Data envelopment analysis models of investment funds

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Abstract

This paper develops theory missing in the sizable literature that uses data envelopment analysis to construct return : risk ratios for investment funds. It explores the production possibility set of the investment funds to identify an appropriate form of returns to scale. It discusses what risk and return measures can justifiably be combined and how to deal with negative risks, and identifies suitable sets of measures. It identifies the problems of failing to deal with diversification and develops an iterative approximation procedure to deal with it. It identifies relationships between diversification, coherent measures of risk and stochastic dominance. It shows how the iterative procedure makes a practical difference using monthly returns of 30 hedge funds over the same time period. It discusses possible shortcomings of the procedure and offers directions for future research.

Keywords: data envelopment analysis, investment fund, diversification, coherent risk measure, returns to scale, stochastic dominance

1. Introduction

Data envelopment analysis (DEA) estimates the technical efficiency of several decision-making units (DMUs) given several inputs and several outputs. Typically the efficiency may be written as

$$\frac{\text{weighted sum of outputs}}{\text{weighted sum of inputs}}. \tag{1}$$

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Authors such as Murthi et al. (1997) and Gregoriou and Zhu (2005) observe the following analogy. If we use risk in place of input and return in place of output in expression (1), we get a ratio that generalises return–risk ratios such as the Sharpe (Sharpe, 1966), Calmar and Sortino (Lhabitant, 2004) ratios, used to compare investment funds. The benefit of DEA is that it lets us account for different investor attitudes to risk and return.

The published literature on DEA for investment funds is not theoretically justified. In using the analogy between output–input and return–risk ratios it usually makes an implicit assumption that fund returns are perfectly correlated. And it often ignores the need for comparable measures of risk and return. We investigate theoretically when and how DEA may be used to compare investment funds, and develop a new method that models diversification directly. To do this we discuss four important issues.

1. DEA depends on technical assumptions of free disposability and convexity. The literature does not discuss whether these reasonably hold when modelling investment funds. We show that, in general, they do not.
2. Originally DEA (Charnes et al., 1978) assumed constant returns to scale (CRS) but later allowed for variable returns to scale (VRS) (Banker et al., 1984). Both are used for investment funds (Gregoriou and Zhu, 2005), but the analogy between output–input and risk–return suggests neither and many authors use one or other with limited or no justification. For example, Gregoriou et al. (2005a) argue for a VRS model only because it compensates for what Section 3 identifies as noncommensurable measures.
3. The literature allows for different investor preferences by combining measures such as mean, median, standard deviation (Gregoriou et al., 2005a), lower and upper semivariance and semiskewness (Gregoriou et al., 2005b), skewness (Wilkens and Zhu, 2001), excess kurtosis (Nguyen-Thi-Thanh, 2006), time horizons (Galagadera and Silvapulle, 2002), percentage of periods with negative returns, skewness (Wilkens and Zhu, 2001), value at risk, conditional value at risk (Chen and Lin, 2006), downside absolute standard deviation, weighted absolute deviation from quantile, and tail value at risk (Lozano and Gutiérrez, 2008). Eling (2006) reviews the measures used and concludes there is no single standard choice. Two problems arise. Some measures use different units from others. And some may take negative values.

4. We need to deal with diversification because a combination of funds may be more efficient than any individual fund. Basso and Funari (2001); Sengupta (2003) account for diversification indirectly using market β (Alexander, 2001) as a risk measure. Morey and Morey (1999); Joro and Na (2006); Briec and Kerstens (2009); Lozano and Gutiérrez (2007) account for it directly using nonlinear versions of DEA. And Lozano and Gutiérrez (2008) account for it directly in linear DEA, limited to models with only one input and one output.

We resolve these issues as follows. Section 2 discusses the assumptions needed for a DEA model of investment funds and how to handle returns to scale. Section 3 introduces commensurability as a way of choosing a set of risk and return measures that can justifiably be combined. Section 4 explains coherent measures of risk and how to satisfy the assumption of convexity and model diversification through a series of linear approximations to an ideal nonlinear model. Section 5 illustrates our method using monthly returns from 30 hedge funds over the same time period. Section 6 discusses possible shortcomings of our method and some future directions for research.

Coherent measures of risk come up again and again in our discussion. They relate to investor preference because they help model stochastic dominance. They allow us to account for distributional shape and also to use multiple risk measures. They also relate closely to what we need in order to deal with diversification. The best-known coherent measure of risk is conditional value at risk (CVaR), which is based on value at risk (VaR). VaR and CVaR are families of risk measures, allowing us to account for many different investor preferences and dsutributional shapes. VaR is the maximum loss that investors might suffer over a time horizon at a specified confidence level. CVaR is the expected loss conditional on this loss exceeding VaR. It is sometimes called expected shortfall, tail conditional expectation, conditional loss or tail loss (Jorion, 2007). Acerbi and Tasche (2002) note some subtle differences among these terms, which we can ignore for continuous distribution functions, and show that CVaR is a coherent measure of risk. Acerbi (2007) shows how to estimate CVaR from a sample. We use this in Section 5.

2. Background

This section describes DEA models for investment funds and the assumptions needed for them and shows an NRS model is appropriate.

A DEA model compares n DMUs using m input and s output measures. An investment fund (or fund for short) produces returns that we can measure at regular time intervals. Section 5 uses examples with average monthly returns. We assume the regular returns are realisations of a real random variable and ignore, for now, the possibility that the random variable is a function of time. We suppose we have funds described by random variables $f_1, \ldots, f_n$. We define a random variable $\sum_{j=1}^{n} \lambda_j f_j$ with $\lambda_j \geq 0$ $(j = 1, \ldots, n)$ and $\sum_{j=1}^{n} \lambda_j \leq 1$ to be a *portfolio* on $f_1, \ldots, f_n$ and we define the *portfolio possibility set* $\mathcal{F}$ to be the set of all such portfolios. Then a *measure* is a function $g : \mathcal{F} \to \mathbb{R}$. Since a measure is a population statistic, we invariably must estimate it with a sample statistic. We call the sample statistic also a measure and use the same notation for both, distinguishing only when the difference is unclear. We classify some measures as return measures and some as risk measures. For example, the mean value is a return measure and the standard deviation a risk measure. Later we describe other risk and return measures and what properties they might have. We suppose we have risk measures $x_1, \ldots, x_m$ and return measures $y_1, \ldots, y_s$. We write $x_{ij} = x_i(f_j)$, $y_{rj} = y_r(f_j)$, $\mathbf{x}_j = (x_{1j}, \ldots, x_{mj})$ and $\mathbf{y}_j = (y_{1j}, \ldots, y_{sj})$ $(i = 1, \ldots, m,$ $r = 1 \ldots, s,\ j = 1, \ldots, n)$ so we can describe DEA compactly.

