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Bounds on Eigenfunctions and Spectral Functions on Manifolds of Negative Curvature

by:

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Abstract

In this dissertation we study the Laplace operator, Δ, acting on functions on a smooth, compact Riemannian manifold M. Our approach is based on the study of the spectrum of the aforementioned operator, i.e. the set of real numbers, $\{\lambda_i\}$, such that $(\Delta - \lambda_i^2)$ is not invertible. Because the spectrum of the Laplace operator is discrete, eigenvalues and eigenfunctions determine the Laplacian completely. Therefore the main object of our interest is the counting function of the Laplacian, i.e. the function which value at some real point λ is the number of eigenvalues, λ_i, of Δ below λ. We also consider a local version of the counting function, which contains information about the corresponding eigenfunctions φ_i:

$$N_x(\lambda) = \sum_{\lambda_i < \lambda} |\varphi_i(x)|^2.$$

The first chapter gives the historical background on the study of the counting function of the Laplace operator, a motivation of our research and the main results.

In the second chapter we give the necessary background, which fixes notation and gives all used definitions. Additionally we describe there the theory of the Fourier transform, homogeneous distributions and unbounded operators.

The third chapter states basic facts about hyperbolic geometry that allow us to proceed to the next chapter where we derive an asymptotic formula for the local counting function with an explicit estimate of the remainder in the case where the manifold is hyperbolic. It is important that these estimates are local and therefore are also valid for non-compact manifolds, whereas in the case of compact manifolds they give us explicit information on the counting function.

The last chapter treats a family of regular modification of the local counting function, i.e. the Riesz means of the counting function:

$$R_k N_x(\lambda) = \lambda^{-k} \int_{-\infty}^{\lambda} (\lambda - \tau)^k \, \mathrm{d}N_x(\tau).$$

We discuss the asymptotics of this function as $\lambda \to \infty$ and we show that it admits an expansion with $k+1$-terms with an error term that is $O(\lambda^{d-1-k}/(log\lambda)^{k+1})$ in the case of manifolds of non-positive curvature.

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Chapter 1

Introduction

Physics always challenges mathematicians to rigorously describe the world around us. Physics motivated the language of partial differential equations and nowadays scientists cannot imagine a description of a moving object without differential equations. Partial differential equations are useful for a description of diffusion, waves and heat for example. PDEs are central to many branches of science, especially physics and chemistry.

This thesis describes one of the best known differential operators - the Laplacian, defined by

$$\Delta u = -\mathrm{div}\,(\mathrm{grad}\,u)\,,$$

where the minus sign is just a convention. The Laplace operator was introduced by the great French mathematician and astronomer Pierre-Simon Laplace (1749-1827) for study of celestial mechanics. He showed that the potential function, u, obeys the Laplace equation, i.e. $\Delta u = 0$. The analogous result was obtained by Leonard Euler for the velocity potential of a fluid in [14].

The Laplace operator on a Riemannian manifold appeared for the first time in a paper of Minakshisundaran and Pleijel in 1949 [38]. In the same year Maaß published a paper about the Laplace operator, but he restricted his study to one-dimensional complex Riemannian manifolds [36]. It is quite surprising that the first papers about the aforementioned operator appeared almost a hundred years after Riemann's lecture in 1854, when he introduced Riemannian geometry. This suggests that it is a very difficult subject of research and even now there are still many questions to be answered. For example, the question if we can explicitly solve the eigenvalue problem

$$\Delta u = \lambda^2 u$$

for a given manifold can be answered positively only in few cases, for example for cubes, disks, ellipses [41], toruses, balls [11, 12]. Until this point, there is no known example of a compact hyperbolic manifold that has completely determined eigendata and only a numerical approximation is available, see e.g. [46].

Although our knowledge is so restricted, there are things we can say about the Laplace operator on compact Riemannian manifolds. Through this chapter M denotes

a compact, smooth Riemannian manifold and n is its dimension. We do know that the spectrum, in this case, is a discrete, countable subset of the real axis, and each eigenspace is of a finite multiplicity. For the proof of these facts we recommend standard textbooks of Bérard [6], Chavel [10], Courant and Hilbert [11, 12].

Before we start asking questions about the Laplacian, let us motivate ourselves. As we have seen, the Laplacian appears in Laplace's equation, $\Delta u = 0$, which describes the potential function without the source of a field, e.g. a gravitational potential with no material points, or a fluid velocity potential in the absence of sources. It has also applications in electrostatics, thermodynamics, electrodynamics and mechanics. In fact it is also the eigenproblem for a zero eigenvalue, which is always the first eigenvalue on closed manifolds.

