Exact natural frequencies of multi-level elastically connected taut strings and related problems

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Introduction The dynamics of a family of simple, but extremely useful structural elements is governed by a second order Sturm-Liouville equation. This equation allows for the uniform distribution of mass and stiffness and enables the motion of strings and shear beams, together with the axial and torsional motion of bars to be described exactly. As a result, each member type in this family has been treated exhaustively when considered as a single member or when joined contiguously to others. However, when such members are linked in parallel by uniformly distributed elastic interfaces, their complexity becomes significantly more intractable and it is this class of structures that has led to renewed interest and which forms the basis of the work that follows.

Initially, differential equations governing the coupled motion of the system are developed from first principles. These are organised into the form of a generalised linear symmetric eigenvalue problem, from which a family of uncoupled differential operators can be established. These operators define a series of exact substitute systems that together describe the complete motion of the original structure. These equations can then be used in either of two ways. In their most powerful form they can be developed into exact dynamic stiffness matrices that enable all the powerful features of the finite element method to be utilised. This subsequently enables sets of members carrying point masses and subject to point spring supports to be analysed easily. Alternatively, the equations are able to yield an exact relational model that links any uncoupled frequency of an original member to the corresponding set of coupled system frequencies. This approach enables 'back of the envelope calculations' to be undertaken quickly and efficiently. The exact mode shapes of the original structure can be recovered in either case. Due to space limitations, only aspects of the first technique are described briefly herein, but both are covered exhaustively elsewhere [1].

Theory The theory below has been developed for an easily envisioned set of taut string members that are connected to each other by elastic interfaces of unequal stiffness, with the top (i = 1) and bottom (i = n) members being additionally connected to foundations. However, the approach applies equally to all the member types previously mentioned. Thus, adopting the assumptions of classical string theory, limiting attention to free vibration and introducing the non-dimensional length parameter, $\xi = x/L$, the equation of motion for a typical elemental length of string, i, and its corresponding constitutive relationship are easily shown to be

$$k_i V_{i-1} - (k_i + k_{i+1})V_i + k_{i+1}V_{i+1} + r_i(D^2 + \gamma_i \omega^2)V_i = 0 \quad \text{and} \quad Q_i = r_i L \; dV_i / d\xi \qquad \text{(1a,b)}$$

where k_i and k_{i+1} are the stiffness / unit length of the elastic interfaces connecting adjacent strings or foundations and V_i and Q_i are the amplitudes of the lateral displacement and vertical component of string tension, respectively, $D^2 = d^2/d\xi^2$, $r_i = T_i/L^2$, $\gamma_i = m_i/r_i$, m_i is the mass / unit length, L is the length of all members comprising the set and ω is the circular frequency. It is now assumed that m_i/T_i is constant for all i and hence that

$$\gamma_i = \gamma = mL^2/T = \text{constant} \qquad (2)$$

This enables the governing equations for the first, last and typical members to be written, respectively, as

$$(k_1 + k_2)V_1 - k_2V_2 - r_1\lambda V_1 = 0\,, \qquad -k_nV_{n-1} + (k_n + k_{n+1})V_n - r_n\lambda V_n = 0 \tag{3a,b}$$

and

$$-k_iV_{i-1} + (k_i + k_{i+1})V_i - k_{i+1}V_{i+1} - r_i\lambda V_i = 0 \tag{3c}$$

where

$$\lambda = D^2 + \gamma\omega^2 \tag{4}$$

and k_1 and k_{n+1} can be zero or non-zero in any combination, thus defining the longitudinal boundary conditions.

Eqs.(3) enable a complete set of equations to be assembled for an n level system, as indicated by Eq.(5a)

$$\left\{\begin{bmatrix} k_1+k_2 & -k_2 & & & \\ & \bullet & & & \\ & -k_i & k_i+k_{i+1} & -k_{i+1} & \\ & & & \bullet & \\ & & & -k_n & k_n+k_{n+1} \end{bmatrix} - \lambda \begin{bmatrix} r_1 & & & & \\ & \bullet & & & \\ & & r_i & & \\ & & & \bullet & \\ & & & & r_n \end{bmatrix}\right\} \begin{bmatrix} V_1 \\ \bullet \\ V_i \\ \bullet \\ V_n \end{bmatrix} = \begin{bmatrix} 0 \\ \bullet \\ 0 \\ \bullet \\ 0 \end{bmatrix} \tag{5a}$$

where zeros have been omitted for clarity. Eq.(5a) can then be written concisely as

$$(\mathbf{k} - \lambda\mathbf{r})\mathbf{V} = \mathbf{0} \tag{5b}$$

The form of Eqs.(5) is that of a generalized symmetric linear eigenvalue problem, for which a number of standard routines are available for calculating the eigenvalues, λ, and corresponding modal vectors, $\mathbf{V}$.