The (input-oriented) DEA model is this. For each DMU $o \in \{1, \ldots, n\}$ choose $\mathbf{u}_o \in \mathbb{R}^s$, $\mathbf{v}_o \in \mathbb{R}^m$, and $\alpha \in \mathbb{R}$ to

maximise

$$\phi_o = \frac{\alpha + \mathbf{y}_o \cdot \mathbf{u}_o}{\mathbf{x}_o \cdot \mathbf{v}_o} \tag{2}$$

subject to

$$\frac{\alpha + \mathbf{y}_j \cdot \mathbf{u}_o}{\mathbf{x}_j \cdot \mathbf{v}_o} \leq 1, \qquad (j = 1, \ldots, n) \tag{3}$$

and

$$\mathbf{u}_o \geq \mathbf{0}, \mathbf{v}_o \geq \mathbf{0}. \tag{4}$$

Note that we solve (2)–(4) separately for each o. The *efficiency* of DMU o is ϕ_o, which must be in $[0, 1]$ provided $\mathbf{x}_j > \mathbf{0}$ and $\mathbf{y}_j \geq \mathbf{0}$ for $j = 1, \ldots, n$. We set $\alpha = 0$ for a CRS model and do not constrain it for a VRS model. We use an input-oriented model because funds are usually compared using return–risk ratios like expression (2). Gregoriou and Zhu (2005) show an

output-oriented model is possible. A slacks-based model (Tone, 2001) is also possible, though we know of none used to model funds.

Model (2)–(4) is sometimes called a ratio model and is usually recast as a linear program. We present the dual form. Gregoriou and Zhu (2005) give details. This form lets us discuss the assumptions needed for DEA, relate them to portfolios and the assumptions needed to model funds, describe the efficient frontier, and discuss what form of returns to scale is appropriate. We find the CRS DEA efficiencies from the following (dual) linear programmes for each $o = 1, \ldots, n$. Choose ϕ_o, $\lambda_{1o}, \ldots, \lambda_{no}$ to

minimise

$$\phi_o \tag{5}$$

subject to

$$\sum_{j=1}^{n} y_{rj}\lambda_{jo} \geq y_{ro}, \qquad (r = 1, \ldots, s) \tag{6}$$

$$\sum_{j=1}^{n} x_{ij}\lambda_{jo} \leq x_{io}\phi_o, \qquad (i = 1, \ldots, m) \tag{7}$$

and

$$\lambda_{jo} \geq 0 \quad (j = 1, \ldots, n). \tag{8}$$

To get the VRS model we add the following constraint to (5)–(8).

$$\sum_{j=1}^{n} \lambda_{jo} = 1. \tag{9}$$

Following Cazals et al. (2002), we define a VRS *production possibility set* as a set $\Psi = \{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}_+^m \times \mathbb{R}_+^s\}$ satisfying

Free disposability: (see, for example, (Shephard, 1970)): if $(\mathbf{x}, \mathbf{y}) \in \Psi$, then $(\mathbf{x}', \mathbf{y}') \in \Psi$ whenever $\mathbf{x}' \geq \mathbf{x}$ and $\mathbf{y}' \leq \mathbf{y}$.

Convexity: if $(\mathbf{x}, \mathbf{y}), (\mathbf{x}', \mathbf{y}') \in \Psi$ then $(t\mathbf{x} + (1-t)\mathbf{x}', t\mathbf{y} + (1-t)\mathbf{y}') \in \Psi$ for $t \in (0, 1)$.

Ψ is the set of $(\mathbf{x}, \mathbf{y})$ such that it is possible to produce $\mathbf{y}$ from $\mathbf{x}$. Free disposability and convexity are usually assumed to be reasonable for DEA models in economics, though Bogetoft (1996); Cazals et al. (2002) discuss cases where convexity is questionable. The *input-oriented efficient frontier* of Ψ is the set $\{(\mathbf{x}, \mathbf{y}) \in \Psi : (\mathbf{x}', \mathbf{y}) \notin \Psi \text{ for } \mathbf{x}' < \mathbf{x}\}$. It is straightforward to show that if $\lambda_{j1}, \ldots, \lambda_{jn}$, ϕ_o solves the VRS DEA model, then $\sum_{j=1}^{n} (\mathbf{x}_j, \mathbf{y}_j)\lambda_j$ is on the efficient frontier of the smallest production possibility set containing all the DMUs and so ϕ_o is a measure of the efficiency of DMU o. It is also straightforward to check that we get the corresponding result for the CRS model if we expand the definition of Ψ so that $(t\mathbf{x}, t\mathbf{y}) \in \Psi$ whenever $(\mathbf{x}, \mathbf{y}) \in \Psi$ and $t > 0$.

Figure 1 about here.

Figure 1 shows the mean (return) and standard deviation (risk) in the monthly returns of 30 funds described in Section 5. The darker shaded area is the production possibility set for the VRS model. Its frontier is shown as a line on the left of this area that is dashed then solid. It represents the possibility, common in economic models, that the marginal return on output eventually decreases as input increases. The total shaded area is the production possibility set for the CRS model. Its frontier is shown as a line on the left of this area that is solid then dashed.

We know of no literature that checks if the standard assumptions about production possibility sets and returns to scale are valid for models of funds. They are not. There is a minor problem with free disposability. Suppose we use $\text{VaR}_{0.2}$ and $\text{VaR}_{0.05}$ as risk measures and have $\text{VaR}_{0.2}(f) = 0.1$ and $\text{VaR}_{0.05}(f) = 0.2$ for fund f. Then the free disposability assumption suggests an investment f' with $\text{VaR}_{0.2}(f') = 0.3$, $\text{VaR}_{0.05}(f') = 0.2$ is possible because it exceeds f in one risk measure. But $\text{VaR}_{0.05} \geq \text{VaR}_{0.2}$ by definition. This does not invalidate DEA because the efficiencies do not depend on free disposability. So we assume from here on that free disposability holds for any production possibility set we consider.

There is a much bigger problem with convexity. Suppose $\mathbf{x}(g)$ is the risk measure and $\mathbf{y}(g)$ the return measure for each fund g. And suppose f and f' are funds. Then, for $t \in (0, 1)$ we can assume that $(\mathbf{x}(tf + (1-t)f'), \mathbf{y}(tf + (1-t)f')) \in \Psi$ because $tf + (1-t)f'$ is a possible portfolio. But the equalities

$$\begin{aligned} t\mathbf{x}(f) + (1-t)\mathbf{x}(f') &= \mathbf{x}(tf + (1-t)f'), \\ t\mathbf{y}(f) + (1-t)\mathbf{y}(f') &= \mathbf{y}(tf + (1-t)f') \end{aligned} \tag{10}$$

only hold for perfectly correlated funds for all measures we know of other than mean value. So convexity is at best an approximation. Section 4 develops a method to deal with this approximation.

We now consider returns to scale in models of funds. A VRS model is implausible because it allows for a minimum level of risk before a positive return is possible. A CRS model is implausible because it allows returns greater than any portfolio can produce. The portfolio possibility set allows us to choose a portfolio that has some proportion uninvested: that is, invested with zero risk and zero return. Assuming equalities (10) are reasonable, this is equivalent to defining the production possibility set Ψ to satisfy

NRS *convexity:* if $(\mathbf{x}, \mathbf{y}), (\mathbf{x}', \mathbf{y}') \in \Psi$ then $(t\mathbf{x} + u\mathbf{x}', t\mathbf{y} + u\mathbf{y}') \in \Psi$ for t, $u \geq 0$, $t + u \leq 1$.

