The Explicit Group TOR Method

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Abstract
The numerical methods for solving partial differential equations have been one of the significant achievements made possible by the digital computers. With the advent of parallel computers, many studies have been performed and a number of new techniques have been investigated in order to develop new methods that are suitable for these computers. One of these techniques is the explicit group iterative methods which have been extensively studied and analysed in the last two decades.

The explicit group iterative methods for the numerical solution of self-adjoint elliptic partial differential equations have been introduced (Evans & Biggins, 1982; Yousif & Evans, 1986) and has been shown to be computationally superior in comparison with other iterative methods. These methods were found to be suitable for parallel computers as they possess independent tasks (Evans & Yousif, 1990). Martins, Yousif & Evans (2002) introduced a new explicit 4-points group accelerated overrelaxation (EGAOR) iterative method, a comparison with the point AOR method has shown its computational advantages. The point TOR method was developed and a number of papers related to the TOR method and its convergence have been presented (Kuang & Ji, 1988; Chang, 1996; Chang, 2001; Martins, Trigo & Evans 2003). In this paper, we formulate a new group method from the TOR family, the explicit 4-points group overrrelaxation (EGTOR) iterative method, the derivation of the new method is presented. Numerical experiments have been carried out and the results obtained confirm the superiority of the new method when compared to the point TOR method.

Keywords: TOR method, AOR method, SOR method, elliptic partial differential equation, group iterative methods.

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1. INTRODUCTION AND PRELIMINARIES

Many physical phenomena in static field problems particularly in electromagnetism field and the incompressible potential flow field are described by elliptic partial differential equations (PDEs). In recent years, improved techniques using explicit group methods have been developed to approximate the solution of these equations.

In this paper we will present the 4-points EGTOR iterative method to approximate the solution of elliptic partial equations. Some numerical experiments will be performed to compare the behaviour of this explicit TOR group method with the corresponding point TOR method.

In section 2, the 4-points EGTOR iterative method is presented and developed, while the point TOR method is given in this section. Some results concerning the analysis of convergence of the interval and point iterative TOR method were already obtained for different classes of matrices, namely H-matrices (Martins, Trigo & Evans, 2003). As it is well-known from literature, this class of matrices involves strictly diagonally dominant matrices, irreducible weakly diagonal matrices, M-matrices and other type of matrices. The study of computational complexity of the point and 4-points EGTOR methods is discussed in section 3. Further, in order to compare the performance of these two iterative methods, numerical experiments have been carried out and the results are summarised in section 4. Finally, concluding remarks are presented in section 5.

Consider the linear self-adjoint elliptic equation,

$$\frac{\partial}{\partial x}\left(A(x,y)\frac{\partial U}{\partial x}\right)+\frac{\partial}{\partial y}\left(B(x,y)\frac{\partial U}{\partial y}\right)-F(x,y)U=G(x,y), \quad (x,y)\in\Omega \qquad (1.1)$$

$$U(x,y)=g(x,y), \qquad (x,y)\in\partial\Omega \qquad (1.2)$$

defined in a bounded region Ω, where $A(x, y) > 0$, $B(x, y) > 0$ and $F(x, y) \geq 0$ and $\partial\Omega$ is the boundary of Ω.

The two dimensional elliptic equation such as Poisson's equation is mathematically represented by

$$\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}=f(x,y), \quad (x,y)\in\Omega, \qquad (1.3)$$

with a Dirichlet boundary condition on a unit square solution domain, $0 \leq x, y \leq 1$, with m^2 internal mesh points. It can be easily concluded that this equation is obtained from (1.2) if we consider $A(x, y) = B(x, y) = 1$ and $F(x, y) = 0$ and $G(x, y) = f(x, y)$. If in (1.3) $f(x,y)=0$ we have Laplace's equation.

It is well-known that the discretisation of (1.3) leads to the linear system (Varga,1962)

$$A\,x=b, \qquad (1.4)$$

where $A\in C^{n,n}$ is a given non-singular, sparse matrix with non vanishing diagonal entries, $b\in C^n$ is a known vector and $x\in C^n$ is the unknown vector.

Hence, the TOR method, defined in the following, can be used if the block diagonal part of the coefficient matrix A of the system (1.4) is non-singular. Some authors have obtained results on the convergence of the interval and point TOR method (Martins, Trigo & Evans, 2003) and other have obtained some convergence conditions for the multisplitting parallel TOR method (Chang, 1996; Chang, 2001).

Split the matrix A of (1.4) such that

$$A=D-L-F-U \qquad (1.5)$$

where D = diag (A), U is a strictly upper triangular matrix, obtained from A, L and F are strictly lower triangular matrices, verifying (1.5).

The corresponding TOR method (Martins, Trigo & Evans, 2003) is given by:

$$(D-\alpha L-\beta F)\, x^{(k+1)} = \omega b+[(1-\omega)D+(\omega-\alpha)L+\omega U+(\omega-\beta)F]\, x^{(k)},\ \ k=0,1,\ldots \quad (1.6)$$

where ω, α and β are real parameters and $\omega \neq 0$.