Substitute systems The n values of λ that satisfy Eqs.(5) define a family of second order differential operators that satisfy the original problem and which are given by Eq.(4) as

$$D^2 + \gamma\omega^2 = \lambda_i \qquad i = 1,2,\ldots\ldots n \tag{6}$$

Eq.(6) can be assigned a physical context by noting that it is a property of such differential operators that they can be written as

$$(D^2 + \gamma\omega^2)V_i = \lambda_i V_i \qquad i = 1,2,\ldots n \tag{7}$$

and hence that

$$(D^2 + \chi_i^2)V_i = 0 \qquad i = 1,2,\ldots n \tag{8}$$

where

$$\chi_i^2 = \gamma\omega^2 - \lambda_i \tag{9}$$

and V_i is a typical lateral displacement amplitude. In this case, each equation now describes the free vibration of a single unified member, but supported on a Winkler foundation of different magnitude in each case. Eqs.(8) therefore represent n substitute systems, each of which yield an infinite number of frequencies that, when arranged in ascending order, comprise the complete spectrum of frequencies of the original problem. It therefore follows that the fundamental frequency of the original problem is given by the single substitute system that yields the lowest frequency, namely the one that incorporates the lowest linear eigenvalue derived from Eqs.(5).

An exact stiffness formulation (exact finite element) is now adopted to solve the i^{th} substitute system and can be expressed as

$$\begin{bmatrix} Q_{i0} \\ Q_{i1} \end{bmatrix} = r_i L \frac{\chi_i}{S_i} \begin{bmatrix} C_i & -1 \\ -1 & C_i \end{bmatrix} \begin{bmatrix} V_{i0} \\ V_{i1} \end{bmatrix} \quad (10)$$

where

$$C_i = \cos\chi_i \text{ and } S_i = \sin\chi_i \text{ for } \chi_i^2 > 0 \text{ and } C_i = \cosh\chi_i \text{ and } S_i = \sinh\chi_i \text{ for } \chi_i^2 < 0 \quad (11a,b)$$

and the subscripts 0 and 1 relate to the left and right hand end of the unified member, respectively.

Identical boundary conditions are now imposed on each substitute system in turn by adding spring supports and/or nodal masses at both $\xi = 0$ and $\xi = 1$. There is no requirement for the masses to be the same at each end and the stiffnesses can be assigned any value between zero (free support) and $+\infty$ (clamped support). The required natural frequencies stemming from each of the *n* substitute systems can then be converged upon to any desired accuracy using the Wittrick-Williams algorithm. All the frequencies thus calculated are natural frequencies of the original system and can be arranged in ascending order to cover any frequency range of interest, which will be guaranteed to be fully populated if the highest frequency is bounded above in each of the substitute systems.

Example Consider now the problem of two identical and parallel taut strings of length 1 m that are linked by an elastic interface of stiffness *k* = 200 N/m^2. The mass/unit length and string tension for both members are 0.01 kg/m and 50 N, respectively. The results are presented in Table 2, where they are compared with those of Oniszczuk [2]. It is interesting to note that in this example the natural frequencies corresponding to anti-symmetric modes are identical to the uncoupled frequencies of the two members, since both members move identically in the same direction and do not extend the massless elastic interface that connects them. Hence $\lambda_1 = 0.0$.

Table 2 Comparison between the natural frequencies given by Oniszczuk [2] and the presented theory for the parallel string problem described above. The frequencies correspond to either A/S (Anti-Symmetric) or S (Symmetric) modes about the horizontal axis of symmetry. * This value has been confirmed as a typing error in the original paper and should be 222.1.

Modal No.	Natural frequencies (rad/sec)			
	[2]		Presented theory	
	A/S	S	Substitute system 1 $\lambda_1 = 0.0$	Substitute system 2 $\lambda_2 = 8.0$
1	221.1*	298.9	222.144	298.911
2	444.3	487.2	444.288	487.229
3	666.4	695.8	666.432	695.796
4	888.6	910.8	888.577	910.806
5	1110.7	1128.6	1110.72	1128.58
6	1332.9	1347.8	1332.86	1347.80

References

[1] Howson, WP, Watson, A and Rafezy, B. Free vibration of multi-level elastically connected members: A simple method for exact solutions. *Int. J. Solids Structs.* (Under review)

[2] Oniszczuk, Z. Transverse vibrations of elastically connected double-string complex system, Part II: Forced vibrations. *J. Sound Vib.* (2000) 232, 367-386.