The solid line in Figure 1 shows the efficient frontier of the smallest production possibility set containing all 30 funds and satisfying this. The corresponding DEA model is (5)–(8) with the extra constraint

$$\sum_{j=1}^{n} \lambda_{jo} \leq 1. \tag{11}$$

We call it a *nonincreasing returns to scale* (NRS) model. Cooper et al. (2007) describe it as a possible relaxation of constraint (9) and Färe and Lovell (1994) discuss how NRS efficiencies can be used to classify returns to scale in a general DEA model. Joro and Na (2006) use NRS with limited explanation in a nonlinear mean–variance–skewness model. It is straightforward to check, using linear programming duality that the NRS model is equivalent to (2)–(4) with $\alpha \leq 0$. We consider only NRS models from here on.

3. Commensurable sets of measures

We call a set of inputs and outputs *commensurable* if each input or output is measured in a positive constant multiple of some common unit. This section argues that any DEA model should ideally use only commensurable sets of inputs and outputs. Later it argues that an input-oriented model of funds should use nonnegative risk measures and shows there are reasonable sets of measures that are commensurable and contain only nonnegative risk measures.

We use commensurable measures for two reasons. First, as (Lovell and Pastor, 1995) note, the efficiency, given by equation (2), should be dimensionless. Second, we want the efficiencies to be units invariant: multiplying the fund returns by a positive constant should not change the efficiencies. Lovell and Pastor (1995) note that most models, including all the models we discuss, have a weaker form of units invariance: multiplying the values of any input or output by a positive constant should not change the efficiencies.

In practice, the measures in a commensurable set usually include the mean and so have the same units as the funds' returns. Some published models use measures such as variance, semivariance, skewness and kurtosis that are not commensurable with the mean return (Wilkens and Zhu, 2001; Gregoriou et al., 2005b; Nguyen-Thi-Thanh, 2006). There are several plausible explanations. First, funds returns are usually expressed as proportions and so it may not be immediately obvious that the efficiencies are not dimensionless. Second, DEA models are often expressed in dual form (like model (5)–(9)) in which the units of the efficiencies are less obvious. Third, some nonlinear models (Morey and Morey, 1999; Joro and Na, 2006; Briec and Kerstens, 2009) must use noncommensurable measures to be practically solvable. For example, Morey and Morey (1999) use mean, variance and covariance as measures in models with quadratic constraints.

Initially DEA did not allow negative values for inputs and outputs. This is usually reasonable for economic models but less so for models of funds. Even the mean return can be negative, as Figure 1 illustrates. If $y_{ro} \leq 0$ for some r and o then $\phi_o = 0$, $\lambda_{1o} = \cdots = \lambda_{no} = 0$ satisfies constraints (6)–(8), (11). But $\phi_o \geq 0$ to satisfy constraints (7) provided $x_{io} > 0$ for some i. So negative return measures are largely unproblematic, and a NRS model with positive inputs and outputs unrestricted in sign is equivalent to one with positive inputs in which any negative output value is replaced by zero.

Negative or zero risk measures are more problematic. If $x_{io} < 0$ for some i it is possible that $\phi_o > 1$. And if $\mathbf{x}_o = \mathbf{0}$, $\phi_o = -\infty$. So we need positive risk measures for a reasonable definition of efficiency. However, to allow risk-free investments, ideally we would like at least the possibility that the risk measure could be zero. One approach to the problem of nonpositive input values is to add a constant to each value. If doing this does not change the efficiencies, we can call the input *translation invariant*. While some DEA models exhibit (at least limited) translation invariance (Lovell and Pastor, 1995), it is easy to check that the inputs in the NRS model are not translation invariant. So ideally we should avoid this approach. Another approach deals

with negative values by measuring efficiency relative to an ideal point (Silva Portela et al., 2004). We eschew it because we have found no way to make it compatible with the production possibility set assumptions for a set of funds. Instead we define the *complete* DEA model as follows. Let $\mathbf{x}_j \geq \mathbf{0}$ and $\mathbf{y}_j \geq \mathbf{0}$ for $j = 1, \ldots, n$. If $\mathbf{x}_o > \mathbf{0}$ define ϕ_o as in model (5)–(8), (11); otherwise put $\phi_o = 1$ or 0 according as $\mathbf{y}_o \geq \mathbf{0}$ or $\mathbf{y}_o < \mathbf{0}$. The complete model has the desirable property that a risk-free fund is efficient whenever the value of at least one return measure is nonnegative. We wish to establish further the consistency in the definition of ϕ_o. To do this we first show how the NRS and VRS models are related.

Suppose that $\mathbf{x}_j > \mathbf{0}$ and $\mathbf{y}_j > \mathbf{0}$ for $j = 1, \ldots, n-1$, and $\mathbf{x}_n = \mathbf{y}_n = \mathbf{0}$. Then for $j = 1, \ldots, n-1$, we can write the programmes of the VRS model (5)–(9) as minimise ϕ_o subject to $\sum_{j=1}^{n-1} y_{rj}\lambda_{jo} \geq y_{ro}$ $(r = 1, \ldots, s)$, $\sum_{j=1}^{n-1} x_{rj}\lambda_{jo} \leq x_{io}\phi_o$ $(i = 1, \ldots, m)$, $\sum_{j=1}^{n-1} \lambda_{jo} \leq 1$, and $\lambda_{jo} \geq 0$ $(j = 1, \ldots, n-1)$. This is just the NRS model with DMUs $1, \ldots, n-1$. It follows that we can find the efficiencies of an NRS model by solving all but one of the programmes of the VRS model we get by adding an extra DMU with all inputs and outputs equal to zero.

We now investigate the consistency of the complete DEA model. This matters because Section 4 creates expanded models that typically include a notional fund o with $\mathbf{x}_o = \mathbf{0}$ and $\mathbf{y}_o > \mathbf{0}$. Ideally we would like to show that ϕ_o, considered as a function of $\mathbf{x}_1, \ldots, \mathbf{x}_n, \mathbf{y}_1, \ldots, \mathbf{y}_n$, is continuous. Although continuity of efficiency measures is desirable (Russell, 1990), Scheel and Scholtes (2003) show for the VRS model that, although ϕ_o is continuous almost everywhere, it may have discontinuities on the frontier and at points where $x_{ij} = 0$. We also deliberately allow a discontinuity at $\mathbf{y}_o = \mathbf{0}$ because it is unlikely that a rational investor would consider a fund with no risk and negative return to be efficient. The following result shows ϕ_o is continuous at $\mathbf{x}_o = \mathbf{0}$ if $\mathbf{y}_o \geq \mathbf{0}$ and $x_{ij} > 0$ for $j \neq o$ and for all i. We write ϕ_o for the efficiency of DMU o in the complete VRS model and ϕ'_o for the efficiency of the oth DMU in the complete VRS model with DMUs given by inputs $\mathbf{x}'_j$ and outputs $\mathbf{y}'_j$ $(j = 1, \ldots, n)$. We use $\mathbf{x}, \mathbf{y}$ as a shorthand for $\mathbf{x}_1, \ldots, \mathbf{x}_n, \mathbf{y}_1, \ldots, \mathbf{x}_n$ and, for $\varepsilon > 0$, write $N_\varepsilon(\mathbf{x}, \mathbf{y})$ for $\{\mathbf{x}', \mathbf{y}' : \|(\mathbf{x}', \mathbf{y}') - (\mathbf{x}, \mathbf{y})\|_2 < \varepsilon\}$, the ε-neighbourhood of $(\mathbf{x}, \mathbf{y})$.

