is C^k and if

$$D^{(k)}f : U \rightarrow L^{(k)}(E, N)$$

is ϵ -Hölder continuous on bounded sets in U .

4. If the RDS (U, θ) is C^∞ , the local stable and unstable manifolds $\mathcal{S}(\omega)$, $\mathcal{U}(\omega)$ are C^∞ .
5. From this theorem, we can essentially view the stable and unstable manifolds. For the stationary solution $Y(\omega)$, we can move points on $\mathcal{U}(\omega)$ from time $+\infty$ back to

time 0 by $U(\cdot, \omega)$ to $Y(\omega)$ with the exponential rate at most λ_{i_0-1} . This also implies the points on $S(\omega)$ can be moved from time $-\infty$ to time 0 by $U(\cdot, \omega)$ to $Y(\omega)$ with the exponential rate at most $-\lambda_{i_0}$.

6. We see a figure below.

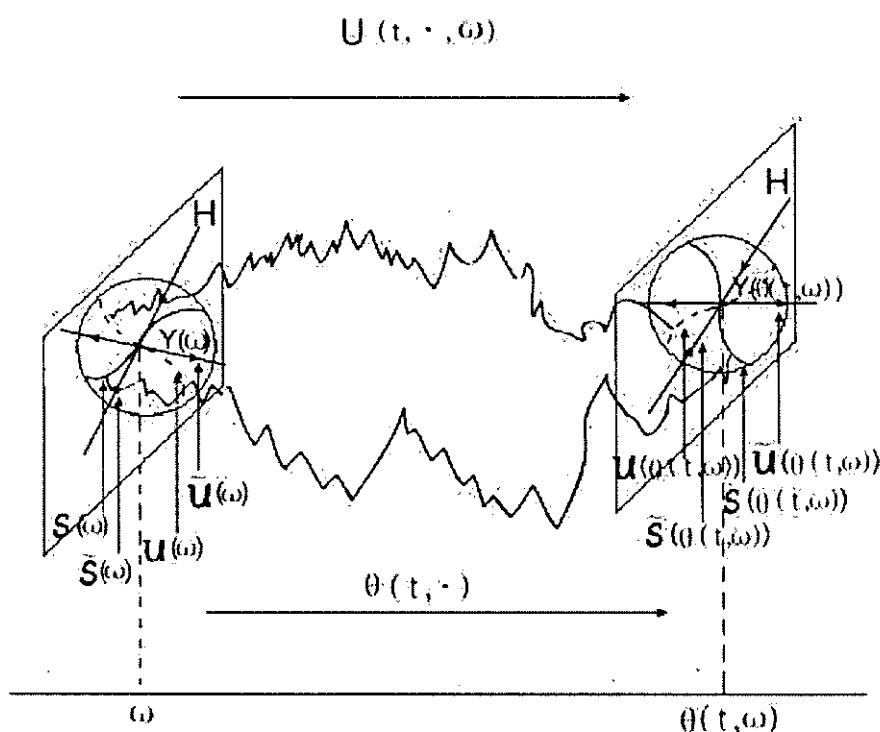


Figure 3.2: Local Invariant Manifold Theorem

§3.1.4 Mohammed, Zhang and Zhao's Results on the Existence of Stationary Solutions

Mohammed, Zhang and Zhao's results are setting on an infinite-dimensional real separable Hilbert space \mathbb{H} . We denote $\{e_n, n \geq 1\}$ a basis for \mathbb{H} . Let A be a self-adjoint operator on \mathbb{H} with a discrete non-vanishing spectrum $\{\mu_n, n \geq 1\}$ which is bounded below. We have $Ae_n = \mu_n e_n$ for $n \geq 1$. Denote μ_m the largest negative eigenvalue of A , and μ_{m+1} is its smallest positive eigenvalue. Hence, we obtain an orthogonal splitting of \mathbb{H} by two parts. One is for the negative eigenvalues $\{\mu_1, \mu_2, \dots, \mu_m\}$. The other one is for the positive corresponding eigenvalues $\{\mu_n : n \geq m + 1\}$. And \mathbb{H} can be written

as

$$\mathbb{H} := \mathbb{H}^+ \oplus \mathbb{H}^-.$$

We see, \mathbb{H}^+ is an infinite-dimensional subspace, \mathbb{H}^- is a finite-dimensional subspace.

We also define the projections onto each subspace by

$$P^+ : \mathbb{H} \rightarrow \mathbb{H}^+$$

$$P^- : \mathbb{H} \rightarrow \mathbb{H}^-.$$

On the Wiener space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$, we denote $W(t)$ be a Brownian motion. Let K be another separable Hilbert space, then assume $B_0 \in L_2(K, \mathbb{H})$.

$$T_t = e^{-At}$$

is a strongly continuous semigroup generated by $-A$. Since \mathbb{H}^- is finite-dimensional, we have T_t on \mathbb{H}^- is invertible for each $t \geq 0$. Therefore, we set $T_{-t} := [T_t]^{-1}$ from $\mathbb{H}^- \rightarrow \mathbb{H}^-$ for each $t \geq 0$.

Now, we consider a semilinear stochastic evolution equation (semilinear see) on \mathbb{H} with the above structure

$$\begin{aligned} du(t) &= [-Au(t) + F(u(t))]dt + B_0 dW(t), \\ u(0) &= x \in \mathbb{H}. \end{aligned}$$

We assume $F : \mathbb{H} \rightarrow \mathbb{H}$ satisfies a globally Lipschitz condition

$$|F(x) - F(y)| \leq L |x - y|,$$

for any $x, y \in \mathbb{H}$, where L is a non-negative constant. Then, the semilinear see has a unique mild solution with the following form

$$u(t, x) = T_t x + \int_0^t T_{t-s} F(u(s, x)) ds + \int_0^t T_{t-s} B_0 dW(s),$$

for $t \geq 0$. In their recent work, Mohammed, Zhang and Zhao [25] has proved the following results for the existence of the stationary solution for this semilinear see. We introduce here two propositions.

Proposition 3.1.5 *Assume the above conditions on A and B_0 , F satisfies the globally bounded and globally Lipschitz conditions. The Lipschitz constant L is with the restriction*

$$L[\mu_{m+1}^{-1} - \mu_m^{-1}] < 1.$$

Then there is a unique \mathcal{F} -measurable map $Y : \Omega \rightarrow \mathbb{H}$ satisfying

$$Y(\omega) = \int_{-\infty}^0 T_{-s} P^+ F(Y(\theta(s, \omega))) ds - \int_0^{\infty} T_{-s} P^- F(Y(\theta(s, \omega))) ds \\ + (\omega) \int_{-\infty}^0 T_{-s} P^+ B_0 dW(s) - (\omega) \int_0^{\infty} T_{-s} P^- B_0 dW(s)$$

for all $\omega \in \Omega$.

Proposition 3.1.6 Assume all the conditions on A , B_0 and F in Proposition 3.1.5.

Then the semilinear see

$$\begin{aligned} du(t) &= [-Au(t) + F(u(t))]dt + B_0 dW(t), \\ u(0) &= x \in \mathbb{H}, \end{aligned}$$

has a unique stationary point $Y : \Omega \rightarrow \mathbb{H}$, such that

$$u(t, \omega, Y(\omega)) = Y(\theta_t \omega)$$

for all $t \geq 0$ and $\omega \in \Omega$, and $Y(\omega)$ is given in Proposition 3.1.5.

We see that proposition 3.1.5 provides us the structure of the stationary point. And proposition 3.1.6 proves it is a unique stationary point. The proofs of these two propositions are very valuable, see Mohammed, Zhang and Zhao [25] for details. Normally, the stationary point in a DS may be non-unique. To obtain the stationary solution in the case that there might be more than one stationary point, we need different techniques.

§3.1.5 The Problems

In this section, we introduce two unsolved problems around the above two propositions.

Problem 1 In proposition 3.1.5, we see that the Lipschitz constant L needs to satisfy the restriction. If we can release it, that will be a significant progress. Actually, it is reasonable since this condition was used by proving the Banach fixed point theorem. If we apply other fixed point theorem or related arguments, this condition may be omitted.

Problem 2 Another one is the boundedness condition for the function F . Our purpose is to weaken it to a weaker condition. This will be quite challenging since this

condition acts an important role in the proof. We are trying to apply the coupling method to weaken such a condition.

We notice, these two problems actually are based on the infinite-dimensional Hilbert space \mathbb{H} . This leads to some trouble. In the next two sections, we will carefully choose the technique tools to deal with it.