The Laplace operator appears also in two classical problems: the heat equation and the wave equation. The heat equation describes the heat distribution over time on a manifold. If we know that $f_0(x)$ is a temperature at point x, then the solution of

$$\begin{cases} \Delta f = -\frac{1}{\nu}\frac{\partial f}{\partial t}, \\ f(0,x) = f_0(x) \end{cases}$$

describes the temperature on the manifold t time units later, where ν is the conductivity of the material. For the wave equation, one can consider a manifold as a vibrating membrane, a "drum". Then the wave equation shows how this drum behaves in time if we deform it. Suppose that f_0 is a function describing deviation from the original shape and f_1 is the initial velocity, then the solution of

$$\begin{cases} \Delta f = -\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}, \\ f(0,x) = f_0(x), \\ \frac{\partial f}{\partial t}(0,x) = f_1(x) \end{cases}$$

describes vibrations of the manifold, where c is the speed of a wave depending on physics, e.g. for electromagnetic waves c is the speed of light.

One can solve these equations by separation of variables:

$$f(t,x) = v(t)u(x).$$

This precisely leads to the eigenvalue problem

$$\Delta u = \lambda^2 u.$$

The time dependent parts of f for the heat equation and the wave equation are given by $e^{-t\nu\lambda^2}$, $e^{itc\lambda}$ respectively. Therefore if we know the eigendata, we can solve these classical problems.

Because the time dependent part of the heat equation decreases exponentially, the most significant contribution to the solution of the heat equation comes from the bottom of the spectrum, thus a study of the first eigenvalue was undertaken by many authors, for example by Lichnerowich in 1958 [32] and Buser [9] in 1978. The former gave a lower

bound on λ_1 depending on the curvature of the manifold, whereas the latter proved also a lower bound on λ_1 in terms of Cheeger's constant. For an upper bound, one needs to read the paper of Bourguignon, Li and Yau [8] from 1994, but their estimate is valid only on $\mathbb{CP}^n$.

So far we presented research only in one direction, i.e. for a given manifold we asked about its spectrum. There are of course many papers which ask an opposite question: does the spectrum of Δ determines a manifold? It is a very natural question, which comes from a simple observation that, for example two bells of a different shape sound different. The first person who asked such a question was Leon Green in 1960. It took four years to get a negative answer, in the paper of Milnor [37]. He showed two isospectral manifolds which are not isometric. The same problem was investigated by Kac in his famous paper: "*Can one hear the shape of a drum?*"[28] from 1966. On the other hand in 1996 Lohkamp [34] showed, that for any compact manifold of dimension grater than two, and for any sequence of positive real numbers, $0 < \lambda_1 < \lambda_2 < \ldots$, one can construct the sequence of metrics g_m on the manifold such that the spectrum of the corresponding Laplacian coincides with the given sequence from 0 to λ_m.

Anther type of spectral problems are Poisson type formulas. This area of research tries to find a link between the lengths of closed geodesics, which is a classical object, with the spectrum of the Laplacian, which can be considered as a quantum object. For example, for a rapidly decreasing function h, one has:

$$\sum_{m\in\mathbb{Z}} h(m) = \sum_{n\in\mathbb{Z}} \int_{\mathbb{R}} h(t)e^{i(2\pi n)t}\,\mathrm{d}t.$$

Note that the left hand side is the sum over square roots of eigenvalues of Δ on a circle, whereas on the right the sum is over lengths of closed geodesics. The first papers on this subject were written by Balian and Bloch [1, 2, 3, 4] in early '70s. They gave the heuristic support for the conjecture that the spectrum determines the lengths of closed geodesics. Guillemin and Melrose [19] in 1979 investigated an inverse relation.

Another important problem is to locate all eigenvalues as accurately as possible. In 1996 Gromov in [17] showed that for any odd dimensional Riemannian manifold eigenvalues satisfy the following

$$\sharp\{\lambda \in [a,b]\} \leq c(b-a)^n \mathrm{Vol}(M),$$

provided that the manifold has bounded curvature, the injectivity radius is large enough and a, b are not too close. Six years later the same author proved in [18] that there is a universal constant, c, which depends on the curvature and dimension only, such that k-th eigenvalue obeys

$$\lambda_k^d \leq \frac{c}{\mathrm{Vol}(M)}k.$$

The problem of our interest is the distribution of large eigenvalues. This problem comes from quantum physics. Consider Schrödinger equation of a free particle

$$\hbar^2 \Delta f = i\hbar \frac{\partial f}{\partial t}$$

where $\hbar$ is Plank's constant. This equation can also be solved by the separation of variables, which gives the eigenvalue problem. Note that the semi-classical limit $\hbar \to 0$ corresponds in Euclidean geometry to rescaling the spatial coordinates outwards, i.e. it gives large scale physics. This leads to the hypothesis that the classical mechanics is a limit (in some sense) of quantum physics. It suggests looking at asymptotics of λ_i as $i \to \infty$. In order to make our description easier, define the counting function of the Laplacian as the number of its eigenvalues, λ_i, below some fixed number λ, i.e.

$$N(\lambda) := \sharp\{\lambda_i : \lambda_i < \lambda\}.$$

Note that this function contains all information about eigenvalues: points where N is discontinuous are exactly square roots of eigenvalues and the size of a jump determines the multiplicity of the eigenvalue.