Theorem 1. *Suppose* $\mathbf{x}_o = \mathbf{0}$, $\mathbf{y}_o \geq \mathbf{0}$ *and* $x_{ij} > 0$ *for all* i *and for all* $j \neq o$. *Then* $\phi'_o = \phi_o$ *for all* $(\mathbf{x}', \mathbf{y}') \geq \mathbf{0}$ *sufficiently close to* $(\mathbf{x}, \mathbf{y})$.

Proof. By definition $\phi_o = 1$. Put $\varepsilon = 0.5 \min\{x_{ij} : i \in \{1, \ldots, m\},\ j \in$

$\{1, \ldots, n\} \setminus \{o\}\} > 0$, and choose $(\mathbf{x}', \mathbf{y}') \in N_\varepsilon(\mathbf{x}, \mathbf{y})$. Then $x'_{ij} \geq x'_{io}$ for all i and j. If $\mathbf{x}'_o = \mathbf{0}$ then $\phi'_o = 1$ by definition, and so $\phi'_o = \phi_o$. Otherwise, choose i such that $x'_{io} > 0$. Then constraint (7) gives

$$\phi'_o x'_{io} \geq \sum_{j=1}^{n} x'_{ij} \lambda_{jo} \geq \sum_{j=1}^{n} \lambda_{jo} x'_{io} = x'_{io} \qquad \text{(using constraint (9))}.$$

Since $x'_{io} > 0$, it follows that $\phi'_o \geq 1$. The feasible solution $\phi'_o = 1$, $\lambda_{oo} = 1$, $\lambda_{jo} = 0$ $(j \neq o)$ implies $\phi'_o \leq 1$, and so $\phi'_o = 1 = \phi_o$ as required. ∎

It follows from the relation between the NRS and VRS models that the result holds *mutatis mutandis* for the complete NRS model.

From here on we consider only DEA models of investment funds and use only models based on the complete NRS model. It remains to show there are commensurable sets of measures suitable for a complete NRS model. Return measures are straightforward since they can be negative: the mean, upper quantiles and upper tail means are in the same units as the data. Risk measures need to be nonnegative. The standard deviation, absolute deviation and lower semideviation are all reasonable. And VaR and CVaR are measured in the same units as the data and are usually positive. Theorem 3 shows how we can deal with negative values.

4. Diversification, convexity, and coherent measures of risk

We wish to compare investment funds to the risks and returns available from any portfolio composed from them. To do this we must consider diversification because diversification may let us reduce the risk of an investment without reducing its return, violating one of the basic assumptions of standard DEA.

This section starts with a mathematical programming model that handles diversification. The model is not usually practically solvable. The section then shows this model is a convex programming problem if its measures have properties similar to the well-known properties of coherent measures of risk. The section continues by presenting an iterative procedure that approximates the solution of the convex programming problem. It finishes with some technical results on convergence and properties of the models discussed.

Diversification: an impractical model

Section 2 defines the production possibility set Ψ for a set of funds as the set of values of $((x_1(f), \ldots, x_m(f)), (y_1(f), \ldots, y_s(f)))$ such that f is in the portfolio possibility set $\mathcal{F}$. If the measures are nonnegative and form a commensurable set, the efficiency ϕ_o with respect to Ψ is given by the complete NRS model if $\mathbf{x}_o = \mathbf{0}$ and by the following mathematical program otherwise. Choose ϕ_o, $\lambda_{1o}, \ldots, \lambda_{no}$ to

minimise ϕ_o subject to

$$y_r \left(\sum_{j=1}^{n} \lambda_{jo} f_j \right) \geq y_r(f_o), \qquad (r = 1, \ldots, s) \tag{12}$$

$$x_i \left(\sum_{j=1}^{n} \lambda_{jo} f_j \right) \leq x_i(f_o)\phi_o, \qquad (i = 1, \ldots, m) \tag{13}$$

$$\sum_{j=1}^{n} \lambda_{jo} \leq 1, \tag{11}$$

and

$$\lambda_{jo} \geq 0 \quad (j = 1, \ldots, n). \tag{8}$$

We call this model the *diversification-consistent* (*input-oriented complete* NRS DEA) *model* because it compares each fund accurately with all possible portfolios and so deals fully with diversification.

The diversification-consistent model is important because it disposes of the assumption, implicit in conventional DEA, that

$$\kappa \left(\sum_{j=1}^{n} \lambda_j f_j \right) = \sum_{j=1}^{n} \lambda_j \kappa(f_j) \tag{14}$$

for each $\kappa \in \{x_1, \ldots, x_m, y_1, \ldots, y_s\}$. This assumption holds if κ is the mean value, but is otherwise unlikely to hold. The main problem with the diversification-consistent model is that is not generally solvable by any known practical method.

The model can use any risk and return measures though we restrict our attention to commensurable sets of nonnegative measures for the reasons Section 3 discusses.

A number of published models, though not in the form of the diversification-consistent model, deal directly with correlation. Lozano and Gutiérrez (2008, 2007) describe diversification-consistent linear DEA models with precisely one input and one output. They estimate additive metrics of the form $\delta_{\mathrm{r}} + \delta_{\mathrm{d}}$ rather than efficiencies, though the models can easily be adjusted to calculate NRS efficiencies. They achieve linearity by specifying particular estimators (and not just particular measures) for the risk measures that depend on having a vector time series of returns assumed to be stationary and without serial autocorrelation. The sets of measures are commensurable though the estimator may be biased for measures such as CVaR (see Acerbi (2007)) if we make the reasonable assumption that the distribution of returns is not discrete. Morey and Morey (1999) describe a DEA models of investment funds that deals with correlation using variance as a risk measure and includes a quadratic constraint. Joro and Na (2006) extend this to include skewness in a model with cubic constraints. Briec et al. (2004); Briec and Kerstens (2009, 2010) use a directional distance function rather than standard DEA and develop nonlinear programming models to allow higher moments as measures. The main drawback of the published nonlinear models is that they use non-commensurable sets of measures to get solvable mathematical programmes.

Measures that make the diversification-consistent model a convex programming problem

The diversification-consistent model is likely to be more tractable if we can find measures that make it a convex programming problem. And it is a convex programming problem whenever Ψ satisfies the assumptions of free disposability and NRS convexity. It is easy to check that NRS convexity is equivalent to requiring

$$x_i \left(\sum_{j=1}^{n} \lambda_j f_j \right) \leq \sum_{j=1}^{n} \lambda_j x_i(f_j) \quad \text{and} \quad y_r \left(\sum_{j=1}^{n} \lambda_j f_j \right) \leq \sum_{j=1}^{n} \lambda_j y_r(f_j) \tag{15}$$

for $i = 1, \ldots, m$ and $r = 1, \ldots, s$ in the diversification-consistent model.

We need measures satisfying inequalities (15). Artzner et al. (1999) call a measure $\kappa : \mathcal{F} \to \mathbb{R}$ a *coherent measure of risk* if it satisfies the following four properties. Here $r \in \mathbb{R}$ represents the total rate of return on a reference instrument.

Translation invariance: for $f \in \mathcal{F}$ and $\alpha \in \mathbb{R}^+$, $\kappa(f + \alpha r) = \kappa(f) - \alpha$.