§3.2 Release Lipschitz Constant Restriction

§3.2.1 Preparations

As previous sections, we also assume \mathbb{H} is a real separable infinite-dimensional Hilbert space. We denote $\{e_n : n \geq 1\}$ a basis for \mathbb{H} . A is a closed linear operator from

$$D(A) \subset \mathbb{H} \rightarrow \mathbb{H}$$

with a discrete non-vanishing spectrum $\{\mu_n, n \geq 1\}$. We have $Ae_n = \mu_n e_n$. we can also denote μ_m the largest negative eigenvalue of A , and μ_{m+1} is its smallest positive eigenvalue. Hence, we obtain an orthogonal splitting of \mathbb{H} by two parts. One is for the negative eigenvalues $\{\mu_1, \mu_2, \dots, \mu_m\}$. The other one is for the positive eigenvalues $\{\mu_{m+1}, \mu_{m+2}, \dots\}$. And \mathbb{H} can be written as

$$\mathbb{H} := \mathbb{H}^+ \oplus \mathbb{H}^-.$$

We see this structure, \mathbb{H}^- is a finite-dimensional subspace, and \mathbb{H}^+ is an infinite-dimensional subspace. We also define the corresponding projections onto each subspace by

$$P^+ : \mathbb{H} \rightarrow \mathbb{H}^+$$

and

$$P^- : \mathbb{H} \rightarrow \mathbb{H}^-.$$

Let $W(t)$, $t \geq 0$ be an \mathbb{H} -valued Brownian motion which is defined on the canonical filtered Wiener space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ and with a separable Hilbert space K as mentioned as before. Suppose $B_0 \in L_2(K, \mathbb{H})$. Suppose $-A$ generates a strongly continuous semigroup

$$T_t = e^{-At}.$$

We now consider the semilinear see on \mathbb{H}

$$du(t) = [-Au(t) + F(u(t))]dt + B_0 dW(t),$$

$$u(0) = x \in \mathbb{H},$$

where $F : \mathbb{H} \rightarrow \mathbb{H}$ satisfies a globally Lipschitz condition

$$|F(x) - F(y)| \leq L |x - y|,$$

for any $x, y \in \mathbb{H}$, where L is a constant. Then, the semilinear see has a unique mild solution with the following form

$$u(t, x) = T_t x + \int_0^t T_{t-s} F(u(s, x)) ds + \int_0^t T_{t-s} B_0 dW(s)$$

for $t \geq 0$. We also recall an important definition here to help the proof in next subsection.

Definition (Equicontinuous) *Let X be a metric space and G is a family of functions on X . The family G is said to be equicontinuous at a point $x_0 \in X$ if for every $\epsilon > 0$, there exists a $\delta > 0$, such that*

$$d(g(x_0), g(x)) < \epsilon$$

for all $g \in G$ and all x such that

$$d(x_0, x) < \delta.$$

The whole family is called equicontinuous if it is equicontinuous at each point of X .

Next we introduce one famous theorem: Arzela-Ascoli theorem.

Arzela-Ascoli Theorem

If S is compact, then a set in $C(S)$ is conditionally compact if and only if it is bounded and equicontinuous.

Here conditionally compact means every infinite subset of $C(S)$ has a limit point which is not necessary in $C(S)$. Hence, when $C(S)$ is closed, conditionally compact is equal to compact.

§3.2.2 Main Results

In this part, we are trying to take off the restriction for the Lipschitz constant L . We will see, to prove the results, Arzela-Ascoli compactness arguments plays an

important role. Firstly, we present a result based on proposition 3.1.5.

Proposition 3.2.1 *Assume the conditions on A , B_0 as above section and F is globally bounded and locally Lipschitz. Then there exist at least one \mathcal{F} -measurable map $Y : \Omega \rightarrow \mathbb{H}$ satisfying*

$$Y(\omega) = \int_{-\infty}^0 T_{-s} P^+ F(Y(\theta(s, \omega))) ds - \int_0^{\infty} T_{-s} P^- F(Y(\theta(s, \omega))) ds \\ + (\omega) \int_{-\infty}^0 T_{-s} P^+ B_0 dW(s) - (\omega) \int_0^{\infty} T_{-s} P^- B_0 dW(s) \quad (3.2)$$

for all $\omega \in \Omega$.

Proof. Firstly, define the \mathcal{F} -measurable map $Y_1 : \Omega \rightarrow \mathbb{H}$ by

$$Y_1(\omega) = (\omega) \int_{-\infty}^0 T_{-s} P^+ B_0 dW(s) - (\omega) \int_0^{\infty} T_{-s} P^- B_0 dW(s). \quad (3.3)$$

Then we have

$$Y_1(\theta_t \omega) = (\theta_t \omega) \int_{-\infty}^0 T_{-s} P^+ B_0 dW(s) - (\theta_t \omega) \int_0^{\infty} T_{-s} P^- B_0 dW(s) \\ = (\omega) \int_{-\infty}^t T_{t-s} P^+ B_0 dW(s) - (\omega) \int_t^{\infty} T_{t-s} P^- B_0 dW(s).$$

Secondly, we denote by $C(T, \mathbb{H})$ the Banach space of all bounded continuous maps from T to \mathbb{H} and for each $\omega \in \Omega$

$$C_B(T, \mathbb{H}) = \{f \in C(T, \mathbb{H}) \text{ and } \|f\|_{\infty} := \sup_{s \in T} |f(s)| \leq B\},$$

where B is a constant with

$$B := \|F\|_{\infty} \left(\frac{1}{\mu_{m+1}} + \frac{1}{\mu_m} \right)$$

and

$$\|F\|_{\infty} := \sup_{v \in \mathbb{H}} |F(v)|.$$

With the above structure, we can consider (3.2) as two parts. We now define $z_0 = 0$, and consider

$$z_{n+1}(\theta_t \omega) = \int_{-\infty}^t T_{t-s} P^+ F(z_n(\theta_s \omega) + Y_1(\theta_s \omega)) ds \\ - \int_t^{+\infty} T_{t-s} P^- F(z_n(\theta_s \omega) + Y_1(\theta_s \omega)) ds.$$

This structure provides us a possibility to solve the problem in an infinite-dimensional space. Our aim is to prove this sequence $\{z_n(\theta, \omega)\}_{n=0}^\infty$ is equicontinuous. For our case, this means we need to prove z is uniformly continuous on t for all n . For this, taking any $t_1, t_2 \in (-\infty, +\infty)$ with $t_1 \leq t_2$, we have

$$\begin{aligned} & | z_{n+1}(\theta_{t_1}\omega) - z_{n+1}(\theta_{t_2}\omega) | \\ & \leq | \int_{-\infty}^{t_1} T_{t_1-s} P^+ F(z_n(\theta_s\omega) + Y_1(\theta_s\omega)) ds \\ & \quad - \int_{-\infty}^{t_2} T_{t_2-s} P^+ F(z_n(\theta_s\omega) + Y_1(\theta_s\omega)) ds | \\ & \quad + | \int_{t_1}^{+\infty} T_{t_1-s} P^- F(z_n(\theta_s\omega) + Y_1(\theta_s\omega)) ds \\ & \quad - \int_{t_2}^{+\infty} T_{t_2-s} P^- F(z_n(\theta_s\omega) + Y_1(\theta_s\omega)) ds | . \end{aligned}$$

For the first term, we have the following estimate,

$$\begin{aligned} & | \int_{-\infty}^{t_1} T_{t_1-s} P^+ F(z_n(\theta_s\omega) + Y_1(\theta_s\omega)) ds \\ & \quad - \int_{-\infty}^{t_2} T_{t_2-s} P^+ F(z_n(\theta_s\omega) + Y_1(\theta_s\omega)) ds | \\ & \leq | \int_{-\infty}^{t_1} T_{t_1-s} P^+ F(z_n(\theta_s\omega) + Y_1(\theta_s\omega)) ds \\ & \quad - \int_{-\infty}^{t_1} T_{t_2-s} P^+ F(z_n(\theta_s\omega) + Y_1(\theta_s\omega)) ds | \\ & \quad + | \int_{-\infty}^{t_1} T_{t_2-s} P^+ F(z_n(\theta_s\omega) + Y_1(\theta_s\omega)) ds \\ & \quad - \int_{-\infty}^{t_2} T_{t_2-s} P^+ F(z_n(\theta_s\omega) + Y_1(\theta_s\omega)) ds | \\ & = | \int_{-\infty}^{t_1} (T_{t_1-s} P^+ - T_{t_2-s} P^+) F(z_n(\theta_s\omega) + Y_1(\theta_s\omega)) ds | \\ & \quad + | \int_{t_1}^{t_2} T_{t_2-s} P^+ F(z_n(\theta_s\omega) + Y_1(\theta_s\omega)) ds | \\ & \leq \| F \|_\infty [\int_{t_1}^{t_2} \| T_{t_2-s} P^+ \| ds \\ & \quad + \int_{-\infty}^{t_1} \| T_{t_1-s} P^+ - T_{t_2-s} P^+ \| ds] \\ & \leq \| F \|_\infty [\int_{t_1}^{t_2} \| T_{t_2-s} P^+ \| ds \\ & \quad + \| I - T_{t_2-t_1} P^+ \| \int_{-\infty}^{t_1} \| T_{t_1-s} P^+ \| ds] \end{aligned}$$

