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Generalized Darboux and Darboux-Levi transformations in 2+1 dimensions

by

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"Yes, they're all fools, gentlemen ....
But the question remains, 'What *kind*
of fools are they?'"

Abstract

Solution generating techniques for 2+1-dimensional nonlinear integrable systems given by the integrability condition of linear problems (Lax pairs) are presented. According to certain symmetries of these linear problems a distinction between generalized Darboux and Darboux-Levi transformations is made. In the 1+1-dimensional limit the link to twisted and untwisted Kac-Moody algebras as prolongation algebras and the well-known N-soliton Ansatz is discussed. It is shown that the Moutard theorem and the dromion solutions for the Davey-Stewartson equation I are contained within this approach. Moreover, the applicability of an extended version of the generalized Darboux-Levi transformation to a Loewner-type system is demonstrated which leads to localized solitonic solutions of a 2+1-dimensional sine-Gordon system (Konopelchenko-Rogers equations).

Contents

Chapter 1

Introduction

During the past twenty years the study of nonlinear integrable systems has been the area of research for legions of scientists. The reason why this field of theoretical and mathematical physics has become so attractive might have its roots in the exact solvability of nonlinear partial differential equations and the behaviour of their simplest solutions called solitons. They describe phenomena in hydrodynamics, nonlinear optics, solid state physics, plasma physics and even general relativity [1, 2]. Solitons, loosely speaking, are one-dimensional localized objects which move at constant velocity, preserve their shape and do not interact with each other except for a phase-shift. J. Scott Russell who first observed 'solitary waves' whilst riding beside the narrow Union canal near Edinburgh described them as follows [3]:

> *"I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped - not so the mass of water in the channel which it had put in motion; it accumulates round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminuation of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the*

month of August 1834, was my first chance interview with that rare and beautiful phenomenon which I have called Wave of Translation ...."

The mathematical tools which have been developed to analyse nonlinear equations governing such phenomena are diverse. The oldest tool which dates back to the 19th century is the Bäcklund transformation [4]. It has its origin in the transformation for the famous sine-Gordon equation. The method from which the modern soliton theory emanated is the inverse spectral transform [5]. Hirota's direct method using bilinear forms of differential equations provides a more elementary tool for generating solutions [6]. Interestingly, the class of multi-soliton solutions for differential equations such as the Korteweg-de Vries equation, modified Korteweg-de Vries equation, sine-Gordon equation, nonlinear Schrödinger equation, Boussinesq equation, Ernst equation and many other nonlinear integrable systems could be constructed by all of these approaches [4] - [11], even though the precise relation between them is still not known.

The common feature of the above-mentioned integrable equations is that they are 1+1-dimensional (or 2+0-dimensional). In this case a necessary condition for the applicability of the inverse scattering scheme is the existence of a linear scattering problem and a compatible time evolution equation (Lax pair [12]). A semi-algorithmic method of finding such a linear problem is the prolongation method of Wahlquist and Estabrook [13, 14], which provides also a means of generating Bäcklund transformations. The content of Chapter 2 is a brief introduction to the Wahlquist-Estabrook approach.

Unfortunately, no comparable procedure is known for differential equations in 2+1 dimensions. In this thesis, however, we shall exploit the fact that the prolongation method yields infinite-dimensional Kac-Moody algebras [15] as prolongation algebras, which can be realized as loop algebras of finite-dimensional ones. Using a suitable representation it turns out that there exists a natural generalization of 1+1-dimensional linear problems and their integrability conditions, the underlying nonlinear differential equations, to 2+1 dimensions.

Different Kac-Moody algebras require different solution generating techniques so that in Chapter 3 linear problems associated with untwisted Kac-Moody algebras will be discussed. The considerations are based on a generalization of the well-known Darboux theorem [16], which in its original formulation provides an invariance of the Schrödinger equation $\lambda\phi = \phi_{xx} + u\phi$.

The twisted Kac-Moody algebras are subalgebras of untwisted ones. In a particular representation this fact can be expressed by certain symmetries of the corresponding linear problems. Since the generalized Darboux transformation does not allow for these symmetries we shall generalize a result by Levi [17] in Chapter 4. His Darboux-Levi transformation is another linear transformation which leaves the Schrödinger equation invariant. It will be shown that in 1+1 dimensions the generalized Darboux-Levi transformation reduces to the usual N-soliton Ansatz [18].

In Chapter 5 we shall prove that this transformation indeed preserves the above-mentioned symmetries of the linear problems provided the associated (adjoint) eigenfunctions satisfy appropriate constraints. First steps concerning a classification of the symmetries are taken, which are endowed with several examples, producing, among other things, well-known results such as the Moutard theorem [19, 20] and the dromion solutions of the Davey-Stewartson equation I [21].

Chapter 6 is devoted to the generalized Loewner system [22], which is not of the type discussed in the previous chapters. It represents in the 1+1-dimensional limit a linear problem which is described by a prolongation algebra involving not only the Taylor part of a loop algebra. It is, however, amenable to an extended version of a generalized Darboux-Levi transformation. Remarkably, the Loewner triad contains a 2+1-dimensional sine-Gordon system (Konopelchenko-Rogers equations [23]) wherein the spatial coordinates occur on an equal footing. Application of the generalized Darboux-Levi transformation will yield localized solitonic solutions.

Chapter 2

Prolongation structures

2.1 The general method

In 1975 Estabrook and Wahlquist [13, 14] introduced a new geometric approach to the study of integrable systems solvable by the inverse scattering technique [5]. Their method is based on Cartan's calculus of differential forms [24, 25] which can be used to express (nonlinear) partial differential equations by an equivalent closed ideal of differential forms. It provides a semi-algorithmic way of finding linear problems (Lax pairs [12]) for a given set of partial differential equations in two dimensions.

Following Estabrook and Wahlquist we begin with a set of first-order differential equations

$$h(u, u_x, u_t, x, t) = 0 \tag{2.1}$$

where u is a vector-valued function depending on x and t the partial derivatives of which being denoted by u_x and u_t. Since we are dealing with systems of differential equations, it is obvious that higher-order equations are contained within this approach.

It was Cartan's merit to point out that in almost all conceivable practical situations, i.e. if $h = 0$ includes in some sense all integrability conditions and is neither under- nor overdetermined, one can find a set of differential two-forms

$$\omega = \alpha_i du^i dx + \beta_i du^i dt + \gamma dx dt$$

(Einstein's summation convention) which is completely equivalent to the system (2.1). $\omega, \alpha_i, \beta_i$ and γ are vector-valued differential two-forms and functions respectively defined on the manifold labelled by (u, x, t). The set $I = \{\omega\}$ has the property that it constitutes a closed ideal of differential forms, i.e. generates?

$$d\omega = 0 \bmod \omega$$

and its maximum dimension integral manifold is two-dimensional. This means that the maximum dimension of a submanifold which annihilates I is $g = 2$. g is called the genus of I [26].

Furthermore, restriction of I to an integral manifold (pull-back), which can be labelled by (x, t) if x and t are involutory, i.e. if the basis one-forms dx and dt are linearly independent, gives the system of differential equations (2.1). Hence

$$\omega|_{u=u(x,t)} = 0 \Leftrightarrow h(u, u_x, u_t, x, t) = 0.$$

The next step in the prolongation method is to seek (vector-valued) one-forms

$$\Omega = -dy + F(u, x, t, y)dx + G(u, x, t, y)dt \tag{2.2}$$

which live on an extended manifold spanned by the primitive variables u, x, t and the so-called pseudopotentials y such that the set $\{\omega, \Omega\}$ is still closed. The condition for this is

$$d\Omega = 0 \bmod (\Omega, \omega). \tag{2.3}$$

This guarantees the existence of two-dimensional integral manifolds of $\{\omega, \Omega\}$, or in other words, the integrability of the Frobenius system [27]

$$\Omega|_{y=y(x,t),u=u(x,t)} = 0.$$

The central task is to evaluate condition (2.3). We shall see that this is by no means algorithmic. However, the difficult part of the process of prolonging I will turn out to be solving 'Lie algebra' equations for the pseudopotentials y. Having found solutions of these equations one can fall back on the widely-studied representation theory of Lie algebras, which, among other things, enables us to get a handle on the as yet unspecified number of pseudopotentials.

After substituting for dy-terms via (2.2) the integrability condition (2.3) reads

$$F_{u^i} du^i dx + G_{u^i} du^i dt - (F_t - G_x + [F, G]) dx dt = 0 \bmod \omega \tag{2.4}$$

where the commutator is the usual Lie bracket between two vectorfields with respect to the pseudopotentials (up to the sign). Finally, we replace as many two-forms as possible by means of the ideal I. The coefficients of the remaining basis two-forms then have to vanish identically. We obtain equations of the general structure

$$\begin{aligned} &[F, G] + F_t - G_x + F_{u^i} a^i + G_{u^i} b^i = 0 \\ &F_{u^i} a^i{}_k + G_{u^i} b^i{}_k = 0 \end{aligned} \tag{2.5}$$

where $a^i, b^i, a^i{}_k$ and $b^i{}_k$ are again functions depending only on u, x and t.

We are now looking for solutions of (2.5) of the form

$$\begin{aligned} F &= f^i(u, x, t) X_i(y) \\ G &= g^i(u, x, t) X_i(y). \end{aligned} \tag{2.6}$$

This may either be a consequence of (2.5), as it is the case for the Korteweg-de Vries equation [13], or is an Ansatz. A prominent example for this is the sine-Gordon equation [28].

Now, if we insert (2.6) into (2.5) and sort with respect to functions of u, x and t we are left with commutator equations for the vectorfields X_i, viz

$$[X_i, X_j] = c_{ij}{}^k X_k, \qquad c_{ij}{}^k = \text{const.} \tag{2.7}$$

Unfortunately, not all commutators will be given a priori since some of the functions f^i and g^i may vanish so that not all vectorfields X_i appear in F and G respectively. (Actually, one is inclined to regard this as a blessing rather than a curse because otherwise one would not be able to construct an infinite-dimensional Lie algebra from (2.7), which, apart from some curious exceptions [29], seems to be a precondition for Bäcklund transformations to exist.)

The question which now arises is whether the vectorfields X_i can be embedded in a(n) (infinite-dimensional) Lie algebra. Assuming that this is the case we alternately introduce new generators for unknown commutators and go through the Jacobi identities, which need to be satisfied. This procedure may determine unknown commutators and generate identities between some vectorfields, which have to be inserted back into the commutator relations that are already known. In doing so we might get further identities.

The process is algorithmic and can be done by computer, which in fact requires recursive programming [30, 31]. However, it either terminates, i.e. there is a finite-dimensional Lie algebra (at least two-dimensional Abelian) related to (2.7) or it seems to be open-ended. If it appears that the commutator table can be enlarged ad infinitum one is forced to interrupt the program and can attempt to deduce the structure of the Lie algebra. Unfortunately, this is a matter of intuition and experience.

Interestingly, on some occasions the incomplete set of commutator relations (2.7) is nothing but the defining relations for Kac-Moody-algebras [15]. They are generated by $3n$ generators e_i, f_i and h_i satisfying

$$\begin{aligned} [h_i, h_j] &= 0, & [e_i, f_j] &= \delta_{ij} h_j \\ [h_i, e_j] &= a_{ij} e_j, & [h_i, f_j] &= -a_{ij} f_j \end{aligned} \tag{2.8}$$

and the Serré relations

$$(\mathrm{ad} e_i)^{1-a_{ij}} e_j = 0, \qquad (\mathrm{ad} f_i)^{1-a_{ij}} f_j = 0 \tag{2.9}$$

where $A = (a_{ij})_{i,j=0,\dots,n-1}$ is an integer-valued (generalized) Cartan matrix. Two cases are of interest to us:

- Finite case. The Cartan matrix is of maximum rank n and corresponds to one of the finite-dimensional simple Lie algebras classified by Killing and Cartan [32].
- Tame case. The Cartan matrix has rank $n-1$. The algebra is an infinite-dimensional affine Kac-Moody algebra.

Under these circumstances the Cartan matrix has the following property

$$\begin{aligned} &a_{ii} = 2, \qquad a_{ij} \leq 0, \qquad i \neq j \\ &a_{ij} = 0 \;\;\Rightarrow\;\; a_{ji} = 0 \end{aligned}$$

and the off-diagonal entries satisfy the conditions

$$\begin{array}{lll} \max(-a_{ij}, -a_{ji}) \leq 4 & & \text{for } n = 2, \mathrm{rank}(A) = 1 \\ a_{ij} = a_{ji} = 0 \quad \vee & \begin{array}{l} \min(-a_{ij}, -a_{ji}) = 1 \\ \max(-a_{ij}, -a_{ji}) \leq 3 \end{array} & \text{otherwise.} \end{array}$$

The algebra of the tame case without derivation and centre can be realized as loop algebra associated with a Lie algebra $\mathcal{G}$ of the first type (or a subalgebra of it, depending on the twist). The commutator relations of

$$L(\mathcal{G}) := \mathcal{G} \otimes \mathbb{R}(\lambda, \lambda^{-1})$$

where $\mathbb{R}(\lambda, \lambda^{-1})$ is the algebra of Laurent polynomials in λ, are simply defined via the ones for its horizontal algebra $\mathcal{G}$. They read

$$[X \otimes \lambda^i, Y \otimes \lambda^j] := [X, Y] \otimes \lambda^{i+j}$$

for arbitrary generators $X, Y \in \mathcal{G}$.

In most cases it is much harder to identify the algebra. However, the prolongation algebras for differential equations which have been derived so far are either (subalgebras of) loop algebras or semi-direct products of loop algebras with the Virasoro algebra [33]. In the following we shall only be concerned with loop algebras as prolongation algebras.

Now, it is always possible to find a faithful matrix representation of the finite-dimensional Lie algebra $\mathcal{G}$, which immediately leads to a representation of its loop algebra. Thus we have found a realization of the vectorfields X_i. They are linear combinations of the basis vectorfields

$$X_i^j := \lambda^j \hat{X}_i y$$

with $\{\hat{X}_i\}$ being a basis of the above-mentioned matrix representation of $\mathcal{G}$. Consequently, the one-form Ω takes the form

$$\Omega = -dy + \hat{F}(\lambda) y dx + \hat{G}(\lambda) y dt$$

where the dependence of $\hat{F}$ and $\hat{G}$ on the primitive variables has been suppressed. It is linear in the pseudopotentials and a Laurent polynomial in the parameter λ. Sectioning on an integral manifold yields the linear problem

$$\begin{aligned} y_x &= \hat{F}(\lambda) y \\ y_t &= \hat{G}(\lambda) y \end{aligned} \tag{2.10}$$

which is guaranteed integrable since the compatibility condition (2.3), i.e.

$$\hat{F}_t - \hat{G}_x + [\hat{F}, \hat{G}] = 0 \tag{2.11}$$

is by construction satisfied on the solution manifold $h = 0$.

Linear problems of the type (2.10) and their compatibility conditions (2.11) will be the starting point for our considerations in this thesis. Before we study their possible generalizations to 2+1 dimensions let us briefly illustrate the prolongation method by its application to the Leznov-Savel'ev system.

2.2 Application to the Leznov-Savel'ev system

The Leznov-Savel'ev system [34]

$$\varphi^i_{xt} = e^{a_{ji}\varphi^j} \tag{2.12}$$

is of some importance to us as it already involves one of the Cartan matrices $A = (a_{ij})$. It contains some of the well-known integrable systems of wave equation-type, e.g.

$$A_1 = (2) \qquad \text{Liouville} \quad \varphi_{xt} = e^{\varphi}$$

$$A_1^{(1)} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \qquad \text{sinh-Gordon} \quad \varphi_{xt} = e^{\varphi} - e^{-\varphi}$$

$$A_2^{(2)} = \begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix} \qquad \text{Dodd-Bullough} \quad \varphi_{xt} = e^{2\varphi} - 2e^{-\varphi}$$

$$A_\infty^{(1)} = \begin{pmatrix} \ddots & \ddots & & & \\ \ddots & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & -1 & 2 & \ddots \\ & & & \ddots & \ddots \end{pmatrix} \qquad \text{Toda lattice} \quad \varphi^i_{xt} = e^{-\varphi^{i-1}+2\varphi^i-\varphi^{i+1}}.$$

If A corresponds to an affine Kac-Moody algebra , i.e. is of rank $n-1$, then (2.12) can be split into a coupled system of $n-1$ equations and an integration for one function, e.g. φ^0, since the kernel of A is one-dimensional. Consequently, the above-mentioned scalar equations are the only scalar equations which can be derived from (2.12).

In order to clarify this further we look more closely at one of the more exotic algebras, the exceptional algebra

$$G_2^{(1)} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -3 & 2 \end{pmatrix}.$$

The Leznov-Savel'ev system adopts the form

$$\begin{aligned} \varphi^0_{xt} &= e^{2\varphi^0-\varphi^1} \\ \varphi^1_{xt} &= e^{-\varphi^0+2\varphi^1-3\varphi^2} \\ \varphi^2_{xt} &= e^{-\varphi^1+2\varphi^2} \end{aligned}$$

which transforms under $\psi^0 := \varphi^0, \psi^1 := 2\varphi^0 - \varphi^1$ and $\psi^2 := -\varphi^1 + 2\varphi^2$ into

$$\begin{aligned} \psi^0_{xt} &= e^{\psi^1} \\ \psi^1_{xt} &= 2e^{\psi^1} - e^{-\frac{1}{2}\psi^1-\frac{3}{2}\psi^2} \\ \psi^2_{xt} &= 2e^{\psi^2} - e^{-\frac{1}{2}\psi^1-\frac{3}{2}\psi^2}. \end{aligned}$$

Thus once one has solved the coupled system for ψ^1 and ψ^2 it remains to integrate for the function ψ^0.

One may now wonder whether the Cartan matrix A has something to do with the prolongation algebra. Following the lines developed in the previous section we shall see that it is indeed possible to identify the prolongation algebra as associated with A. To this end we write (2.12) in terms of a closed ideal. One possibility is

$$\begin{aligned} \omega^i_1 &= -d\varphi^i dt + \chi^i dx dt \\ \omega^i_2 &= d\chi^i dx + e^{a_{ji}\varphi^j} dx dt. \end{aligned} \tag{2.13}$$

It is easy to verify that (2.13) is closed and has genus 2. The first set of two-forms ω^i_1 defines the functions $\chi^i = \varphi^i_x$ on an integral manifold (x and t are involutory) whereas the second one reproduces the Leznov-Savel'ev system.

Replacing $d\varphi^i dt$- and $d\chi^i dx$-terms in (2.4) and sorting with respect to the remaining basis two-forms we end up with

$$\begin{aligned} [F, G] &= -F_{\chi^i} e^{a_{ji}\varphi^j} + G_{\varphi^i} \chi^i \\ F_{\varphi^i} &= G_{\chi^i} = 0. \end{aligned} \tag{2.14}$$

We have neglected the dependence of F and G on the coordinates x and t. This can be done without loss of generality for autonomous differential equations, i.e. for those which do not depend explicitly on the coordinates.

Similar as for the sine-Gordon equation we are not able to extract any useful information about the remaining dependence of F and G on the primitive variables from these equations. Hence we make the more or less natural Ansatz (cf. [28]) As in the case of

$$\begin{aligned} F &= X_i \chi^i + Y \\ G &= Z_i e^{a_{ji}\varphi^j}. \end{aligned}$$

Finally, (2.14) has to be satisfied identically in χ^i and the exponentials. We conclude

$$\begin{aligned} [X_i, Z_j] &= a_{ij} Z_j \\ [Y, Z_i] &= -X_i. \end{aligned} \tag{2.15}$$

Let us now come back to the defining relations (2.8). Summation of the second equation over f_j yields

$$[e_i, \sum_j f_j] = h_i.$$

Hence the identifications

$$X_i := h_i, \qquad Y := \sum_j f_j, \qquad Z_i := e_i \tag{2.16}$$

provide a possible solution of our problem, viz

$$\begin{aligned} \Omega &= -dy + (h_i \chi^i + \sum_j f_j) dx + e_i e^{a_{ji}\varphi^j} dt \\ d\Omega &= 0 \bmod (\Omega, \omega_1, \omega_2). \end{aligned} \tag{2.17}$$

We should mention that in the case of finite-dimensional Kac-Moody algebras this solution seems too restrictive. The prolongation algebra for example of the Liouville equation is in fact not the finite-dimensional $A_1 = sl(2, \mathbb{R})$ but its loop algebra. To see this we emphasize that (2.16) is only a particular case of the more general solution $X_0 := h_0, Y := f_0 + \lambda e_0, Z_0 := e_0$, namely $\lambda = 0$. Thus if we regard the arbitrary parameter λ as the indeterminate of the Laurent polynomials $\mathbb{R}(\lambda, \lambda^{-1})$ we rediscover the prolongation algebra $sl(2, \mathbb{R}) \otimes \mathbb{R}(\lambda, \lambda^{-1})$. We shall not pursue this point any further as the general solution of the Leznov-Savel'ev system in the finite case has already been given anyway [35].

We are therefore interested in the affine Kac-Moody algebras. It has turned out that (2.17) reproduces for example the well-known prolongation structure for the sinh-Gordon equation [36]. Since the Dodd-Bullough equation will be mentioned later in connection with the Novikov-Veselov equation, we derive, as an example, its linear problem (2.10) in detail. All we need is a matrix representation of the generators $\{e_i, f_i, h_i\}$ associated with $A_2^{(2)}$. The simplest one is given in terms of the 3×3-matrices

$$\hat{e}_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \lambda^{-1} & 0 & 0 \end{pmatrix}, \quad \hat{f}_0 = \begin{pmatrix} 0 & 0 & \lambda \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \hat{h}_0 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\hat{e}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \hat{f}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}, \quad \hat{h}_1 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

which immediately leads to the linear problem

$$\begin{aligned} y_x &= \begin{pmatrix} -\varphi_x & 0 & \lambda \\ 2 & 0 & 0 \\ 0 & 2 & \varphi_x \end{pmatrix} y \\ y_t &= \begin{pmatrix} 0 & e^{-\varphi} & 0 \\ 0 & 0 & e^{-\varphi} \\ \lambda^{-1} e^{2\varphi} & 0 & 0 \end{pmatrix} y. \end{aligned} \tag{2.18}$$

As can be seen it involves only one function $\varphi := \varphi^0 - 2\varphi^1$ which reflects the decoupling of the Leznov-Savel'ev system in the affine case. Finally, its integrability condition (2.11) is indeed the Dodd-Bullough equation.

We wish to complete the chapter with the remark that each equation of the Leznov-Savel'ev system has its compatible counterpart of modified Korteweg-de Vries-type [34]. Compatible is meant in the sense that one can find a closed ideal of differential two-forms which represents both equations and has genus 3. As examples we mention the known relation between the sinh-Gordon equation and the modified Korteweg-de Vries equation in potential form, viz

$$\begin{aligned}\varphi_{xt} &= e^{\varphi} - e^{-\varphi}\\ \varphi_z &= \varphi_{xxx} - \tfrac{1}{2}\varphi_x^3\end{aligned}$$

and the compatibility of the Dodd-Bullough equation and a fifth-order equation found by Konopelchenko and Dubrovsky [37, 38], namely

$$\begin{aligned}\varphi_{xt} &= e^{2\varphi} - 2e^{-\varphi}\\ \varphi_z &= \varphi_{xxxxx} - 5\varphi_{xxx}\varphi_{xx} - 5\varphi_{xxx}\varphi_x^2 - 5\varphi_{xx}^2\varphi_x + \varphi_x^5.\end{aligned}$$

The fact that each couple of equations share three-dimensional integral manifolds can formally be verified by cross-differentiation.

Chapter 3

Darboux transformations

In this and the following chapters we shall try to find 2+1-dimensional generalizations of integrable systems which are given by the integrability condition (2.11). The idea is to stick as close as possible to the original systems in 1+1 dimensions. For this purpose an appropriate extension of certain symmetries which may characterize the linear problems (2.10) needs to be found. It is clear that these symmetries make the underlying partial differential equations in question differ from each other. They can most conveniently be described in terms of the prolongation algebras. Furthermore, it will turn out that the kind of algebra is closely related to the kind of solution generating technique. In the case of untwisted Kac-Moody algebras generalized Darboux transformations [39] will be the suitable approach, whereas linear problems corresponding to twisted Kac-Moody algebras are amenable to an extended version of the Darboux-Levi transformation [17]. Finally, we shall show that either of those transformations reduce in the 1+1-dimensional limit to the well-known N-soliton Ansatz of Neugebauer, Kramer and Meinel [18, 40].