As $D-\alpha L-\beta F$ is a non-singular matrix for any choice of the parameters α and β, with $E_1 = D^{-1}L$, $E_2 = D^{-1}F$ and $U_1 = D^{-1}U$, the equation (1.6) takes the form

$$x^{(k+1)} = T_{\omega,\alpha,\beta}x^{(k)} + \omega(D-\alpha L-\beta F)^{-1}\, b, \qquad k=0,1,... \qquad (1.7)$$

where

$$T_{\omega,\alpha,\beta} = (I-\alpha E_1-\beta E_2)^{-1}\left[(1-\omega)I+(\omega-\alpha)E_1+\omega U_1+(\omega-\beta)E_2\right]. \qquad (1.8)$$

Some special well-known iterative methods can be derived from the TOR iterative method by assigning special values to the parameters ω, α and β. The Jacobi(J), Gauss-Seidel (GS), Simultaneous Overrelaxation (JOR), Successive Overrelaxation (SOR) and Accelerated Overrelaxation (AOR) iterative methods are special cases of the TOR iterative method as shown in Table 1.

ω	α	β	*Method*
1	0	0	J
1	1	1	GS
ω	0	0	JOR
ω	ω	ω	SOR
ω	α	α	AOR

Table 1: The derivation of some iterative methods from the TOR method.

In what follows we consider the linear system (1.4) where the matrix A has Property $A^{(\pi)}$ and is π-consistently ordered. Thus, we present some definitions (Young, 1971).

Definition 1.1

An ordered grouping π of $W = \{1, 2,\ldots, n\}$ is a subdivision of W into disjoint subsets R_1, R_2, …, R_q such that $R_1 + R_2 + \ldots + R_q = W$.

Given a matrix A and an ordered grouping π we define the submatrices $A_{r,s}$ for $r, s = 1, 2,\ldots, q$ as follows: $A_{r,s}$ is obtained from A by deleting all rows except those corresponding to R_r and all columns except those corresponding to R_s.

Definition 1.2

Let π be an ordered grouping with q groups. A matrix A has Property $A^{(\pi)}$ if the $q \times q$ matrix $Z = (z_{r,s})$ defined by

$$z_{r,s} = \begin{cases} 0 & if\ A_{r,s} = 0 \\ 1 & if\ A_{r,s} \neq 0 \end{cases} \qquad (1.9)$$

has Property A.

Definition 1.3
A matrix A of order n is consistently ordered if for some t there exist disjoint subsets S_1, S_2, …, S_t of $W = \{1, 2, \ldots, n\}$ such that $\sum_{k=1}^{t} S_k = W$ and such that if i and j are associated, then $j \in S_{k+1}$ if $j > i$ and $j \in S_{k-1}$ if $j < i$, where S_k is the subset containing i.

Definition 1.4
A matrix A is π-consistently ordered if the matrix Z of (1.9) is consistently ordered.

2. THE 4-POINTS GROUP EXPLICIT TOR ITERATIVE METHOD

In this section we will present an explicit set of equations for the 4-points EGTOR iterative method, where each group is formed from 4 points of the net region, according to Figure 1, where $t = (qm+1)$, step 2, $(q+1)m-1$, m is an even number and $q = 0$, step 2, m-2. Each group G_k, $k = 1, 2, \ldots, m^2/4$ contains only four elements $\{t, t+1, t+m, t+m+1\}$.

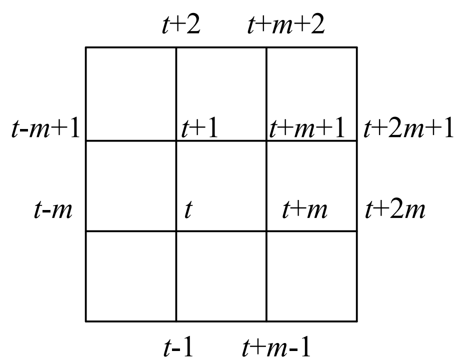


Figure 1

Let us suppose that the groups are ordered in red-black ordering, if we use the five-point approximation scheme, the finite difference equation at the point P (see Figure 2) has the form

$$u_p + \alpha_1 u_{B,P} + \alpha_2 u_{R,P} + \alpha_3 u_{T,P} + \alpha_4 u_{L,P} = b_P, \tag{2.1}$$

where B, R, T and L denote *B*ottom, *R*ight, *T*op and *L*eft of the point P, respectively.