Subadditivity: for $f,\, g \in \mathcal{F}$, $\kappa(f+g) \leq \kappa(f) + \kappa(g)$.

Positive homogeneity: for $\lambda \geq 0$ and $f \in \mathcal{F}$, $\kappa(\lambda f) = \lambda\kappa(f)$.

Monotonicity: for $f,\, g \in \mathcal{F}$ with $f \leq g$, $\kappa(g) \leq \kappa(f)$.

Translation invariance ensures that if we add $r\kappa(f)$ to f we get a random variable with risk measured as zero. Subadditivity ensures a sum of random variables has no greater risk than the sum of the risks of the individual random variables. Positive homogeneity ensures the risk is proportional to the size of investment. And monotonicity ensures that coherent risk measures are consistent with a general stochastic ordering. We define an additional property that is useful for return measures.

Superadditivity: for $f,\, g \in \mathcal{F}$, $\kappa(f+g) \geq \kappa(f) + \kappa(g)$.

The following theorem shows how some of these properties are useful. We define a measure to be *convexity consistent* if it is positively homogeneous and either: (i) a risk measure and subadditive; or (ii) a return measure and superadditive.

Theorem 2. *Suppose that each measure in a diversification-consistent model is convexity consistent. Then its production possibility set Ψ satisfies* NRS *convexity.*

Proof. Suppose $(\mathbf{x}_1, \mathbf{y}_1)$, $(\mathbf{x}_2, \mathbf{y}_2) \in \Psi$ and $t,\, u \in (0,1)$ with $t + u \leq 1$. Then, for some $f_1, f_2 \in \mathcal{F}$ we have $\mathbf{x}(f_1) \leq \mathbf{x}_1$, $\mathbf{y}(f_1) \geq \mathbf{y}_1$, $\mathbf{x}(f_2) \leq \mathbf{x}_2$ and $\mathbf{y}(f_2) \geq \mathbf{y}_2$.

Put $\mathbf{x}' = (x'_1, \ldots, x'_m) = \mathbf{x}(tf_1 + uf_2)$ and for $j = 1, 2$ write $\mathbf{x}_j = (x_{1j}, \ldots, x_{mj})$. Then, for $i = 1, \ldots, m$,

$$\begin{aligned} x'_i &= x_i(tf_1 + uf_2) \\ &\leq x_i(tf_1) + x_i(uf_2) \quad \text{(subadditivity)} \\ &= tx_i(f_1) + ux_i(f_2) \quad \text{(positive homogeneity)} \\ &\leq tx_{i1} + ux_{i2}. \end{aligned}$$

Hence $t\mathbf{x}_1 + u\mathbf{x}_2 \geq \mathbf{x}'$.

Similarly, put $\mathbf{y}' = (y'_1, \dots, y'_s) = \mathbf{y}(tf_1 + uf_2)$ and for $j = 1, 2$ write $\mathbf{y}_j = (y_{1j}, \dots, y_{sj})$. Then, for $r = 1, \dots, s$,

$$\begin{aligned} y'_r &= y_r(tf_1 + uf_2) \\ &\geq y_r(tf_1) + y_r(uf_2) \quad \text{(superadditivity)} \\ &= ty_r(f_1) + uy_r(f_2) \quad \text{(positive homogeneity)} \\ &\geq ty_{r1} + uy_{r2}. \end{aligned}$$

Hence $t\mathbf{y}_1 + u\mathbf{y}_2 \leq \mathbf{y}'$.

Since we defined $\mathbf{x}'$ and $\mathbf{y}'$ so that $(\mathbf{x}', \mathbf{y}') \in \Psi$, it follows from the free disposability of Ψ that $(t\mathbf{x}_1 + u\mathbf{x}_2, t\mathbf{y}_1 + u\mathbf{y}_2) \in \Psi$. It follows that Ψ satisfies NRS convexity. ∎

We now show there are practical convexity-consistent measures. It is easy to check that standard deviation is convexity consistent, and it is nonnegative by definition. Theorem 2 shows any nonnegative coherent measure of risk is convexity consistent. Krause (2002) and Acerbi and Tasche (2002) describe a range of coherent risk measures including lower semideviation and CVaR_α. We can use several values of α to get several risk measures and so model much of the shape of the distribution. It is easy to check the mean and $-$CVaR are convexity-consistent return measures. Typically we might use CVaR_α as a risk measure for $\alpha \leq 0.4$ and $-\text{CVaR}_\alpha$ as a return measure for $\alpha \geq 0.6$.

Coherent measures of risk may have negative values, but we need nonnegative risk measures. The following result shows we can construct a nonnegative convexity-consistent risk measure from any coherent measure.

Theorem 3. *Suppose $\mathcal{F}$ is a class of random variables and $x : \mathcal{F} \to \mathbb{R}$ a coherent measure of risk. Then $\hat{x} = \max(x, 0)$ is positively homogeneous, subadditive and monotonic.*

Proof. Let f and $g \in \mathcal{F}$. Then, for $\lambda \geq 0$,

$$\hat{x}(\lambda f) = \max(x(\lambda f), 0)) = \max(\lambda x(f), 0) = \lambda \hat{x}(f).$$

So $\hat{x}$ is positively homogeneous. And

$$\begin{aligned} \hat{x}(f + g) &= \max(x(f + g), 0) \leq \max(x(f) + x(g), 0) \\ &\leq \max(x(f), 0) + \max(x(g), 0) = \hat{x}(f) + \hat{x}(g). \end{aligned}$$

So $\hat{x}$ is subadditive. It follows that $\hat{x}$ is convexity consistent. Finally,

$$\begin{aligned} f \leq g &\Rightarrow x(g) \leq x(f) \Rightarrow \max(x(g), 0) \leq \max(x(f), 0) \\ &\Rightarrow \hat{x}(g) \leq \hat{x}(f). \end{aligned}$$

So $\hat{x}$ is monotonic. ∎

Note that the newly-constructed measure is not, in general, translation invariant. The main benefit of translation invariance is that it lets us to include a reference instrument in the measure. This does not matter when we can include it in the model.

A practical iterative solution method

The production possibility set of the complete NRS model is the smallest set containing $(\mathbf{x}_j, \mathbf{y}_j)$ $(j = 1, \ldots, n)$ and satisfying free disposability and NRS convexity. So it is contained in the production possibility set Ψ of the diversification-consistent model whenever Ψ is convex. This happens whenever all the measures are convexity consistent. Then the efficiencies of the complete NRS model estimate the efficiencies of the diversification-consistent model, but may do so poorly.