3.1 2+1-dimensional equations

In this section we shall assume that the prolongation algebra is the Taylor part of an untwisted Kac-Moody algebra $A_n^{(1)}$. Prominent members of this class are nonlinear Schrödinger equation, Korteweg-de Vries equation, modified Korteweg-de Vries equation ($A_1^{(1)}$) or Boussinesq equation ($A_2^{(1)}$) [36, 41].

As mentioned in the previous chapter its realization is

$$A_{n-1}^{(1)+} := sl(n, \mathbb{R}) \otimes \mathbb{R}(\lambda).$$

Furthermore, we choose the $n^2 - 1$ traceless matrices of dimension n as representation and suppose that the linear problem (2.10)

$$\begin{aligned} y_x &= F(\lambda)y \\ y_t &= G(\lambda)y \end{aligned} \tag{3.1}$$

is polynomial of finite degree. (We have dropped the hat since it is obvious that F and G are matrices.) Thus, so far, the matrices F and G are characterized by their degree, dimension and the absence of trace-terms. Unfortunately, this is not sufficient to distinguish between the underlying partial differential equations. In most cases the highest or second highest powers of F and G have a particular structure. Experience, however, has shown that the N-soliton Ansatz has enough degrees of freedom to allow for that special choice of matrices. The reason for this fact has not been unveiled yet. The problem is that one does not know in terms of which quantities one might classify these reductions.

A natural way of generalizing the linear problem (3.1) to 2+1 dimensions is now to replace the parameter λ by a derivative ∂_z and regard F and G as linear differential operators in ∂_z depending on x, t and the additional independent variable z. The associated linear problem therefore reads

$$\begin{aligned} \phi_x &= F(\partial_z)\phi \\ \phi_t &= G(\partial_z)\phi \end{aligned} \tag{3.2}$$

where ϕ is called the (vector-valued) eigenfunction of (3.2). It is now necessary to drop the conditon that the operators F and G are traceless. In 1+1 dimensions this did not constitute a restriction since trace-terms could be gauged away. Thus the matrix-valued differential operators F and G are only determined by their order and dimension.

Finally, the nonlinear equations in 2+1 dimensions are given by the operator equation

$$F_t(\partial_z) - G_x(\partial_z) + [F(\partial_z), G(\partial_z)] = 0 \tag{3.3}$$

which is the compatibility condition of (3.2). If we assume that $F(\partial_z)$ and $G(\partial_z)$ are parametrized by some functions u^i we can write (3.3) as

$$h(u, u_x, u_t, u_z, u_{zz}, \ldots) = 0. \tag{3.4}$$

The 1+1-dimensional limit with respect to ∂_z is performed by assuming F and G not to depend on z any more and setting

$$\phi(x, t, z) = e^{\lambda z}\phi(x, t).$$

Consequently, (3.2) and (3.3) turn into (2.10) and(2.11) respectively.

Even though the substitution $\lambda \to \partial_z$ seems quite naïve we shall see that it is a powerful method for the purpose of creating integrable equations which have given 1+1-dimensional equations as dimensional reductions. On the other hand, if we assume that an equation in 2+1 dimensions can be represented by the integrability condition (3.3) its symmetry reduced equations with respect to any Lie-point symmetry must be amenable to the prolongation method. Hence it is certainly helpful not to tackle a 2+1-dimensional equation directly but to find a linear problem for a symmetry reduced equation such that the generalization in the above-mentioned manner gives the equation with which we started.

3.2 A generalized Darboux theorem

We are now looking for linear transformations which leave the linear problem (3.2) invariant. To this end we briefly recall an old theorem given by Darboux in 1882 [16].

Theorem 1 (Darboux Theorem). The nonlinear Schrödinger equation

$$\lambda\phi = \phi_{zz} + u\phi \tag{3.5}$$

is invariant under

$$\begin{aligned} \phi \to \tilde{\phi} &= \Big(-\frac{\mathring{\phi}_z}{\mathring{\phi}} + \partial_z\Big)\phi \\ u \to \tilde{u} &= u + 2(\ln \mathring{\phi})_{zz} \end{aligned}$$

where $\mathring{\phi}$ is an arbitrary solution of (3.5) with the parameter λ_0.

The essential point is that the new field $\tilde{u}$ depends only on the eigenfunction $\overset{\circ}{\phi}$ and not on ϕ. It is therefore possible to iterate the above transformation. The iterated version of the Darboux theorem was proven by Crum [42]. It reads

Theorem 2. The function

$$\tilde{\phi} = \frac{W(\phi_1, \ldots, \phi_N, \phi)}{W(\phi_1, \ldots, \phi_N)} \tag{3.6}$$

solves the Schrödinger equation (3.5) with the field

$$\tilde{u} = u + 2[\ln W(\phi_1, \ldots, \phi_N]_{zz}$$

for linearly independent eigenfunctions $\phi_1, \ldots, \phi_N$ and the usual abbreviation of the Wronskian determinant

$$W(f_1, \ldots, f_k) := \det \left(\frac{d^{j-1} f_i}{dz^{j-1}} \right)_{i,j=1,\ldots,k}.$$

In order to generalize the Darboux theorem we observe that in Crum's result the new eigenfunction $\tilde{\phi}$ can equivalently be characterized by

$$\begin{aligned} \tilde{\phi} &= P(\partial_z)\phi = (P_j \partial_z^j + \partial_z^N)\phi \\ P(\partial_x)\phi_i &= 0, \qquad i = 1, \ldots, N. \end{aligned}$$

Having this in mind we are now in the position to generalize the Darboux theorem. The only thing we have to do is to replace in the above formula the scalar eigenfunctions by matrix-valued ones.

Theorem 3 (Generalized Darboux transformation). Let $F(\partial_z)$ be a matrix-valued linear differential operator of finite order and Φ be a matrix-valued solution of

$$\Phi_x = F(\partial_z)\Phi. \tag{3.7}$$

Further, let $\tilde{\Phi}$ be defined by

$$\tilde{\Phi} := P(\partial_z)\Phi$$

where the linear differential operator $P(\partial_z)$ of order N is given by

$$P(\partial_z)\Phi_i := (P_j\partial_z^j + \partial_z^N)\Phi_i = 0 \tag{3.8}$$

with N linearly independent solutions $\Phi_1, \ldots, \Phi_N$ of (3.7).
Then there exists an operator $\tilde{F}(\partial_z)$ of the same order as $F(\partial_z)$ such that the pair $\{\tilde{F}(\partial_z), \tilde{\Phi}\}$ again satisfies (3.7).

We note that the operator $P(\partial_z)$ is given in a purely algebraic manner. The system of linear equations (3.8) may be solved via Cramer's rule. In the scalar case its solution can nicely be written in terms of Wronskian determinants (cf. Crum theorem).

The constructive proof of this theorem is quite simple. It is based on the factorizability of linear differential operators. By assumption the column vectors of $\{\Phi_i\}$ are a basis of the kernel of P. (From now on we omit the explicit dependence of operators on ∂_z in the formulae.) From that we compute

$$0 = (P\Phi_i)_x = (P_x + PF)\Phi_i$$

i.e. the kernel of P is a subspace of the kernel of $Q := P_x + PF$. Q is of order $N + n_F$, where n_F is the order of F. We now define the operator $\tilde{F}$ by the requirement that the operator

$$R := Q - \tilde{F}P$$

is of order $N-1$. On the other hand, we know that

$$R\Phi_i = Q\Phi_i - \tilde{F}P\Phi_i = 0$$

for N eigenfunctions Φ_i. Consequently $R \equiv 0$.

This kind of factorizability is well-known in the scalar case and, as shown above, extendable to matrix-operators. It is clear that $\tilde{F}$ has the same order as F. Furthermore, we obtain

$$\tilde{\Phi}_x = (P\Phi)_x = (P_x + PF)\Phi = Q\Phi = \tilde{F}P\Phi = \tilde{F}\tilde{\Phi}$$

which establishes Theorem 3. q.e.d.

It is now obvious that Theorem 3 is applicable to the second equation of the linear problem (3.2) as well. Thus we are left with

$$
\begin{aligned}
\tilde{\Phi}_x &= \tilde{F}\tilde{\Phi} \\
\tilde{\Phi}_t &= \tilde{G}\tilde{\Phi}.
\end{aligned}
\tag{3.9}
$$

The corresponding integrability condition is

$$
(\tilde{F}_t - \tilde{G}_x + [\tilde{F}, \tilde{G}])\tilde{\Phi} = 0 \tag{3.10}
$$

which is equivalent to the operator equation (3.3) for the twiddled quantities since at any point (x, t) the general solution of (3.9) is an arbitrary function of z. Hence the ordinary differential operator within the brackets of (3.10) vanishes identically. A direct way of proving the operator equation (3.3) will be formulated in the next section.

To summarize: We have proven that the generalized Darboux transformation as defined in Theorem 3 leaves the linear problem (3.2) invariant. The coefficients $\tilde{u}^i$ which parametrize the new operators $\tilde{F}$ and $\tilde{G}$ are given in terms of the old ones and the arbitrary solutions Φ_i of (3.2),viz

$$
\tilde{u} = \tilde{u}(u, \Phi_1, \ldots, \Phi_N). \tag{3.11}
$$

For $N = 1$ it is often possible to solve (3.11) for Φ_1. Insertion of Φ_1 into the linear problem (3.2) then results in two equations of the form

$$
\begin{aligned}
&f(\tilde{u}, \tilde{u}_x, \tilde{u}_t, \tilde{u}_z, \tilde{u}_{\ldots}, u, u_x, u_t, u_z, u_{\ldots}) = 0 \\
&g(\tilde{u}, \tilde{u}_x, \tilde{u}_t, \tilde{u}_z, \tilde{u}_{\ldots}, u, u_x, u_t, u_z, u_{\ldots}) = 0.
\end{aligned}
$$

which are called a Bäcklund transformation or more precise an auto-Bäcklund transformation [4] of the nonlinear equations (3.4). $\tilde{u}_{\ldots}$ and $u_{\ldots}$ denote some elements of the local jet bundle. For lack of a better expression we also use this term for the relation (3.11) ($N = 1$) analogously to the 1+1-dimensional case [13]. The iterated version is then called the N-fold Bäcklund transformation.

Finally, we wish to stress that in the 1+1-dimensional limit, i.e. $\partial_z \to \lambda$, the Darboux transformation reduces to the N-soliton Ansatz

$$
\begin{aligned}
&\tilde{\phi} = P(\lambda)\phi \\
&P(\lambda_i)\phi_i = 0, \qquad i = 1, \ldots, nN \\
&\tilde{F}(\lambda) = P(\lambda)F(\lambda)P^{-1}(\lambda) + P_x(\lambda)P^{-1}(\lambda) \\
&\tilde{G}(\lambda) = P(\lambda)G(\lambda)P^{-1}(\lambda) + P_t(\lambda)P^{-1}(\lambda).
\end{aligned}
$$

Here, ϕ_i are n-dimensional vector-valued eigenfunctions. The second equation guarantees that the transformed matrices $\tilde{F}$ and $\tilde{G}$ are again polynomial in λ as pointed out in [18]. We have therefore found a natural generalization of the N-soliton Ansatz, which has successfully been applied to the entire AKNS-system or in a modified version to Ernst's equation of general relativity [40].

3.3 Explicit formulae

So far, the transformed operator F is only given implicitly. (The same goes for G.) The $n_F + 1$ highest orders in the defining equation

$$\tilde{F}P = PF + P_x \tag{3.12}$$

(cf. the proof of Theorem 3) constitute recursive relations for the coefficients of $\tilde{F}$. In order to find an explicit expression for $\tilde{F}$ it is convenient to introduce the concept of pseudo-differential operators as, for example, employed in the Sato theory [43]. We shall be using the pseudo-differential symbol only as a means of book-keeping, even though its area of application is much wider.

We consider the space of operators

$$\{\sum_{i<+\infty} m_i \partial_z^i\} \tag{3.13}$$

where m_i are matrix-valued functions depending on z. In this context the independent variables x and t can be regarded as parameters. The set of operators (3.13) is provided with an algebraic structure by means of multiple application of the generalized Leibniz rules

$$\begin{aligned} &\partial_z^i \partial_z^j := \partial_z^{i+j} \\ &\partial_z m := m\partial_z + m_z \\ &\partial_z^{-1} m := m\partial_z^{-1} - m_z\partial_z^{-2} + m_{zz}\partial_z^{-3} \mp \ldots . \end{aligned} \tag{3.14}$$

We emphasize that for $k \geq 0$, ∂_z^k is a usual differential operator whereas ∂_z^{-1} is no~~t~~ more than the formal inverse of ∂_z. It cannot act on a function in the sense of an integration! A useful formula deduced from (3.14) is

$$m\partial_z^{-1} = \partial_z^{-1} m + \partial_z^{-2} m_z + \partial_z^{-3} m_{zz} + \ldots . \tag{3.15}$$

The benefit which we get from the introduction of the pseudo-differential operators is the formal invertibility of differential operators. This enables us to solve the recursion relations (3.12). First of all we prove

Lemma 4. The inverse of the operator P as defined in Theorem 3 is given by

$$P^{-1} = \Phi_i \partial_z^{-1} Q^i$$

where the matrices Q^i are the solution of the linear equations

$$(\partial_z^i \Phi_j)_0 Q^j = \delta^{iN-1}. \tag{3.16}$$

In the following $(\mathcal{O}m)_0$ is an abbreviation of the function which is given by the action of an operator $\mathcal{O}$ on the function m. However, in order not to clutter up the formulae with too many symbols we try to suppress the index whenever it is obvious that $\mathcal{O}m$ is a function and not an operator. Similarly, $\mathcal{O}_{\geq 0}$ denotes the differential part of $\mathcal{O}$.

For the proof we note that $P\Phi_i\partial_z^{-1}Q^i$ is purely differential since $(P\Phi_i)_0 = 0$. Application of the formula (3.15) yields

$$\begin{aligned}\Phi_i\partial_z^{-1}Q^i &= \sum_{j\geq 0} \partial_z^{-1-j}(\partial_z^j\Phi_i)_0 Q^i \\ &= \partial_z^{-N} + \sum_{j\geq N} \partial_z^{-1-j}(\partial_z^j\Phi_i)_0 Q^i\end{aligned}$$

because of (3.16). Hence

$$\begin{aligned}P\Phi_i\partial_z^{-1}Q^i &= (P\Phi_i\partial_z^{-1}Q^i)_{\geq 0} \\ &= [(\partial_z^N + P_k\partial_z^k)(\partial_z^{-N} + \sum_{j\geq N} \partial_z^{-1-j}(\partial_z^j\Phi_i)_0 Q^i)]_{\geq 0} \\ &= 1.\end{aligned}$$

Thus multiplication of (3.12) by P^{-1} from the right and projection onto the differential part yields

$$\tilde{F} = (PFP^{-1})_{\geq 0}$$

since P_x is of order $N-1$ and for any differential operator R of order $N-1$ the differential part of RP^{-1} vanishes.

To formulate the most general Darboux-type transformation which leaves the linear problem (3.2) invariant we remark that we still have one more degree of freedom, namely a gauge transformation of the operators F and G with an arbitrary matrix depending on x, t and z. The following theorem takes this into account.

Theorem 5. The linear problem

$$\begin{aligned} \Phi_x &= F\Phi \\ \Phi_t &= G\Phi \end{aligned}$$

and its integrability condition

$$F_t - G_x + [F, G] = 0$$

is invariant under

$$\begin{aligned} \Phi \rightarrow \tilde{\Phi} &:= \Upsilon P \Phi \\ F \rightarrow \tilde{F} &:= \Upsilon (PFP^{-1})_{\geq 0} \Upsilon^{-1} + \Upsilon_x \Upsilon^{-1} \\ G \rightarrow \tilde{G} &:= \Upsilon (PGP^{-1})_{\geq 0} \Upsilon^{-1} + \Upsilon_t \Upsilon^{-1} \end{aligned}$$

with P as defined in Theorem 3 and an arbitrary matrix Υ depending on x, t and z.

Even though it has already been proven that the integrability condition for the transformed quantities is satisfied, we wish to mention, for reasons which become apparent later, that this property can be regarded as embedded in a more general framework. The general formulation reads

Lemma 6. Let the pseudo-differential operators L_1 and L_2 satisfy the condition

$$L_{1t} - L_{2x} + [L_1, L_2] = 0. \tag{3.17}$$

The gauge transformation

$$\begin{aligned} \tilde{L}_1 &:= TL_1T^{-1} + T_xT^{-1} \\ \tilde{L}_2 &:= TL_2T^{-1} + T_tT^{-1} \end{aligned}$$

with an arbitrary pseudo-differential operator T then generates a new solution of (3.17).

Lemma 6 may directly be proven by inserting $\tilde{L}_1$ and $\tilde{L}_2$ into (3.17).

The essential feature of the Darboux operator P is that it transforms differential operators into differential operators. We shall see that the operator which corresponds to the generalized Darboux-Levi transformation shares this property.

3.4 The Davey-Stewartson equation II

The aim of the following is to show the practical applicability of the methods developed in the previous sections. Before we discuss an example in extenso we wish to mention the classical Kadomtsev-Petviashvili equation

$$u_{zz} \pm (u_t + u_{xxx} + uu_x)_x = 0. \tag{3.18}$$

It arises naturally in plasma physics, gas dynamics and hydrodynamics. In the latter case it describes long gravity waves in shallow water, which move predominantly in one direction with a small perturbation in the perpendicular one. Aside from this physical importance the Kadomtsev-Petviashvili equation is the generic member of a hierarchy of the same name [43]. The 2+1-dimensional equations which can be derived from this hierarchy are integrable in the sense of admitting a non-trivial prolongation structure (Lax formulation), infinitely many conservation laws, Bäcklund transformations, multi-soliton solutions and other interesting structures.

Coming back to the Kadomtsev-Petviashvili equation itself it has yet another property which seems typical of integrable equations. Its Lie-point symmetry group [44] is infinite-dimensional. To be explicit, its symmetry algebra is a subalgebra of the loop algebra $sl(5, \mathbb{R}) \otimes \mathbb{R}(\lambda, \lambda^{-1})$. It is known that symmetry reduction with respect to any generator leads, apart from the trivial equation $u_{zz} = 0$, either to the Korteweg-de Vries equation ($u_z = 0$)

$$u_t + u_{xxx} + uu_x = 0$$

or to the Boussinesq equation ($u_t = 0$)

$$u_{zz} \pm (u_{xxx} + uu_x)_x = 0$$

[45]. Furthermore, starting with one of those 1+1-dimensional equations and generalizing them as described before (the prolongation algebra is $A_1^{(1)}$ and

$A_2^{(1)}$ respectively) one can show that one recovers equation (3.18). We stress that in this case the gauge matrix Υ turns out not to be trivial.

As another example, which will be discussed in detail, we now choose the nonlinear Schrödinger equation. Its 'real' version has the form

$$\begin{aligned} &a_t + a_{xx} - 2aba = 0 \\ &b_t - b_{xx} + 2bab = 0 \end{aligned} \tag{3.19}$$

By letting $t \to -it$ and identifying a and b with the complex functions u and $\pm\bar{u}$ respectively we obtain the nonlinear Schrödinger equation

$$iu_t + u_{xx} \mp 2u|u|^2 = 0$$

which plays an important role, for example, in nonlinear optics [46]. The bar denotes complex conjugation.

It has been pointed out that the prolongation algebra for system (3.19) is the untwisted Kac-Moody algebra $(A_1^{(1)})$ [29, 47]. The degree of the matrix F is $\deg F = 1$ whereas $\deg G = 2$. In our particular representation their dimension is 2.

Let us now have a look at the Darboux transformation for $\Upsilon = 1$ (Theorem 3). The highest coefficient of the operators transform as

$$\tilde{H}_k = H_k$$

where $H = F$ for $k = 1$ and $H = G$ for $k = 2$. Since H_k is invariant, we can choose it to be constant. Consequently,

$$\begin{aligned} \tilde{H}_{k-1} &= H_{k-1} \ + [P_{N-1}, H_k] \\ \tilde{H}_{k-2} &= H_{k-2} \ + [P_{N-1}, H_{k-1}] + [P_{N-2}, H_k] \\ &\quad - [P_{N-1}, H_k]P_{N-1} + NH_{k-1z} - kH_kP_{N-1z}. \end{aligned} \tag{3.20}$$

The first equation shows that only the projection of H_{k-1} onto the image of $\operatorname{ad} H_k$ changes. Thus the Darboux transformation allows the choice

$$H_k \sim \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad H_{k-1} \sim \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix} \tag{3.21}$$

which is exactly the additional requirement of the type mentioned in the introduction of this chapter. It is the particular feature of the real version of the nonlinear Schrödinger equation.

We are now in the position to calculate its 2+1-dimensional extension. We consider the linear problem

$$\begin{aligned}\Phi_x &= (F_0 + F_1\partial_z)\Phi \\ \Phi_t &= (G_0 + G_1\partial_z + G_2\partial_z^2)\Phi.\end{aligned} \tag{3.22}$$

The integrability condition (3.3) gives the following constraints on the matrices F_i and G_i:

$$\begin{aligned}&[F_1, G_2] = 0 \\ &[F_0, G_2] + [F_1, G_1] = 0 \\ &-G_{1x} + [F_0, G_1] + [F_1, G_0] - 2G_2F_{0z} + F_1G_{1z} = 0 \\ &F_{0t} - G_{0x} + [F_0, G_0] - G_1F_{0z} - G_2F_{0zz} + F_1G_{0z} = 0.\end{aligned} \tag{3.23}$$

The parametrization

$$F_1 = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad F_0 = Q = \begin{pmatrix} 0 & b \\ a & 0 \end{pmatrix}$$

immediately leads to

$$G_2 = 2\sigma_3, \qquad G_1 = 2Q, \qquad G_0 = R$$

because of the first two equations of (3.23). (The scalar factor in G_2 is arbitrary.) The remaining equations then simplify to

$$2Q_x + 2\sigma_3Q_z - [\sigma_3, R] = 0 \tag{3.24}$$

$$Q_t - R_x + [Q, R] - 2QQ_z - 2\sigma_3Q_{zz} + \sigma_3R_z = 0. \tag{3.25}$$

(3.24) is an equation only for the off-diagonal entries of R. Hence we decompose R into its diagonal part and off-diagonal part, i.e. $R = R^d + R^o$ and obtain

$$R^o = \sigma_3Q_x + Q_z.$$

Furthermore, decomposition of (3.25) results in

$$R_x^d - \sigma_3R_z^d - [Q, \sigma_3Q_x] + [Q, Q_z]_+ = 0 \tag{3.26}$$

(diagonal part) and

$$Q_t - \sigma_3(Q_{xx} + Q_{zz}) + [Q, R^d] = 0 \tag{3.27}$$

(off-diagonal part), where $[\,.\,,\,.\,]_+$ is the anti-commutator.

In order to evaluate (3.26) it is now convenient to make the decomposition

$$R^d = p + q\sigma_3$$

which leads to a coupled system for p and q, viz

$$\begin{aligned} p_x + (ab)_z - q_z &= 0 \\ q_x + (ab)_x - p_z &= 0. \end{aligned}$$

After introducing a potential χ and writing (3.27) explicitly we finally conclude

$$\begin{aligned} a_t + \Delta a + a\Delta\chi &= 0 \\ b_t - \Delta b - b\Delta\chi &= 0 \\ \Box\chi + 2ab &= 0 \end{aligned} \tag{3.28}$$

with the parametrization

$$\begin{aligned} p &= \chi_{xz} \\ q &= \chi_{zz} - ab \end{aligned} \tag{3.29}$$

and the abbreviations $\Delta := \partial_x^2 + \partial_z^2$ and $\Box := \partial_x^2 - \partial_z^2$. As it can easily be seen, dimensional reduction of (3.28) with respect to ∂_z leads to the system (3.19) with which we started.