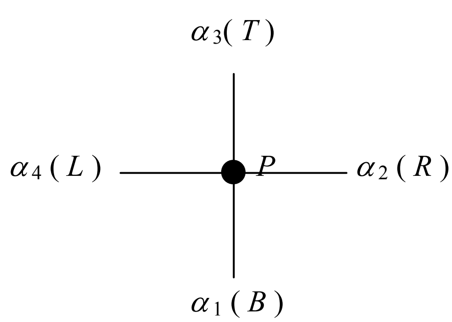


Figure 2

If this scheme is used, for all the mesh points, then in the case where the mesh is the unit square and $\Delta x = \Delta y = h = 1/7$, we have the linear system

$$A_1 u = b_1,$$

with,

$$A_1 = \left[\begin{array}{ccccc|cccc}
R_0 & 0 & 0 & 0 & 0 & R_2 & R_3 & 0 & 0 \\
0 & R_0 & 0 & 0 & 0 & R_4 & 0 & R_3 & 0 \\
0 & 0 & R_0 & 0 & 0 & R_1 & R_4 & R_2 & R_3 \\
0 & 0 & 0 & R_0 & 0 & 0 & R_1 & 0 & R_2 \\
0 & 0 & 0 & 0 & R_0 & 0 & 0 & R_1 & R_4 \\
\hline
R_4 & R_2 & R_3 & 0 & 0 & R_0 & 0 & 0 & 0 \\
R_1 & 0 & R_2 & R_3 & 0 & 0 & R_0 & 0 & 0 \\
0 & R_1 & R_4 & 0 & R_3 & 0 & 0 & R_0 & 0 \\
0 & 0 & R_1 & R_4 & R_2 & 0 & 0 & 0 & R_0
\end{array}\right] \tag{2.2}$$

where $R_0 = \left(\begin{array}{cc|cc} 1 & \alpha_3 & \alpha_2 & 0 \\ \alpha_1 & 1 & 0 & \alpha_2 \\ \hline \alpha_4 & 0 & 1 & \alpha_3 \\ 0 & \alpha_4 & \alpha_1 & 1 \end{array}\right)$, $R_1 = \left(\begin{array}{cc|cc} 0 & \alpha_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \alpha_1 \\ 0 & 0 & 0 & 0 \end{array}\right)$, $R_2 = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline \alpha_2 & 0 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 \end{array}\right)$,

$$R_3 = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ \alpha_3 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_3 & 0 \end{array}\right) \text{ and } R_4 = \left(\begin{array}{cc|cc} 0 & 0 & \alpha_4 & 0 \\ 0 & 0 & 0 & \alpha_4 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right). \tag{2.3}$$

If the groups are taken in the natural row ordering, then the coefficient matrix A_1 has the block structure

$$A_1 = \left[\begin{array}{ccc|ccc|ccc}
R_0 & R_2 & 0 & R_3 & 0 & 0 & 0 & 0 & 0 \\
R_4 & R_0 & R_2 & 0 & R_3 & 0 & 0 & 0 & 0 \\
0 & R_4 & R_0 & 0 & 0 & R_3 & 0 & 0 & 0 \\
\hline
R_1 & 0 & 0 & R_0 & R_2 & 0 & R_3 & 0 & 0 \\
0 & R_1 & 0 & R_4 & R_0 & R_2 & 0 & R_3 & 0 \\
0 & 0 & R_1 & 0 & R_4 & R_0 & 0 & 0 & R_3 \\
\hline
0 & 0 & 0 & R_1 & 0 & 0 & R_0 & R_2 & 0 \\
0 & 0 & 0 & 0 & R_1 & 0 & R_4 & R_0 & R_2 \\
0 & 0 & 0 & 0 & 0 & R_1 & 0 & R_4 & R_0
\end{array}\right] \tag{2.4}$$

The matrix A_1 (in (2.2) or (2.4)), has Property $A^{(\pi)}$ and is π- consistently ordered.

To derive the explicit group TOR method, we evaluate the transformed matrix A_2 and the modified vector b_2, where

$$A_2 = T^{-1} A_1, \tag{2.5}$$

and

$$b_2 = T^{-1} b, \tag{2.6}$$

where $T = \mathrm{diag}\{R_0\}$.

As T^{-1} is equal to diag$\{R_0^{-1}\}$ and the matrix R_0^{-1} is given by

$$R_0^{-1} = \frac{1}{(\alpha_6+1)^2+2\alpha_5-1}\begin{bmatrix} \alpha_5 & \alpha_3\alpha_6 & \alpha_2\alpha_7 & 2\alpha_2\alpha_3 \\ \alpha_1\alpha_6 & \alpha_5 & 2\alpha_1\alpha_2 & \alpha_2\alpha_7 \\ \alpha_4\alpha_7 & 2\alpha_3\alpha_4 & \alpha_5 & \alpha_3\alpha_6 \\ 2\alpha_1\alpha_4 & \alpha_4\alpha_7 & \alpha_1\alpha_6 & \alpha_5 \end{bmatrix}, \tag{2.7}$$

where,

$$\alpha_5 = 1-\alpha_1\alpha_3-\alpha_2\alpha_4,\ \ \alpha_6 = \alpha_1\alpha_3-\alpha_2\alpha_4-1,\ \ \alpha_7 = \alpha_2\alpha_4-\alpha_1\alpha_3-1.$$

Therefore

$$A_2 = \begin{bmatrix} I & C \\ B & I \end{bmatrix}, \tag{2.8}$$

where C and B can be evaluated easily.