$$\begin{aligned}
 &\leq \|F\|_\infty \left[\int_{t_1}^{t_2} \|T_{t_2-s}P^+\| ds \right. \\
 &\quad \left. + \|I - T_{t_2-t_1}P^+\| \int_{-\infty}^{t_1} e^{-(t_1-s)\mu_{m+1}} ds \right] \\
 &\leq \|F\|_\infty \left[\int_{t_1}^{t_2} \|T_{t_2-s}P^+\| ds \right. \\
 &\quad \left. + \|I - T_{t_2-t_1}P^+\| \frac{1}{\mu_{m+1}} \right],
 \end{aligned}$$

and by a similar argument to the second part, we have

$$\begin{aligned}
 &\left| \int_{t_1}^{+\infty} T_{t_1-s}P^- F(z_n(\theta_s\omega) + Y_1(\theta_s\omega)) ds \right. \\
 &\quad \left. - \int_{t_2}^{+\infty} T_{t_2-s}P^- F(z_n(\theta_s\omega) + Y_1(\theta_s\omega)) ds \right| \\
 &\leq \left| \int_{t_1}^{+\infty} T_{t_1-s}P^- F(z_n(\theta_s\omega) + Y_1(\theta_s\omega)) ds \right. \\
 &\quad \left. - \int_{t_2}^{+\infty} T_{t_1-s}P^- F(z_n(\theta_s\omega) + Y_1(\theta_s\omega)) ds \right| \\
 &\quad + \left| \int_{t_2}^{+\infty} T_{t_1-s}P^- F(z_n(\theta_s\omega) + Y_1(\theta_s\omega)) ds \right. \\
 &\quad \left. - \int_{t_2}^{+\infty} T_{t_2-s}P^- F(z_n(\theta_s\omega) + Y_1(\theta_s\omega)) ds \right| \\
 &= \left| \int_{t_2}^{+\infty} (T_{t_1-s}P^- - T_{t_2-s}P^-) F(z_n(\theta_s\omega) + Y_1(\theta_s\omega)) ds \right| \\
 &\quad + \left| \int_{t_1}^{t_2} T_{t_1-s}P^- F(z_n(\theta_s\omega) + Y_1(\theta_s\omega)) ds \right| \\
 &\leq \|F\|_\infty \left[\int_{t_1}^{t_2} \|T_{t_1-s}P^-\| ds \right. \\
 &\quad \left. + \int_{t_2}^{+\infty} \|T_{t_1-s}P^- - T_{t_2-s}P^-\| ds \right] \\
 &\leq \|F\|_\infty \left[\int_{t_1}^{t_2} \|T_{t_1-s}P^-\| ds \right. \\
 &\quad \left. + \|T_{t_1-t_2}P^- - I\| \int_{t_2}^{+\infty} \|T_{t_2-s}P^-\| ds \right] \\
 &\leq \|F\|_\infty \left[\int_{t_1}^{t_2} \|T_{t_1-s}P^-\| ds \right. \\
 &\quad \left. + \|T_{t_1-t_2}P^- - I\| \int_{t_2}^{+\infty} e^{-(t_2-s)\mu_m} ds \right]
 \end{aligned}$$

$$\begin{aligned} &\leq \|F\|_\infty \left[\int_{t_1}^{t_2} \|T_{t_1-s}P^-\| ds \right. \\ &\quad \left. + \|T_{t_1-t_2}P^- - I\| \left(-\frac{1}{\mu_m}\right) \right]. \end{aligned}$$

Therefore, by combining two parts, we have

$$\begin{aligned} &|z_{n+1}(\theta_{t_1}\omega) - z_{n+1}(\theta_{t_2}\omega)| \\ &\leq \|F\|_\infty \left[\int_{t_1}^{t_2} \|T_{t_2-s}P^+\| ds + \int_{t_1}^{t_2} \|T_{t_1-s}P^-\| ds \right. \\ &\quad \left. + \frac{1}{\mu_{m+1}} \|I - T_{t_2-t_1}P^+\| - \frac{1}{\mu_m} \|T_{t_1-t_2}P^- - I\| \right]. \end{aligned}$$

Note when $t_1 \leq s \leq t_2$ as $\mu_m < 0$ and $\mu_{m+1} > 0$,

$$\|T_{t_2-s}P^+\| \leq e^{-(t_2-s)\mu_{m+1}} \leq 1,$$

and

$$\|T_{t_1-s}P^-\| \leq e^{-(t_1-s)\mu_m} \leq 1.$$

We also know that T_t is a strongly continuous semigroup. Thus, from the above arguments, we can easily check $z_{n+1}(\theta_t\omega)$ is uniformly continuous on t for all n . Then we say the sequence $\{z_n(\theta_t\omega)\}_{n=0}^\infty$ is equicontinuous. Moreover, for the boundedness of this sequence, it is easy to see that

$$\begin{aligned} |z_{n+1}(\theta_t\omega)| &\leq \|F\|_\infty \left[\int_{-\infty}^t \|T_{t-s}P^+\| ds + \int_t^{+\infty} \|T_{t-s}P^-\| ds \right] \\ &\leq \|F\|_\infty \left[\int_{-\infty}^t e^{-(t-s)\mu_{m+1}} ds + \int_t^{+\infty} e^{-(t-s)\mu_m} ds \right] \\ &\leq \|F\|_\infty \left(\frac{1}{\mu_{m+1}} - \frac{1}{\mu_m} \right) \\ &< \infty. \end{aligned}$$

Hence, we can use the Arzela-Ascoli Theorem on the sequence $\{z_n(\theta_t\omega)\}_{n=0}^\infty$. For arbitrarily large $N > 0$, we firstly have that the time set $T = [-N, N]$ is a compact set. Then the set $\{z_n(\theta_t\omega)\}_{n=0}^\infty$ is conditionally compact. This means there exists at least one subsequence $z_{n_k}(\theta_t\omega)$ such that

$$z_{n_k}(\theta_t\omega) \rightarrow z(\theta_t\omega)$$

as $k \rightarrow \infty$, for any $t \in [-N, N]$. Next, we need to lift the limit from $T = [-N, N]$ to $T = (-\infty, +\infty)$. For this, we see that

$$z_{n+1}(\omega) = \int_{-\infty}^0 T_{-s}P^+ F(z_n(\theta_s\omega) + Y_1(\theta_s\omega)) ds$$

$$\begin{aligned}
 & - \int_0^{+\infty} T_{-s} P^- F(z_n(\theta_s \omega) + Y_1(\theta_s \omega)) ds \\
 = & \int_{-N}^0 T_{-s} P^+ F(z_n(\theta_s \omega) + Y_1(\theta_s \omega)) ds \\
 & - \int_0^N T_{-s} P^- F(z_n(\theta_s \omega) + Y_1(\theta_s \omega)) ds \\
 & + \int_{-\infty}^{-N} T_{-s} P^+ F(z_n(\theta_s \omega) + Y_1(\theta_s \omega)) ds \\
 & - \int_N^{+\infty} T_{-s} P^- F(z_n(\theta_s \omega) + Y_1(\theta_s \omega)) ds.
 \end{aligned}$$

So

$$\begin{aligned}
 |z_{n+1}(\omega) & - \int_{-N}^0 T_{-s} P^+ F(z_n(\theta_s \omega) + Y_1(\theta_s \omega)) ds \\
 & + \int_0^N T_{-s} P^- F(z_n(\theta_s \omega) + Y_1(\theta_s \omega)) ds| \\
 \leq & | \int_{-\infty}^{-N} T_{-s} P^+ F(z_n(\theta_s \omega) + Y_1(\theta_s \omega)) ds | \\
 & + | \int_N^{+\infty} T_{-s} P^- F(z_n(\theta_s \omega) + Y_1(\theta_s \omega)) ds | \\
 \leq & \|F\|_\infty [\int_{-\infty}^{-N} \|T_{-s} P^+\| ds + \int_N^{+\infty} \|T_{-s} P^-\| ds] \\
 \leq & \|F\|_\infty \left(\frac{1}{\mu_{m+1}} e^{-\mu_{m+1} N} - \frac{1}{\mu_m} e^{\mu_m N} \right).
 \end{aligned}$$