Similar to the 1+1-dimensional case, we can identify a and b with a complex field u. We are interested in the following identifications

- $t \to -it, \quad a = u, \quad b = -\bar{u}$:

$$\begin{aligned} iu_t + \Delta u + u\Delta\chi &= 0 \\ \Box\chi - 2|u|^2 &= 0 \end{aligned} \qquad \text{Davey-Stewartson I} \tag{3.30}$$

- $t \to -it, \quad z \to -iz, \quad a = u, \quad b = \pm\bar{u}$:

$$\begin{aligned} iu_t + \Box u + u\Box\chi &= 0 \\ \Delta\chi \pm 2|u|^2 &= 0 \end{aligned} \qquad \text{Davey-Stewartson II.} \tag{3.31}$$

The systems (3.30) and (3.31) are different versions of the Davey-Stewartson equation. (3.31) with the plus sign between the Laplace operator and the nonlinear term has a physical application. It describes the evolution of long water waves of slowly varying amplitude under gravity [48, 49]. The version (3.30) is the one that possesses localized coherent structures, the so-called dromions [21, 50]. It remains to prove which of the reductions are compatible with the Darboux transformation.

For the purpose of getting explicit formulae for the new solutions of the system (3.28) we note that the defining relations for P can be written in terms of $2N$ linearly independent vector-valued solutions of (3.22), namely

$$P\phi_i = 0, \qquad i = 1, \dots, 2N. \tag{3.32}$$

The relevant coefficients are therefore found to be

$$\begin{aligned}
(P_{N-1})_{11} &= -W(\phi^1, \phi^2, \dots, \partial_z^{N-2}\phi^1, \partial_z^{N-2}\phi^2, \partial_z^{N}\phi^1, \partial_z^{N-1}\phi^2)/W \\
(P_{N-1})_{22} &= -W(\phi^1, \phi^2, \dots, \partial_z^{N-2}\phi^1, \partial_z^{N-2}\phi^2, \partial_z^{N-1}\phi^1, \partial_z^{N}\phi^2)/W \\
(P_{N-1})_{12} &= -W(\phi^1, \phi^2, \dots, \partial_z^{N-2}\phi^1, \partial_z^{N-2}\phi^2, \partial_z^{N-1}\phi^1, \partial_z^{N}\phi^1)/W \\
(P_{N-1})_{21} &= -W(\phi^1, \phi^2, \dots, \partial_z^{N-2}\phi^1, \partial_z^{N-2}\phi^2, \partial_z^{N}\phi^2, \partial_z^{N-1}\phi^2)/W
\end{aligned}$$

with the definition

$$W := W(\phi^1, \phi^2, \dots, \partial_z^{N-1}\phi^1, \partial_z^{N-1}\phi^2).$$

The quantities $W(\dots)$ denote double Wronski-type determinants of matrices with columns of the kind

$$\begin{pmatrix} \partial_z^k \phi_1^\alpha \\ \vdots \\ \partial_z^k \phi_{2N}^\alpha \end{pmatrix}$$

for some k and $\alpha = 1, 2$. Since the first equation of the linear problem (3.22) enables us to replace derivatives of ϕ_i with respect to z of any order by derivatives with respect to x via

$$\begin{aligned}
\phi_{ix}^1 &= b\phi_i^2 + \phi_{iz}^1 \\
\phi_{ix}^2 &= a\phi_i^1 - \phi_{iz}^2
\end{aligned}$$

we obtain the useful formulae

$$\begin{aligned}(P_{N-1})_{11}+(P_{N-1})_{22}&=-(\ln W)_z\\(P_{N-1})_{11}-(P_{N-1})_{22}&=-(\ln W)_x.\end{aligned}\tag{3.33}$$

We are now left with the explicit calculation of the new fields $\tilde{a},\tilde{b}$ and $\tilde{\chi}$ in terms of the generalized Wronskian determinants. The first equation of (3.20) gives

$$\begin{aligned}\tilde{a}&=a+2(P_{N-1})_{21}\\\tilde{b}&=b-2(P_{N-1})_{12}.\end{aligned}$$

Taking the trace of the second equation on the one hand and doing the same after multiplying it by σ_3 on the other hand we compute

$$\begin{aligned}\mathrm{tr}\tilde{R}&=\mathrm{tr}R-4\mathrm{tr}(\sigma_3P_{N-1})_z\\\mathrm{tr}(\sigma_3\tilde{R})&=\mathrm{tr}(\sigma_3R)+\mathrm{tr}Q^2-\mathrm{tr}(Q+[P_{N-1},\sigma_3])^2-4\mathrm{tr}P_{N-1z}.\end{aligned}$$

If we now take into account the parametrization of R we obtain two equations for $\tilde{\chi}$ of the form

$$\begin{aligned}\tilde{\chi}_{xz}&=\chi_{xz}-2\mathrm{tr}(\sigma_3P_{N-1})_z\\\tilde{\chi}_{zz}&=\chi_{zz}-2\mathrm{tr}P_{N-1z}\end{aligned}$$

which can be integrated by means of (3.33). The final result is

$$\tilde{\chi}=\chi+2\ln W.$$

Trivial functions of integration have been set to zero whereas the nontrivial one has to vanish since the linear problem (3.22) and its Darboux transformation is invariant under $(x,z,b,\tilde{a},\tilde{b})\to(z,x,-b,\pm\tilde{a},\mp\tilde{b})$ depending on N even or odd.

In closing this chapter we remark that the Davey-Stewartson equation II can be characterized by the two equivalent properties

- The operators F and G have the symmetry $\bar{F}\sigma=\sigma F,\bar{G}\sigma=\sigma G$.
- If ϕ is an eigenfunction of (3.22), so is $\sigma\bar{\phi}$. (3.34)

with

$$\sigma := \begin{pmatrix} 0 & \pm 1 \\ 1 & 0 \end{pmatrix}.$$

The sign in σ corresponds to the sign in (3.31). Hence the choice

$$\phi_{i+N} = \sigma\bar{\phi}_i, \qquad i = 1, \ldots, N$$

in the defining relations (3.32) produces the symmetry

$$\bar{P}\sigma = \sigma P$$

which has the consequence

$$\sigma\bar{\tilde{\phi}} = \sigma\bar{P}\bar{\phi} = P\sigma\bar{\phi}$$

because of $\sigma^{-1} = \pm\sigma$. Thus the transformed eigenfunction $\tilde{\phi}$ satisfies the property (3.34) as well as ϕ. Consequently, the Darboux transformation allows for the reduction (3.31).

Without going into details we state that for the Davey-Stewartson equation II with the bright soliton solution (minus sign) the seed solution $u = \chi = 0$ and a suitable choice of eigenfunctions (essentially exponentials) lead to the class of solutions found by Hirota, Ohta and Satsuma [51]. It has been derived via Hirota's direct method using bilinear equations [6] and contains two-dimensional multi-soliton solutions.

Finally, it has turned out that the remaining system (3.30) is not compatible with the Darboux transformation. However, we shall see in Chapter 5 that there is a symmetry associated with the corresponding linear problem which is preserved under a generalized Darboux-Levi transformation.

Chapter 4

Darboux-Levi transformations

The previous chapter has been devoted to the extension of linear problems and their integrability conditions (underlying nonlinear differential equations) in 1+1 dimensions which have no symmetry in a sense which will be specified later. The precise form of the linear problems has been given by the prolongation algebras $A_n^{(1)+}$ and a particular representation of the simple Lie algebras $sl(n+1,\mathbb{R})$. As seen at the end of the last section, certain symmetries may appear which are not preserved under a Darboux transformation. Thus another transformation has to be sought which allows for these symmetries. Since the theory of this new kind of transformation can be developed independently of invariances, we shall first turn our attention to the general theory (this chapter) and then give the precise definition of the above-mentioned symmetries (next chapter). In the 1+1-dimensional limit they will be closely related to twisted Kac-Moody algebras as prolongation algebras so that our very first intention of preserving all characteristic properties of the 1+1-dimensional originals will be met.

4.1 A generalized Darboux-Levi transformation

As motivation and justification of the title of this section we review a theorem given by Levi in 1988 [17].

Theorem 7 (Darboux-Levi transformation). The Schrödinger equation

$$\lambda\phi = \phi_{zz} + u\phi \tag{4.1}$$

is invariant under

$$\phi \to \tilde{\phi} := \phi - \mathring{\phi}\, \frac{M(\mathring{\psi}, \phi)}{M(\mathring{\psi}, \mathring{\phi})}$$

$$u \to \tilde{u} := u + 2[\ln M(\mathring{\psi}, \mathring{\phi})]_{zz}$$

where $\mathring{\phi}$ and $\mathring{\psi}$ are arbitrary solutions of (4.1) and its adjoint

$$-\mu\psi = \psi_{zz} + u\psi$$

respectively and a bilinear potential has been introduced according to

$$M_z(\psi, \phi) := \psi\phi.$$

We note that the new eigenfunction $\tilde{\phi}$ is again a linear functional of ϕ which satisfies $\tilde{\phi}[\mathring{\phi}] = 0$. Furthermore, the new field $\tilde{u}$ depends only on the eigenfunction and adjoint eigenfunction $\mathring{\phi}$ and $\mathring{\psi}$ respectively which 'drive' the transformation of ϕ. What is different from the Darboux transformation is the introduction of a potential. We shall see that a similar bilinear potential will have a key position in the extended version of the Darboux-Levi transformation.

We consider, as usual, matrix-operators F and G which are polynomial in ∂_z and of finite order. They define the linear problem

$$\begin{aligned} \phi_x &= F\phi \\ \phi_t &= G\phi \end{aligned} \tag{4.2}$$

where ϕ is a vector-valued eigenfunction. Its adjoint is of the form

$$\begin{aligned} \psi_x &= -F^*\psi \\ \psi_t &= -G^*\psi \end{aligned} \tag{4.3}$$

where the adjoint of any operator $\mathcal{O}$ is defined by

$$\mathcal{O}^* = (\mathcal{O}_i\partial_z^i)^* := (-\partial_z)^i\mathcal{O}_i^T. \tag{4.4}$$

The superscript ^{T} denotes transposition. ψ is called an adjoint eigenfunction. Since $(\mathcal{O}_1\mathcal{O}_2)^* = \mathcal{O}_2^*\mathcal{O}_1^*$ it is clear that the integrability conditions of (4.2) and (4.3) are the adjoints of each other, i.e. the underlying nonlinear differential equations are the same.

Now, from the definition of the adjoint operator it immediately follows that $\psi^T F\phi - (F^*\psi)^T\phi$ is a total z-derivative, viz

$$\psi^T F\phi - (F^*\psi)^T\phi = X_z(\psi,\phi).$$

X emerges from throwing over z-derivatives in $\psi^T F\phi$ to the left-hand side by substracting total derivatives. It can nicely be expressed in terms of residues of pseudo-differential operators, i.e.

$$\mathrm{res}(\mathcal{O}_i\partial_z^i) := \mathcal{O}_{-1}.$$

All we need are the useful formulae

$$\begin{aligned}\mathrm{res}\mathcal{O}^* &= -(\mathrm{res}\mathcal{O})^T\\ (\mathrm{res}\mathcal{O})_z &= \mathrm{res}(\partial_z\mathcal{O} - \mathcal{O}\partial_z)\end{aligned}$$

with which we compute

$$\begin{aligned}\psi^T F\phi - (F^*\psi)^T\phi &= \mathrm{res}(\psi^T F\phi\partial_z^{-1}) - \mathrm{res}(\phi^T F^*\psi\partial_z^{-1})\\ &= \mathrm{res}(\psi^T F\phi\partial_z^{-1} - \partial_z^{-1}\psi^T F\phi)\\ &= [\mathrm{res}(\partial_z^{-1}\psi^T F\phi\partial_z^{-1})]_z.\end{aligned}$$

On the other hand if we combine (4.2) and (4.3) we obtain

$$(\psi^T\phi)_x = \psi^T\phi_x + \psi_x^T\phi = \psi^T F\phi - (F^*\psi)^T\phi = X_z(\psi,\phi). \tag{4.5}$$

We are therefore in the position to formulate

Lemma 8. The linear problem (4.2) and its adjoint (4.3) admit the potential

$$\begin{aligned}M_z(\psi,\phi) &= \psi^T\phi\\ M_x(\psi,\phi) &= \mathrm{res}(\partial_z^{-1}\psi^T F\phi\partial_z^{-1})\\ M_t(\psi,\phi) &= \mathrm{res}(\partial_z^{-1}\psi^T G\phi\partial_z^{-1}).\end{aligned}$$

For the proof it remains to show that the two potentials which are defined by (4.5) and its counterpart for G coincide. Indeed, cross-differentiation and use of the identity

$$\operatorname{res}(\partial_z^{-1}\mathcal{O}_1\mathcal{O}_2\partial_z^{-1}) = \operatorname{res}(\partial_z^{-1}\mathcal{O}_1(\mathcal{O}_2)_0\partial_z^{-1}) + \operatorname{res}(\partial_z^{-1}(\mathcal{O}_1^*)_0^T\mathcal{O}_2\partial_z^{-1})$$

for any differential operators $\mathcal{O}_1$ and $\mathcal{O}_2$ [52] yields

$$\begin{aligned}
[M_x(\psi,\phi)]_t - [M_t(\psi,\phi)]_x = \; & \operatorname{res}(\partial_z^{-1}\psi^T(F_t - G_x)\phi\partial_z^{-1}) \\
& - \operatorname{res}((G^*\psi)_0^T F\phi\partial_z^{-1}) \\
& + \operatorname{res}(\partial_z^{-1}\psi^T F(G\phi)_0\partial_z^{-1}) \\
& + \operatorname{res}((F^*\psi)_0^T G\phi\partial_z^{-1}) \\
& - \operatorname{res}(\partial_z^{-1}\psi^T G(F\phi)_0\partial_z^{-1}) \\
= \; & \operatorname{res}(\partial_z^{-1}\psi^T(F_t - G_x + [F, G])\phi\partial_z^{-1})
\end{aligned}$$

which vanishes due to the integrability condition of (4.2).

We remark that in the scalar case the potential $M(\psi,\phi)$ is precisely the one which has been introduced in Theorem 7. The transformation of ϕ is equivalently given by

$$\tilde{\phi} = \phi + AM(\mathring{\psi}, \phi)$$

where the function A is determined via

$$\tilde{\phi}[\mathring{\phi}] = 0.$$

Analogous to the Darboux transformation it turns out that this is the suitable characterization for a generalized Darboux-Levi transformation.

Theorem 9 (Generalized Darboux-Levi transformation). Let $\phi_1, \ldots, \phi_N$ be linearly independent solutions of

$$\phi_x = F\phi \tag{4.6}$$

and $\psi_1, \ldots, \psi_N$ linearly independent solutions of the adjoint problem

$$\psi_x = -F^*\psi.$$

Then there exists a differential operator $\tilde{F}$ of the same order as F which satisfies (4.6) with the new eigenfunction

$$\tilde{\phi} := \phi + A^i M(\psi_i, \phi)$$

where the vector-valued functions A^i are the solution of the linear algebraic equations

$$\tilde{\phi}[\phi_i] = 0.$$

The proof will be given constructively by defining the linear functional

$$R[\phi] := \tilde{\phi}_x - \tilde{F}\tilde{\phi} \tag{4.7}$$

for an as yet unspecified differential operator $\tilde{F}$. The first observation is that

$$R[\phi_i] = 0 \tag{4.8}$$

as a direct consequence of $\tilde{\phi}[\phi_i] = 0$. If we evaluate (4.7) and separate the 'differential part' we obtain

$$R[\phi] = Q[\phi] + (A^i_x - (\tilde{F}A^i)_0)M(\psi_i, \phi) \tag{4.9}$$

with

$$\begin{aligned} Q[\phi] := {} & F\phi + A^i X(\psi_i, \phi) \\ & - \tilde{F}[\phi + A^i M(\psi_i, \phi)] + (\tilde{F}A^i)_0 M(\psi_i, \phi). \end{aligned} \tag{4.10}$$

$Q[\phi]$ can be regarded as a differential operator acting on ϕ. Hence the condition $Q = 0$ defines $\tilde{F}$ uniquely, having the same order as F. Finally, since Q vanishes identically, (4.8) constitutes a linear homogeneous system of N equations for N vector-valued coefficients. Its determinant $\det M(\psi_i, \phi_j)$ can be chosen non-zero as the potentials are determined only up to arbitrary constants. We therefore conclude

$$A^i_x = \tilde{F}A^i \tag{4.11}$$

and hence $R[\phi] = 0$. q.e.d.

We observe that the vectors A^i are obviously eigenfunctions of the twiddled linear problem (4.2), which reflects the fact that in Theorem 9 $\tilde{\phi}$ is only defined up to arbitrary constants stemming from the potentials $M(\psi_i, \phi)$.

It is now convenient to derive two different explicit expressions for the new operator $\tilde{F}$. To this end we note that insertion of the definition of $X(\psi_i, \phi)$ into $Q = 0$ yields

$$[\tilde{F}(1 + A^i \partial_z^{-1} \psi_i^T)]_{\geq 0} = [(1 + A^i \partial_z^{-1} \psi_i^T) F]_{\geq 0} \tag{4.12}$$

which can immediately be solved for $\tilde{F}$. We get

$$\begin{aligned} \tilde{F} &= (DFD^{-1})_{\geq 0} \\ D &:= 1 + A^i \partial_z^{-1} \psi_i^T. \end{aligned} \tag{4.13}$$

On the other hand, we can drop the subscripts $_{\geq 0}$ in (4.12) if we subtract the corresponding negative orders. (4.12) becomes

$$\begin{aligned} \tilde{F}D - (\tilde{F}A^i)_0 \partial_z^{-1} \psi_i^T &= DF - (A^i \partial_z^{-1} \psi_i^T F)_{<0} \\ &= DF + (F^* \psi_i \partial_z^{-1} A^{iT})^*_{<0} \\ &= DF + [(F^* \psi_i)_0 \partial_z^{-1} A^{iT}]^* \\ &= DF - A^i \partial_z^{-1} (F^* \psi_i)_0^T \\ &= DF + A^i \partial_z^{-1} \psi_{ix}^T \end{aligned}$$

and hence

$$\tilde{F} = DFD^{-1} + D_x D^{-1} \tag{4.14}$$

having used (4.11).

The next step in the procedure is to find an explicit expression for D^{-1}. After some trial and error we obtain

Lemma 10. The inverse of the operator

$$D = 1 + A^i \partial_z^{-1} \psi_i^T$$

is given by

$$D^{-1} = 1 - \phi_i \partial_z^{-1} B^{iT}.$$

The vector-valued functions B^i are the solution of the linear equations

$$\psi_j + B^i M(\psi_j, \phi_i) = 0.$$

The proof of the lemma is very simple if one makes use of the formula

$$m_z = \partial_z m - m\partial_z$$

which is true for any matrix m depending on z. A brief calculation shows

$$\psi_j^T \phi_i = M_z(\psi_j, \phi_i) = \partial_z M(\psi_j, \phi_i) - M(\psi_j, \phi_i)\partial_z$$

from which we conclude

$$\begin{aligned} D(1 - \phi_i \partial_z^{-1} B^{iT}) &= 1 + A^j \partial_z^{-1} \psi_j^T - \phi_i \partial_z^{-1} B^{iT} \\ &\quad - A^j \partial_z^{-1} \psi_j^T \phi_i \partial_z^{-1} B^{iT} \\ &= 1 + A^j \partial_z^{-1} [\psi_j^T + B^{iT} M(\psi_j, \phi_i)] \\ &\quad - [\phi_i + A^j M(\psi_j, \phi_i)] \partial_z^{-1} B^{iT} \\ &= 1. \end{aligned}$$

To round off we wish to find the transformation law of the adjoint eigenfunction. Taking the adjoint of (4.13) we obtain

$$F^* = [(1 + B^i \partial_z^{-1} \phi^{iT}) F (1 - \psi_j \partial_z^{-1} A^{jT})]_{\geq 0}$$

which is nothing but (4.13) itself after interchanging eigenfunctions and adjoint eigenfunctions as well as A^j and B^i. Doing the same in Theorem 9 we end up with

$$\tilde{\psi} = \psi + B^i M(\psi, \phi_i).$$

It is now clear that all of the above considerations for the operator F hold mutatis mutandis for G. Furthermore, since we have been able to write the transformation of the operators as gauge transformation with the pseudo-differential operator D (4.14), we can apply Lemma 6, which implies that the integrability condition for the transformed operators is again satisfied. A further gauge transformation with an arbitrary (non-constant) matrix Υ results in

Theorem 11. The linear problem

$$\begin{aligned} \phi_x &= F\phi \\ \phi_t &= G\phi \end{aligned}$$

together with its adjoint

$$\psi_x = -F^*\psi$$
$$\psi_t = -G^*\psi$$

and their compatibility condition

$$F_t - G_x + [F, G] = 0$$

possess the invariance

$$\begin{aligned}
\phi \to \tilde{\phi} &:= \Upsilon[\phi + A^i M(\psi_i, \phi)] \\
\psi \to \tilde{\psi} &:= \Upsilon^{-1T}[\psi + B^i M(\psi, \phi_i)] \\
F \to \tilde{F} &:= \Upsilon(DFD^{-1})_{\geq 0}\Upsilon^{-1} + \Upsilon_x\Upsilon^{-1} \\
G \to \tilde{G} &:= \Upsilon(DGD^{-1})_{\geq 0}\Upsilon^{-1} + \Upsilon_t\Upsilon^{-1}
\end{aligned}$$

where the quantities involved are defined in Lemma 8, Theorem 9 and Lemma 10.

For practical purposes it is helpful to be aware of the following relation between the coefficients A^i and B^i:

Lemma 12. The vectors A^i and B^i satisfy the matrix identity

$$A^i\psi_i^T = \phi_i B^{iT}.$$

To prove this identity we have to consider only the definitions of A^i and B^i. We compute

$$\begin{aligned}
0 &= [\phi_i + A^j M(\psi_j, \phi_i)]B^{iT} \\
&= \phi_i B^{iT} + A^j B^{iT} M(\psi_j, \phi_i) \\
&= \phi_i B^{iT} - A^j \psi_j^T.
\end{aligned}$$

Résumé: The (generalized) Darboux-Levi transformation is driven by N pairs of (adjoint) eigenfunctions (ψ_i, ϕ_i). The new operators and therefore the parametrizing fields $\tilde{u}$ depend only on these pairs, i.e.

$$\tilde{u} = \tilde{u}(u, \psi_1, \phi_1, \ldots, \psi_N, \phi_N).$$

Both Darboux transformation and Darboux-Levi transformation have the property that the new eigenfunction $\tilde{\phi}$ is a linear functional of ϕ. It is algebraically determined by the basis $\{\phi_i\}$ of its kernel.

Let us now turn to the question whether the Darboux-Levi transformation in the 1+1-dimensional limit $\partial_z \longrightarrow \lambda$ has anything to do with what is already known in 1+1 dimensions.

4.2 The 1+1-dimensional limit

For simplicity we shall discuss only the case $N = 1$ and $\Upsilon = 1$. (In fact one can show that this implies no loss of generality.) As mentioned in the previous chapter the 1+1-dimensional limit is performed by letting

$$\phi(x,t,z) \longrightarrow e^{\lambda z}\phi(x,t)$$
$$\psi(x,t,z) \longrightarrow e^{\mu z}\psi(x,t).$$

The crucial point is that we are now able to integrate for the potential $M(\psi_1, \phi)$ because

$$M_z(\psi_1,\phi) = \psi_1^T \phi e^{(\mu_1+\lambda)z}$$

(cf. Lemma 8) which leads to

$$M(\psi_1,\phi) = (\mu_1+\lambda)^{-1}\psi_1^T\phi e^{(\mu_1+\lambda)z} + f(x,t)$$

with an arbitrary function of integration $f(x,t)$. Since the right-hand sides of the two remaining equations for $M(\psi_1, \phi)$ in Lemma 8 are bilinear in the exponentials, it immediately follows that f is constant. The constant is arbitrary and can therefore be set to zero. Hence we obtain for the new eigenfunction $\tilde{\phi}$

$$\tilde{\phi} = (\mu_1+\lambda)^{-1}[\mu_1 - (\mu_1+\lambda_1)\frac{\phi_1\psi_1^T}{\psi_1^T\phi_1} + \lambda]\phi. \tag{4.15}$$

On the other hand, the 1+1-dimensional limit of the linear problem (4.2) and its adjoint (4.3) reads

$$\begin{aligned} \phi_x &= F(\lambda)\phi, & \psi_x &= -F^T(-\mu)\psi \\ \phi_t &= G(\lambda)\phi, & \psi_t &= -G^T(-\mu)\psi \end{aligned} \tag{4.16}$$

from which we conclude

$$
\begin{aligned}
&[\psi^T(\mu_1)\phi(-\mu_1)]_x = 0 \\
&[\psi^T(\mu_1)\phi(-\mu_1)]_t = 0.
\end{aligned}
$$

Since (4.16) constitute~~s~~ Frobenius systems, we can find $n-1$ linearly independent eigenfunctions $\phi_1(-\mu_1), \ldots, \phi_{n-1}(-\mu_1)$ ~~being~~ orthogonal to $\psi(\mu_1)$. n is the dimension of the matrices F and G. If we now apply the N-soliton Ansatz

$$
\tilde{\phi} = P(\lambda)\phi = (P_0 + \lambda)\phi
$$

with

$$
\begin{aligned}
&P(\lambda_1)\phi(\lambda_1) = 0 \\
&P(-\mu_1)\phi_i(-\mu_1) = 0
\end{aligned}
$$

where $\phi(\lambda_1)$ is an arbitrary eigenfunction, we reproduce (4.15) up to a constant and hence irrelevant factor, viz

$$
\tilde{\phi} = [\mu_1 - (\mu_1 + \lambda_1)\frac{\phi(\lambda_1)\psi^T(\mu_1)}{\psi^T(\mu_1)\phi(\lambda_1)} + \lambda]\phi.
$$

One can easily verify that $\tilde{\phi}$ vanishes for $\phi \in \{\phi(\lambda_1), \phi_k(-\mu_1)\}$.