We want to improve the estimates of the efficiencies of the diversification-consistent model. The following model helps. It is no more than the complete NRS model with some extra funds, which are portfolios composed of the original funds. We call these portfolios *notional funds.*

For $k = 1, \ldots, \hat{n}$ and $(j = 1, \ldots, n)$ choose $\mu_{jk} \geq 0$ so that $\sum_{j=1}^{n} \mu_{jk} \leq 1$. Write $\hat{f}_k = \sum_{j=1}^{n} \mu_{jk} f_j$. Put $\hat{x}_{ij} = x_i(\hat{f}_j)$, $(i = 1, \ldots, m,\ j = 1, \ldots, \hat{n})$ and $\hat{y}_{rj} = y_r(\hat{f}_j)$, $(r = 1, \ldots, s,\ j = 1, \ldots, \hat{n})$. If $\mathbf{x}_o \leq \mathbf{0}$ put $\phi_o = 1$ or 0 according as $\mathbf{y}_o \geq \mathbf{0}$ or $\mathbf{y}_o < \mathbf{0}$. Otherwise, for $o = 1, \ldots, n$, choose ϕ_o, $\lambda_{1o}, \ldots, \lambda_{no}$, $\hat{\lambda}_{1o}, \ldots, \hat{\lambda}_{\hat{n}o}$ to

minimise ϕ_o subject to

$$\sum_{j=1}^{n} y_{rj}\lambda_{jo} + \sum_{j=1}^{\hat{n}} \hat{y}_{rj}\hat{\lambda}_{jo} \geq y_{ro}, \qquad (r = 1, \ldots, s) \tag{16}$$

$$\sum_{j=1}^{n} x_{ij}\lambda_{jo} + \sum_{j=1}^{\hat{n}} \hat{x}_{ij}\hat{\lambda}_{jo} \leq x_{io}\phi_o, \qquad (i = 1, \ldots, m) \tag{17}$$

$$\sum_{j=1}^{n} \lambda_{jo} + \sum_{j=1}^{\hat{n}} \hat{\lambda}_{jo} \leq 1, \tag{18}$$

and

$$\lambda_{jo} \geq 0 \quad (j = 1, \ldots, n), \qquad \hat{\lambda}_{jo} \geq 0 \quad (j = 1, \ldots, \hat{n}). \tag{19}$$

We call this model the *expanded* (*complete* NRS) *model.*

If Ψ_{c}, Ψ_{e}, and Ψ are the production possibility sets of the complete, expanded and diversification-consistent models then we have $\Psi_{\mathrm{c}} \subseteq \Psi_{\mathrm{e}} \subseteq \Psi$ because each notional fund must be contained in Ψ but need not be contained in Ψ_{c}. So if we can find notional funds in $\Psi \setminus \Psi_{\mathrm{c}}$ we can use them to construct an expanded model that better estimates the efficiencies of the diversification-consistent model.

Figure 2 about here.

Figure 2 describes an iterative procedure that constructs successively better approximations to the diversification-consistent model. In this procedure F represents the original funds, $\hat{F}$ the notional funds and P the pairs of funds that have not yet been considered for some improving solution. Initially we impose no extra stopping condition. Section 5 shows what we might use in practice. At each iteration $\hat{F}$ may change and we consider a fund e efficient or inefficient according as $\phi_e = 1$ or not in the expanded model ((16)–(19)) with $\{\hat{f}_1, \ldots, \hat{f}_{\hat{n}}\} = \hat{F}$.

The following observation shows why we need only consider $0.5(f + g)$ in the iterative procedure. It is a standard result, which is easy to show holds for the complete and expanded models. Put $\hat{\mathbf{x}}_j = (\hat{x}_1, \ldots, \hat{x}_m)$ and $\hat{\mathbf{y}}_j = (\hat{y}_1, \ldots, \hat{y}_s)$ $(j = 1, \ldots, \hat{n})$. Suppose $\sum_{j=1}^{n} \lambda_{jo}(\mathbf{x}_j, \mathbf{y}_j) + \sum_{j=1}^{\hat{n}} \hat{\lambda}_{jo}(\hat{\mathbf{x}}_j, \hat{\mathbf{y}}_j)$ is on the efficient frontier. Then, $\lambda_{jo} = 0$ unless $(\mathbf{x}_j, \mathbf{y}_j)$ is on the frontier $(j = 1, \ldots, n)$ and $\hat{\lambda}_{jo} = 0$ unless $(\hat{\mathbf{x}}_j, \hat{\mathbf{y}}_j)$ is on the frontier $(j = 1, \ldots, \hat{n})$.

We want to choose f and g so that h is reasonably likely to be added to S. To achieve this we maintain the pairs of funds P as a priority queue.We expect more effieicent funds to be better choices for f and g. And we expect inequalities (15) to get further from equality as the correlations between the funds get further from -1. These suggest $\phi_j^2 \phi_k^2 (1 - \rho(f_j, f_k))$ should be a reasonable priority measure and we find it works well in practice.

Section 5 shows how the iterative procedure works in practice.

Convergence of the iterative procedure and properties of its solution

To see why we might reasonably expect the iterative procedure to give efficiencies that converge to the values in the diversification-consistent model, consider the following result.

Theorem 4. *Suppose $\tilde{f}_1, \ldots, \tilde{f}_{m+s}$ are funds in an expanded model with convexity-consistent measures. Suppose also $\lambda_1 \geq 0, \ldots, \lambda_{m+s} \geq 0$ satisfy $\sum_{j=1}^{m+s} \lambda_j \leq 1$. Then*

1. *if* $x_i\left(\sum_{j=1}^{m+s}\lambda_j\tilde{f}_j\right) < \sum_{j=1}^{m+s}\lambda_j x_i\left(\tilde{f}_j\right)$ *for some* $i \in \{1,\ldots,m\}$, $x_i\left(\sum_{j=1}^{m+s}\mu_j\tilde{f}_j\right) < \sum_{j=1}^{m+s}\mu_j x_i\left(\tilde{f}_j\right)$ *whenever* $\mu_j > 0$ $(j = 1,\ldots,n)$ *and* $\sum_{j=1}^{m+s}\mu_j = 1$*;*
2. *if* $y_r\left(\sum_{j=1}^{m+s}\lambda_j\tilde{f}_j\right) > \sum_{j=1}^{m+s}\lambda_j y_r\left(\tilde{f}_j\right)$ *for some* $r \in \{1,\ldots,s\}$, $y_r\left(\sum_{j=1}^{m+s}\mu_j\tilde{f}_j\right) > \sum_{j=1}^{m+s}\mu_j y_r\left(\tilde{f}_j\right)$ *whenever* $\mu_j > 0$ $(j = 1,\ldots,n)$ *and* $\sum_{j=1}^{m+s}\mu_j = 1$.

Proof. Put $\lambda = \min\{\lambda_j/\mu_j : j = 1,\ldots,n\}$ and $\lambda_j > 0$. Then $\lambda > 0$ and $\mu_j - \lambda\lambda_j \geq 0$ for $j = 1,\ldots,n$. If $x_i\left(\sum_{j=1}^{m+s}\lambda_j\tilde{f}_j\right) < \sum_{j=1}^{m+s}\lambda_j x_i\left(\tilde{f}_j\right)$ for some $i \in \{1,\ldots,m\}$, $\mu_j > 0$ $(j = 1,\ldots,n)$ and $\sum_{j=1}^{m+s}\mu_j = 1$ then

$$\begin{aligned}
x_i\left(\sum_{j=1}^{m+s}\mu_j\tilde{f}_j\right) &= x_i\left(\lambda\sum_{j=1}^{m+s}\lambda_j\tilde{f}_j + \sum_{j=1}^{m+s}(\mu_j - \lambda\lambda_j)\tilde{f}_j\right)\\
&\leq \lambda x_i\left(\sum_{j=1}^{m+s}\lambda_j\tilde{f}_j\right) + \sum_{j=1}^{m+s}(\mu_j - \lambda\lambda_j)x_i\left(\tilde{f}_j\right)\\
&< \lambda\sum_{j=1}^{m+s}\lambda_j x_i\left(\tilde{f}_j\right) + \sum_{j=1}^{m+s}(\mu_j - \lambda\lambda_j)x_i\left(\tilde{f}_j\right)\\
&= \sum_{j=1}^{m+s}\mu_j x_i\left(\tilde{f}_j\right).
\end{aligned}$$

The second part follows by a similar argument. ∎

Since the production possibility set Ψ_e of an expanded model with convexity-consistent measures is an unbounded convex polytope, the practical consequence of Theorem 4 is that we need only check one point in the interior of each facet on the frontier of Ψ_e to find a new notional fund or show none exists. For $m + s > 2$ the iterative procedure of Figure 2 only checks points on the boundary of each facet. However, it is reasonable to expect that if the interior points of a facet give new notional funds, then so should the boundary.