The matrices A_1 and A_2 have the same block structures. The unique difference is that instead of the matrices R_0 and R_i, $i = 1,\ldots,4$ we have the identity matrices and $R_0^{-1}R_i$, respectively.

If we consider the model problem (1.4) and a square grid, we have

$$\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = -\frac{1}{4}.$$

Hence, from (2.7)

$$R_0^{-1} = \frac{1}{6}\begin{bmatrix} 7 & 2 & 2 & 1 \\ 2 & 7 & 1 & 2 \\ 2 & 1 & 7 & 2 \\ 1 & 2 & 2 & 7 \end{bmatrix}, \tag{2.9}$$

and hence,

$$R_0^{-1}R_1 = -\frac{1}{24}\begin{bmatrix} 0 & 7 & 0 & 2 \\ 0 & 2 & 0 & 1 \\ 0 & 2 & 0 & 7 \\ 0 & 1 & 0 & 2 \end{bmatrix}. \tag{2.10}$$

Thus we can set up the computational molecule at the point P as is shown in Figure 3. Similarly, we can obtain $R_0^{-1}R_i$, $i = 2, 3, 4$.

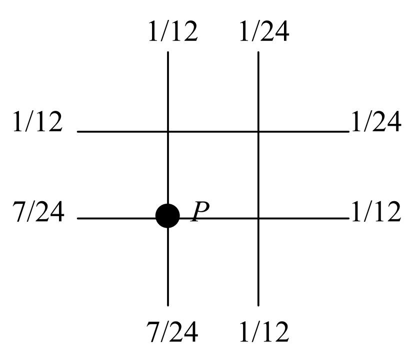


Figure 3

Therefore we can derive the explicit group TOR iterative method, by using this molecule. Here, we have considered in the implicit version the partition of matrix in the form (1.5) where U is associated with the elements u_{t+2}, u_{t+m+2}, u_{t+2m+1} and u_{t+2m}, L is associated the elements u_{t-m} and u_{t-1}, and finally F is associated with u_{t-m+1} and u_{t+m-1}. This version will be called, in the sequel, Variant A. Then we obtain the following formulas:

$$\begin{aligned} u_t^{(k+1)} &= \frac{1}{24}\{7\left[\omega(u_{t-m}^{(k)} + u_{t-1}^{(k)} - h^2 f_t) + \alpha(u_{t-m}^{(k+1)} + u_{t-1}^{(k+1)} - u_{t-m}^{(k)} - u_{t-1}^{(k)})\right] \\ &+ 2\left[\omega(u_{t+m-1}^{(k)} + u_{t-m+1}^{(k)} + u_{t+2m}^{(k)} + u_{t+2}^{(k)} - h^2(f_{t+1} + f_{t+m})) + \beta(u_{t+m-1}^{(k+1)} + u_{t-m+1}^{(k+1)} - u_{t+m-1}^{(k)} - u_{t-m+1}^{(k)})\right] \\ &+ \omega(u_{t+2m+1}^{(\mathrm{k})} + u_{t+m+2}^{(k)} - h^2 f_{t+m+1})\} + (1-\omega)u_t^{(k)} \end{aligned} \tag{2.11a}$$

$$\begin{aligned} u_{t+1}^{(k+1)} &= \frac{1}{24}\{7\left[\omega(u_{t-m+1}^{(k)} + u_{t+2}^{(k)} - h^2 f_{t+1}) + \beta(u_{t-m+1}^{(k+1)} + u_{t-m+1}^{(k)})\right] \\ &+ 2\left[\omega(u_{t-m}^{(k)} + u_{t-1}^{(k)} + u_{t+2m+1}^{(k)} + u_{t+m+2}^{(k)} - h^2(f_t + f_{t+m+1})) + \alpha(u_{t-m}^{(k+1)} + u_{t-1}^{(k+1)} - u_{t-m}^{(k)} - u_{t-1}^{(k)})\right] \\ &+ \omega(u_{t+2m}^{(k)} + u_{t+m-1}^{(k)} - h^2 f_{t+m}) + \beta(u_{t+m-1}^{(k+1)} - u_{t+m-1}^{(k)})\} + (1-\omega)u_{t+1}^{(k)} \end{aligned} \tag{2.11b}$$

$$\begin{aligned} u_{t+m}^{(k+1)} &= \frac{1}{24}\{7\left[\omega(u_{t+m-1}^{(k)} + u_{t+2m}^{(k)} - h^2 f_{t+m}) + \beta(u_{t+m-1}^{(k+1)} - u_{t+m-1}^{(k)})\right] \\ &+ 2\left[\omega(u_{t-m}^{(k)} + u_{t-1}^{(k)} + u_{t+2m+1}^{(k)} + u_{t+m+2}^{(k)} - h^2(f_t + f_{t+m+1})) + \alpha(u_{t-m}^{(k+1)} + u_{t-1}^{(k+1)} - u_{t-m}^{(k)} - u_{t-1}^{(k)})\right] \\ &+ \omega(u_{t-m+1}^{(k)} + u_{t+2}^{(k)} - h^2 f_{t+1}) + \beta(u_{t-m+1}^{(k+1)} - u_{t-m+1}^{(k)})\ \} + (1-\omega)u_{t+m}^{(k)} \end{aligned} \tag{2.11c}$$