For the above inequality, we firstly take the limit for the terms on the left side. when $n \rightarrow \infty$, we have

$$\begin{aligned}
 |z(\omega) & - \int_{-N}^0 T_{-s} P^+ F(z(\theta_s \omega) + Y_1(\theta_s \omega)) ds \\
 & + \int_0^N T_{-s} P^- F(z(\theta_s \omega) + Y_1(\theta_s \omega)) ds| \\
 \leq & \|F\|_\infty \left(\frac{1}{\mu_{m+1}} e^{-\mu_{m+1} N} - \frac{1}{\mu_m} e^{\mu_m N} \right).
 \end{aligned}$$

Then taking the limit $N \rightarrow \infty$, noticing $z(\theta_s \omega)$ is well defined for all $s \in (-\infty, +\infty)$ since N can be arbitrarily big. Thus,

$$\begin{aligned}
 z(\omega) & = \int_{-\infty}^0 T_{-s} P^+ F(z(\theta_s \omega) + Y_1(\theta_s \omega)) ds \\
 & - \int_0^{+\infty} T_{-s} P^- F(z(\theta_s \omega) + Y_1(\theta_s \omega)) ds.
 \end{aligned}$$

Finally, we add Y_1 defined by the integral equation (3.3) to the above equation and also assume

$$Y(\omega) := z(\omega) + Y_1(\omega).$$

Then we have the following expression

$$\begin{aligned} Y(\omega) &= \int_{-\infty}^0 T_{-s} P^+ F(Y(\theta(s, \omega))) ds - \int_0^{\infty} T_{-s} P^- F(Y(\theta(s, \omega))) ds \\ &\quad + (\omega) \int_{-\infty}^0 T_{-s} P^+ B_0 dW(s) - (\omega) \int_0^{\infty} T_{-s} P^- B_0 dW(s) \end{aligned}$$

for all $\omega \in \Omega$. This is the end of the proof. $\#$

Proposition 3.2.2 *Assume all the conditions on A , B_0 and F in Proposition 3.2.1. Then the semilinear see*

$$\begin{aligned} du(t) &= [-Au(t) + F(u(t))]dt + B_0 dW(t), \\ u(0) &= x \in \mathbb{H}, \end{aligned}$$

has at least one stationary point $Y : \Omega \rightarrow \mathbb{H}$, such that

$$u(t, \omega, Y(\omega)) = Y(\theta_t \omega)$$

for all $t \geq 0$ and $\omega \in \Omega$.

Proof. By the last proof, we have

$$\begin{aligned} Y(\theta_t \omega) &= \int_{-\infty}^t T_{t-s} P^+ F(Y(\theta(s, \omega))) ds - \int_t^{\infty} T_{t-s} P^- F(Y(\theta(s, \omega))) ds \\ &\quad + (\omega) \int_{-\infty}^t T_{t-s} P^+ B_0 dW(s) - (\omega) \int_t^{\infty} T_{t-s} P^- B_0 dW(s) \\ &= \int_{-\infty}^0 T_{t-s} P^+ F(Y(\theta(s, \omega))) ds - \int_0^{\infty} T_{t-s} P^- F(Y(\theta(s, \omega))) ds \\ &\quad + (\omega) \int_{-\infty}^0 T_{t-s} P^+ B_0 dW(s) - (\omega) \int_0^{\infty} T_{t-s} P^- B_0 dW(s) \\ &\quad + \int_0^t T_{t-s} P^+ F(Y(\theta(s, \omega))) ds - \int_t^0 T_{t-s} P^- F(Y(\theta(s, \omega))) ds \\ &\quad + (\omega) \int_0^t T_{t-s} P^+ B_0 dW(s) - (\omega) \int_t^0 T_{t-s} P^- B_0 dW(s) \\ &= T_t Y(\omega) + \int_0^t T_{t-s} F(Y(\theta_s \omega)) ds + (\omega) \int_0^t T_{t-s} B_0 dW(s). \end{aligned}$$

Therefore, $Y(\theta_t\omega)$, $t \geq 0$, $\omega \in \Omega$ is a stationary solution with the starting point $x = Y(\omega)$, since by the uniqueness of the solution, we have $u(t, \omega, Y(\omega))$ is also a solution and

$$u(t, \omega, Y(\omega)) = Y(\theta_t\omega)$$

for all $t \geq 0$ and $\omega \in \Omega$. This stationary point maybe non-unique. This is because in Proposition 3.2.1, the Arzela-Ascoli compactness argument can not guarantee the uniqueness. This finishes the proof. \sharp

§3.3 Weaken the Condition of F

In Mohammed, Zhang and Zhao's paper [25], it is difficult to remove both the restriction of Lipschitz constant and the globally boundedness condition in the same time. Our purpose in this section is to push the results of last section further to find a weaker condition to replace the globally bounded condition for F . Now consider the following equation with a standard cut off function F_n ,

$$\begin{aligned} z(t) = & \int_{-\infty}^t T_{t-s}P^+ F_n(z(s) + Y_1(s))ds \\ & - \int_t^{\infty} T_{t-s}P^- F_n(z(s) + Y_1(s))ds \end{aligned} \quad (3.4)$$

for all $z(\theta_t\omega) \in C_B(T, \mathbb{H})$ and all $\omega \in \Omega$. Here

$$F_n := \begin{cases} F, & \text{if } |F| \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

And F is a function from

$$\mathbb{H} \rightarrow \mathbb{H}.$$

Then we see that F_n is bounded whatever F is. By the previous proof, we have, as F_n is bounded, there exists at least one $z(t)_{t \in T}$. And the existence property depends on n , such that

$$\|z\|_{\infty} \leq B_n,$$

where B_n is the radius of a closed ball which depends on n and is dominated by F_n such that

$$B_n := \|F_n\|_{\infty} \left(\frac{1}{\mu_{m+1}} - \frac{1}{\mu_m} \right).$$

Here comes a new idea. If we can prove $z(t)_{t \in T}$ exists and does not depend on n , such that

$$\|z\|_{\infty} \leq B'.$$

If such kind of B' exists, this is to say we can always choose n big enough to cover every F such that

$$F_n = F,$$

and the globally bounded condition for F will be possible to be omitted. This is the idea we are going to work out. Before we start our hard trip, it is necessary to enhance us with a powerful weapon. We introduce next with the famous Gronwall inequality in forward form and backward form, respectively.

§3.3.1 Gronwall Inequality

Gronwall inequality is a famous tool in many fields of mathematics. Here we present a generalized one-dimensional form. We start with the one-dimensional ODE in the inhomogeneous case,

$$x' = \gamma(t)x + f(t).$$

To solve this ODE, we use the variation of constant method. See Hartman [13] for the elementary proof. The Gronwall inequality comes from this proof. Actually, for the inequality proof, we only need require x to be non-negative. Here comes the generalized Gronwall inequality.

Forward Gronwall Inequality

Let $x(t)$ be a \mathbb{R}^+ -valued function on $[a, b]$, $\beta(t)$ and $\gamma(t)$ are \mathbb{R} -valued functions, α is a constant and if

$$x(t) \leq \alpha + \beta(t) + \int_a^t \gamma(s)x(s)ds,$$

for all $a \leq t \leq b$. Then we have

$$x(t) \leq [\alpha + \beta(a)]e^{\int_a^t \gamma(s)ds} + \int_a^t \beta'(s)e^{\int_s^t \gamma(r)dr} ds,$$

for all $a \leq t \leq b$.

The proof is elementary, similar to the proof of finding ODE solution. We omit it here. For our research, we are also interesting in the backward type Gronwall inequality. We next deduce it from the forward one.

Backward Gronwall Inequality

Let $x(t)$ be a \mathbb{R}^+ -valued function on $[b, a]$, $\beta(t)$ and $\gamma(t)$ are \mathbb{R} -valued functions, α is a constant and if

$$x(t) \leq \alpha + \beta(t) + \int_t^a \gamma(s)x(s)ds,$$

for all $b \leq t \leq a$. Then we have

$$x(t) \leq [\alpha + \beta(a)]e^{\int_t^a \gamma(s)ds} - \int_t^a \beta'(s)e^{\int_t^s \gamma(r)dr} ds,$$

for all $b \leq t \leq a$.