Thus the Darboux-Levi transformation reduces in the 1+1-dimensional limit to the well-known N-soliton Ansatz for suitably chosen eigenfunctions. In the reduction $\psi(\mu_1) = \psi[\phi(\lambda_1 = \mu_1)]$, which will be discussed in the following chapter, this is precisely the choice which generates the Bäcklund transformation for equations such as the potential modified Korteweg-de Vries equation, Dodd-Bullough equation, the fifth-order equation mentioned at the end of Chapter 2 and many other equations [53].

4.3 The time-dependent Schrödinger equation

The example which we wish to discuss in this section is the 'time-dependent' Schrödinger equation. We consider the real scalar operator

$$
F = \partial_x^2 + u. \tag{4.17}
$$

Here, we are not interested in a pair of linear equations. We shall investigate only the equation

$$\phi_x = \phi_{zz} + u\phi$$

and its adjoint

$$-\psi_x = \psi_{zz} + u\psi.$$

The Darboux-Levi transformation provides according to Theorem 9 an invariance of the more general operator

$$F_2\partial_z^2 + F_1\partial_z + F_0.$$

Evaluation of (4.12) or (4.13) yields

$$\begin{aligned}
\tilde{F}_2 &= F_2 \\
\tilde{F}_1 &= F_1 \\
\tilde{F}_0 &= F_0 - 2F_2(A^i\psi_i)_z + F_2A^i\psi_{iz} - A^i(\psi_iF_2)_z
\end{aligned}$$

which allows for the specialization (4.17). The field u transforms as

$$\tilde{u} = u - 2(A^i\psi_i)_z.$$

The following lemma helps simplify the above expression.

Lemma 13. The coefficients A^i satisfy the identity

$$\psi_i^T A^i = -(\ln|M|)_z$$

with the abbreviation $|M| := \det M(\psi_i, \phi_j)$.

For the proof it is convenient to define the vectors

$$M^i[v_1, \ldots, v_N] := (M^i[v_1^j, \ldots, v_N^j])_{j=1,\ldots,N}$$

where $M^i[v_1^j, \ldots, v_N^j]$ denotes the determinant of the matrix which is obtained by replacing the i.th row in the matrix M by $(v_1^j, \ldots, v_N^j)$. $\{v_k\}$ are vectors of arbitrary dimension. These determinants are those which appear when solving the algebraic equations for A^i via Cramer's rule. We obtain

$$A^i = -\frac{M^i[\phi_1, \ldots, \phi_N]}{|M|}.$$

Hence

$$\begin{aligned}
\psi_i^T A^i &= -\psi_i^T M^i[\phi_1, \dots, \phi_N]/|M| \\
&= -\sum_i M^i[M_z(\psi_i, \phi_1), \dots, M_z(\psi_i, \phi_N)]/|M| \\
&= -(\ln|M|)_z
\end{aligned}$$

which proves the lemma.

The final result for the new potential of the Schrödinger equation is then

$$\tilde{u} = u + 2(\ln|M|)_{zz}$$

which has, in the 1+1-dimensional limit and $N = 1$, exactly the form given in the original Darboux-Levi transformation (Theorem 7).

Chapter 5

Reductions and symmetries

Our intention in this chapter is to reduce the number of fields u which parametrize the operators F and G in the linear problems (4.2) and (4.3) and therefore in the associated nonlinear differential equations such that the Darboux-Levi transformation is still applicable. We shall endow the operators with symmetries relating them to their adjoints. As a consequence the adjoint eigenfunctions will turn out to be linear functionals of eigenfunctions so that the Darboux-Levi transformation will only be driven by an arbitrary number of eigenfunctions. The symmetries can be expressed as gradation of the prolongation algebras in the 1+1-dimensional limit. After a suitable application of isomorphisms the prolongation algebras can be identified with twisted Kac-Moody algebras. For this reason we wish to begin with a brief introduction to graded loop algebras.

5.1 Twisted Kac-Moody algebras

We know already that untwisted Kac-Moody algebras can be realized as loop algebras of finite-dimensional simple Lie algebras. A twisted Kac-Moody algebra is associated with an outer automorphism of a simple Lie algebra $\mathcal{G}$. Let us assume that $\mathcal{G}$ admits an automorphism σ of finite order τ, i.e.

$$[\sigma(X),\sigma(Y)] = \sigma([X,Y]), \qquad \sigma^{\tau} = 1$$

for $X, Y \in \mathcal{G}$. It is then possible to decompose $\mathcal{G}$ into the eigenspaces of σ as

$$\mathcal{G} = \bigoplus_{\bar{k}\in Z_{\tau}} \mathcal{G}_{\bar{k}}, \qquad [\mathcal{G}_{\bar{k}},\mathcal{G}_{\bar{l}}] \subset \mathcal{G}_{\overline{k+l}} \tag{5.1}$$

where $Z_\tau := Z \bmod \tau$ and $\mathcal{G}_{\bar{k}}$ are the eigenspaces of σ for the eigenvalues $\exp(2i\pi k/\tau)$ with $\bar{k} = k \bmod \tau$. It is clear that $\mathcal{G}_{\bar{0}}$ is a Lie algebra, the so-called horizontal algebra. The gradation (5.1) now defines in a natural manner the loop algebra

$$L(\mathcal{G},\sigma) = \bigoplus_{k\in Z} (\mathcal{G}_{\bar{k}} \otimes \lambda^k) \tag{5.2}$$

associated with σ. The twisted Kac-Moody algebras are central extensions of these loop algebras. For our purposes it is sufficient to identify them with the latter algebras. Taking into account that different outer automorphisms of $\mathcal{G}$ may generate identical loop algebras one finds the following classes of twisted Kac-Moody algebras [15]:

- Twist $\tau = 2$. $A_2^{(2)}, A_{2n}^{(2)}, A_{2n'-1}^{(2)}, D_{n-1}^{(2)}, E_6^{(2)}$ for $n \geq 2$ and $n' \geq 3$.
- Twist $\tau = 3$. $D_4^{(3)}$.

In fact these classes correspond to the symmetries of the Dynkin diagrams of the finite simple Lie algebras A_n, D_n and E_6 which are isomorphic to the conjugacy classes of their outer automorphisms. The conjugacy classes are given by the factor group

$$\frac{\text{automorphisms}}{\text{inner automorphisms}}.$$

The inner automorphisms are 'divided out' since they generate isomorphic loop algebras. The factor group is finite and discrete.

Similar as in Chapter 3 we shall discuss only the loop algebras associated with the simple Lie algebras A_l, i.e. $A_2^{(2)}, A_{2n}^{(2)}$ and $A_{2n-1}^{(2)}$. The representation we choose is given in terms of their horizontal algebras

$$\mathcal{G}_{\bar{0}} = B_n \qquad \text{for } A_{2n}^{(2)}, \qquad n \geq 1$$

and

$$\mathcal{G}_{\bar{0}} = D_n \qquad \text{for } A_{2n-1}^{(2)}, \qquad n \geq 3.$$

The simple Lie algebras B_n and D_n can be regarded as matrices being antisymmetric relative to

$$\Xi_{B_n} = \begin{pmatrix} 1 & & \\ & & 1_n \\ & 1_n & \end{pmatrix} \qquad , \Xi_{D_n} = \begin{pmatrix} & 1_n \\ 1_n & \end{pmatrix}$$

respectively [32]. 1_n denotes the n-dimensional unit matrix. A representation of the graded loop algebras $L(\mathcal{G},\sigma)$ is then obtained by means of the defining relation

$$X^T(-\lambda)\Xi = -\Xi X(\lambda) \tag{5.3}$$

where Ξ is one of the matrices above. Different representations are obtained by applying similarity transformations with a constant matrix $\Theta(\lambda)$ to the matrices $X(\lambda)$, which leads to the invariance of (5.3)

$$\begin{aligned} X &\to \Theta^{-1}(\lambda) X \Theta(\lambda) \\ \Xi &\to \Theta^T(-\lambda)\Xi\Theta(\lambda). \end{aligned} \tag{5.4}$$

We stress that after a similarity transformation Ξ may depend on λ. Moreover, Ξ is only defined up to a constant factor.

It is easily seen that (5.3) is still meaningful for $\mathcal{G} = D_n$ with $n = 1, 2$. This is interesting because the associated loop algebras are of course isomorphic to the untwisted Kac-Moody algebras $A_1^{(1)}$ and $A_3^{(1)}$ respectively. On the other hand, the Taylor parts, in which we are only interested, are not. Thus it makes sense to introduce the algebras $A_1^{(2)}$ and $A_3^{(2)}$.

Furthermore, one can construct a representation of $A_{2n}^{(2)}$ in another gradation corresponding to an outer automorphism of order $\tau = 4$ [54]. The horizontal algebra is the symplectic algebra C_n [32]. It is clear that a representation of this algebra cannot be given by (5.3) if λ is supposed to be associated with the gradation. For, if the parameter λ in (5.3) coincides with the indeterminate of the Laurent polynomials in (5.2), the twist of the corresponding loop algebra has to be $\tau = 2$.

How do we find, in some sense, a natural representation of the algebra in question? To this end we perform a similarity transformation (5.4) to $\Xi = \Xi_{B_n}$ with the matrix

$$\Theta(\lambda) = \begin{pmatrix} 1 & & \\ & 1_n & \\ & & -\lambda^{-1}1_n \end{pmatrix} \tag{5.5}$$

which results in the block matrix

$$\Xi_{C_n} := \begin{pmatrix} \lambda & & \\ & & -1_n \\ & 1_n & \end{pmatrix}.$$

We observe that C_n is indeed antisymmetric relative to the λ-free part of Ξ_{C_n}, which we denote by A. The solution of (5.3) decomposes naturally into four classes

$$\lambda^{2k}\begin{pmatrix} 0 & 0 \\ 0 & Q_+ \end{pmatrix}, \qquad \lambda^{2k}\begin{pmatrix} 0 & u_-^T A \\ \lambda u_- & 0 \end{pmatrix}$$

$$\lambda^{2k}\begin{pmatrix} \lambda a & 0 \\ 0 & \lambda Q_- \end{pmatrix}, \qquad \lambda^{2k}\begin{pmatrix} 0 & -\lambda u_+^T A \\ \lambda^2 u_+ & 0 \end{pmatrix}$$

with the restrictions

$$Q_\pm^T A = \mp A Q_\pm, \qquad a := -\mathrm{tr} Q_-.$$

After a further similarity transformation with

$$\Theta(\lambda) = \begin{pmatrix} \lambda^{-\frac{1}{2}} & \\ & 1_{2n} \end{pmatrix}$$

and the substitution $\lambda \to \lambda^2$ we finally end up with the decomposition of the representation

$$\bigoplus_{k \in Z} (X_0 \lambda^{4k} \oplus X_1 \lambda^{4k+1} \oplus X_2 \lambda^{4k+2} \oplus X_3 \lambda^{4k+3}). \tag{5.6}$$

The matrices X_k are obviously the matrices above for $\lambda = 1$. They constitute a representation of the eigenspaces $\mathcal{G}_{\bar{k}}$ of the outer automorphism of order $\tau = 4$ with which we started. Hence we shall regard the representation given by $\Xi = \Xi_{C_n}$ as *the* representation of the Kac-Moody algebra $A_{2n}^{(2)}$ in this particular gradation. We introduce the notation $A_{2n}^{(4)}$ for this algebra, taking into account that the similarity transformation (5.5) maps the Taylor part of $A_{2n}^{(4)}$ onto 'negative powers' of $A_{2n}^{(2)}$. Hence their Taylor parts are not isomorphic.

An analogous analysis can be done for $A_{2n-1}^{(2)}$. There exists an outer automorphism of order $\tau = 4$ which leads to a gradation with horizontal algebra $\mathcal{G}_{\bar{0}} = C_{n-1} \oplus \mathcal{A}_1$, i.e. a direct sum of the Lie algebra C_{n-1} and the one-dimensional Abelian Lie algebra $\mathcal{A}_1$. *The* representation of $A_{2n-1}^{(4)}$ can be computed from (5.3) with the matrix

$$\Xi_{CA_n} := \begin{pmatrix} \lambda & & & -1 \\ & & -1_{n-1} & \\ & 1_{n-1} & & \\ 1 & & & \end{pmatrix}.$$

We shall see later that the algebra $A_1^{(4)}$ generates the well-known Moutard theorem [55].

Let us come back to the algebras $A_{2n}^{(2)}$ and $A_{2n-1}^{(2)}$ and their corresponding linear problems. Hence the matrices F and G satisfy (5.3) for $\Xi = \Xi_{B_n}$ or $\Xi = \Xi_{D_n}$. The similarity transformation (5.4) is then nothing but a gauge transformation of the linear problem (3.1), i.e.

$$\phi \rightarrow \Theta^{-1}\phi.$$

A gauge transformation has no influence on the integrability condition of a linear problem since the gauge matrix drops out. Hence linear problens are characterized only by the conjugacy classes Ξ modulo gauge transformations.

As long as we are dealing with Lie algebras over the set of the complex numbers $\mathbb{C}$ the classification of Kac is valid and any real symmetric matrix $\Xi = S_k$ ($\det S_k \neq 0$) is equivalent to one of the matrices Ξ_{B_n} or Ξ_{D_n}, viz

$$\Xi_{B_n} \cong S_{2n+1}$$
$$\Xi_{D_n} \cong S_{2n}$$

with $\dim S_k = k$. The reason for this is the following: Any non-singular symmetric matrix can be brought into the canonical form

$$\Xi_{p,q} = \begin{pmatrix} 1_p & \\ & -1_q \end{pmatrix}$$

via a similarity transformation with a real orthogonal matrix Θ. The integers p and q depend on the signs of the eigenvalues of the matrix S_{p+q}, i.e. its signature. Moreover, the minus-signs can be gauged away if we apply another similarity transformation with

$$\Theta = \begin{pmatrix} 1_p & \\ & i1_q \end{pmatrix}.$$

We emphasize that, strictly speaking, (5.4) is only a similarity transformation for the representation of the algebra and not for the defining matrix Ξ. Otherwise the latter transformation would not have been possible.

The situation changes if we are dealing with real Lie algebras. In this case Ξ_{B_n} and Ξ_{D_n} are no longer equivalent to any symmetric matrix (of suitable dimension, of course) since only real gauge matrices Θ are permitted under these circumstances. In fact the conjugacy classes are now given by the diagonal matrices $\Xi_{p,q}$, which define the real forms $so(p,q)$ of B_n and D_n for $p+q=2n+1$ and $p+q=2n$ respectively. In both cases we can assume $p \geq q$ since the algebras $so(p,q)$ and $so(q,p)$ are isomorphic. Hence we only find

$$\begin{aligned} B_n &\cong so(n+1,n) \\ D_n &\cong so(n,n). \end{aligned}$$

This way of looking at the prolongation algebras opens up the possibility of extending the transposition ^{T} to Hermitian conjugation † in the defining relation (5.3). The condition

$$X^\dagger(-\lambda)\Xi = -\Xi X(\lambda) \tag{5.7}$$

for $\Xi_{p,q}$ now defines loop algebras of the real Lie algebra $sl(p+q,\mathbb{C})$ with twist $\tau = 2$ and horizontal algebras $\mathcal{G}_{\bar{0}} = su(p,q)$. It is clear that all of the previous considerations concerning the Darboux-Levi transformation are completely unaffected by this extension.

The above list of graded loop algebras is by no means exhaustive. It should only be considered as an indication of what can be done. It is evident that a classification of loop algebras for example with regard to their Taylor parts is required. Whether this will be sufficient is not obvious. The class of linear problems, however, which is covered by the algebras $A_{2n}^{(4)}$ and $A_{2n-1}^{(4)}$ and the (real) algebras related to $A_{2n}^{(2)}$ and $A_{2n-1}^{(2)}$ will turn out to be of considerable importance. Its 2+1-dimensional extension will be the subject of the following sections. It is related among other things to the above-mentioned Moutard theorem and therefore the Novikov-Veselov equation [55], the dromion solutions of the Davey-Stewartson equation I [21], a 2+1-dimensional sine-Gordon system found by Konopelchenko and Rogers [23] and its localized solitonic solutions.

5.2 The algebras $A_{2n}^{(2)}$ and $A_{2n-1}^{(2)}$

Here, we assume that the prolongation algebra associated with the linear problem (3.1) is related to either $A_{2n}^{(2)}$ or $A_{2n-1}^{(2)}$. In our particular representation F and G then satisfy the relation (5.3) for the symmetric matrices $\Xi = \Xi_{p,q}$. Following the recipe $\lambda \to \partial_z$ it converts into

$$H^*(\partial_z)\Xi = -\Xi H(\partial_z) \tag{5.8}$$

for $H = F, G$. It is quite obvious that its 1+1-dimensional limit is indeed (5.3). In order to show this it is sufficient to let (5.8) act on eigenfunctions, viz

$$\begin{aligned}[H^*(\partial_z)\Xi + \Xi H(\partial_z)]\phi(x,t,z) &= [H^*(\partial_z)\Xi + \Xi H(\partial_z)]e^{\lambda z}\phi(x,t) \\ &= e^{\lambda z}[H^T(-\lambda)\Xi + \Xi H(\lambda)]\phi(x,t).\end{aligned}$$

Hence choosing two linearly independent eigenfunctions $\phi(x,t)$ we obtain (5.3). Analogous to the 1+1-dimensional case the following two conditions are equivalent:

- The operators F and G satisfy (5.8).
- If ϕ is an eigenfunction then $\psi := \Xi\phi$ is an adjoint eigenfunction. (5.9)

For the proof we compute (for F and analogously for G)

$$(\Xi\phi)_x + F^*\Xi\phi = (\Xi F + F^*\Xi)\phi.$$

Thus if (5.8) is satisfied then the right-hand side of the above equation vanishes and therefore $\Xi\phi$ is an adjoint eigenfunction. On the other hand, if the left-hand side is zero then the kernel of the linear differential operator $\Xi F + F^*\Xi$ includes the infinite-dimensional class of eigenfunctions ϕ. This is only possible if the operator vanishes identically, which establishes (5.8).

It now remains to check whether the Darboux-Levi transformation allows for the symmetry (5.8). It indeed does under the restrictions of

Theorem 14. The Darboux-Levi transformation (Theorem 11) preserves the symmetry

$$\begin{aligned}F^*S &= -SF \\ G^*S &= -SG\end{aligned}$$

for a constant symmetric matrix $S = S^T$ if the adjoint eigenfunctions ψ_i are related to the eigenfunctions ϕ_i through

$$\psi_i = S\phi_i$$

and the matrix M is chosen to be symmetric, i.e.

$$M(\psi_i, \phi_j) = M(\psi_j, \phi_i).$$

Furthermore, the gauge matrix Υ is restricted by

$$S = \Upsilon^T S \Upsilon.$$

The verification of Theorem 14 can be divided into two steps. We first prove the symmetry condition on M. The definition of the potentials in Lemma 8 yields

$$M_z(\psi_i, \phi_j) = \psi_i^T \phi_j = \phi_i^T S \phi_j = M_z(\psi_j, \phi_i)$$

since the matri~~c~~ S is symmetric. The second equation reads x

$$\begin{aligned} M_x(\psi_i, \phi_j) &= \operatorname{res}(\partial_z^{-1}\phi_i^T S F \phi_j \partial_z^{-1}) \\ &= -\operatorname{res}(\partial_z^{-1}\phi_j^T F^* S \phi_i \partial_z^{-1}) \\ &= \operatorname{res}(\partial_z^{-1}\phi_j^T S F \phi_i \partial_z^{-1}) \\ &= M_x(\psi_j, \phi_i) \end{aligned}$$

having used (5.8). Hence taking into account the arbitrariness of the integration constants in the potentials the symmetry condition can be fulfilled.

We shall now show that for any new eigenfunction $\tilde{\phi}$ ~~it~~ exists an adjoint there eigenfunction $\tilde{\psi}$ satisfying

$$\tilde{\psi} = S\tilde{\phi}$$

which will prove Theorem 14 (cf. (5.9)). To this end we introduce the linear functional

$$\begin{aligned} R[\phi] &:= \tilde{\psi} - S\tilde{\phi} \\ &= \Upsilon^{-1T}[\psi + B^i M(\psi, \phi_i)] - S\Upsilon[\phi + A^i M(\psi_i, \phi)] \\ &= (\Upsilon^{-1T} S - S\Upsilon)\phi + (\Upsilon^{-1T} B^i - S\Upsilon A^i) M(\psi_i, \phi) \end{aligned}$$

where we have chosen $\psi = S\phi$ and $M(\psi, \phi_i) = M(\psi_i, \phi)$. The first term on the right-hand side of the last equation is supposed to vanish whereas the second one is zero for non-vanishing determinant $\det M(\psi_i, \phi_j)$ since, by definition, $R[\phi_i] = 0$ (cf. the argument which has been used in the proof of Theorem 9). q.e.d.

We finish this section with an identity which is useful for explicit calculations.

Lemma 15. The following identity holds:

$$A^i \phi_i^T = \phi_j A^{jT}.$$

A brief calculation, which is typical of the derivation of formulae like the one above (cf. Lemma 12), yields

$$\begin{aligned} 0 &= A^i[\phi_i^T + A^{jT} M(\psi_j, \phi_i)] \\ &= A^i \phi_i^T + A^i A^{jT} M(\psi_i, \phi_j) \\ &= A^i \phi_i^T - \phi_j A^{jT}. \end{aligned}$$

We have again made use of the fact that the matrix of the potentials is symmetric.

5.3 The Davey-Stewartson equation I

In Chapter 3 we had already shown that the Davey-Stewartson equation II is amenable to a generalized Darboux transformation. The Davey-Stewartson equation I had defied this approach and we have postponed the problem of finding a transformation which leaves it invariant. We are now in the position to solve this problem. The appropriate mathematical tools have been developed in the last chapter and the previous sections.

First of all we note that the prolongation algebra for the nonlinear Schrödinger equation is $A_1^{(1)}$. The generators appearing in the 1+1-dimensional version of the linear problem (3.22) generate a complete basis of the Taylor part of the loop algebra $sl(2, \mathbb{R}) \otimes \mathbb{R}(\lambda, \lambda^{-1})$. However, in the form which can be generalized to the Davey-Stewartson equation I (3.30) the linear problem consists only of matrices satisfying (5.7) for $\Xi = \Xi_{2,0}$. The prolongation

algebra can therefore be regarded as real twisted loop algebra of $sl(2, \mathbb{C})$ a or the
with horizontal algebra $su(2)$.