$$\begin{aligned} u_{t+m+1}^{(k+1)} &= \frac{1}{24}\{7\left[\omega(u_{t+2m+1}^{(k)} + u_{t+m+2}^{(k)} - h^2 f_{t+m+1})\right] \\ &+ 2\left[\omega(u_{t+m-1}^{(k)} + u_{t-m+1}^{(k)} + u_{t+2m}^{(k)} + u_{t+2}^{(k)} - h^2(f_{t+1} + f_{t+m})) + \beta(u_{t+m-1}^{(k+1)} + u_{t-m+1}^{(k+1)} - u_{t+m-1}^{(k)} - u_{t-m+1}^{(k)})\right] \\ &+ \omega(u_{t-m}^{(k)} + u_{t-1}^{(k)} - h^2 f_t) + \alpha(u_{t-m}^{(k+1)} + u_{t-1}^{(k+1)} - u_{t-m}^{(k)} - u_{t-1}^{(k)})\ \} + (1-\omega)u_{t+m+1}^{(k)} \end{aligned} \tag{2.11d}$$

where $t = (pm + 1)$, step 2, $(p + 1)m - 1$ and $p = 0$, step 2, m-2.

Obviously, equation (2.15) in (Yousif & Evans, 1986), can be obtained from (2.11) if we let $\alpha = \beta = 0$ and $\omega = 1$.

Different versions of the 4-points EGTOR iterative method can be obtained considering other association of elements in matrices L and F for the partition (1.5). For instance, a second version of the 4-points group TOR method was obtained associating L with the elements u_{t-m} and u_{t-m+1} and F with u_{t-1} and u_{t+m-1}, this version is denoted by Variant B.

3. ANALYSIS OF THE COMPUTATIONAL COMPLEXITY OF THE POINT AND 4- POINTS EGTOR METHODS

The computational effort measured by the number of operations needed to obtain an approximation of the solution of (1.1) using the two methods presented in section 1 and section 2 will be discussed. We assume that a multiplication takes the same computer time as an addition.

3.1 The Point TOR Method

The finite difference solution of the model problem by the point TOR method is given by

$$
\begin{aligned}
u_t^{(k+1)} &= \alpha_2 \times u_{t-1}^{(k+1)} + \beta_2 \times u_{t-m}^{(k+1)} + \omega_2 \times (u_{t+1}^{(k)} + u_{t+m}^{(k)} - h_2 \times f_t) + \\
&\quad b_1 \times u_{t-1}^{(k)} + b_2 \times u_{t-m}^{(k)} + \omega_1 \times u_t^{(k)},
\end{aligned}
\tag{3.1}
$$

where

$\omega_1 = 1 - \omega,\ \omega_2 = \dfrac{\omega}{4},\ \alpha_2 = \dfrac{\alpha}{4},\ \beta_2 = \dfrac{\beta}{4},\ h_2 = 2h^2,\ b_1 = \omega_2 - \alpha_2$ and $b_2 = \omega_2 - \beta_2$.

It can be observed that the number of operations required (excluding the convergence test) for the point TOR method is $14m^2$ operations per iteration.

3.2 The 4-Points EGTOR Iterative Method

From equations (2.11), it can be seen that the number of operations required (excluding the convergence test) for the 4-points EGTOR iterative method is $29.5m^2$ operations per iteration. However, by making use of the fact that not all the elements involved in the calculations of the four points are different, we can reduce the work requirement to $15.25m^2$ operations per iteration as shown bellow.

Let $\omega_1 = 1 - \omega$, $\omega_4 = \omega - \alpha$, $\omega_5 = \omega - \beta$, $b_1 = 1/24$, $b_2 = 2b_1$, $b_3 = 7b_1$, $b_4 = \omega_4 b_1$, $b_5 = \omega_4 b_2$, $b_6 = \alpha b_1$, $b_7 = \alpha b_2$, $b_8 = \omega b_2$, $b_9 = \omega b_1$, $b_{10} = \omega_5 b_1$, $b_{11} = \omega_5 b_2$, $b_{12} = \beta\, b_1$, $b_{13} = \beta\, b_2$, $b_{14} = -h^2 b_9$ these need only be calculated once.