Proof. We see that the inequality is changed to

$$x(t) \leq \alpha + \beta(t) + \int_t^a \gamma(s)x(s)ds,$$

for all $t \leq a$. Define

$$x(t) = x(a - (a - t)) := z(a - t).$$

Then

$$z(t) = x(a - t)$$

for $t \geq 0$. Moreover,

$$\begin{aligned} z(t) &= x(a - t) \\ &\leq \alpha + \beta(a - t) + \int_{a-t}^a \gamma(s)x(s)ds. \end{aligned}$$

Changing variable by applying $s = a - \tau$, we then have

$$\begin{aligned} z(t) &\leq \alpha + \beta(a - t) + \int_t^0 \gamma(a - \tau)x(a - \tau)d(-\tau) \\ &\leq \alpha + \beta(a - t) + \int_0^t \gamma(a - s)z(s)ds. \end{aligned}$$

So by forward Gronwall inequality

$$z(t) \leq [\alpha + \beta(a)]e^{\int_0^t \gamma(a-s)ds} + \int_0^t \frac{d}{ds}\beta(a-s)e^{\int_s^t \gamma(a-\tau)d\tau} ds.$$

Thus,

$$\begin{aligned} x(t) &= z(a - t) \\ &\leq [\alpha + \beta(a)]e^{\int_0^{a-t} \gamma(a-s)ds} + \int_0^{a-t} \frac{d}{ds}\beta(a-s)e^{\int_s^{a-t} \gamma(a-\tau)d\tau} ds \end{aligned}$$

$$\begin{aligned}
 &= [\alpha + \beta(a)]e^{\int_a^t \gamma(s)d(-s)} + \int_a^t [-\beta'(r)]e^{\int_a^{-r} \gamma(a-\tau)d\tau} d(-r) \\
 &= [\alpha + \beta(a)]e^{\int_t^a \gamma(s)ds} + \int_t^a [-\beta'(r)]e^{\int_t^r \gamma(\tau)d\tau} dr \\
 &= [\alpha + \beta(a)]e^{\int_t^a \gamma(s)ds} - \int_t^a \beta'(s)e^{\int_t^s \gamma(\tau)d\tau} ds
 \end{aligned}$$

This finishes the proof. \square

§3.3.2 Main Results

In this part, our purpose is to find an alternative condition to replace the global boundedness condition for F . We start by considering the structure of the infinite-dimensional Hilbert space

$$\mathbb{H} = \mathbb{H}^+ \oplus \mathbb{H}^-.$$

From (3.4), for $z \in \mathbb{H}$ and a given $Y_1 \in \mathbb{H}$, we have

$$\begin{aligned}
 z(t) &:= (z^+(t), z^-(t)), \\
 Y_1(t) &:= (Y_1^+(t), Y_1^-(t))
 \end{aligned}$$

where $z^+(t), Y_1^+(t) \in \mathbb{H}^+$ and $z^-(t), Y_1^-(t) \in \mathbb{H}^-$. Then (3.4) can be expressed by two parts

$$z^+(t) = \int_{-\infty}^t T_{t-s} P^+ F_n(z^+(s) + Y_1^+(s), z^-(s) + Y_1^-(s)) ds \quad (3.5)$$

and

$$z^-(t) = - \int_t^{\infty} T_{t-s} P^- F_n(z^+(s) + Y_1^+(s), z^-(s) + Y_1^-(s)) ds. \quad (3.6)$$

Denote by $\{e_n, n \geq 1\}$ a basis for \mathbb{H} , by $\{\mu_n, n \geq 1\}$ the discrete non-vanishing spectrum of the operator $-A$. μ_m is the largest negative eigenvalue and μ_{m+1} is the smallest positive eigenvalue. Then assume

$$\begin{aligned}
 z_1^-(t) &= (z^-(t), e_1), \\
 z_2^-(t) &= (z^-(t), e_2), \\
 &\vdots \\
 z_m^-(t) &= (z^-(t), e_m), \\
 z_{m+1}^+(t) &= (z^+(t), e_{m+1}), \\
 z_{m+2}^+(t) &= (z^+(t), e_{m+2}),
 \end{aligned}$$

⋮

Consider the differential forms of (3.5) and (3.6) according to each eigenvalue of A , we have

$$\begin{aligned}
 \frac{dz_1^-(t)}{dt} &= -\mu_1 z_1^-(t) + (F_n(z^+(t) + Y_1^+(t), z^-(t) + Y_1^-(t)), e_1), \\
 \frac{dz_2^-(t)}{dt} &= -\mu_2 z_2^-(t) + (F_n(z^+(t) + Y_1^+(t), z^-(t) + Y_1^-(t)), e_2), \\
 &\vdots \\
 \frac{dz_m^-(t)}{dt} &= -\mu_m z_m^-(t) + (F_n(z^+(t) + Y_1^+(t), z^-(t) + Y_1^-(t)), e_m), \\
 \frac{dz_{m+1}^+(t)}{dt} &= -\mu_{m+1} z_{m+1}^+(t) + (F_n(z^+(t) + Y_1^+(t), z^-(t) + Y_1^-(t)), e_{m+1}), \\
 \frac{dz_{m+2}^+(t)}{dt} &= -\mu_{m+2} z_{m+2}^+(t) + (F_n(z^+(t) + Y_1^+(t), z^-(t) + Y_1^-(t)), e_{m+2}), \\
 &\vdots
 \end{aligned}$$

Multiplying with $z_1^-(t), z_2^-(t), \dots, z_m^-(t), z_{m+1}^+(t), z_{m+2}^+(t) \dots$ for each equation respectively, we have

$$\begin{aligned}
 \frac{1}{2} \frac{d(z_1^-)^2(t)}{dt} &= -\mu_1 (z_1^-)^2(t) + z_1^-(t) (F_n(z^+(t) + Y_1^+(t), z^-(t) + Y_1^-(t)), e_1), \\
 \frac{1}{2} \frac{d(z_2^-)^2(t)}{dt} &= -\mu_2 (z_2^-)^2(t) + z_2^-(t) (F_n(z^+(t) + Y_1^+(t), z^-(t) + Y_1^-(t)), e_2), \\
 &\vdots \\
 \frac{1}{2} \frac{d(z_m^-)^2(t)}{dt} &= -\mu_m (z_m^-)^2(t) + z_m^-(t) (F_n(z^+(t) + Y_1^+(t), z^-(t) + Y_1^-(t)), e_m), \\
 \frac{1}{2} \frac{d(z_{m+1}^+)^2(t)}{dt} &= -\mu_{m+1} (z_{m+1}^+)^2(t) + z_{m+1}^+(t) (F_n(z^+(t) + Y_1^+(t), z^-(t) \\
 &\quad + Y_1^-(t)), e_{m+1}) \\
 \frac{1}{2} \frac{d(z_{m+2}^+)^2(t)}{dt} &= -\mu_{m+2} (z_{m+2}^+)^2(t) + z_{m+2}^+(t) (F_n(z^+(t) + Y_1^+(t), z^-(t) \\
 &\quad + Y_1^-(t)), e_{m+2}) \\
 &\vdots
 \end{aligned}$$

Since the spectrum $\{\mu_d, d \geq 1\}$ is non-vanishing, this means

$$\mu_1 < \mu_2 < \dots < \mu_m < 0 < \mu_{m+1} < \mu_{m+2} < \dots,$$

we then have

$$\frac{1}{2} \frac{d(z_1^-)^2(t)}{dt} \geq -\mu_m (z_1^-)^2(t) + z_1^-(t) (F_n(z^+(t) + Y_1^+(t), z^-(t) + Y_1^-(t)), e_1),$$

$$\begin{aligned}
 \frac{1}{2} \frac{d(z_2^-)^2(t)}{dt} &\geq -\mu_m(z_2^-)^2(t) + z_2^-(t)(F_n(z^+(t) + Y_1^+(t), z^-(t) + Y_1^-(t)), e_2), \\
 &\vdots \\
 \frac{1}{2} \frac{d(z_m^-)^2(t)}{dt} &= -\mu_m(z_m^-)^2(t) + z_m^-(t)(F_n(z^+(t) + Y_1^+(t), z^-(t) + Y_1^-(t)), e_m), \\
 \frac{1}{2} \frac{d(z_{m+1}^+)^2(t)}{dt} &= -\mu_{m+1}(z_{m+1}^+)^2(t) + z_{m+1}^+(t)(F_n(z^+(t) + Y_1^+(t), z^-(t) \\
 &\quad + Y_1^-(t)), e_{m+1}) \\
 \frac{1}{2} \frac{d(z_{m+2}^+)^2(t)}{dt} &\leq -\mu_{m+1}(z_{m+2}^+)^2(t) + z_{m+2}^+(t)(F_n(z^+(t) + Y_1^+(t), z^-(t) \\
 &\quad + Y_1^-(t)), e_{m+2}) \\
 &\vdots
 \end{aligned}$$