This observation is now reflected by the fact that the linear problem (3.22) is generated by antisymmetric operators F and G, i.e.

$$\begin{aligned} F^* &= -F \\ G^* &= -G. \end{aligned} \tag{5.10}$$

We can therefore apply Theorem 14 for $S = 1$. In order to investigate the remaining question whether the Darboux-Levi transformation allows for the specialization (3.21) of F and G we have a look at the three highest orders of the transformed operators $\tilde{F}$ and $\tilde{G}$ in Theorem 11. Using Lemma 12 and setting $\Upsilon = 1$ we get

$$\begin{aligned} \tilde{H}_k &= H_k \\ \tilde{H}_{k-1} &= H_{k-1} + [A^i \psi_i^\dagger, H_k] \\ \tilde{H}_{k-2} &= H_{k-2} + [A^i \psi_i^\dagger, H_{k-1}] - [A^i \psi_{iz}^\dagger, H_k] \\ &\quad - [A^i \psi_i^\dagger, H_k] A^j \psi_j^\dagger - k H_k (A^i \psi_i^\dagger)_z \end{aligned} \tag{5.11}$$

where again $H = F$ for $k = 1$ and $H = G$ for $k = 2$. Interestingly, the above structure is almost the same as the one for the Darboux transformation (3.20). We therefore conclude that the specialization (3.21) is preserved. Moreover, if we identify $A^i \psi_i^\dagger$ with P_{N-1} in the formulae (3.20) we obtain precisely the same expressions for the new fields $\tilde{u}$ and $\tilde{\chi}$ as in Section 3.4. They read

$$\tilde{u} = u + 2(A^i)^2 \psi_i^1$$

and

$$\begin{aligned} \tilde{\chi}_{xz} &= \chi_{xz} - 2(\psi_i^\dagger \sigma_3 A^i)_z \\ \tilde{\chi}_{zz} &= \chi_{zz} - 2(\psi_i^\dagger A^i)_z. \end{aligned}$$

In Lemma 13 we have already proven that $\psi_i^\dagger A^i = -(\ln |M|)_z$. On the other hand, if we take into consideration that

$$M_x(\psi_i, \phi_j) = \operatorname{res}(\partial_z^{-1} \psi_i^\dagger F \phi_j \partial_z^{-1}) = \psi_i^\dagger \sigma_3 \phi_j \tag{5.12}$$

it is easy to see that a similar identity is true for $\psi_i^\dagger \sigma_3 A^i$ as well, namely

$$\psi_i^\dagger \sigma_3 A^i = -(\ln |M|)_x.$$

Hence we immediately obtain

$$\tilde{\chi} = \chi + 2\ln|M| + f(x,t) + g(t)z. \tag{5.13}$$

The function of integration g is trivial since the Davey-Stewartson equation is invariant under $\chi \to \chi + g(t)z$. Moreover, if one interchanges the roles of x and z in the linear problem (3.22), the symmetry (5.10) converts into

$$\begin{aligned} F'^{*}(\partial_x)\sigma_3 &= -\sigma_3 F'(\partial_x) \\ G'^{*}(\partial_x)\sigma_3 &= -\sigma_3 G'(\partial_x). \end{aligned}$$

The adjoint of F' and G' is now taken with respect to ∂_x. We then observe that the new potentials $M'(\psi_i', \phi_j')$ for $\psi_i' = \sigma_3\psi_i$ and $\phi_i' = \phi_i$ are the same as before (cf. (5.12)). A direct consequence is that the Darboux-Levi transformation applied to the operators F' and G' is identical with the one discussed above. It now turns out that the new formula for the field $\tilde{\chi}$ is given by (5.13) after interchanging x and z. Consequently, $f(x,t)$ has to have the form $f(x,t) = f(t)x + h(t)$ and can therefore be neglected as well as $g(t)$.

As seed solution we take the most simple solution of the Davey-Stewartson equation I: simplest

$$u = 0, \qquad \Box\chi = 0.$$

For this solution the linear problem (3.22) simplifies to

$$\begin{aligned} \phi_x &= \sigma_3\phi_z \\ \phi_t &= -i(\chi_{xz} + \chi_{zz}\sigma_3)\phi - 2i\sigma_3\phi_{zz}. \end{aligned} \tag{5.14}$$

The corresponding potentials are then given by the defining relations

$$\begin{aligned} M_z(\psi_i, \phi_j) &= \phi_i^\dagger \phi_j \\ M_x(\psi_i, \phi_j) &= \phi_i^\dagger \sigma_3 \phi_j \\ M_t(\psi_i, \phi_j) &= 2i\phi_{iz}^\dagger \sigma_3 \phi_j - 2i\phi_i^\dagger \sigma_3 \phi_{jz}. \end{aligned} \tag{5.15}$$

We immediately observe that system (5.14) has decoupled with respect to the components ϕ^1 and ϕ^2 of the eigenfunction, which enables us to choose the solutions

$$\begin{aligned} &\phi_i^1 = \phi_i^1(\alpha, t), \qquad \phi_i^2 = 0, \qquad i = 1, \ldots, N_0 \\ &\phi_i^2 = \phi_i^1(\beta, t), \qquad \phi_i^1 = 0, \qquad i = N_0 + 1, \ldots, N \end{aligned}$$

with $\alpha := z + x$ and $\beta := z - x$. They satisfy two copies of the Schrödinger equation

$$
\begin{aligned}
&i\phi^1_t - 2\phi^1_{\alpha\alpha} - f_{\alpha\alpha}\phi^1 = 0 \\
&i\phi^2_t + 2\phi^2_{\beta\beta} + g_{\beta\beta}\phi^2 = 0
\end{aligned}
$$

with the potentials $f = f(\alpha, t)$ and $g = g(\beta, t)$, which are related to the field χ through $\chi = \frac{1}{2}(f + g)$. Moreover, let us introduce the notation

$$
\begin{aligned}
M^1_{ij} &:= M(\psi_i, \phi_j) \qquad && \text{for } i, j \leq N_0 \\
M^2_{ij} &:= M(\psi_i, \phi_j) \qquad && \text{for } i, j > N_0 \\
C_{ij} &:= M(\psi_i, \phi_j) \qquad && \text{for } i > N_0, j \leq N_0.
\end{aligned}
$$

It follows from (5.15) that the matrices M^1 and M^2 are independent of β and α respectively and that the matrix C is constant. Finally, the integration constants in the potentials M^1_{ij} and M^2_{ij} have to be chosen such that M^1 and M^2 are Hermitian. This is the symmetry condition of Theorem 14 extended to complex matrices.

Putting everything together we obtain the absolute value of the complex field $\tilde{u}$ in the form

$$
|\tilde{u}|^2 = \Box \ln \det \begin{pmatrix} M^1 & C^\dagger \\ C & M^2 \end{pmatrix}.
$$

This class of solution is not new. It has been derived by Gilson and Nimmo [21] via a direct approach using the Hirota form of the Davey-Stewartson equation. It contains the dromion solution found by Boiti, Leon, Martina and Pempinelli [50] and more generally the interaction of $N_0(N - N_0)$ dromions. For the asymptotic analysis and properties of their interaction see [21]. Localized solutions of the same functional form have also been found for the Novikov-Veselov equation. Their derivation was based on the Moutard theorem [55]. We shall see that the Moutard transformation can be regarded as a generalized Darboux-Levi transformation.

Before we close this section we wish to remark that Theorem 14 is obviously also applicable to scalar linear problems. $S = 1$ can formally be associated with $\Xi = \Xi_{B_0}$. Without proof we state that the prolongation algebra of the 1+1-dimensional Kaup-Kuperschmidt equation [56] is $A^{(2)}_2$. The Darboux-Levi transformation with non-trivial gauge matrix Υ can be applied to its 2+1-dimensional version. On the other hand, the corresponding linear problem admits a reduction to the scalar problem

$$
\begin{aligned}
\phi_x &= (\partial_z^3 + 2u\partial_z + u_z)\phi \\
\phi_t &= (9\partial_z^5 + 30u\partial_z^3 + 45u_z\partial_z^2 \\
&\qquad + (35u_{zz} + 20u^2 + 10p_x)\partial_z + 20uu_z + 5u_x + 10u_{zzz})\phi.
\end{aligned}
$$

The structure of the scalar operators F and G seems complicated. On the contrary, it is not. F and G are completely determined by the assumption that their highest orders are constant and that they are antisymmetric. The integrability condition yields the 2+1-dimensional Kaup-Kuperschmidt equation

$$
u_t + u_{zzzzz} + 10uu_{zzz} + 25u_zu_{zz} + 20u^2u_z = 5(p_{xx} + p_{xzzz} + 2(p_xp_z)_z)
$$

for $u = p_z$. Finally, the new field $\tilde{u}$ is given by the simple formula

$$
\tilde{u} = u + \tfrac{3}{2}(\ln|M|)_{zz}
$$

which reproduces the result of the above-mentioned Darboux-Levi transformation applied to the linear problem involving 3×3-matrices.

5.4 The algebras $A_{2n}^{(4)}$ and $A_{2n-1}^{(4)}$

This chapter is devoted to linear operators $H = F, G$ which possess the symmetry

$$
H^*\Xi = -\Xi H \tag{5.16}
$$

where Ξ is now an operator associated with the twisted algebras $A_{2n}^{(4)}$ or $A_{2n-1}^{(4)}$. Ξ is obtained by the substitution $\lambda \to \partial_z$ in the definition for the matrices Ξ_{C_n} and $\Xi_{\mathcal{C}A_n}$. Using the same notation the operators Ξ have the form

$$
\Xi_{C_n} = \begin{pmatrix} \partial_z & & \\ & & -1_n \\ & 1_n & \end{pmatrix}, \qquad \Xi_{\mathcal{C}A_n} = \begin{pmatrix} \partial_z & & & -1 \\ & & -1_{n-1} & \\ & 1_{n-1} & & \\ 1 & & & \end{pmatrix}.
$$

Firstly, we observe that the operator equation (5.16) is well-defined in the sense that its adjoint does not give anything new. To see this we decompose Ξ in a natural manner into

$$\Xi = S\partial_z + A$$

from which we conclude that Ξ is antisymmetric since $S = S^T$ and $A = -A^T$. Hence

$$(H^*\Xi + \Xi H)^* = \Xi^* H + H^*\Xi^* = -(H^*\Xi + \Xi H).$$

Secondly, a similar argument analogous to the one given in Section 5.2 shows that the 1+1-dimensional reduction of (5.16) is indeed (5.3). Furthermore, it is clear that the condition (5.16) on the operators F and G is equivalent to the requirement that if ϕ is an eigenfunction then $\Xi\phi$ is an adjoint eigenfunction. Hence we have to prove this relation between the new eigenfunction $\tilde{\phi}$ and the new adjoint eigenfunction $\tilde{\psi}$ to verify

Theorem 16. Let the adjoint eigenfunctions ψ_i be given by

$$\psi_i = \Xi\phi_i$$

and let the potentials $M(\psi_i, \phi_j)$ be restricted by

$$M(\psi_i, \phi_j) + M(\psi_j, \phi_i) = \phi_i^T S \phi_j. \tag{5.17}$$

Then the Darboux-Levi transformation leaves

$$F^*\Xi = -\Xi F$$
$$G^*\Xi = -\Xi G$$

invariant if the gauge matrix Υ has the form

- $\Xi = \Xi_{C_n}$:

$$\Upsilon = \begin{pmatrix} 1 & 0 \\ 0 & \Lambda \end{pmatrix} \begin{pmatrix} 1 & 0 \\ v & 1_{2n} \end{pmatrix}$$

with $v = (A^i)^1 \phi_i^\circ$ and $A^\circ = \Lambda^T A^\circ \Lambda$.

- $\Xi = \Xi_{\mathcal{C}A_n}$:

$$\Upsilon = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \Lambda & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1_{2n-2} & 0 \\ \alpha & v^T & \beta \end{pmatrix}$$

with $v = (A^i)^1 A^\circ \phi_i^\circ$ and $A^\circ = \Lambda^T A^\circ \Lambda$. The constant β is defined as $\beta := 1 + (A^i)^1 \phi_i^1 = \pm 1$ whereas α is arbitrary.

The matrix A° and the vectors ϕ_i° are the projections of A and ϕ_i respectively onto the subspace associated with Λ.

The existence of potentials restricted by (5.17) is connected with a particular choice of the integration constants. According to the definition of the potentials we compute on the one hand

$$\begin{aligned} M_z(\psi_i, \phi_j) + M_z(\psi_j, \phi_i) &= (\Xi\phi_i)^T \phi_j + (\Xi\phi_j)^T \phi_i \\ &= \phi_{iz}^T S \phi_j + \phi_{jz}^T S \phi_i \\ &= (\phi_i^T S \phi_j)_z \end{aligned}$$

since A is antisymmetric. On the other hand, from the symmetry (5.16) it follows

$$\mathrm{res}(\partial_z^{-1} \phi_i^T \Xi F \phi_j \partial_z^{-1}) = -\mathrm{res}(\partial_z^{-1} \phi_j^T \Xi F \phi_i \partial_z^{-1})$$

and consequently

$$\begin{aligned} & M_x(\psi_i, \phi_j) + M_x(\psi_j, \phi_i) \\ &= \mathrm{res}(\partial_z^{-1} (\Xi\phi_i)_0^T F \phi_j \partial_z^{-1}) + \mathrm{res}(\partial_z^{-1} (\Xi\phi_j)_0^T F \phi_i \partial_z^{-1}) \\ &= -\mathrm{res}(\partial_z^{-1} \phi_j^T F^* (\Xi\phi_i)_0 \partial_z^{-1}) - \mathrm{res}(\partial_z^{-1} \phi_i^T F^* (\Xi\phi_j)_0 \partial_z^{-1}) \\ &= -\mathrm{res}(\partial_z^{-1} \phi_j^T F^* \Xi \phi_i \partial_z^{-1}) + \mathrm{res}(\partial_z^{-1} \phi_j^T F^* S \phi_i) \\ &\quad - \mathrm{res}(\partial_z^{-1} \phi_i^T F^* \Xi \phi_j \partial_z^{-1}) + \mathrm{res}(\partial_z^{-1} \phi_i^T F^* S \phi_j) \\ &= \phi_i^T S (F\phi_j)_0 + \phi_j^T S (F\phi_i)_0 \\ &= (\phi_i^T S \phi_j)_x \end{aligned}$$

where we have used the identity

$$(\Xi m)_0 = \Xi m - S m \partial_z$$

for any matrix m depending on z.

Having established that (5.17) can be imposed on the potentials it remains to show that $\Xi\tilde{\phi}$ is an adjoint eigenfunction. To this end we need

Lemma 17. The constraint (5.17) leads to the identity

$$\phi_i A^{iT} + A^j \phi_j^T + A^j \phi_j^T S \phi_i A^{iT} = 0.$$

For the proof see Lemma 15.

The $(1,1)$-component of the matrix identity above reads

$$(A^i)^1 \phi_i^1 (2 + (A^i)^1 \phi_i^1) = 0$$

and hence

$$\beta := 1 + (A^i)^1 \phi_i^1 = \pm 1. \tag{5.18}$$

It is now convenient to introduce the linear functional

$$R[\phi] := \tilde{\psi} - \Xi\tilde{\phi}$$

with $\psi = \Xi\phi$. Insertion of the definition of $\tilde{\phi}$ and $\tilde{\psi}$ yields

$$\begin{aligned} R[\phi] &:= \Upsilon^{-1T}[\psi + B^i M(\psi, \phi_i)] - \Xi\Upsilon[\phi + A^i M(\psi_i, \phi)] \\ &= \Upsilon^{-1T}[\Xi\phi + B^i(-M(\psi_i, \phi) + \phi_i^T S\phi)] - \Xi\Upsilon[\phi + A^i M(\psi_i, \phi)] \\ &= Q\phi - [\Upsilon^{-1T} B^i + (\Xi\Upsilon A^i)_0] M(\psi_i, \phi). \end{aligned}$$

The differential operator Q has the form

$$Q := \Upsilon^{-1T}(\Xi + \psi_i A^{iT} S) - \Xi\Upsilon - S\Upsilon A^i \psi_i^T$$

where we have used Lemma 12.

Let us for the moment consider only the case $\Lambda = 1$. It is then straight forward to show that Q vanishes identically for the particular choice of Υ given in Theorem 16. Assuming that the determinant of the potentials $|M|$ is non-zero, we conclude that R vanishes as well, since $R[\phi_i] = 0$.

For the case $\Lambda \neq 1$ we denote that matrix which contains Λ in Theorem 16 by Υ_0. By construction it commutes with the differential part of the operator Ξ, i.e.

$$\Xi\Upsilon_0 = A\Upsilon_0 + \Upsilon_0 S \partial_z$$

from which follows

$$\tilde{\psi} - \Xi\tilde{\phi} = \Upsilon_0^{-1T}\tilde{\psi}|_{\Lambda=1} - \Xi\Upsilon_0\tilde{\phi}|_{\Lambda=1} = (\Upsilon_0^{-1T}A - A\Upsilon_0)\tilde{\phi}|_{\Lambda=1}.$$

This finishes the proof of Theorem 16. q.e.d.

We wish to stress that the form of the gauge matrix Υ has been derived in a purely algebraic manner. There are still some degrees of freedom left since the matrix Λ has not yet been entirely determined. In the next chapter we shall see by way of an example in which manner Λ may be specified if one imposes further restrictions on the operators F and G.

5.5 The 2+1-dimensional Sawada-Kotera equation

The examples concerning the Darboux-transformation and the Darboux-Levi transformation which have been discussed so far have not completely exhibited the variety of these approaches. In order to illustrate the points not touched upon so far we shall discuss the 2+1-dimensional extension of the Sawada-Kotera equation [56]. Its prolongation algebra is $A_2^{(2)}$. Unfortunately, in this gradation the corresponding linear problem contains negative powers of λ. The crucial point is that one can apply a gauge transformation to the linear problem such that the new matrices F and G satisfy (5.3) with $\Xi = \Xi_{C_1}$ and that they are polynomial in λ. In this sense it is reasonable to regard the prolongation algebra as $A_2^{(4)}$, which allows us to make use of the results developed in the last section.

We start with the linear problem of the form

$$\phi_x = (\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -u & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}\lambda)\phi \tag{5.19}$$

$$\phi_t = (\begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix} + 3\begin{pmatrix} 2u & 0 & 3 \\ * & -u & 0 \\ * & 0 & -u \end{pmatrix}\lambda + 9\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\lambda^2)\phi.$$

The asterisks denote some functions which are uniquely determined by the compatibility condition. It turns out that they can be expressed in terms of derivatives of the field u. What remains is a single equation for u, the Sawada-Kotera equation

$$u_t + u_{xxxxx} + 5uu_{xxx} + 5u_x u_{xx} + 5u^2 u_x = 0. \tag{5.20}$$

We observe that the highest powers in (5.19) are constant. Let us investigate whether we can maintain this special form in 2+1 dimension. To this end it is noted that the transformation of the operators F and G under the Darboux-Levi transformation is given by (5.11) 'plus' a non-trivial gauge transformation with Υ. We find

$$\begin{aligned} \tilde{H}_k &= \Upsilon H_k \Upsilon^{-1} \\ \tilde{H}_{k-1} &= \Upsilon(H_{k-1} + [A^i\psi_i^T, H_k])\Upsilon + \delta_{k1}\Upsilon_x\Upsilon^{-1} \end{aligned} \tag{5.21}$$

with the usual notation $H = F$ for $k = 1$ and $H = G$ for $k = 2$. From the first equation it is readily seen that $\tilde{F}_1 = F_1$ and $\tilde{G}_2 = G_2$ iff

$$\Lambda = \begin{pmatrix} 1 & 0 \\ v^1 & 1 \end{pmatrix}. \tag{5.22}$$

Hence the gauge matrix Υ has been determined completely by algebraic conditions. From the second equation we conclude that the second highest orders of F and G can be assumed to be traceless as

$$\mathrm{tr}(\Upsilon_x\Upsilon^{-1}) = (\ln\det\Upsilon)_x = 0.$$

It should be emphasized that this trace condition and the special structure of the highest orders of F and G are the only conditions to impose on the linear problem (aside from the symmetry (5.16), of course). Thus we are left with a bunch of algebraic equations for the coefficients of F and G, namely the algebraic part of the symmetry (5.16) and the algebraic part of the integrability condition, i.e. the first two equations of (3.23). Solving these equations it turns out that the linear problem in 2+1 dimensions has exactly the same form as the 1+1-dimensional version (5.19).

The third and fourth equation of the integrability condition (3.23) then determine the remaining coefficients in terms of the field u and an auxilliary field q, say. We finally end up with the 2+1-dimensional Sawada-Kotera equation

$$u_t + u_{xxxxx} + 5uu_{xxx} + 5u_x u_{xx} + 5u^2 u_x = 5(q_{zz} + q_{xxxz} + (q_x q_z)_x)$$

with $u = q_x$, which indeed yields (5.20) in the 1+1-dimensional limit $q_z = 0$.

The best way of calculating the new field $\tilde{u}$ is multiplying the second equation of (5.21) by Υ from the right and then take the $(3,2)$-component. The result is

$$\tilde{u} = u - (A_i)^1\phi_i^3 - v^2 - v_x^1. \tag{5.23}$$

where v is given by

$$v = (A^i)^1\Lambda\phi_i^\circ$$

(cf. Theorem 16 and (5.22)) or, more explicitly,

$$\begin{aligned} v^1 &= (A^i)^1\phi_i^2 \\ v^2 &= (v^1)^2 + (A^i)^1\phi_i^3. \end{aligned}$$

In principle we could stop at this point since we have already managed to find a new solution of the 2+1-dimensional Sawada-Kotera equation in terms of the seed solution and an arbitrary number of eigenfunctions and corresponding potentials. We shall, however, try to simplify the relation (5.23).

The first observation is that the components of an arbitrary eigenfunction ϕ are simply related to each other via the linear problem (5.19), viz

$$\begin{aligned} \phi^2 &= \phi_x^1 \\ \phi^3 &= \phi_{xx}^1. \end{aligned} \tag{5.24}$$

Secondly, an appropriate combination of the $(1,1)$-, $(2,1)$- and $(2,2)$-component of the matrix identity in Lemma 17 immediately gives

$$(v^1)^2 + 2(A^i)^2\phi_i^2 = 0.$$

Finally, the relations (5.24) must also hold true for the new eigenfunction $\tilde{\phi}$. Hence the two expressions

$$\tilde{\phi}^2 = v^1[\phi^1 + (A^i)^1M(\psi_i,\phi)] + \phi^2 + (A^i)^2M(\psi_i,\phi)$$

and

$$\tilde{\phi}_x^1 = \phi_x^1 + (A^i)_x^1M(\psi_i,\phi) + (A^i)^1\phi_i^2\phi^1$$

must coincide. Consequently,

$$(A^i)^1_x = (A^i)^2 + v^1(A^i)^1.$$

The reason for this is again the fact that the potentials $M(\psi_i, \phi)$ are only defined up to arbitrary integration constants. It is therefore permitted to sort with respect to them. Multiplication by ϕ_i^2 and summation then yields

$$(A^i)^1\phi_i^3 = [(A^i)^1\phi_i^2]_x - \tfrac{1}{2}(v^1)^2$$

where we have used (5.24).

The last step is to find a simple expression for v^1. To this end we note that

$$M_x(\psi_i, \phi_j) = \psi_i^3\phi_j^1 = \phi_i^2\phi_j^1.$$

Having in mind the proof of Lemma 13 we deduce

$$v^1 = -(\ln|M|)_x.$$

Putting everything together we find

$$\tilde{u} = u + 3(\ln|M|)_{xx}. \tag{5.25}$$

This result is not very surprising. In fact (5.24) suggests eliminating ϕ^2 and ϕ^3 in the linear problem (5.19) (of course after $\lambda \to \partial_x$) via ϕ^1 and write (5.19) as scalar linear problem. The corresponding operators are then polynomials in ∂_x which explains the occurrence of x-derivatives in (5.25).

In this case the reduction to a scalar linear problem leads again to equations of the type (3.2). In the next section we shall discuss a hyperbolic differential equation which is a priori not amenable to the Darboux-Levi transformation. It can, however, be converted into an evolution equation associated with $A_1^{(4)}$.

5.6 The Moutard transformation

We now wish to show that the classical Moutard theorem [19, 20] can be derived from the Darboux-Levi transformation applied to a linear operator having the symmetry (5.16) for $\Xi = \Xi_{\mathcal{C}A_1}$. In the first place we are not interested in a couple of linear equations which define nonlinear partial differential equations, rather we focus on the single equation

$$\phi_{x'} = (2\begin{pmatrix} 0 & -1 \\ u & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\partial_z)\phi \tag{5.26}$$

and the behaviour of the potential u under a Darboux-Levi transformation.

We verify by direct computation that the operator F is associated with the algebra $A_1^{(4)}$. The relations (5.21) show that for $\alpha = 0$ in Theorem 16 the form of F_1 is preserved and that the coefficient $\tilde{F}_0$ has the form

$$\tilde{F}_0 = 2\begin{pmatrix} 0 & \tilde{v} \\ \tilde{u} & 0 \end{pmatrix}.$$

Hence it remains to prove that $\tilde{v} = -1$. To do so we evaluate the second equation of (5.21). It becomes

$$\begin{aligned} \tilde{u} &= \beta(u + (A^i)^2\psi_i^1) \\ \tilde{v} &= -\beta(1 + (A^i)^1\psi_i^2). \end{aligned}$$

From $\psi_i^2 = \phi_i^1$ and (5.18) it follows $\tilde{v} = -\beta^2 = -1$. We could have proven this directly because the upper right entry of F_0 is uniquely fixed by F_1 via the symmetry (5.16).