Thus if we set

$$
\begin{aligned}
&s_1 = u_{t-m}^{(k)} + u_{t-1}^{(k)}, \quad s_2 = u_{t-m}^{(k+1)} + u_{t-1}^{(k+1)}, \quad s_3 = u_{t+m-1}^{(k)} + u_{t-m+1}^{(k)}, \\
&s_4 = u_{t+m-1}^{(k+1)} + u_{t-m+1}^{(k+1)}, \quad s_5 = u_{t+2m}^{(k)} + u_{t+2}^{(k)}, \quad s_6 = u_{t+2m+1}^{(k)} + u_{t+m+2}^{(k)}, \\
&s_7 = b_{14} f_t,\ s_8 = b_{14} f_{t+1},\ s_9 = b_{14} f_{t+m},\ s_{10} = b_{14} f_{t+m+1},\ s_{11} = 2(s_8 + s_9), \\
&s_{12} = 2(s_7 + s_{10}),
\end{aligned}
$$

and

$$
\begin{aligned}
&t_1 = b_4 \times s_1 + b_6 \times s_2,\ t_2 = b_{11} \times s_3 + b_{13} \times s_4 + b_8 \times s_5, \\
&t_3 = b_{10} \times u_{t+m-1}^{(k)} + b_{12} \times u_{t+m-1}^{(k+1)} + b_9 \times u_{t+2m}^{(k)}, \\
&t_4 = b_9 \times u_{t+2}^{(k)} + b_{10} \times u_{t-m+1}^{(k)} + b_{12} \times u_{t-m+1}^{(k+1)}, \\
&t_5 = b_9 \times s_6,\ t_6 = 2 \times (t_1 + t_5),\ t_7 = t_1 + s_7,\ t_8 = t_4 + s_8,\ t_9 = t_3 + s_9, \\
&t_{10} = t_5 + s_{10},\ t_{11} = t_2 + s_{11},\ t_{12} = t_6 + s_{12},
\end{aligned}
\tag{3.2}
$$

then we have

$$
\begin{aligned}
u_t^{(k+1)} &= 7t_7 + t_{10} + t_{11} + \omega_1 u_t^{(k)}, \\
u_{t+1}^{(k+1)} &= 7t_8 + t_9 + t_{12} + \omega_1 u_{t+1}^{(k)}, \\
u_{t+m}^{(k+1)} &= 7t_9 + t_8 + t_{12} + \omega_1 u_{t+m}^{(k)}, \\
u_{t+m+1}^{(k+1)} &= 7t_{10} + t_7 + t_{11} + \omega_1 u_{t+m+1}^{(k)}.
\end{aligned}
\tag{3.3}
$$

These equations require an average of $15.25m^2$ operations per iteration.

If we consider Laplace's equation then it can be shown that the number of operations required for the point TOR and the 4-points EGTOR iterative methods are $12m^2$ and $11.75m^2$ operations per iteration respectively.

4. NUMERICAL RESULTS

We now present some numerical experiments in order to compare the point and 4-points EGTOR iterative methods.

Problem 1. Consider Laplace's equation,

$$\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}=0 \qquad (x,y)\in\Omega=(0,1)\times(0,1) \tag{4.1}$$

and the Dirichlet boundary conditions

$$\begin{aligned} &U(x,0)=\sin\pi x, && 0\le x\le 1,\\ &U(0,y)=U(1,y)=U(x,1)=0, && 0\le x,y\le 1 \end{aligned} \tag{4.2}$$

The numerical experiments have been performed using Matlab 7.9, on Core 2 Duo, 2.26 GHZ (4GM RAM), laptop (MacBook Pro) with Macintosh system. The methods have been compared in terms of number of iterations, computing effort and CPU time (in seconds). Throughout the experiments the convergence test used was the average error test with tolerance error $\varepsilon = 10^{-7}$.

The numerical solution for the problem (4.1)-(4.2), using the 4-points EGTOR method whit $h = 1/13$ is illustrated in Figure 4.

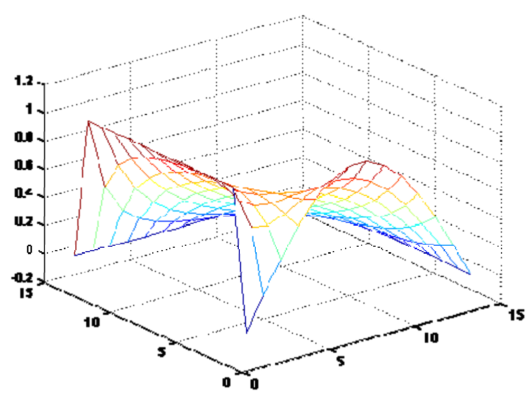


Figure 4: Numerical solution of the problem (4.1)-(4.2) obtained with $h = 1/13$

The coefficient matrix for the two methods possesses Property $A^{(\pi)}$ and are π-consistently ordered. Therefore the theory of block SOR is valid and can be used to predict ω. Hence, in this example, we start with an experimental values of ω very close to the optimal parameter of the SOR method with α and β close to ω.

In Table 2 and Table 3, we sum up the computational results for the point TOR and the 4-points EGTOR method applied to the problem (4.1)-(4.2) respectively. From these two tables and Figures 5a and 5b, it can be noted that the 4-points EGTOR method is more efficient when compared to the point TOR method.