We consider the above differential inequalities according to the positive and negative eigenvalues. For the first m differential inequalities, we consider the backward integral equations. For the rest inequalities, we consider the forward integral equations. Then we have

$$\begin{aligned}
 (z_1^-)^2(t) &\leq \int_t^\infty 2\mu_m(z_1^-)^2(s)ds - \int_t^\infty 2z_1^-(s)(F_n(z^+(s) + Y_1^+(s), z^-(s) \\
 &\quad + Y_1^-(s)), e_1)ds, \\
 (z_2^-)^2(t) &\leq \int_t^\infty 2\mu_m(z_2^-)^2(s)ds - \int_t^\infty 2z_2^-(s)(F_n(z^+(s) + Y_1^+(s), z^-(s) \\
 &\quad + Y_1^-(s)), e_2)ds, \\
 &\vdots \\
 (z_m^-)^2(t) &= \int_t^\infty 2\mu_m(z_m^-)^2(s)ds - \int_t^\infty 2z_m^-(s)(F_n(z^+(s) + Y_1^+(s), z^-(s) \\
 &\quad + Y_1^-(s)), e_m)ds, \\
 (z_{m+1}^+)^2(t) &= \int_{-\infty}^t -2\mu_{m+1}(z_{m+1}^+)^2(s)ds + \int_{-\infty}^t 2z_{m+1}^+(s)(F_n(z^+(s) + Y_1^+(s), z^-(s) \\
 &\quad + Y_1^-(s)), e_{m+1})ds, \\
 (z_{m+2}^+)^2(t) &\leq \int_{-\infty}^t -2\mu_{m+1}(z_{m+2}^+)^2(s)ds + \int_{-\infty}^t 2z_{m+2}^+(s)(F_n(z^+(s) + Y_1^+(s), z^-(s) \\
 &\quad + Y_1^-(s)), e_{m+2})ds, \\
 &\vdots
 \end{aligned}$$

Then applying the forward and backward Gronwall inequality for each differential inequality, we have

$$(z_1^-)^2(t) \leq -2 \int_t^\infty e^{-(t-s)2\mu_m} z_1^-(s)(F_n(z^+(s) + Y_1^+(s), z^-(s) + Y_1^-(s)), e_1)ds,$$

$$\begin{aligned}
 (z_2^-)^2(t) &\leq -2 \int_t^\infty e^{-(t-s)2\mu_m} z_2^-(s) (F_n(z^+(s) + Y_1^+(s), z^-(s) + Y_1^-(s)), e_2) ds, \\
 &\vdots \\
 (z_m^-)^2(t) &= -2 \int_t^\infty e^{-(t-s)2\mu_m} z_m^-(s) (F_n(z^+(s) + Y_1^+(s), z^-(s) + Y_1^-(s)), e_m) ds, \\
 (z_{m+1}^+)^2(t) &= 2 \int_{-\infty}^t e^{-(t-s)2\mu_{m+1}} z_{m+1}^+(s) (F_n(z^+(s) + Y_1^+(s), z^-(s) + Y_1^-(s)), e_{m+1}) ds, \\
 (z_{m+2}^+)^2(t) &\leq 2 \int_{-\infty}^t e^{-(t-s)2\mu_{m+1}} z_{m+2}^+(s) (F_n(z^+(s) + Y_1^+(s), z^-(s) + Y_1^-(s)), e_{m+2}) ds, \\
 &\vdots
 \end{aligned}$$

Now we combine them into two types by writing

$$(z^+)^2(t) = (z_{m+1}^+)^2(t) + (z_{m+2}^+)^2(t) + \dots$$

$$(z^-)^2(t) = (z_1^-)^2(t) + (z_2^-)^2(t) + \dots + (z_m^-)^2(t)$$

and

$$\begin{aligned}
 (z^+(s) &, P^+ F_n(z^+(s) + Y_1^+(s), z^-(s) + Y_1^-(s))) \\
 &= z_{m+1}^+(s) (F_n(z^+(s) + Y_1^+(s), z^-(s) + Y_1^-(s)), e_{m+1}) \\
 &\quad + z_{m+2}^+(s) (F_n(z^+(s) + Y_1^+(s), z^-(s) + Y_1^-(s)), e_{m+2}) + \dots,
 \end{aligned}$$

$$\begin{aligned}
 (z^-(s) &, P^- F_n(z^+(s) + Y_1^+(s), z^-(s) + Y_1^-(s))) \\
 &= z_1^-(s) (F_n(z^+(s) + Y_1^+(s), z^-(s) + Y_1^-(s)), e_1) \\
 &\quad + z_2^-(s) (F_n(z^+(s) + Y_1^+(s), z^-(s) + Y_1^-(s)), e_2) + \dots \\
 &\quad + z_m^-(s) (F_n(z^+(s) + Y_1^+(s), z^-(s) + Y_1^-(s)), e_m).
 \end{aligned}$$

Then (3.5) and (3.6) change to

$$\begin{aligned}
 (z^+)^2(t) &\leq 2 \int_{-\infty}^t e^{-(t-s)2\mu_{m+1}} (z^+(s), P^+ F_n(z^+(s) + Y_1^+(s), z^-(s) \\
 &\quad + Y_1^-(s))) ds \tag{3.7}
 \end{aligned}$$

and

$$\begin{aligned}
 (z^-)^2(t) &\leq -2 \int_t^\infty e^{-(t-s)2\mu_m} (z^-(s), P^- F_n(z^+(s) + Y_1^+(s), z^-(s) \\
 &\quad + Y_1^-(s))) ds. \tag{3.8}
 \end{aligned}$$

From the inequalities (3.7) and (3.8), we see a hope to weaken the condition for F . This will become a coupling problem if we can do a Monotone change for different part of F_n . Then, in the later discussion, we will assume the function F_n need to satisfy the conditions as follows.

Assumption 1

$$\begin{aligned} (x, (P^+)F_n(x+a, y+b)) &\leq L_1x^2 + L_2y^2 + A_1 \\ (y, (-P^-)F_n(x+a, y+b)) &\leq L_3x^2 + L_4y^2 + B_1, \end{aligned}$$

where

$$\begin{aligned} L_1 &< \mu_{m+1}, \\ L_4 &< -\mu_m, \\ L_2, L_3 &\geq 0, \end{aligned}$$

and $A_1, B_1 \geq 0$ are constants.

We notice that L_1, L_2, L_3, L_4 and A_1, B_1 can be chosen to be independent on n since we can deduce this from F . Thus, we have (3.7) and (3.8) change to

$$(z^+)^2(t) \leq 2 \int_{-\infty}^t e^{-(t-s)2\mu_{m+1}} [L_1(z^+)^2(s) + L_2(z^-)^2(s) + A_1] ds$$

and

$$(z^-)^2(t) \leq 2 \int_t^{\infty} e^{-(t-s)2\mu_m} [L_3(z^+)^2(s) + L_4(z^-)^2(s) + B_1] ds.$$

This will lead to

$$(z^+)^2(t) \leq 2 \int_{-\infty}^t e^{-(t-s)2\mu_{m+1}} [L_1(z^+)^2(s) + L_2(z^-)^2(s)] ds + \frac{A_1}{\mu_{m+1}} \quad (3.9)$$

and

$$(z^-)^2(t) \leq 2 \int_t^{\infty} e^{-(t-s)2\mu_m} [L_3(z^+)^2(s) + L_4(z^-)^2(s)] ds - \frac{B_1}{\mu_m}. \quad (3.10)$$

In the next step we will apply the forward and backward Gronwall inequalities and coupling method. This leads to

$$e^{t2\mu_{m+1}} (z^+)^2(t) \leq \int_{-\infty}^t e^{s2\mu_{m+1}} 2L_2(z^-)^2(s) ds + \frac{A_1}{\mu_{m+1}} e^{t2\mu_{m+1}}$$

$$+ \int_{-\infty}^t [e^{s2\mu_{m+1}}(z^+)^2(s)]2L_1 ds.$$

Then applying the forward Gronwall inequality to the above inequality, we immediately have

$$e^{t2\mu_{m+1}}(z^+)^2(t) \leq \int_{-\infty}^t e^{s2\mu_{m+1}}2L_2(z^-)^2(s)e^{2L_1(t-s)} ds + \int_{-\infty}^t 2A_1 e^{s2\mu_{m+1}} e^{2L_1(t-s)} ds.$$