Having established the invariance of the linear equation (5.26) we can now formulate

Theorem 18 (Generalized Moutard theorem). The linear hyperbolic differential equation

$$\phi_{xy} = u\phi$$

is invariant under

$$\begin{aligned} \phi \to \tilde{\phi} &= \phi + A^i M(\psi_i, \phi) \\ u \to \tilde{u} &= u - (\ln|M|)_{xy} \end{aligned}$$

where the potentials $M(\psi_i, \phi)$ are defined by

$$M_x(\psi_i, \phi) = \phi_{ix}\phi$$
$$M_y(\psi_i, \phi) = \phi_i \phi_y$$

satisfying $M(\psi_i, \phi_j) + M(\psi_j, \phi_i) = \phi_i \phi_j$. The quantities A^i and $|M|$ are defined as usual.

For the proof we observe that the first component of (5.26) can be solved for ϕ^2, viz

$$\phi^2 = \tfrac{1}{2}(\phi^1_z - \phi^1_{x'}).$$

Insertion into the second component of (5.26) yields the scalar hyperbolic equation

$$\Box\phi = 4u\phi \tag{5.27}$$

with the revised definition $\phi := \phi^1$ and $\Box := \partial_z^2 - \partial_{x'}^2$. It is clear that we can express the potentials $M(\psi_i, \phi_j)$ and the transformation formula for $\tilde{u}$ in terms of ϕ_i. To this end we compute

$$\psi^1 = \phi^1_z - \phi^2 = \tfrac{1}{2}(\phi_z + \phi_{x'}).$$

If we now introduce the new coordinates $x := z + x'$ and $y := z - x'$ (5.27) transforms into

$$\phi_{xy} = u\phi \tag{5.28}$$

and the potentials obey the equations

$$\begin{aligned} M_x(\psi_i, \phi) &= \phi_{ix}\phi \\ M_y(\psi_i, \phi) &= \phi_i \phi_y. \end{aligned} \tag{5.29}$$

The most elegant way of simplifying the equation for $\tilde{u}$ is deriving it in a different way. We know already that (5.28) is invariant under

$$\phi \to \tilde{\phi} = \phi + A^i M(\psi_i, \phi) \tag{5.30}$$

where $A^i := (A^i)^1$. Hence the only thing we have to do is to insert (5.30) into the twiddled version of (5.28) and sort with respect to ϕ, first derivatives of ϕ and potentials. We are allowed to do so because this decomposition is equivalent to the way we have defined the new operator $\tilde{F}$ in (5.26). Thus

$$\begin{aligned}0 &= \tilde{\phi}_{xy} - \tilde{u}\tilde{\phi} \\ &= \phi_{xy} + [(A^i\phi_{ix})_y - \tilde{u}]\phi + (A^i\phi_i)_x\phi_y + (A^i_{xy} - \tilde{u}A^i)M(\psi_i, \phi).\end{aligned}$$

Finally, substitution for ϕ_{xy} by means of (5.28) and collection of terms linear in ϕ results in

$$\tilde{u} = u - (\ln|M|)_{xy} \tag{5.31}$$

where we have again made use of $A^i\phi_{ix} = -(\ln|M|)_x$. q.e.d.

We shall now show that for $N = 1$ Theorem 18 is a reformulation of Moutard's theorem. For this we solve (5.30) for the potential $M(\psi_1, \phi)$:

$$M(\psi_1, \phi) = -\tfrac{1}{2}\phi_1(\tilde{\phi} - \phi)$$

and insert it into the definitions (5.29). We obtain, together with the formula (5.31) for the new field $\tilde{u}$,

$$\begin{aligned}\frac{1}{\phi_1}(\phi_1\tilde{\phi})_x &= -\phi_1\left(\frac{\phi}{\phi_1}\right)_x \\ \frac{1}{\phi_1}(\phi_1\tilde{\phi})_y &= \phi_1\left(\frac{\phi}{\phi_1}\right)_y\end{aligned} \tag{5.32}$$

$$\tilde{u} = u - 2(\ln\phi_1)_{xy}. \tag{5.33}$$

The formulae (5.32) and (5.33) together with the hyperbolic equation (5.28) constitute Moutard's theorem. It may be phrased as follows:

For a given potential u and a pair of eigenfunctions ϕ and ϕ_1 solving (5.28) a new eigenfunction $\tilde{\phi}$ is obtained by integration of the Frobenius system (5.32). Its integrability condition is satisfied modulo (5.28). The new potential $\tilde{u}$ is given by (5.33).

Let us investigate whether the Moutard theorem is of any practical use. To this end it is worth going back to Chapter 2 and having a look at the linear problem of the Dodd-Bullough equation (2.18). We are interested in the second component of the pseudopotential y and denote it by $\phi := y^2$. Solving the 'x-part' of the linear problem for y^1 and y^2 we obtain

$$y^1 = \tfrac{1}{2}\phi_x$$
$$y^2 = \tfrac{1}{2}\lambda^{-1}(\phi_{xx} + \varphi_x\phi_x).$$

Insertion into the first equation of the t-part and the third equation of the x-part then yields the Lax-pair [12]

$$\phi_{xt} = 2e^{-\varphi}\phi$$
$$4\lambda\phi = \phi_{xxx} + (\varphi_{xx} - \varphi_x^2)\phi_x.$$

Interestingly, the hyperbolic equation (5.28) has turned up during this procedure. (From now on we set $t = y$.)

After the substitution $4\lambda \to \partial_s$, a simple calculation reveals that the second equation of the 2+1-dimensional Lax-pair

$$\begin{aligned} \phi_{xy} &= u\phi \\ \phi_s &= \phi_{xxx} + 3\chi_{xx}\phi_x \end{aligned} \tag{5.34}$$

has the symmetry

$$F^*\partial_x = -\partial_x F \tag{5.35}$$

with

$$F = \partial_x^3 + 3\chi_{xx}\partial_x$$

which can formally be identified with (5.16) for $\Xi = \Xi_{C_0}$. The compatibility condition for the Lax-pair (5.34) yields a 2+1-dimensional generalization of the Dodd-Bullough equation in a disguised form, viz

$$u_s = (u_{xx} + 3\chi_{xx}u)_x, \qquad u = -\chi_{xy}. \tag{5.36}$$

It now turns out that the Darboux-Levi transformation applied to the second equation of the Lax pair (5.34) coincides with the one appearing in the generalized Moutard theorem. This is hardly surprising because in the coordinates z and x' we can bring (5.34) into the form of an evolution equation, which is, of course, compatible with (5.26).

The connection with the Novikov-Veselov equation is as follows: Exploiting the symmetry of (5.28) in x and y, it easy to see that one can add a third equation

$$\phi_r = \phi_{yyy} + 3\chi_{yy}\phi_y$$

to the Lax pair (5.34) without lo~~o~~sing compatibility. As expected the additional nonlinear equation reads

$$u_r = (u_{yy} + 3\chi_{yy}u)_y.$$

A combination with (5.36) results in the symmetrized version

$$u_t = u_{xxx} + u_{yyy} + 3(\chi_{xx}u)_x + 3(\chi_{yy}u)_y$$

with $t := \frac{1}{2}(s+r)$, the Novikov-Veselov equation [57].

In [55] Athorne and Nimmo have generated dromion solutions for this equation of the same functional form as for the Davey-Stewartson equation I. They are contained within a more general class of solutions which are characterized by expressions involving antisymmetric bilinear forms $S(\phi_i, \phi)$. They are related to the potentials $M(\psi_i, \phi)$ through

$$S(\phi_i, \phi) = 2M(\psi_i, \phi) - \phi_i\phi.$$

Even though Athorne and Nimmo regard the original Darboux-Levi transformation extended to 2+1 dimensions (cf. Section 4.3) as generalized Moutard transformation, we are inclined to put it the other way round. As we have seen in this section the Moutard transformation is a particular case of, what we call, generalized Darboux-Levi transformations. Time will show which point of view will prevail.

5.7 Another symmetry

In finishing this chapter we shall discuss another reduction of the generators F and G which may be associated with the algebras $A_{2n}^{(2)}$ and $A_{2n-1}^{(2)}$. Even so, one could be under the impression that the connection with these algebras is somewhat artificial. It will turn out, however, that the content of this section is useful for the considerations in the next chapter.

Let us reconsider the naïve way of 'lifting' the relation (5.3) for $\Xi = \Xi_{p,q}$ to 2+1 dimensions. Why not first multiply (5.3) by λ and then replace λ by ∂_z? We should obtain

$$H^* S\partial_z = -S\partial_z H \tag{5.37}$$

for the symmetric matrix $S = \Xi_{p,q}$.

Let us see whether this Ansatz is reasonable. First of all, (5.37) is equal to its adjoint. Secondly, we observe that $H_z = 0$, which suggests assuming H to be in the subalgebra of differential operators

$$H = H_{\geq 1}. \tag{5.38}$$

It is evident that the integrability condition associated with two of them, say F and G,

$$F_t - G_x + [F, G] = 0$$

is compatible with the assumption above. Moreover, there are some reasons for the Darboux-Levi transformation to preserve (5.37). If we look back at the proof of Theorem 16 we realize that the only assumptions which have entered into the proof for the restriction of the potentials have been the symmetry properties of A and S. Thus setting formally $A = 0$ there immediately follows the existence of potentials obeying

$$M(\psi_i, \phi_j) + M(\psi_j, \phi_i) = \phi_i^T S \phi_j$$

for $\psi_i = S\phi_{iz}$. It remains to show that $\tilde{\psi} = S\tilde{\phi}_z$ is a new adjoint eigenfunction with $\psi = S\phi_z$ and an appropriate gauge matrix Υ to be found. Whether the new operators $\tilde{F}$ and $\tilde{G}$ will again be contained within the class (5.38) has to be examined afterwards.

The procedure is as usual. We introduce the linear functional

$$\begin{aligned} R[\phi] &:= \tilde{\psi} - S\tilde{\phi}_z \\ &= \Upsilon^{-1T}[\psi + B^i M(\psi_i, \phi)] - S\partial_z \Upsilon[\phi + A^i M(\psi_i, \phi)] \\ &= Q\phi - [\Upsilon^{-1T} B^i + S(\Upsilon A^i)_z] M(\psi_i, \phi) \end{aligned}$$

where

$$Q := \Upsilon^{-1T}(S\partial_z + \psi_i A^{iT} S) - S\Upsilon_z - S\Upsilon(\partial_z + A^i \phi_{iz}^T S).$$

The functional R is supposed to be zero. This is again equivalent to the requirement that the differential operator Q vanishes identically, which constitutes (differential) equations for Υ. Their solution is given in

Lemma 19. The matrix equation

$$\Upsilon^T S \Upsilon_z = S(\phi_{iz} A^{iT} - A^i \phi_{iz}^T) S \tag{5.39}$$

under the constraint

$$S = \Upsilon^T S \Upsilon \tag{5.40}$$

has the general solution

$$\begin{aligned} \Upsilon &= \Lambda(1 + \phi_i A^{iT} S) \\ S &= \Lambda^T S \Lambda \end{aligned}$$

for a constant matrix Λ.

As a side remark we note that (5.39) can be written as linear equation in Υ by means of (5.40). Hence it is sufficient to prove Lemma 19 for $\Lambda = 1$.

It need not be proven that the general solution satisfies (5.40) since this is precisely the matrix identity of Lemma 17. The differential equation (5.39) may be derived as follows:

$$\begin{aligned} 0 &= (A^i \phi_i^T + A^i M(\psi_j, \phi_i) A^{jT})_z \\ &= (A^i \phi_i^T)_z + (\phi_j + A^i \phi_i^T S \phi_j) A_z^{jT} + A^i \phi_i^T S \phi_{jz} A^{jT} - A_z^i \phi_i^T \\ &= (1 + A^i \phi_i^T S)(\phi_j A^{jT})_z + A^i \phi_{iz}^T - \phi_{jz} A^{jT} \\ &= S^{-1} \Upsilon^T S \Upsilon_z S^{-1} + A^i \phi_{iz}^T - \phi_{jz} A^{jT}. \end{aligned}$$

It is now relatively easy to verify that the particular choice of Υ given in ~~Theorem~~ 19 allows for the condition (5.38) if we consider an alternative form Lemma
of the Darboux-Levi transformation.

Theorem 20. In the reduction

$$\begin{aligned} F^* S \partial_z &= -S \partial_z F \\ G^* S \partial_z &= -S \partial_z G \\ \psi &= S \phi_z \\ M(\psi_i, \phi) + M(\psi, \phi_i) &= \phi_i^T S \phi \end{aligned}$$

the transformation law for the eigenfunction under the generalized Darboux-Levi transformation (Theorem 11) can be converted into

$$\phi \to \tilde{\phi} = \Lambda[\phi + C^i M(\psi, \phi_i)] \tag{5.41}$$

where the vectors C^i are the solution of $\tilde{\phi}[\phi_i] = 0$ and the matrix Λ satisfies the constraint

$$S = \Lambda^T S \Lambda$$

for a non-singular symmetric matrix S. Furthermore, the transformation acts within the subalgebra (5.38).

We should mention the reverse order of the arguments in the potentials in (5.41).

All we have to show is the identity

$$\begin{aligned} \Upsilon[\phi + A^i M(\psi_i, \phi)] &= \Lambda[\phi + C^i M(\psi, \phi_i)] \\ &= \Lambda[\phi + C^i \phi_i^T S \phi - C^i M(\psi_i, \phi)]. \end{aligned}$$

Again, it is sufficient to take into consideration only the terms not involving the potentials since both sides of the equations vanish for $\phi = \phi_i$. Hence once we have established

$$\phi_i A^{iT} = C^i \phi_i^T \tag{5.42}$$

we get the relations

$$\Lambda C^i + \Upsilon A^i = 0 \tag{5.43}$$

for free. We briefly calculate

$$0 = [\phi_i + C^j M(\psi_i, \phi_j)] A^{iT} = \phi_i A^{iT} - C^j \phi_j^T.$$

In order to complete the proof of Theorem 20 we insert (5.41) into the linear problem for the new eigenfunction $\tilde{\phi}$. The equation for $\tilde{F}$ reads

$$\begin{aligned} 0 &= \tilde{\phi}_x - \tilde{F}\tilde{\phi} \\ &= \Lambda[F\phi + C^i_x M(\psi, \phi_i) + C^i \mathrm{res}(\partial_z^{-1} \phi_z^T S F \phi_i \partial_z^{-1})] \\ &\qquad - \tilde{F}\Lambda[\phi + C^i M(\psi, \phi_i)]. \end{aligned}$$

In our construction the new operator F has been defined via the 'differential part' of the equation above. The terms of zeroth order simply yield

$$\Lambda F_0 = \tilde{F}_0 \Lambda$$

which proves that if F_0 is zero, so is $\tilde{F}_0$. q.e.d.

The explicit form of the new operators F and G is now given in

Theorem 21. Under the assumptions made in Theorem 20 the new operators $\tilde{F}$ and $\tilde{G}$ have the form

$$\begin{aligned}\tilde{F} &= \Lambda(TFT^{-1})_{\geq 1}\Lambda^{-1}\\ \tilde{G} &= \Lambda(TGT^{-1})_{\geq 1}\Lambda^{-1}\end{aligned}$$

where

$$\begin{aligned}T &= 1 + C^i\partial_z^{-1}\phi_i^T S\partial_z\\ T^{-1} &= 1 + \phi_i\partial_z^{-1}C^{iT}S\partial_z.\end{aligned}$$

From Theorem 11 we conclude that we have to show

$$\begin{aligned}0 &= \Lambda T - \Upsilon D\\ &= \Lambda(1 + C^i\phi_i^T S - C^i\partial_z^{-1}\phi_{iz}^T S) - \Upsilon(1 + A^i\partial_z^{-1}\phi_{iz}^T S)\end{aligned}$$

which is true because of (5.42) and (5.43). Finally, the explicit form of the inverse operator T^{-1} may be verified by repeating the arguments given in the proof of Lemma 10.

Summary: We have explicitly constructed a gauge matrix Υ such that the symmetry (5.37) is preserved under the Darboux-Levi transformation. It has turned out that the additional requirement (5.38) is then satisfied automatically. In fact it is possible to drop the symmetry condition (5.37) without losing the nice property (5.38). Unfortunately, the formula for Υ is more complicated, which could be an indication that one has to look at this problem from a different viewpoint.

Chapter 6

The generalized Loewner system

So far we have proceeded on the basic assumption that the linear problem (3.1) is polynomial in the parameter λ. That is the reason why the way of extending linear problems and their corresponding Bäcklund transformations in 1+1 dimensions to their 2+1-dimensional counterparts has been rather straight forward.

From now on we shall drop this condition, which makes it difficult to develop a general theory of 2+1-dimensional linear problems and Darboux (-Levi) transformations. This chapter is therefore devoted to a particular linear system which can be regarded as an extension of usual linear problems in 1+1 dimensions. It will contain, for example, the Leznov-Savel'ev system (2.12) for $A_n^{(1)}$ [61], i.e. in particular the sinh-Gordon equation. Hence the associated prolongation algebras will then evidently be more than only the Taylor parts of Kac-Moody algebras.

6.1 The Loewner system

Ignoring the genesis of the Loewner system [22] we start with a linear problem in 1+1 dimensions of the form

$$\begin{aligned} \phi_x &= \lambda R\phi \\ \phi_t &= (V + \lambda^{-1}W)\phi \end{aligned} \tag{6.1}$$

with matrices R, V and W of arbitrary dimension. The question which immediately arises is the following: How do we have to interpret the inverse of λ in the substitution $\lambda \to \partial_y$? We shall not give an answer to this question in that we get rid of this problem by performing the most simple and naïve transformation to the second equation of (6.1). We first multiply it by λ and then replace λ by ∂_y. For purely aesthetic reasons we denote in this chapter the third independent variable by y. We obtain the linear triad

$$\begin{aligned} \phi_x &= R\phi_y \\ \phi_{yt} &= (V\partial_y + W)\phi \\ \phi_{xt} &= (V\partial_x + W')\phi \end{aligned} \tag{6.2}$$

where the third equation, which can be derived from the other two equations, has been added to underline the invariance of the above triad under

$$\begin{aligned} &x \leftrightarrow y \\ &R \to R^{-1} \\ &W \leftrightarrow W'. \end{aligned} \tag{6.3}$$

This indicates already that it might be possible to derive nonlinear differential equations from (6.2) wherein the 'spatial coordinates' x and y occur on an equal footing.

The integrability condition of the system (6.2) reads

$$\begin{aligned} &R_t = [V, R] \\ &V_x - V_y R + [W, R] = 0 \\ &W_x - (RW)_y = 0 \end{aligned} \tag{6.4}$$

with $W' = RW$. We emphasize that (6.4) is founded on the assumption that the class of solutions of (6.2) is as large as possible, i.e. we have sorted with respect to ϕ, ϕ_y and ϕ_{yy}. There exist~~s~~ examples of linear triads (in 1+1 dimensions) for which a similar assumption is too restrictive in order to get non-trivial nonlinear differential equations [62].

In the original formulation of Loewner, given in a gasdynamical context, the compatibility conditions (6.4) together with the last two equations of (6.2) constitute an infinitesimal Bäcklund transformation for the remaining equation

$$\phi_x = R\phi_y \tag{6.5}$$

if the independent variable y is reinterpreted as continuous parameter on which the vector-valued function ϕ and the matrices R, V and W are assumed to depend. We should mention that the systems (6.2) and (6.4) are not the most general ones discussed by Loewner in [22]. For a more detailed study of the Loewner system we refer to [23, 61, 63]. a

In the following we are interested in a special reduction of the triad (6.2). We observe that the first equation is of the kind discussed in the last section of the previous chapter if we impose the constraint

$$F^* S\partial_y = -S\partial_y F \tag{6.6}$$

on the operator $F := R\partial_y$ (cf. Theorem 20). Consequently, for this operator the work has already been done and the Darboux-Levi transformation is established.

How can we find the corresponding symmetry of the second equation

$$\phi_{ty} = (V\partial_y + W)\phi. \tag{6.7}$$

To this end we formally associate with (6.7) the operator

$$L := \partial_y^{-1}(V\partial_y + W)$$

and assume that the symmetry condition (6.6) holds for the pseudo-differential operator L as well. We compute

$$\begin{aligned} 0 &= L^* S\partial_y + S\partial_y L \\ &= -(-\partial_y V^T + W^T)S + S(V\partial_y + W) \end{aligned}$$

which together with (6.6) leads to the constraints

$$\begin{aligned} R^T S &= SR \\ V^T S &= -SV \\ W^T S &= S(W - V_y). \end{aligned} \tag{6.8}$$

It is easy to verify that these constraints are compatible with the nonlinear equations (6.4). Transposition and substitution for the appropriate quantities by means of (6.8) does not change the equations.

In order to proceed we now need to find a suitable extension of the potentials $M(\psi, \phi_i)$ given in Theorem 20. So far they are only defined by a contour integration in the (x, y)-plane, i.e.

$$\begin{aligned} M_y(\psi, \phi_i) &= \phi_y^T S \phi_i \\ M_x(\psi, \phi_i) &= \operatorname{res}(\partial_y^{-1} \psi^T R \partial_y \phi_i \partial_y^{-1}) = \phi_x^T S \phi_i \end{aligned} \tag{6.9}$$

with $\psi = S\phi_y$. After some trial and error we find that the 'time'-derivative of the potentials is given by

$$M_t(\psi, \phi_i) = \phi^T S \phi_{it} - \phi^T S V \phi_i. \tag{6.10}$$

The compatibility condition of (6.9) and (6.10) is directly proven to be satisfied modulo the linear system (6.2) and the constraints (6.8).

Having found the appropriate extension of our potentials we can formulate the following

Theorem 22. The linear triad (6.2) and the constraints (6.8) are invariant under

$$\begin{aligned} &\phi \to \tilde{\phi} = \phi + C^i M(\psi, \phi_i) \\ &R \to \tilde{R} = \Upsilon R \Upsilon^{-1} \\ &V \to \tilde{V} := \Upsilon V \Upsilon^{-1} + \Upsilon_t \Upsilon^{-1} \\ &W \to \tilde{W} := W + [C^i(\phi_{it}^T S + \phi_i^T S V)]_y \end{aligned}$$

where the matrix Υ is given in Lemma 19 for $\Lambda = 1$, i.e.

$$\begin{aligned} &\Upsilon = 1 + C^i \phi_i^T S \\ &\Upsilon^{-1} = 1 + \phi_i C^{iT} S. \end{aligned}$$

The invariance of equation (6.5) has already been proven in Theorem 20. The transformation law for R may be read off Theorem 21. We shall verify the invariance of (6.7) in a constructive manner. For that we introduce the linear functional

$$\begin{aligned} Q[\phi] &:= \tilde{\phi}_{ty} - \tilde{V}\tilde{\phi}_y - \tilde{W}\tilde{\phi} \\ &= [(1 + C^i\phi_i^T S)V + (C^i\phi_i^T S)_t - \tilde{V}(1 + C^i\phi_i^T S)]\phi_y \\ &\quad + [W + (C^i(\phi_{it}^T S + \phi_i^T S V))_y - \tilde{W}]\phi \\ &\quad + (C^i_{ty} - \tilde{V}C^i_y - \tilde{W}C^i)M(\psi, \phi_i). \end{aligned}$$

The new matrices $\tilde{V}$ and $\tilde{W}$ are now defined such that the differential part of Q vanishes. Taking into account that $Q[\phi_i] = 0$ we conclude once again that $Q \equiv 0$.