Theorem 3 shows we can choose risk measures that are monotonic. We do not require this for convexity-consistent measures. The following result shows how monotonicity might be useful.

Theorem 5. *Suppose* f_p *and* f_q *are funds in a diversification-consistent model with nonnegative convexity-consistent monotonic measures. Then* $f_p \geq f_q \Longrightarrow \phi_p \geq \phi_q$.

Proof. Suppose $f_p \geq f_q$. If $x_i(f_p) = 0$ for $i = 1, \ldots, m$ then $\phi_p = 1 \geq \phi_q$ and so the result holds. Otherwise suppose $x_i(f_p) > 0$ for some i. Then, for $r = 1, \ldots, s$,

$$y_r \left(\sum_{j=1}^{n} \lambda_{jp} f_j \right) \geq y_r(f_p) \geq y_r(f_q).$$

Similarly, for $i = 1, \ldots, m$,

$$x_i \left(\sum_{j=1}^{n} \lambda_{jp} f_j \right) \leq x_r(f_p)\phi_p \leq x_i(f_q)\phi_p.$$

So $\lambda_{jq} = \lambda_{jp}$, $(j = 1, \ldots, n)$, $\phi_q = \phi_p$ satisfies constraints (12)–(13), (11) and (8) and so $\phi_p \geq \phi_q$ as required. ∎

Theorem 5 shows we can construct DEA models consistent with a general stochastic ordering. We note that, for any stochastic ordering $\preccurlyeq$, we can replace monotonicity with the condition that $\kappa(g) \leq \kappa(f)$ for f, $g \in \mathcal{F}$ with $f \preccurlyeq g$ for each measure κ. So, for example, we can construct diversification-consistent models that are consistent with second or third degree stochastic dominance (Levy, 1992), as Lozano and Gutiérrez (2008, 2007) do for simple DEA models with one risk and one return measure. Some caution is needed. The definitions of consistency with stochastic ordering (Ogryczak and Ruszczyński, 2002; Lozano and Gutiérrez, 2008) give implications of the form $f \preccurlyeq g \Longrightarrow$ some conditions on the measures $\Longrightarrow \phi_f \leq \phi_g$while we typically want $\phi_f \leq \phi_g \Longrightarrow f \preccurlyeq g$.We may be able to do no better than choose measures that make this reverse implication reasonably likely.

5. A practical illustration

We show here that the procedure of Figure 2 works well in practice and show it converges reasonably quickly to good estimates of the efficiencies of the diversification-consistent model.

We need some data. We use 60 monthly returns from 30 hedge funds between 2000 and 2204, taken from Center for International Securities and Derivatives Markets (2010). The data set contains ten of each of three classes of fund. These are market neutral (MN), global macro (GM) and long/short equity (LS). Global macro funds usually adopt a riskier strategy than long/short or market neutral ones. So we expect diverse risk–return

characteristics in different fund types. We also expect substantial asymmetry in the distributions of returns, justifying a model that incorporates more than mean and standard deviation. The columns on the right of Table 1 summarise the main features of the fund returns and indicate substantial departure from normality. Table 2 shows the correlations among six funds. These reflect the pattern in the 870 correlations between pairs of funds: at 5% significance level, 38 are significantly negatively correlated and only 137 significantly positively correlated.

We use C++ code to maintain the data structures and implement the iterative procedure, and parallel CPLEX (Ilog, 2008) to solve the linear programmes. We use a 2.66 GHz Intel Core 2 duo processor and a Gnu/Linux operating system. The iterative procedure can run indefinitely. So we introduce a stopping criterion. We check the average reduction in efficiency after a small fixed number of improving iterations and stop when this average reduction falls below a prespecified tolerance. In practice we find that 10 improving iterations and a tolerance level of 10^{-6} works well and gives a solution in under ten seconds for the examples we consider.

We consider two examples. Both use a commensurable set of convexity-consistent measures including only nonnegative risk measures. Our first example illustrates two things. First it shows the improvement available from diversification. Second, it shows that, even without a risk-free investment, the diversification-consistent model may easily find portfolios with positive return and zero measured risk. It uses the mean return of the 30 funds as a return measure and $\max(\text{CVaR}_{0.4}, 0)$ as a risk measure. Figure 3 shows the production possibility set and efficient frontier from our approximation to a diversification-consistent NRS model. The dashed line shows the frontier from the conventional NRS model. There is clear evidence that the conventional model substantially overestimates the efficiency of many funds.

Figure 3 about here.

Our second example uses risk measures $\max(\text{CVaR}_{0.05}, 0)$, $\max(\text{CVaR}_{0.1}, 0)$, $\max(\text{CVaR}_{0.2}, 0)$ and $\max(s_2 - \overline{y}, 0)$, where s_2 is the lower semideviation and $\overline{y}$ the mean, and return measures $\overline{y}$ and $-\text{CVaR}_{0.9}$. (Ogryczak and Ruszczyński (2002) show $\max(s_2 - \overline{y}, 0)$ is coherent and consistent with second degree stochastic dominance.) Six performance measures is typical for a DEA model of investment funds (Gregoriou et al., 2005a). They are enough to model most of the features of the data: much of the variance, skewness and kurtosis. They

give $f_p \leq f_q \Longrightarrow \phi_p \leq \phi_q$ for $p, q \in \{1, \ldots, n\}$ in the diversification-consistent model.

Table 1 about here

Column NRS shows the efficiencies from the NRS model described at the end of Section 2. Column DC shows the efficiencies found by the iterative procedure of Figure 2, which approximate closely those of the diversification-consistent model. Column RATIO shows the mean : standard deviation ratio for each fund. The diversification-consistent model has lower efficiencies than the NRS model for all but one fund and changes the rank order of the efficiencies. The rank order of the mean : standard deviation ratio shows greater difference, indicating the weakness of a measure that ignores the shape of funds' distributions.

The efficient notional funds we find are portfolios of six of the ten most efficient funds. Table 2 shows the correlations between these six funds. These are far from the perfect correlation needed for the NRS model.

Table 2 about here

6. Discussion

We have identified the returns to scale and measures needed for a DEA model of investment funds and shown how to handle scope for diversification. However, a number of issues remain. Although the procedure of Figure 2 works well in practice, there may be a much more efficient method to estimate a set of frontier portfolios. This would help for large data sets or (see below) if we want to repeat the procedure many thousands of times.