So it is trivial to see that

$$\begin{aligned} (z^+)^2(t) &\leq \int_{-\infty}^t e^{(t-s)2(L_1-\mu_{m+1})}2L_2(z^-)^2(s) ds \\ &\quad + 2A_1 \int_{-\infty}^t e^{(t-s)2(L_1-\mu_{m+1})} ds \\ &\leq 2 \int_{-\infty}^t e^{(t-s)2(L_1-\mu_{m+1})} L_2(z^-)^2(s) ds + \frac{A_1}{\mu_{m+1} - L_1}. \end{aligned} \quad (3.11)$$

From (3.10) we have

$$\begin{aligned} e^{t2\mu_m}(z^-)^2(t) &\leq \int_t^{\infty} e^{s2\mu_m}2L_3(z^+)^2(s) ds - \frac{B_1}{\mu_m} e^{t2\mu_m} \\ &\quad + \int_t^{\infty} [e^{s2\mu_m}(z^-)^2(s)]2L_4 ds. \end{aligned}$$

Applying the backward Gronwall inequality, we have

$$\begin{aligned} e^{t2\mu_m}(z^-)^2(t) &\leq \int_t^{\infty} e^{s2\mu_m}2L_3(z^+)^2(s)e^{2L_4(s-t)} ds \\ &\quad + \int_t^{\infty} 2B_1 e^{s2\mu_m} e^{2L_4(s-t)} ds. \end{aligned}$$

So it is trivial to see that

$$\begin{aligned} (z^-)^2(t) &\leq \int_t^{\infty} e^{(s-t)2(\mu_m+L_4)}2L_3(z^+)^2(s) ds \\ &\quad + 2B_1 \int_t^{\infty} e^{(s-t)2(\mu_m+L_4)} ds \\ &\leq 2 \int_t^{\infty} e^{(s-t)2(\mu_m+L_4)} L_3(z^+)^2(s) ds - \frac{B_1}{\mu_m + L_4}. \end{aligned} \quad (3.12)$$

Observing (3.11) and (3.12), we see that if we prove one of $(z^+)(t)$ and $(z^-)(t)$ is bounded, the other one can be deduced to be bounded automatically. Next, we substitute the term $(z^-)^2(s)$ in (3.11) by the inequality (3.12). Then we can use the change of integration order to get

$$(z^+)^2(t) \leq 2 \int_{-\infty}^t e^{(t-s)2(L_1-\mu_{m+1})} L_2 [2 \int_s^{\infty} e^{(r-s)2(\mu_m+L_4)} L_3(z^+)^2(r) dr$$

$$\begin{aligned}
 & -\frac{B_1}{\mu_m + L_4}]ds + \frac{A_1}{\mu_{m+1} - L_1} \\
 \leq & 4 \int_{-\infty}^t e^{(t-s)2(L_1 - \mu_{m+1})} L_2 \int_s^\infty e^{(r-s)2(\mu_m + L_4)} L_3 (z^+)^2(r) dr ds + M \\
 = & 4L_2 L_3 \left[\int_{-\infty}^t \int_{-\infty}^r e^{2(L_1 - \mu_{m+1})(t-s) - 2(\mu_m + L_4)(s-r)} ds (z^+)^2(r) dr \right. \\
 & \left. + \int_t^\infty \int_{-\infty}^t e^{2(L_1 - \mu_{m+1})(t-s) - 2(\mu_m + L_4)(s-r)} ds (z^+)^2(r) dr \right] + M \\
 = & 4L_2 L_3 \left[\int_{-\infty}^t e^{2(L_1 - \mu_{m+1})t + 2(\mu_m + L_4)r} \int_{-\infty}^r e^{2(\mu_{m+1} - L_1 - \mu_m - L_4)s} ds (z^+)^2(r) dr \right. \\
 & \left. + \int_t^\infty e^{2(L_1 - \mu_{m+1})t + 2(\mu_m + L_4)r} \int_{-\infty}^t e^{2(\mu_{m+1} - L_1 - \mu_m - L_4)s} ds (z^+)^2(r) dr \right] + M \\
 \leq & \frac{2L_2 L_3}{\mu_{m+1} - L_1 - \mu_m - L_4} \\
 & \left[\int_{-\infty}^t e^{2(L_1 - \mu_{m+1})t + 2(\mu_m + L_4)r + 2(\mu_{m+1} - L_1 - \mu_m - L_4)r} (z^+)^2(r) dr \right. \\
 & \left. + \int_t^\infty e^{2(L_1 - \mu_{m+1})t + 2(\mu_m + L_4)r + 2(\mu_{m+1} - L_1 - \mu_m - L_4)t} (z^+)^2(r) dr \right] + M \\
 = & \lambda \left[\int_{-\infty}^t e^{2(L_1 - \mu_{m+1})(t-s)} (z^+)^2(s) ds \right. \\
 & \left. + \int_t^\infty e^{2(\mu_m + L_4)(s-t)} (z^+)^2(s) ds \right] + M, \tag{3.13}
 \end{aligned}$$

where

$$M := \frac{A_1}{\mu_{m+1} - L_1} - \frac{L_2 B_1}{(\mu_{m+1} - L_1)(\mu_m + L_4)} > 0,$$

and

$$\lambda := \frac{2L_2 L_3}{\mu_{m+1} - L_1 - \mu_m - L_4} > 0.$$

Denote

$$\alpha := \max\{2(\mu_{m+1} - L_1), -2(\mu_m + L_4)\},$$

$$\beta := \min\{2(\mu_{m+1} - L_1), -2(\mu_m + L_4)\}.$$

Then $\alpha, \beta > 0$, and

$$(z^+)^2(t) \leq M + \lambda \int_{-\infty}^t e^{-\beta(t-s)} (z^+)^2(s) ds$$

$$+ \int_t^\infty e^{-\beta(s-t)}(z^+)^2(s)ds.$$

For the above inequality, we consider a variable change for term $\int_t^\infty e^{-\beta(s-t)}(z^+)^2(s)ds$, then

$$\begin{aligned} \int_t^\infty e^{-\beta(s-t)}(z^+)^2(s)ds &= \int_{-\infty}^{-t} e^{-\beta(-\tau-t)}(z^+)^2(-\tau)d\tau \\ &= \int_{-\infty}^{-t} e^{-\beta(-s-t)}(z^+)^2(-s)ds. \end{aligned}$$

Hence

$$\begin{aligned} (z^+)^2(t) &\leq M + \lambda \left(\int_{-\infty}^t e^{-\beta(t-s)}(z^+)^2(s)ds \right. \\ &\quad \left. + \int_{-\infty}^{-t} e^{-\beta(-s-t)}(z^+)^2(-s)ds \right). \end{aligned} \quad (3.14)$$

Replacing t by $-t$ into (3.14), we have a new form

$$\begin{aligned} (z^+)^2(-t) &\leq M + \lambda \left(\int_{-\infty}^{-t} e^{-\beta(-t-s)}(z^+)^2(s)ds \right. \\ &\quad \left. + \int_{-\infty}^t e^{-\beta(t-s)}(z^+)^2(-s)ds \right). \end{aligned} \quad (3.15)$$

Adding (3.14) and (3.15) together, we have

$$\begin{aligned} (z^+)^2(t) + (z^+)^2(-t) &\leq 2M + \lambda \left[\int_{-\infty}^t e^{-\beta(t-s)}((z^+)^2(s) + (z^+)^2(-s))ds \right. \\ &\quad \left. + \int_{-\infty}^{-t} e^{-\beta(-s-t)}((z^+)^2(s) + (z^+)^2(-s))ds \right]. \end{aligned}$$

Observing the above inequality, we find that it becomes an induction problem. Let

$$G'(t) = (z^+)^2(t) + (z^+)^2(-t).$$

Then $G'(t) \geq 0$ and

$$G'(t) \leq 2M + \lambda \left(\int_{-\infty}^t e^{-\beta(t-s)}G'(s)ds + \int_{-\infty}^{-t} e^{-\beta(-s-t)}G'(s)ds \right). \quad (3.16)$$