The first equation of the constraints (6.8) is preserved due to Theorem 20. With the help of the identity

$$S = \Upsilon^T S \Upsilon \tag{6.11}$$

(cf. Lemma 19) we calculate

$$\begin{aligned} \tilde{V}^T S &= \Upsilon^{-1T} V^T \Upsilon^T S + \Upsilon^{-1T} \Upsilon_t^T S \\ &= -S(\Upsilon V \Upsilon^{-1} + \Upsilon_t \Upsilon^{-1}) \\ &= -S\tilde{V}. \end{aligned}$$

The second step in this calculation has been performed by pulling S through to the left-hand side via (6.11) and $V^T S = -SV$. To finish the proof of Theorem 22 we shall show that the constraint

$$\tilde{W}^T S = S(\tilde{W} - \tilde{V}_y)$$

is nothing but the time evolution of Υ compatible with its spatial counterpart considered in Lemma 19. An appropriate calculation may be

$$\begin{aligned} \tilde{W}^T S - S(\tilde{W} - \tilde{V}_y) &= (W^T + \Gamma_y^T)S - S(W + \Gamma_y - \tilde{V}_y) \\ &= (S\tilde{V} - SV - S\Gamma + \Gamma^T S)_y \end{aligned} \tag{6.12}$$

with the abbreviation

$$\Gamma := C^i(\phi_{it}^T S + \phi_i^T SV).$$

We suppose for the moment that the expression in brackets on the right-hand side of (6.12) already vanishes. Insertion of the definition of $\tilde{V}$ and Γ then yields

$$S\Upsilon_t\Upsilon^{-1} = S(C^i\phi_{it}^T - \phi_{it}C^{iT})S - SC^i\phi_i^T SV\phi_j C^{jT} S$$

which can be verified by copying the proof of Lemma 19. q.e.d.

We have demonstrated by way of an example that an extended Darboux-Levi transformation is applicable to linear problems which do not have the form of evolution equations. A more general approach geared to the Loewner system (6.2) without any symmetry does exist. Up to now, however, we have not succeeded in gaining a deeper insight into how linear problems involving higher derivatives with respect to all coordinates x, y and t may be tackled in the most general case.

6.2 A 2+1-dimensional sine-Gordon system

We have already mentioned that the invariance (6.3) of the Loewner system might admit nonlinear integrable equations wherein x and y occur on an equal footing. This is an interesting observation since for quite a long time the only integrable equations sharing this property had been the Davey-Stewartson equation as 2+1-dimensional generalization of the nonlinear Schrödinger equation and the Novikov-Veselov equation, which may be regarded as generalization of the Korteweg-de Vries equation or Dodd-Bullough equation (cf. Section 5.6).

Very recently Konopelchenko and Rogers proposed a strong generalization of the classical sine-Gordon equation (sometimes also called Enneper equation) [23]. Konopelchenko and Dubrovsky have shown its integrability in the sense that it is amenable to the inverse spectral transform method. Their solutions have been constructed by the dressing method based on the $\bar{\partial}$-∂-problem [58]. Recently Nimmo has exploited the Moutard transformation to determine a broad class of solutions of the 2+1-dimensional sine-Gordon system [60]. As expected (cf. Section 5.6), his approach can be translated into the language of the Darboux-Levi transformation given in the previous section.

In order to derive the Konopelchenko-Rogers equations we investigate the structure of the Darboux-Levi transformation in Theorem 22. We immediately observe that it preserves the constraints

$$\begin{aligned} &R^2 = 1 \\ &\mathrm{tr} R = 0. \end{aligned} \tag{6.13}$$

We shall show that for 2×2-matrices R, V and W the constraints (6.8) and (6.13) determine completely a sinh-Gordon system, which is gauge-equivalent to the sine-Gordon system if we choose

$$S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

From (6.13) and the first of the symmetry conditions (6.8) it follows that the matrix R is parametrized by one function only. We choose

$$R = \begin{pmatrix} 0 & e^{-\varphi-\varphi'} \\ e^{\varphi+\varphi'} & 0 \end{pmatrix}. \tag{6.14}$$

The first equation of (6.4) then determines uniquely

$$V = -\frac{1}{2}\begin{pmatrix} \varphi_t + \varphi'_t & 0 \\ 0 & -\varphi_t - \varphi'_t \end{pmatrix}.$$

The parametrization of W is more tricky. First of all we note that the third symmetry constraint (6.8) allows to express the antisymmetric part of SW in terms of V. Secondly, the third nonlinear equation of (6.4) admits the introduction of a scalar potential according to

$$\begin{aligned} \mathrm{tr}W &= -\tfrac{1}{2}(\varphi - \varphi')_{ty} \\ \mathrm{tr}(RW) &= -\tfrac{1}{2}(\varphi - \varphi')_{tx}. \end{aligned}$$

Finally, the second equation of (6.4) determines algebraically the remaining coefficients of W. It becomes

$$W = -\frac{1}{2}\begin{pmatrix} \varphi_{ty} & -e^{-\varphi-\varphi'}\varphi'_{tx} \\ e^{\varphi+\varphi'}\varphi_{tx} & -\varphi'_{ty} \end{pmatrix}.$$

The rest of the nonlinear system (6.4) then produces the 2+1-dimensional sinh-Gordon system [61]

$$\begin{aligned} (e^{\varphi+\varphi'}\varphi_{tx})_x &= (e^{\varphi+\varphi'}\varphi_{ty})_y \\ (e^{-\varphi-\varphi'}\varphi'_{tx})_x &= (e^{-\varphi-\varphi'}\varphi'_{ty})_y. \end{aligned} \tag{6.15}$$

We observe that the system above is characterized by the feature that the spatial coordinates appear on an equal footing. The following gauge-equivalent forms of (6.15) will share this property. Setting

$$\begin{aligned} \chi &:= -\tfrac{1}{2}i(\varphi + \varphi') \\ \rho &:= \tfrac{1}{2}(\varphi - \varphi')_t \end{aligned}$$

we obtain the alternative form

$$\begin{aligned} &\chi_{tuv} + \rho_v\chi_u + \rho_u\chi_v = 0 \\ &\rho_{uv} - (\chi_u\chi_v)_t = 0 \end{aligned}$$

with $x =: u + v, y =: u - v$, which has been found by Konopelchenko and Dubrovsky [58] and has been used in the above-mentioned approach by Nimmo.

In order to obtain the sine-Gordon system we perform a gauge transformation to the Loewner system (6.2), viz

$$\phi \to \Theta^{-1}\phi, \qquad \Theta := \frac{1}{\sqrt{2}}\begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix}. \tag{6.16}$$

The factor in front of the matrix has been chosen such that the potentials $M(\psi, \phi_i)$ are identical in both gauges. The transformation of the symmetric matrix S then reads

$$S \to \Theta^T S \Theta = 1.$$

As derived in [61] the auxiliary triad (6.2) converts under this gauge transformation into

$$\begin{aligned}
\phi_x &= -\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}\phi_y \\
\phi_{ty} &= (\frac{1}{2}\begin{pmatrix} 0 & -\theta_t \\ \theta_t & 0 \end{pmatrix}\partial_y + \frac{1}{4}\begin{pmatrix} K & -\theta_{ty} + \theta'_y \\ \theta_{ty} + \theta'_y & L \end{pmatrix})\phi \\
\phi_{tx} &= (\frac{1}{2}\begin{pmatrix} 0 & -\theta_t \\ \theta_t & 0 \end{pmatrix}\partial_x + \frac{1}{4}\begin{pmatrix} K' & -\theta_{tx} + \theta'_x \\ \theta_{tx} + \theta'_x & L' \end{pmatrix})\phi
\end{aligned} \tag{6.17}$$

where we have introduced the abbreviations

$$\begin{aligned}
K &= (\theta_{ty} + \theta'_y)\cot\theta - (\theta_{tx} + \theta'_x)\sin^{-1}\theta \\
K' &= (\theta_{tx} + \theta'_x)\cot\theta - (\theta_{ty} + \theta'_y)\sin^{-1}\theta \\
L &= (\theta_{ty} - \theta'_y)\cot\theta + (\theta_{tx} - \theta'_x)\sin^{-1}\theta \\
L' &= (\theta_{tx} - \theta'_x)\cot\theta + (\theta_{ty} + \theta'_y)\sin^{-1}\theta.
\end{aligned}$$

The connection between the parametrizations in the two gauges is achieved by means of

$$\begin{aligned}
\theta &= -i(\varphi + \varphi') \\
\theta'_x &= \theta_{ty}\cos\theta + \sigma_{ty}\sin\theta \\
\theta'_y &= \theta_{tx}\cos\theta + \sigma_{tx}\sin\theta
\end{aligned} \tag{6.18}$$

with $\sigma := \varphi - \varphi'$. The integrability of (6.18) is guaranteed modulo the sinh-Gordon system (6.15). The linear triad (6.17) has as compatibility condition the Konopelchenko-Rogers equations

$$\begin{aligned} &\left(\frac{\theta_{tx}}{\sin\theta}\right)_x - \left(\frac{\theta_{ty}}{\sin\theta}\right)_y + \frac{\theta_x\theta'_y - \theta_y\theta'_x}{\sin^2\theta} = 0 \\ &\left(\frac{\theta'_x}{\sin\theta}\right)_x - \left(\frac{\theta'_y}{\sin\theta}\right)_y + \frac{\theta_x\theta_{ty} - \theta_y\theta_{tx}}{\sin^2\theta} = 0. \end{aligned} \tag{6.19}$$

It is noted that under the reduction

$$\theta' = \theta_t \tag{6.20}$$

the system (6.19) takes the simple form

$$\left(\frac{\theta_{tx}}{\sin\theta}\right)_x - \left(\frac{\theta_{ty}}{\sin\theta}\right)_y + \frac{\theta_x\theta_{ty} - \theta_y\theta_{tx}}{\sin^2\theta} = 0 \tag{6.21}$$

which represents a 2+1-dimensional generalization of the sine-Gordon equation. In the one-dimensional limit $\theta_y = 0$ or $\theta_x = 0$ we obtain indeed the sine-Gordon equation after a suitable transformation of the independent variable t. Moreover, in this reduction there is a link between the triad (6.17) and the scalar Zakharov-Manakov system [64, 65]. On insertion of (6.20) into (6.17) it is readily seen that one can integrate for the component ϕ^1. We obtain

$$\phi_t^2 = \frac{\theta_t}{2}\phi^1 \tag{6.22}$$

so that the Loewner system (6.17) reduces to the scalar triad

$$\begin{aligned} \alpha_{tu} &= \frac{\Sigma_{tu}}{\Sigma_t}\,\alpha_t - \Sigma_t \tan\Sigma\,\alpha_u \\ \alpha_{tv} &= \frac{\Sigma_{tv}}{\Sigma_t}\,\alpha_t + \Sigma_t \cot\Sigma\,\alpha_v \\ \alpha_{uv} &= -\Sigma_v \tan\Sigma\,\alpha_u + \Sigma_u \cot\Sigma\,\alpha_v \end{aligned} \tag{6.23}$$

with the change of the dependent variables $\alpha := \phi^2$ and $\Sigma := \theta/2$. The independent variables u and v are defined as before. In this representation the sine-Gordon equation (6.21) becomes

$$\Sigma_{tuv} + \Sigma_{tu}\Sigma_v \tan\Sigma - \Sigma_{tv}\Sigma_u \cot\Sigma = 0 \tag{6.24}$$

which is, indeed, the integrability condition for the triad (6.23).

The general scalar Zakharov-Manakov triad is given by (6.23) if one replaces the coefficients, which are in the case of the sine-Gordon equation parametrized by a single field Σ, by arbitrary functions. In the most general case α is a vector-valued function and the coefficients are matrices.

Let us now come back to the Darboux-Levi transformation and calculate explicitly the new fields $\tilde{\phi}$ and $\tilde{\phi}'$ in the very first gauge. From the constraint $S = \Upsilon^T S \Upsilon$ in Lemma 19 we immediately deduce

$$\begin{aligned} &\Upsilon_{11}\Upsilon_{22} + \Upsilon_{12}\Upsilon_{21} = 1 \\ &\Upsilon_{11}\Upsilon_{21} = 0 = \Upsilon_{22}\Upsilon_{12} \end{aligned} \tag{6.25}$$

and hence from $\tilde{R} = \Upsilon R \Upsilon^{-1}$ and the particular parametrization (6.14)

$$\begin{pmatrix} e^{-\tilde{\varphi}-\tilde{\varphi}'} \\ e^{\tilde{\varphi}+\tilde{\varphi}'} \end{pmatrix} = \begin{pmatrix} \Upsilon_{11}^2 e^{-\varphi-\varphi'} + \Upsilon_{12}^2 e^{\varphi+\varphi'} \\ \Upsilon_{22}^2 e^{\varphi+\varphi'} + \Upsilon_{21}^2 e^{-\varphi-\varphi'} \end{pmatrix}.$$

Using (6.25) we finally obtain

$$\begin{aligned} &\tilde{\varphi} + \tilde{\varphi}' = -\varphi - \varphi' + 2\ln[(C^i)^2\phi_i^2] \qquad &&\text{for } \Upsilon_{12} = \Upsilon_{21} = 0 \\ &\tilde{\varphi} + \tilde{\varphi}' = \varphi + \varphi' + 2\ln[1 + (C^i)^2\phi_i^1] \qquad &&\text{for } \Upsilon_{11} = \Upsilon_{22} = 0. \end{aligned}$$

We wish to mention that in general the Darboux-Levi transformation for arbitrary N can be regarded as N-fold iteration of the simple Darboux-Levi transformation for $N = 1$. This generalizes the well-known result for the Darboux transformation. Consequently,

$$\begin{aligned} &\Upsilon_{12} = \Upsilon_{21} = 0 \qquad &&\text{for } N \text{ odd} \\ &\Upsilon_{11} = \Upsilon_{22} = 0 \qquad &&\text{for } N \text{ even} \end{aligned}$$

since iteration yields $\tilde{\varphi} + \tilde{\varphi}' = (-1)^N(\varphi + \varphi') + \ldots$.

For the derivation of the new field $\tilde{\sigma} = \tilde{\varphi} - \tilde{\varphi}'$ we note that the Loewner triad (6.2) is completely symmetric in the independent variables x and y.

Thus it is clear that a formula similar to the one given in Theorem 22 has to hold for $\tilde{W}'$. Hence we can also make use of

$$\tilde{W}' = W' + [C^i(\phi_{it}^T S + \phi_i^T SV)]_x.$$

If we now take the trace of this equation and its counterpart in Theorem 22 we find

$$\tilde{\sigma}_t = \sigma_t - 2[C^{iT}(S\phi_{it} - SV\phi_i)] + f(t).$$

Integration by means of an identity analogous to the one given in Lemma 13 yields

$$\tilde{\sigma} = \sigma + 2\ln|M|$$

where functions of integration have been neglected due to the obvious invariance $\sigma \to \sigma + h(t) + g(x,y)$.

Even though we have been able to integrate the Frobenius system (6.18) for the new field $\tilde{\theta}'$ we are only interested in giving the transformation formula for the function θ in the gauge (6.17). For this purpose it is necessary to mention that the coefficients C^i transform under the gauge transformation (6.16) as

$$C^i \to \Theta^{-1}C^i.$$

Consequently, a short calculation, using the identities (6.25), shows

$$\tilde{\theta} = \theta + 4\arctan\frac{(C^i)^2\phi_i^1}{(C^i)^2\phi_i^2}. \tag{6.26}$$

The distinction between even and odd N could be dropped as we have applied the invariance of the sine-Gordon system $(\tilde{\theta}, \tilde{\theta}') \to (-\tilde{\theta}, -\tilde{\theta}')$ for N odd. It is clear that we could have derived this relation from Theorem 22 for $S = 1$.

The next question which arises is whether the reduction of the sine-Gordon system to the single equation (6.21) is preserved by the Darboux-Levi transformation. It is quite laborious to verify directly that the constraint (6.20) is satisfied by the twiddled quantities. A more elegant method is to exploit the fact that (6.20) is equivalent to the condition for the components of the eigenfunction (6.22) if we take into account the invariance $\theta' \to \theta' + f(t)$. The calculations, however, are lengthy but straight forward.

We shall suppress them and state that they lead to the required result. In fact the transformation for the single sine-Gordon equation had first been found by tackling the Zakharov-Manakov system directly. It has turned out later that it coincides with the Darboux-Levi transformation for the sine-Gordon system under the reduction (6.22).

It is of some interest that the Darboux-Levi transformation ($N = 1$) for the sine-Gordon equation in the form (6.24) admits a formulation as auto-Bäcklund transformation in the original sense. In terms of the field $\Sigma = \theta/2$ (6.26) reads for $N = 1$

$$\tilde{\Sigma} = \Sigma + 2 \arctan \Gamma$$

where

$$\Gamma := \frac{\phi_1^1}{\phi_1^2} = \frac{\alpha_{1t}}{\Sigma_t \alpha_1}.$$

The triad system (6.23) written in terms of $\Gamma = \tan(\tilde{\Sigma} - \Sigma)/2$ now provides the following auto-Bäcklund transformation for (6.24):

$$\begin{aligned}
\Sigma_{uv}^- &= \Sigma_u^-(\ln \sin \Sigma^+)_v + \Sigma_v^-(\ln \cos \Sigma^+)_u \\
\Sigma_{tu}^+ \tan \Sigma^+ + \Sigma_{tu}^- \cot \Sigma^- &= \Sigma_t^+ \Sigma_u^-(\cot \Sigma^+ \cot \Sigma^- - \tan \Sigma^+ \tan \Sigma^-) \\
\Sigma_{tv}^+ \cot \Sigma^+ - \Sigma_{tv}^- \cot \Sigma^- &= \Sigma_t^+ \Sigma_v^-(\tan \Sigma^+ \cot \Sigma^- - \cot \Sigma^+ \tan \Sigma^-)
\end{aligned} \tag{6.27}$$

with the definition of the quantities

$$\Sigma^\pm := \frac{\tilde{\Sigma} \pm \Sigma}{2}.$$

The elimination process of α_1 via Γ in the triad (6.23) has been carried out by taking mixed second derivatives of Γ and replacing second and third derivatives of α_1 by means of (6.23). Interestingly, the remaining terms have turned out to depend only on Γ.

The link to the well-known auto-Bäcklund transformation in 1+1 dimensions [66] comes about as follows. In the 1+1-dimensional limit $\Sigma^\pm$ are supposed to depend only on $x = u + v$. Hence we can integrate the first equation of (6.27) and obtain

$$(\tilde{\Sigma} - \Sigma)_x = \lambda(t) \sin(\tilde{\Sigma} + \Sigma).$$

The two other equations coincide iff λ is constant. They then become

$$(\tilde{\Sigma}+\Sigma)_t = g(t)\sin(\tilde{\Sigma}-\Sigma).$$

Hence we have recovered the usual Bäcklund transformation for the sine-Gordon equation if we take into account that (6.21) is written in a form which admits the invariance $t \to h(t)$.

We wish to close this section with two remarks. ~~At~~ first the relation between the Darboux-Levi transformation for the sinh-Gordon system and the Moutard transformation ~~shall~~ be worked out. To this end we apply will another gauge transformation to the Loewner system (6.2). This time the gauge transformation will be generated by a non-constant matrix Θ so that the operator F does not transform as simply as before. In fact with

$$\Theta = \begin{pmatrix} e^{\frac{\varphi+\varphi'}{2}} & 0 \\ 0 & e^{-\frac{\varphi+\varphi'}{2}} \end{pmatrix}$$

the equation (6.5) becomes

$$(\partial_x - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\partial_y + \frac{1}{2}\begin{pmatrix} -\varphi_x - \varphi'_x & -\varphi_y - \varphi'_y \\ \varphi_y + \varphi'_y & \varphi_x + \varphi'_x \end{pmatrix})\phi = 0. \qquad (6.28)$$

Now if we solve (6.28) for ϕ^1_x and ϕ^1_y and take the compatibility condition we obtain, after repeating the same procedure for ϕ^2,

$$\Box\phi^1 = e^{-\frac{\varphi+\varphi'}{2}}(\Box e^{\frac{\varphi+\varphi'}{2}})\phi^1$$
$$\Box\phi^2 = e^{\frac{\varphi+\varphi'}{2}}(\Box e^{-\frac{\varphi+\varphi'}{2}})\phi^2$$

with $\Box := \partial_y^2 - \partial_x^2$. The system above consists of two copies of the scalar hyperbolic equation (5.27) which is amenable to the Moutard transformation (cf. Section 5.6). Hence it is not surprising that, in this gauge, a transformation of the Loewner system which is based on Moutard's theorem can be found [60].

Secondly, it is worth mentioning that the reduction of the sine-Gordon system to the sine-Gordon equation (6.21) reflects the more general fact that the matrix-valued Zakharov-Manakov system can be written as Loewner a system with matrices of double the size and special form. Even though the transformation from the Zakharov-Manakov system to the (particular)

Loewner system is explicitly known [67] it is not clear whether the Darboux-Levi transformation given in this chapter allows ~~for~~ this specialization of the ? Loewner system in general. It is certainly necessary to study the properties of the Zakharov-Manakov system since in the 1+1-dimensional limit it contains or is equivalent to physically interesting systems such as the Ernst equation of general relativity or the self-induced transparency equations [67] respectively.

6.3 A class of solutions

In this section it is our intention to give explicitly a class of solutions of the Konopelchenko-Rogers equations (6.19). For this purpose it is useful to introduce the notation

$$|D^{ij}| := (C^k)^i \phi^j_k |M|$$

so that the matrix D^{ij} is given by

$$D^{ij} = \begin{pmatrix} 0 & \phi^j_1 & \cdots & \phi^j_N \\ \phi^i_1 & & & \\ \vdots & & M & \\ \phi^i_N & & & \end{pmatrix}$$

which follows from the definition of the coefficients C^k in Theorem 20 by means of Cramer's rule. In this notation the solution (6.26) becomes

$$\tilde{\theta} = \theta + 4 \arctan \frac{|D^{21}|}{|D^{22}|}. \tag{6.29}$$

We start with the most simple seed solution of the sine-Gordon system simplest

$$\theta = \theta(s,t)$$
$$\theta' = 0$$

with the argument $s := (\mu_0 + \lambda_0)x + (\mu_0 - \lambda_0)y$ and some constants λ_0 and μ_0. Integration of (6.19) then yields the 1+1-dimensional sine-Gordon equation

$$\theta_{ts} = 4 \sin \theta.$$

In the limit $\theta \to 0$ the triad system (6.17) becomes

$$\phi_x = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \phi_y$$

$$\phi_{ty} = \begin{pmatrix} -2\lambda_0 & 0 \\ 0 & 2\mu_0 \end{pmatrix} \phi$$

$$\phi_{tx} = \begin{pmatrix} 2\lambda_0 & 0 \\ 0 & 2\mu_0 \end{pmatrix} \phi$$

which can easily be solved to give the simplest solutions

$$\phi_i = \begin{pmatrix} e^{\lambda_i(x-y)+2\lambda_0\lambda_i^{-1}t+\rho_i} \\ e^{\mu_i(x+y)+2\mu_0\mu_i^{-1}t+\sigma_i} \end{pmatrix}.$$

The constants λ_i, μ_i, ρ_i and σ_i are as yet unspecified. Insertion into the defining relations for the potentials (6.9) and (6.10)

$$\begin{aligned} M_y(\psi_i, \phi_j) &= \phi_{iy}^T \phi_j \\ M_x(\psi_i, \phi_j) &= \phi_{ix}^T \phi_j \\ M_t(\psi_i, \phi_j) &= \phi_i^T \phi_{jt} \end{aligned}$$

immediately yields

$$M(\psi_i, \phi_j) = \frac{\lambda_i}{\lambda_i + \lambda_j} e^{(\lambda_i+\lambda_j)(x-y)+2\lambda_0(\lambda_i^{-1}+\lambda_j^{-1})t+\rho_i+\rho_j} + \frac{\mu_i}{\mu_i + \mu_j} e^{(\mu_i+\mu_j)(x+y)+2\mu_0(\mu_i^{-1}+\mu_j^{-1})t+\sigma_i+\sigma_j}$$

where the integration constants have been set to zero for simplicity.

Fortunately, only the quotient of determinants enter into the formula (6.29). Hence we can multiply the rows and columns of the matrices D^{21} and D^{22} by common factors without distorting the equation for $\tilde{\theta}$. Thus we obtain the simple formula

$$\tilde{\theta} = 4 \arctan \frac{|D^+|}{|D^-|} \tag{6.30}$$

where

$$D^{\pm} := \begin{pmatrix} 0 & e^{\pm\alpha_1} & \cdots & e^{\pm\alpha_N} \\ e^{-\alpha_1} & & & \\ \vdots & & M^0 & \\ e^{-\alpha_N} & & & \end{pmatrix}$$

$$M^0_{ij} := \frac{\lambda_i}{\lambda_i + \lambda_j} e^{\alpha_i + \alpha_j} + \frac{\mu_i}{\mu_i + \mu_j} e^{-\alpha_i - \alpha_j}$$

and

$$\alpha_i := \tfrac{1}{2}(\lambda_i - \mu_i)x - \tfrac{1}{2}(\lambda_i + \mu_i)y + (\lambda_0\lambda_i^{-1} - \mu_0\mu_i^{-1})t + \nu_i. \tag{6.31}$$

For real solutions we choose real constants $\lambda_i, \mu_i, \lambda_0, \mu_0$ and complex constants ν_i the imaginary parts of which are half-integer multiples of π.