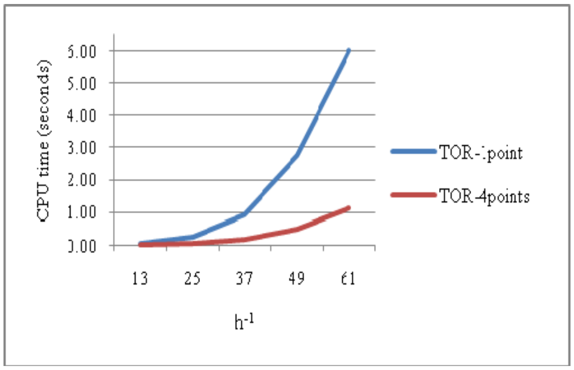
h^{-1}	ω	α	β	No. of Iterations	Computing effort	CPU time
13	1.63	1.62 1.63 1.64	1.62 1.61 1.60	38	456 m^2	0.05
25	1.77- 1.78	1.68 1.69 : 1.89	1.88 1.87 : 1.67	74	888 m^2	0.25
37	1.85	1.77 1.78	1.92 1.91	109	1308 m^2	0.91
49	1.88	1.82 1.83 : 1.91	1.94 1.93 : 1.85	145	1740 m^2	2.77
61	1.90	1.85 1.86 : 1.95	1.96 1.95 : 1.86	176	2112 m^2	6.00

Table 2: Computational results for the point TOR method in problem (4.1)-(4.2)

h^{-1}	ω	α	β	No. of Iterations	Computing effort	CPU time
13	1.50- 1.51	1.49 1.50 : 1.81	1.53 1.52 : 1.21	26	305.5 m^2	0.02
25	1.65- 1.68	1.42 1.43 : 1.99	1.99 1.98 : 1.42	51	599.2 m^2	0.07
37	1.74- 1.75	1.59 1.60 : 1.99	1.99 1.98 : 1.59	74	869.5 m^2	0.17
49	1.72	1.69 1.70 : 1.99	1.99 1.98 : 1.69	100	1175 m^2	0.50
61	1.84	1.74 1.75 : 1.99	1.99 1.98 : 1.74	123	1445.2 m^2	1.11

Table 3: Computational results for the 4-points EGTOR method (problem (4.1)-(4.2))

The plots of the CPU computation time *vs* the mesh size for the two methods is given in Figure 5a. Also, the logarithm of the number of iterations *vs* logh^{-1} for the two methods is plotted, these graphs are shown in Figure 5b. As expected, the plots for the two methods were straight lines with a slope of unity, thus verifying the SOR theory.



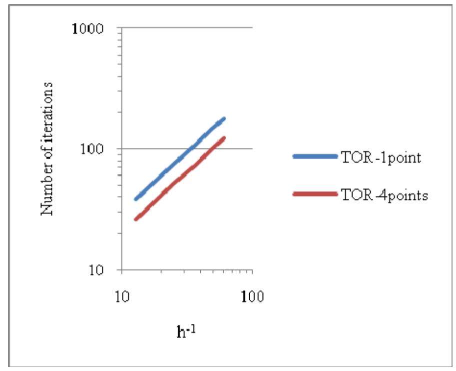


Figure 5a **Figure 5b**

Figures 5a and 5b: Computational results for the point TOR and the 4-points EGTOR methods (problem (4.1)-(4.2))

In Table 4 we present the computational results obtained with the variant of the 4-points EGTOR method described at the end of section 2 (Variant B). The results are very similar in terms of number of iterations, however as it requires a higher number of operations per iteration it is not competitive with Variant A, given by equations 2.11, even when it reaches the solution with less iterations.

	Variant A			Variant B		
h^{-1}	No. of Iterations	Computing effort	CPU time	No. of Iterations	Computing effort	CPU time
13	26	305.5 m^2	0.02	27	344.3 m^2	0.06
25	51	599.2 m^2	0.07	51	650.2 m^2	0.08
37	74	869.5 m^2	0.17	74	943.5 m^2	0.23
49	100	1175 m^2	0.50	97	1236.8 m^2	0.63
61	123	1445.2 m^2	1.11	121	1542.8 m^2	1.39

Table 4: Comparison results for the two variants of the 4-points group TOR method (problem (4.1)-(4.2))

Problem 2. The Laplace equation (4.1) was also considered with another Dirichlet boundary conditions

$$U(0, y) = 100, \qquad 0 \le y \le 1, \qquad (4.3)$$
$$U(x, 0) = U(x, 1) = U(1, y) = 0, \qquad 0 \le x, y \le 1,$$

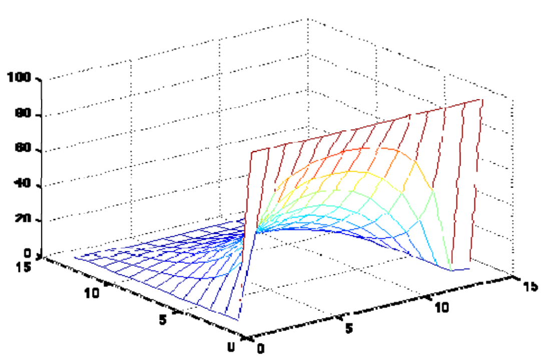


Figure 6: Numerical solution of the problem (4.1)-(4.3) obtained with h = 1/13

The numerical solution for the problem (4.1) with the boundary conditions (4.3), using the 4-points EGTOR method whit h = 1/13 is illustrated in Figure 6. The computational results for the point TOR and 4-points EGTOR methods applied to the problem (4.1)-(4.3) are summarised in Table 5 and Table 6 respectively.