Like all applications of DEA to modelling funds we know of, ours finds efficiencies that describe past rather than predict future performance. We treat measures as fixed quantities rather than random variables. This issue needs further investigation. We suggest an approach based on the bootstrap, which Simar and Wilson (2000); Dyson and Shale (2010) discuss for general DEA models. The multiple observations for each fund allow another bootstrap approach and we have some promising preliminary results. We need to repeat the iterative procedure many times but find mean efficiencies and confidence intervals. We also find the bootstrap resolves a problem with deterministic DEA: the fund with highest mean return (GM2640) is given efficiency 1 no matter how great its risk measures are.

Our data come from a time series. We have ignored autocorrelation, which may affect any estimate of future efficiency. We have also ignored the time horizon of investors (Galagadera and Silvapulle, 2002). Different funds may be at their most efficient at different time horizons, and both autocorrelation and time horizons could be investigated further.

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FUND	NRS	DC	RATIO	MEAN	SD	SKEW	KURT	AC
GM2640	1.000	1.000	0.239	0.022	0.094	0.487	**1.982**	**0.364**
LS90	1.000	0.912	0.420	0.020	0.048	0.458	**1.641**	**0.211**
GM2695	1.000	0.635	0.346	0.019	0.055	**0.606**	−0.329	0.013
GM45	1.000	0.626	0.794	0.015	0.019	0.308	−0.594	**0.320**
LS40	0.668	0.553	0.215	0.021	0.098	0.206	**1.249**	0.087
MN2681	0.535	0.247	0.648	0.014	0.022	−0.214	**−0.241**	−0.121
GM366	0.860	0.234	0.695	0.011	0.016	**0.725**	**2.219**	0.157
GM2693	0.881	0.231	0.700	0.011	0.016	**0.703**	**2.170**	0.166
MN629	0.720	0.186	0.673	0.008	0.012	0.053	0.899	**0.310**
MN2540	0.618	0.182	0.589	0.009	0.015	0.455	0.872	**0.393**
MN2547	0.470	0.178	0.545	0.007	0.013	**0.690**	1.870	0.043
MN2704	0.262	0.102	0.315	0.004	0.011	**1.094**	**2.155**	**0.330**
GM2889	0.315	0.088	0.419	0.010	0.024	0.000	**1.624**	0.066
LS78	0.171	0.065	0.250	0.007	0.028	0.192	0.379	0.098
LS57	0.141	0.058	0.219	0.008	0.037	0.001	−0.646	**−0.221**
MN299	0.131	0.045	0.180	0.005	0.028	**2.786**	**16.492**	**−0.306**
LS6	0.101	0.042	0.151	0.009	0.063	**0.941**	**1.906**	−0.099
MN301	0.117	0.041	0.164	0.005	0.028	**2.760**	**16.290**	**−0.312**
LS64	0.102	0.037	0.144	0.006	0.041	**2.355**	**11.882**	−0.181
GM6876	0.010	0.033	0.186	0.006	0.030	−0.081	**2.067**	0.195
GM59	0.079	0.026	0.139	0.006	0.046	0.095	**1.901**	0.020
GM3434	0.069	0.024	0.108	0.006	0.056	**1.469**	**7.007**	−0.099
LS32	0.046	0.019	0.070	0.003	0.037	**1.752**	**7.259**	−0.140
LS20	0.044	0.018	0.067	0.003	0.037	**1.759**	**7.297**	−0.142
MN147	0.043	0.017	0.085	0.001	0.015	0.139	**1.268**	−0.072
LS79	0.037	0.012	0.068	0.006	0.082	**−1.146**	**6.987**	−0.088
MN2639	0.027	0.010	0.053	0.001	0.020	0.063	**1.378**	−0.168
GM2068	0.023	0.008	0.051	0.002	0.038	**−1.417**	**3.012**	0.124
LS39	0.007	0.003	0.013	0.001	0.083	0.493	0.550	0.020
MN2777	0.000	0.000	−0.018	−0.001	0.044	−0.381	0.297	0.010

NRS: efficiency from NRS model; DC: efficiency from expanded model approximating diversification-consistent model; RATIO: mean–standard-deviation ratio; MEAN: mean return; SD: standard deviation in return; SKEW: skewness in return; KURT: kurtosis in return; AC: serial autocorrelation in return series. Boldface indicates differences from zero significant at 5% level.

Table 1: Efficiencies and summary statistics for 30 funds

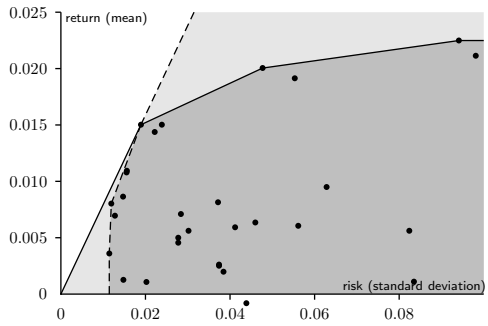

Figure 1: Comparison of different returns to scale

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put $F = \{f_1, \ldots, f_n\}$;
put $\hat{F} = \emptyset$;
put $P = \{\{f_i, f_j\} : i, j \in \{1, \ldots, n\}, i < j\}$;
while $P \neq \emptyset$ and no other *stopping condition* is met:
  choose $\{f, g\} \in P$ and put $P = P \setminus \{f, g\}$;
  choose a pair of distinct funds f and g in S not considered before;
  for $e \in \{f, g\}$:
    if $e \in \hat{F}$ and e is *inefficient:*
      put $\hat{F} = \hat{F} \setminus \{e\}$;
      for $p \in P$: if $e \in p$: put $P = P \setminus \{p\}$;
  if neither f nor g was removed from $\hat{F}$:
    put $h = 0.5(f + g)$;
    if h is *efficient* and $\mathbf{x}(h) \neq 0.5(\mathbf{x}(f)+\mathbf{x}(g))$ and $\mathbf{y}(h) \neq 0.5(\mathbf{y}(f)+\mathbf{x}(g))$:
      for $e \in F \cup \hat{F}$: put $P = P \cup \{e, h\}$;
      put $\hat{F} = \hat{F} \cup \{h\}$;

Figure 2: Iterative procedure to generate expanded models

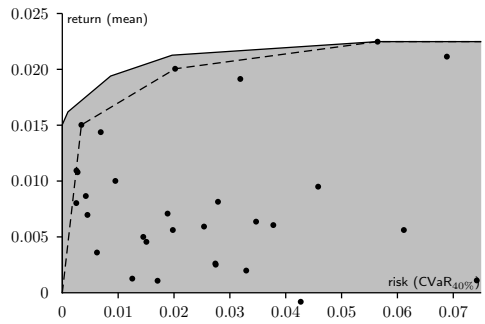

Figure 3: Approximating a diversification-consistent NRS model

FUND	LS90	GM2695	GM45	GM2693	MN2540
GM2640	*−0.144*	0.303	***−0.306***	*−0.084*	*−0.136*
LS90		*0.063*	0.295	*0.121*	*0.390*
GM2695			0.387	*0.252*	*−0.074*
GM45				0.357	*0.160*
GM2693					*0.030*

Italics indicates correlation not significantly different from zero at 5% level; bold italics indicates correlation negative and significantly different from zero at 5% level.

Table 2: Correlation coefficients for six funds