For the estimation of this inequality, we use the induction method by assuming the starting point $G'_1(t) \leq 2M$, then

$$\begin{aligned} G'_1(t) &\leq 2M \\ G'_2(t) &\leq 2M + \lambda \int_{-\infty}^t e^{-\beta(t-s)}(2M)ds + \lambda \int_{-\infty}^{-t} e^{-\beta(-s-t)}(2M)ds \end{aligned}$$

$$\begin{aligned}
 &\leq 2M + 2M\left(\frac{2\lambda}{\beta}\right) \\
 G'_3(t) &\leq 2M + \lambda \int_{-\infty}^t e^{-\beta(t-s)}(2M + 2M\left(\frac{2\lambda}{\beta}\right))ds \\
 &\quad + \lambda \int_{-\infty}^{-t} e^{-\beta(-s-t)}(2M + 2M\left(\frac{2\lambda}{\beta}\right))ds \\
 &\leq 2M + 2M\left(\frac{2\lambda}{\beta}\right) + 2M\left(\frac{2\lambda}{\beta}\right)^2 \\
 &\quad \vdots \\
 G'_m(t) &\leq 2M + 2M\left(\frac{2\lambda}{\beta}\right) + 2M\left(\frac{2\lambda}{\beta}\right)^2 + \dots + 2M\left(\frac{2\lambda}{\beta}\right)^{m-1} \\
 &\quad \vdots
 \end{aligned}$$

We see from the induction, if $G'_m(t)$ has a uniform bound, we require $\frac{2\lambda}{\beta} < 1$. This means we need

$$\frac{\frac{2L_2L_3}{\mu_{m+1}-L_1-\mu_m-L_4}}{\beta} = \frac{4L_2L_3}{(\alpha + \beta)\beta} < 1.$$

And this leads to

$$L_2L_3 < \frac{1}{4}(\alpha + \beta)\beta.$$

Hence, with this condition, we have that $G'_m(t)$ has a uniform bound which does not depend on n . This means $(z^+)^2(t) + (z^+)^2(-t)$ is bounded uniformly in n . And since $(z^+)^2(t)$ and $(z^+)^2(-t)$ must be non-negative, we have $(z^+)^2(t)$ has a uniform bound. Replacing this bound into (14), we obtain a bound for $(z^-)^2(t)$. Then we have a bound for $|z(t)|$, since

$$|z(t)| = ((z^+(t))^2 + (z^-(t))^2)^{\frac{1}{2}}.$$

And this bound completely does not depend on n of F_n . Hence, we can choose n big enough such that

$$F_n = F.$$

Then, the globally boundedness condition for F can be omitted now. Instead, the Assumption 1 changes to

Assumption 2

$$\begin{aligned}
 (x, (P^+)F(x + a, y + b)) &\leq L_1x^2 + L_2y^2 + A_1 \\
 (y, (-P^-)F(x + a, y + b)) &\leq L_3x^2 + L_4y^2 + B_1
 \end{aligned}$$

where

$$\begin{aligned} L_1 &< \mu_{m+1}, \\ L_4 &< -\mu_m, \\ L_2 &, \quad L_3 \geq 0, \\ L_2 L_3 &< \frac{1}{4}(\alpha + \beta)\beta, \end{aligned}$$

and $A_1, B_1 \geq 0$ are constants.

To conclude, we have the following.

Proposition 3.3.1 *Assume conditions on A, B_0 in Proposition 3.1.5 and locally Lipschitz with Assumption 2 for F . Then there exists at least one \mathcal{F} -measurable map $Y : \Omega \rightarrow \mathbb{H}$ satisfying*

$$\begin{aligned} Y(\omega) &= \int_{-\infty}^0 T_{-s} P^+ F(Y(\theta(s, \omega))) ds - \int_0^{\infty} T_{-s} P^- F(Y(\theta(s, \omega))) ds \\ &\quad + (\omega) \int_{-\infty}^0 T_{-s} P^+ B_0 dW(s) - (\omega) \int_0^{\infty} T_{-s} P^- B_0 dW(s) \end{aligned}$$

for all $\omega \in \Omega$.

Proof. This follows from the above arguments. We only need to change the Assumption 1 by Assumption 2 and F_n by F from the beginning. $\#$

Proposition 3.3.2 *Assume all the conditions on A, B_0 and F in Proposition 3.3.1. Then the semilinear see*

$$\begin{aligned} du(t) &= [-Au(t) + F(u(t))]dt + B_0 dW(t), \\ u(0) &= x \in \mathbb{H}, \end{aligned}$$

has at least one stationary point $Y : \Omega \rightarrow \mathbb{H}$, such that

$$u(t, \omega, Y(\omega)) = Y(\theta_t \omega)$$

for all $t \geq 0$ and $\omega \in \Omega$.

Proof. This is straightforward to follow the proof of Proposition 3.2.2. $\#$

§3.4 Invariant Manifold

In this section, we present an invariant manifold theorem according to the previous results. Here we recall the setting and hypotheses in Section 3.2.1. We will see the existence of local stable and unstable manifolds for the RDS which is generated by the mild solution of the semilinear see (3.1) of the form

$$\begin{aligned} du(t) &= [-Au(t) + F(u(t))]dt + B_0dW(t), \\ u(0) &= x \in \mathbb{H}. \end{aligned}$$

Next, the theorem follows from the previous results.

Theorem 3.4.1 (Invariant Manifold Theorem)

Assume the hypotheses on the coefficients of the semilinear see (3.1) in Section 3.2.1. Assume that the stationary solution $Y(\omega)$ obtained in Proposition 3.3.2 of the following RDS

$$U : \mathbb{R}^+ \times \Omega \times \mathbb{H} \rightarrow \mathbb{H}$$

generated by mild solutions of (3.1) is hyperbolic. Then the random dynamical system (U, θ) has a local stable and unstable manifolds satisfying all the assertions of Theorem 3.1.3 in an infinite-dimensional manner.

Proof. To prove this, we need to check two points.

1. The hyperbolic property of Y , such that

$$E \log^+ |Y| < \infty.$$

2. The integrability condition of Theorem 3.1.3 which is

$$\int_{\Omega} \log^+ \sup_{0 \leq t_1, t_2 \leq a} \|U(t_2, \theta(t_1, \omega), Y(\theta(t_1, \omega)))\| d\mathbb{P}(\omega) < \infty$$

for any $0 < \rho, a < \infty$.

For (1), in Proposition 3.2.1, we have the facts

$$Y_1 \in L^p(\Omega, \mathbb{H})$$

for all $p \geq 1$, and

$$z_0 \in L^\infty(\Omega, \mathbb{H}).$$

These lead to

$$Y \in L^p(\Omega, \mathbb{H})$$

for all $p \geq 1$. Hence, assertion (1) is satisfied. For (2), we firstly have the hyperbolic stationary point Y is integrable. This has been shown above as

$$Y \in L^p(\Omega, \mathbb{H}).$$

In Mohammed, Zhang and Zhao [25], Theorem 1.2.6 has proved, in a semilinear see with a linear noise case, such kind of RDS (U, θ) exists and be integrable in \mathbb{H} , such that

$$E \log^+ \left\{ \sup_{0 \leq t_1, t_2 \leq a} \frac{|U(t_2, \theta(t_1, \cdot), x)|}{(1 + |x|)} \right\} < \infty$$

for all $\omega \in \Omega$, all $a > 0$ and $x \in \mathbb{H}$. Our case is the semilinear see with the additive noise. We can regard it as a special case. Then all the results in Theorem 1.2.6 of Mohammed, Zhang and Zhao [25] still hold. Therefore, the integrable condition is satisfied by the above inequality and the integrability of Y , and the conclusion of Theorem 3.4.1 follows immediately from Theorem 3.1.3. This finishes the proof. $\#$

§3.5 Further Research

In this chapter, we removed the Lipschitz constant restriction. However, for the global boundedness condition for F , it can be replaced by a weaker Assumption 2. Although they are weaker than the previous boundedness condition, we can not completely drop the constant restrictions for L_1, L_2, L_3 and L_4 . Further research is needed to relax these restrictions for L_1, L_2, L_3 and L_4 . One may find a weaker condition for F by using a different method.

In this whole chapter, we consider the semilinear stochastic evolution equation with additive noise. Actually, we may also possibly consider a semilinear see with the linear noise such as

$$\begin{aligned} du(t) &= [-Au(t) + F(u(t))]dt + Bu(t)dW(t), \\ u(0) &= x \in \mathbb{H}, \end{aligned}$$

where

$$B : \mathbb{H} \rightarrow L_2(K, \mathbb{H})$$

is a bounded linear operator. In Mohammed, Zhang and Zhao [25], they proved the existence of flows of different type semilinear stochastic evolution equations (semilinear see's). In Zhang and Zhao [40], they studied the non-linear noise case. With their work, it is possible to extend our work to different type of semilinear see's. Also, with great courage and hard work, it is reasonable to consider stochastic partial differential equations under our settings. This need more work.

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