The simplest solutions within this class are those describing the interaction of plane wave solitons [58, 59]. A different choice of eigenfunctions solving the triad (6.17) leads to breather solutions [58, 60].

Here, we wish to present a new class of solutions which may represent the interaction of an arbitrary number of two-dimensional localized solitons. To this end we restrict the parameters in (6.31) to $\mu_i = -\lambda_i$ for $i = 1, \ldots, N_0$ and $\mu_i = \lambda_i$ for $i = N_0 + 1, \ldots, N$, i.e.

$$\begin{aligned} \alpha_i &= \lambda_i x + (\lambda_0 + \mu_0)\lambda_i^{-1} t + \nu_i, \qquad & i &= 1, \ldots, N_0 \\ \alpha_i &= -\lambda_i y + (\lambda_0 - \mu_0)\lambda_i^{-1} t + \nu_i, \qquad & i &= N_0 + 1, \ldots, N. \end{aligned}$$

For $N = 1, N_0 = 1$ we obtain the well-known one-soliton solution

$$\tilde{\theta} =: a = 4 \arctan e^{2\alpha_1}$$

of the 1+1-dimensional sine-Gordon equation

$$a_{tx} = 4(\mu_0 + \lambda_0) \sin a. \tag{6.32}$$

There are two cases to distinguish. a is called a kink solution if it increases by 2π in positive x-direction and an antikink if it decreases by 2π.

Similarly, for $N = 1, N_0 = 0$ the solution (6.30) represents an (anti)kink moving in y-direction. It becomes

$$\tilde{\theta} =: b = 4 \arctan e^{2\alpha_2} \tag{6.33}$$

and solves the sine-Gordon equation in the form

$$b_{ty} = 4(\mu_0 - \lambda_0)\sin b. \tag{6.34}$$

For reasons which will become apparent soon we have committed a slight impropriety in denoting the argument in (6.33) by α_2 and not α_1.

The simplest 2+1-dimensional solution is obtained in the case $N = 2$, $N_0 = 1$. It reads

$$\tilde{\theta} = 4\arctan\frac{A_{21}}{e^{\alpha_1-\alpha_2} - e^{\alpha_2-\alpha_1}}$$

with the definitions

$$A_{21} := c_{21}e^{\alpha_1+\alpha_2} + c_{21}^{-1}e^{-\alpha_1-\alpha_2}$$

$$c_{ij} := \frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j}.$$

After absorbing the factor c_{21} into ν_1 and ν_2 we obtain

$$\tilde{\theta} = 4\arctan\frac{\cosh(\alpha_1+\alpha_2)}{\sinh(\alpha_1-\alpha_2)} \tag{6.35}$$

which is reminiscent of the two-soliton solution in 1+1 dimensions.

The quantity

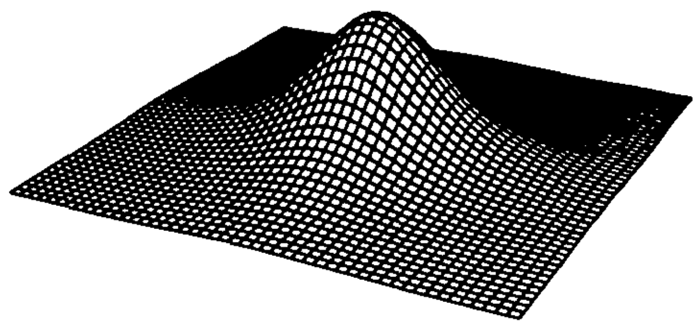
$$\tilde{\theta}_x\tilde{\theta}_y = 16\frac{\lambda_1\lambda_2}{\cosh 2\alpha_1 \cosh 2\alpha_2}$$

now provides a localized object since it decays to zero exponentially as $(x,y) \to \infty$ in any direction. The sign of its amplitude depends on the imaginary parts of ν_1 and ν_2. Interestingly, it can be regarded as nonlinear superposition of two one-dimensional (anti)kinks for positive amplitude and a kink and an antikink for negative amplitude moving in x-direction and y-direction respectively. This can be seen from

$$\tilde{\theta}_x\tilde{\theta}_y = a_x b_y$$

where a and b are the one-soliton solutions of (6.32) and (6.34) repectively. Hence we shall call the solution with positive amplitude a two-dimensional kink and the solution with negative amplitude a two-dimensional antikink. A kink solution at a fixed time t is plotted in Figure 1.

Figure 1. Kink



We shall now show that for $N = 3, N_0 = 2$ the solution (6.30) decomposes asymptotically into two two-dimensional (anti)kinks. It can be brought into the form

$$\tilde{\theta} = 4 \arctan \frac{A_{32}e^{\alpha_1} + A_{21}e^{\alpha_3} + A_{13}e^{\alpha_2}}{A_{32}e^{-\alpha_1} + A_{21}e^{-\alpha_3} + A_{13}e^{-\alpha_2}} \tag{6.36}$$

with

$$\begin{aligned} A_{21} &:= c_{21}e^{\alpha_1+\alpha_2} + c_{21}e^{-\alpha_1-\alpha_2} \\ A_{13} &:= c_{13}e^{\alpha_3+\alpha_1} + c_{13}^{-1}e^{-\alpha_3-\alpha_1} \\ A_{32} &:= c_{32}e^{\alpha_2+\alpha_3} + c_{32}^{-1}e^{-\alpha_2-\alpha_3}. \end{aligned}$$

For the asymptotic analysis we consider the quantity $|D^+|/|D^-|$ in a frame moving with the (anti)kink associated with α_2 and α_3 as $t \to \infty$. We compute for $\alpha_1 \to +\infty$

$$\frac{|D^+|}{|D^-|} \to \frac{(c_{32} + c_{13} + c_{21})e^{\alpha_2+\alpha_3} + c_{32}^{-1}e^{-\alpha_2-\alpha_3}}{c_{13}e^{\alpha_3-\alpha_2} + c_{21}e^{\alpha_2-\alpha_3}}$$

and for $\alpha_1 \to -\infty$

$$\frac{|D^+|}{|D^-|} \to \frac{c_{21}e^{\alpha_3-\alpha_2} + c_{13}^{-1}e^{\alpha_2-\alpha_3}}{(c_{32}^{-1} + c_{13}^{-1} + c_{21})e^{-\alpha_2-\alpha_3} + c_{32}e^{\alpha_2+\alpha_3}}.$$

Hence to obtain asymptotically solutions of the form (6.35) the constraints

$$(c_{32}+c_{13}+c_{21})c_{32}c_{13}c_{21} < 0$$
$$(c_{32}^{-1}+c_{13}^{-1}+c_{21})c_{32}c_{13}c_{21} < 0$$

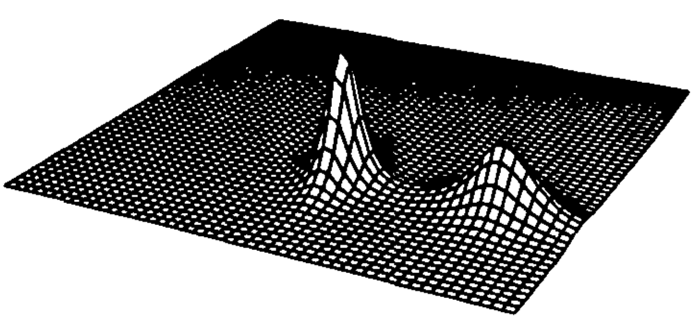
need to be imposed on the parameters λ_i. An analogous analysis associated with the frame of reference corresponding to the second (anti)kink does not give further conditions.

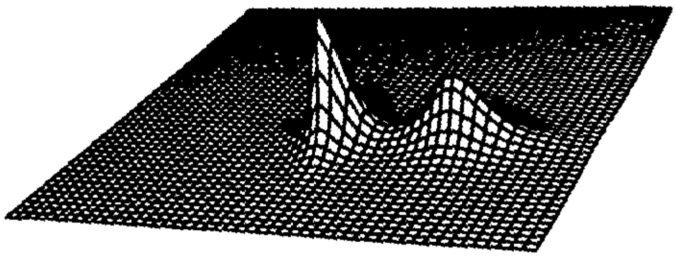
Moreover, we observe that a(n) (anti)kink cannot alter its character during the interaction. The amplitude remains unchanged. The only indication that an interaction has occured is that the objects are phase-shifted. In that sense they behave like one-dimensional solitons. As can be read off the above limits, the phase-shifts are complicated functions of the coefficients c_{ij}.

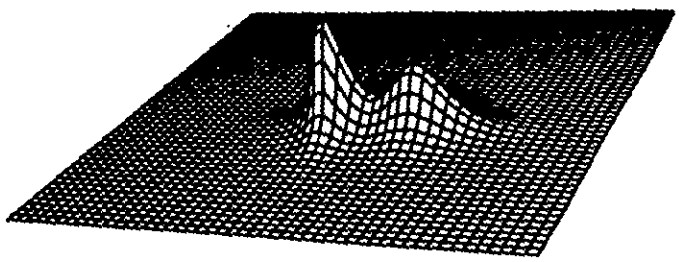
In Figures 2 and 3 snapshots are shown at various times during the interaction of a kink with a kink and an antikink respectively.

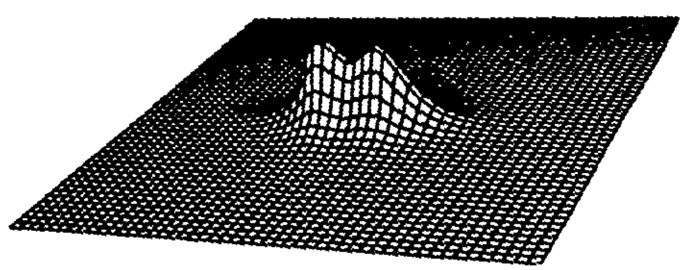
We emphasize that the properties of these solitonic solutions differ from those of the dromion solutions [21]. The dromion solutions admit head-on collisions and a change of their amplitudes. It is even possible to create and annihilate dromions. Head-on collisions cannot appear in the case of two-dimensional (anti)kinks since the signs of the components of their velocities are determined by λ_0 and μ_0. Whether for arbitrary N and N_0 creation and annihilation of localized solitons can be generated has to be left to a more detailed study of the class of solutions discussed in this section.

Figure 2. Kink-kink interaction









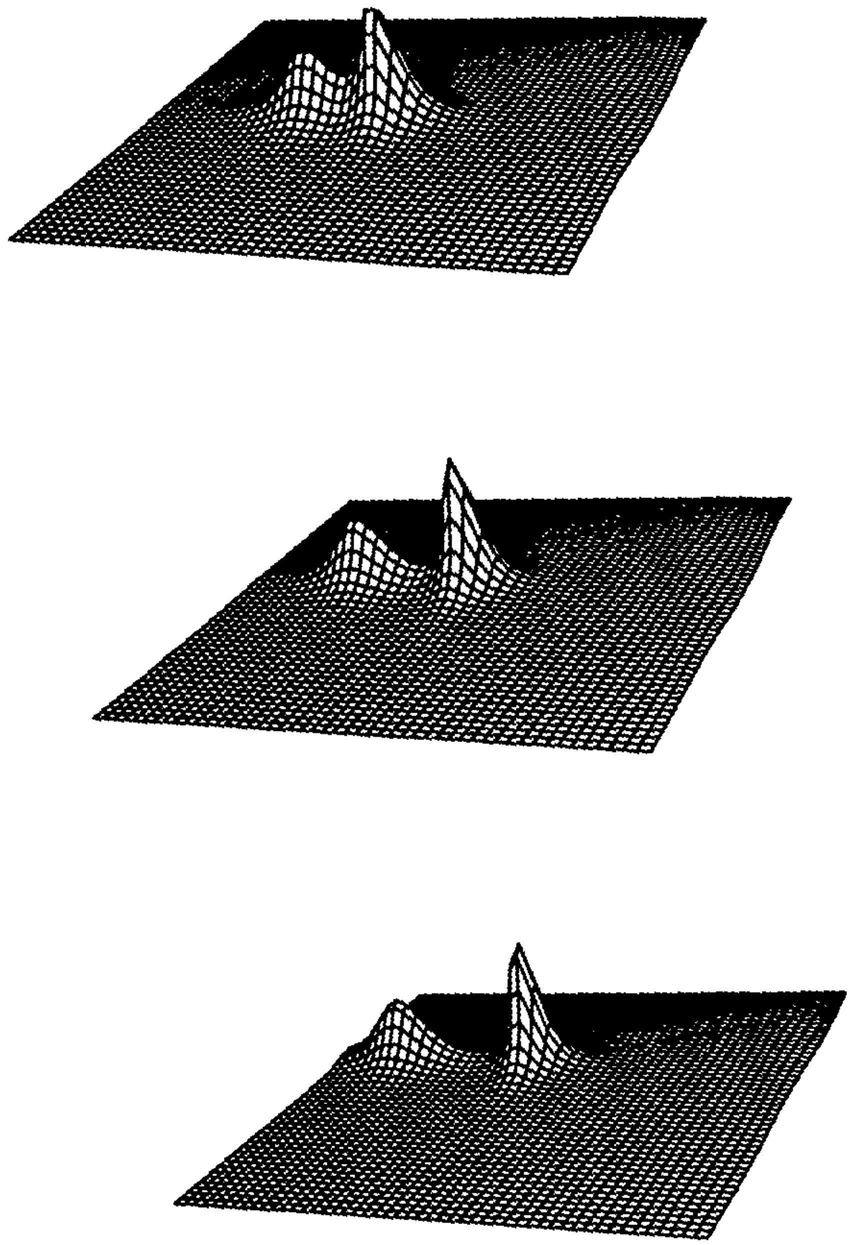
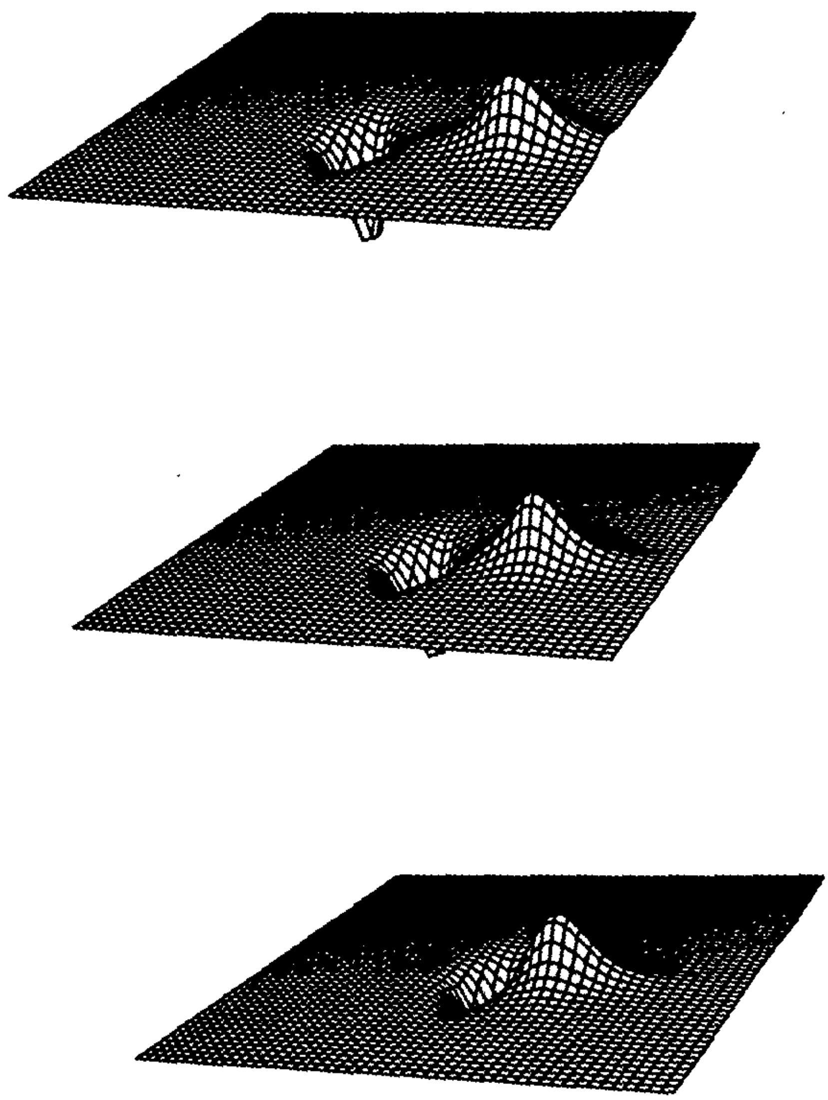
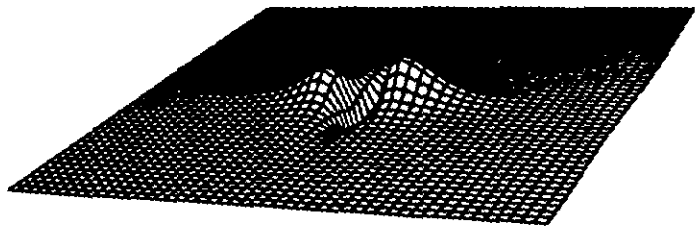
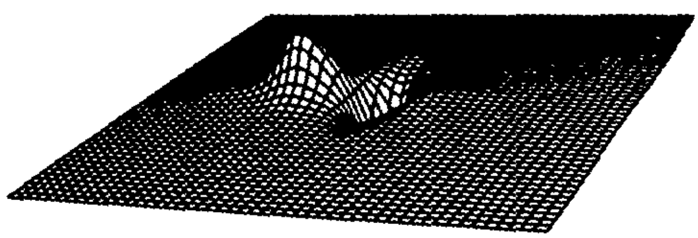
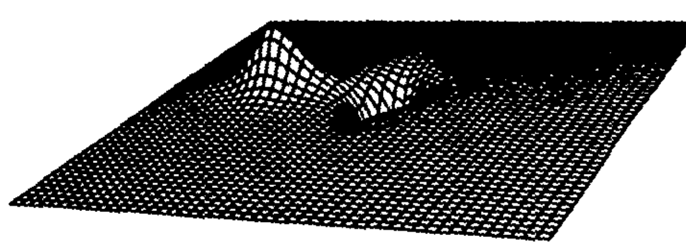


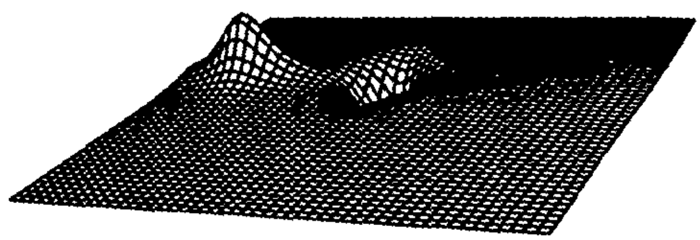
Figure 3. Kink-antikink interaction











Chapter 7

Concluding remarks

In this thesis we have developed solution generating techniques for nonlinear partial differential equations in 2+1 dimensions which can be represented as integrability conditions of auxiliary linear equations. A distinction between linear problems with and without a symmetry of the kind discussed in Chapter 5 has been necessary, which has led us to the Darboux-Levi and the Darboux transformation respectively. In both cases the orders of the associated linear differential operators could be preserved. Unfortunately, symmetries and the orders of the operators have not been sufficient to characterize the underlying differential equations. In order to maintain the additional constraints of the type (3.21) or (5.19) it has been inevitable to introduce an additional gauge transformation associated with the matrix Υ. Analogous to the 1+1-dimensional case it is still unclear how the gauge matrix and the constraints are in general related to each other. Interestingly, so far, Υ could always be determined in a purely algebraic manner, which in principle confirms our philosophy that 'everything' can be derived without solving non-trivial differential equations.

There exist, however, classes of linear problems for which the gauge matrix Υ can be ignored. One of those is associated with the (multi-component) Kadomtsev-Petviashvili hierarchy. Its construction is based on the generalized Sato theory [43, 68], which parametrizes a collection of operators $F, G, \ldots$ corresponding to different 'times' $t_F = x, t_G = t, \ldots$ such that the integrability conditons associated with any pair of operators are, in some sense, automatically satisfied. To this end one introduces the dressing operator

$$W = 1 + w_1\partial_z^{-1} + w_2\partial_z^{-2} + \ldots$$

and calculates its inverse according to

$$W^{-1} = 1 - w_1\partial_z^{-1} + (w_1^2 - w_2)\partial_z^{-2} + \dots .$$

Furthermore, one defines a hierarchy of differential operators

$$L_A := (WAW^{-1})_{\geq 0}$$

where A belongs to a set of commuting constant differential operators. The multi-component Kadomtsev-Petviashvili hierarchy is now defined by the evolution equations

$$W_{t_A} = -(WAW^{-1})_{<0}W$$

which satisfy the hierarchy of integrability conditions. It includes the Kadomtsev-Petviashvili equation for the scalar pseudo-differential operator W and in the case of 2×2-matrices the Davey-Stewartson equation.

As we have seen in Chapter 3 and 5, the gauge matrix Υ could be assumed to be trivial, i.e. $\Upsilon = 1$. In [69] it has been shown that this is possible for the entire Kadomtsev-Petviashvili hierarchy. Moreover, one can express the Darboux(-Levi) transformation in terms of its action on the pseudo-differential operator W, viz

$$\begin{aligned} &\tilde{W} = PW\partial_z^{-N} && \text{Darboux} \\ &\tilde{W} = DW && \text{Darboux-Levi.} \end{aligned}$$

Even though the structure of the operators L_A is explicitly known, e.g. in the simplest case

$$L_A = a\partial_z^n + [w_1, a]\partial_z^{n-1} + \dots$$

it is not obvious how this parametrization is directly given in terms of the orders of L_A. This is closely related to the question what kind of subalgebra of a Kac-Moody algebra is described by the matrices L_A in the 1+1-dimensional limit (if it is a subalgebra).

The second outstanding problem is a classification of the symmetries of the differential operators given by the condition

$$H^*\Xi = -\Xi H. \tag{7.1}$$

For which operators Ξ can constraints on the adjoint eigenfunctions be found such that the Darboux-Levi transformation preserves the above reduction? It is, for example, clear that if a linear operator Θ and its inverse are polynomial in ∂_z the reduction (7.1) and the integrability condition is invariant under the generalized gauge transformation

$$
\begin{aligned}
&H \rightarrow \Theta^{-1} H \Theta \\
&\Xi \rightarrow \Theta^{*} \Xi \Theta.
\end{aligned}
$$

The next item deals with a generalization of the results given in this thesis to prolongation algebras which are semi-direct products of loop algebras and the Virasoro algebra. A prominent example for an integrable system which admits this particular prolongation algebra is the Ernst equation

$$
\mathcal{E}_{z\bar{z}} + \frac{1}{2(z+\bar{z})}(\mathcal{E}_z + \mathcal{E}_{\bar{z}}) = \frac{\mathcal{E}_z \mathcal{E}_{\bar{z}}}{\mathrm{Re}\mathcal{E}}
$$

governing axially symmetric stationary gravitational fields in general relativity [33]. The linear problem involves in this case a non-constant parameter λ. It has the form

$$
\begin{aligned}
\phi_z &= F(\lambda)\phi, \qquad & \lambda_z &= f(\lambda) \\
\phi_{\bar{z}} &= G(\lambda)\phi, \qquad & \lambda_{\bar{z}} &= g(\lambda).
\end{aligned}
$$

Obviously, the difficulty is to find a suitable extension to 2+1 dimensions as λ is not constant. Interestingly, one can circumvent this problem if one considers the linear problem which has been used to generate the Hoenselaers-Kinnersley-Xanthopoulos transformations of the Ernst equation [70]. An explicit transformation which relates the two linear problems has been given by Kramer [71]. As mentioned in the previous chapter, the latter one is a special 1+1-dimensional reduction of the matrix-valued Zakharov-Manakov system. Unfortunately, there exists a symmetry of the linear problem which is rather complicated and not of the type considered here.

Without claiming to have been exhaustive we wish to conclude the thesis in the hope that we have made at least some contributions to the theory of nonlinear integrable systems.

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