h^{-1}	ω	α	β	No. of Iterations	Computing effort	CPU time (seconds)
13	1.57 1.58 1.59 1.60 1.61 1.62 1.63	1.63 1.59 1.55 1.52 1.49 1.47 1.45	1.61 1.65 1.69 1.72 1.75 1.77 1.79	38	456 m^2	0.03
25	1.66- 1.67	1.69 1.70 : 1.73	1.87 1.86 : 1.83	71	852 m^2	0.24
37	1.82- 1.83	1.70 1.71 1.72	1.99 1.98 1.97	100	1200 m^2	0.80
49	1.85- 1.86	1.77 1.78	1.99 1.98	134	1608 m^2	2.58
61	1.81 1.82 1.83	1.84 1.83 1.82	1.97 1.98 1.99	171	2052 m^2	5.85

Table 5: Computational results for the point TOR method (problem (4.1)-(4.3))

h^{-1}	ω	α	β	No. of Iterations	Computing effort	CPU time (seconds)
13	1.47- -1.52	1.26 1.27 : 1.68	1.75 1.74 : 1.33	27	317.2 m^2	0.01
25	1.57	1.42 1.43 : 1.99	1.99 1.98 1.42	50	587.5 m^2	0.06
37	1.63- -1.64	1.59 1.60 : 1.99	1.99 1.98 : 1.59	73	857.8 m^2	0.16
49	1.79- -1.80	1.68 1.69 : 1.99	1.99 1.98 : 1.68	96	1128 m^2	0.48
61	1.82- -1.83	1.74 1.75 : 1.99	1.99 1.98 : 1.74	119	1398.2 m^2	1.07

Table 6: Computational results for the 4-points EGTOR method (problem (4.1)-(4.3))

The plots of the CPU computation time *vs* the mesh size and the logarithm of the number of iterations *vs* logh^{-1} for the two methods are shown in Figures 7a. and 7b, respectively.

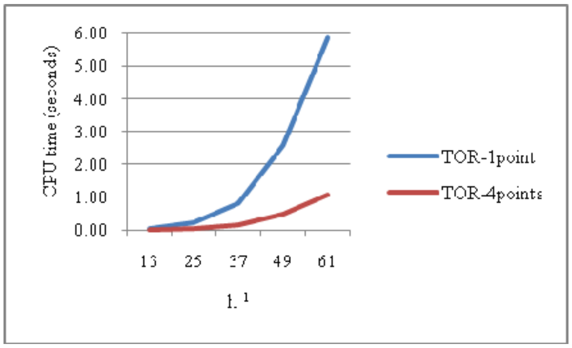


Figure 7a

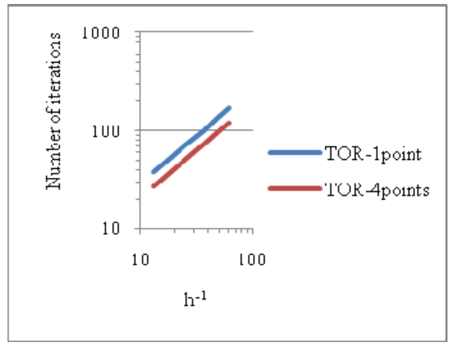


Figure 7b

Figures 7a and 7b: Computational results for the point TOR and the 4-points EGTOR methods (problem (4.1)-(4.3))

From the results presented in Table 5, Table 6 and Figures 7a and 7b, it is clear that the 4-points EGTOR method offers significant economies over the point TOR method.

In Table 7 we compare the two variants of the 4-points EGTOR methods described in this paper. The results are similar to the results given for Problem 1, and hence, we reach a conclusion similar to the one given for the previous problem.

	Variant A			Variant B		
h^{-1}	No. of Iterations	Computing effort	CPU time (seconds)	No. of Iterations	Computing effort	CPU time (seconds)
13	27	317.2 m^2	0.01	26	331.5 m^2	0.01
25	50	587.5 m^2	0.06	49	624.8 m^2	0.07
37	73	857.8 m^2	0.16	73	930.7 m^2	0.23
49	96	1128 m^2	0.48	94	1198.5 m^2	0.60
61	119	1398.2 m^2	1.07	116	1479.0 m^2	1.32

Table 7: Comparison results for two variants of the 4-points EGTOR method (problem (4.1)-(4.3))

5. CONCLUSIONS

From our analysis of the two methods, amount of computational work and minimum complexity, and the results given in Table 2, Table 3, Table 5 and Table 6 indicates that the new 4-points EGTOR method appears to be more efficient than the point TOR method.

Further, the group explicit algorithm is suitable for parallel computers as it possesses separate and independent tasks, as the groups of 4-points can be executed concurrently. Other blocks (groups) are also possible, i.e., the 2, 6, 9 or 16 point group and will be matter for further research.

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