If we want to add data about eigenfunctions to the counting function we define the local counting function:

$$N_x(\lambda) := \sum_{\lambda_i < \lambda} |\varphi_i(x)|^2.$$

Then the local counting function of the Laplace operator on M satisfies the famous local Weyl law:

$$N_x(\lambda) = \frac{\omega_n}{(2\pi)^n}\lambda^n + O(\lambda^{n-1}),$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$. This was proved by Herman Weyl [49] in 1911 for the Dirichlet Laplacian on domains in $\mathbb{R}^n$. The proof of this result in the case of closed Riemannian manifolds is due to Levitan [30]. In 1968 Hörmander [24] generalised it to the case of pseudodifferential operators of order m.

Various improvements on the remainder term are known in the case of compact manifolds with negative sectional curvature or under assumptions on the nature of the dynamics of the geodesic flow. In particular, in 1975 Duistermaat and Guillemin [13] proved that the estimate $O(\lambda^{n-1})$ cannot be improved when the geodesic flow is periodic. They also showed that for boundary-less manifolds the remainder is $o(\lambda^{n-1})$, when the set of periodic geodesics has Liouville measure 0. In 1977 Bérard [5] obtained a logarithmic improvement for compact manifolds with non-positive sectional curvature (for surfaces it suffices to have no conjugate points):

$$\left| N_x(\lambda) - \frac{\omega_n}{(2\pi)^n}\lambda^n \right| \leq C\frac{\lambda^{n-1}}{\log \lambda}. \tag{1.1}$$

Note that the original local Weyl law immediately implies a uniform bound on eigenfunctions:

$$\|\varphi_j\|_{L^\infty(M)} \leq C\lambda_j^{\frac{n-1}{2}},$$

for some $C > 0$ that depends on a geometry of the manifold, in the case that Δ has compact resolvent. This estimate is sharp without additional assumptions, as the example

of the sphere shows. Further Sogge [45] showed that without any additional assumptions on M we have the following bound on L^p norms:

$$\|\varphi_j\|_{L^p(M)} \leq C_{L^p} \lambda_j^{\delta(n,p)},$$
$$\delta(n,p) = \begin{cases} \frac{n-1}{2} - \frac{n}{p}, & \frac{2(n+1)}{n-1} \leq p \leq \infty, \\ \frac{n-1}{4} - \frac{n-1}{2p}, & 2 \leq p \leq \frac{2(n+1)}{n-1}. \end{cases}$$

It is natural that Bérard's logarithmic improvement of the local counting function leads to the logarithmic improvements of the L^p estimates. In [48] Hassell and Tacy proved that under the same assumptions as in Bérard [5], one has:

$$\|\varphi_j\|_{L^p(M)} \leq C_{L^p} \frac{\lambda_j^{\frac{n-1}{2} - \frac{n}{p}}}{(\log \lambda_j)^{\frac{1}{2} - \frac{n+1}{p(n-1)}}}.$$

Similarly, one can derive a bound on the C^l-norm

$$\|\varphi_j\|_{C^l(M)} \leq C_l \lambda_j^{l + \frac{n-1}{2}}.$$

While such estimates of eigenfunctions are very useful for proofs and general considerations, for purposes of numerical analysis one often needs an explicit value of the constant.

The work presented here provides explicit estimates of the eigenfunctions and local counting function of the Laplacian for manifolds that are hyperbolic near the point in question (Corollary 4.2.6, Theorem 4.2.1) . In particular it gives an exact formulas for C_l in the estimate above. We also present an algorithm for bounding $C^l(M)$ norms of eigenfunctions.

The thesis is organised in the following way: the second chapter provides necessary background on the theory of unbounded operators. Although it is well-known theory, we present it in order to fix notation. The third chapter is an introduction to the theory of hyperbolic manifolds; we state there facts that are needed for an analysis of the Laplace operator on negatively curved spaces. This also give a direct link to the fourth chapter, where, as we already mentioned, we discuss questions about the local counting function of the Laplacian. The main result there is Theorem 4.2.1, which gives explicit estimates on the local counting function. The estimates we obtain have applications, such as estimates on eigenfunctions (Corollary 4.2.6) and estimates on the local heat kernel (Theorem 4.2.7). They are also used to derive estimates for the determinant of Δ (Corollary 4.3.2). Numerical bounds of the determinant of Laplacian on the Bolza surface that confirm the most accurate known estimate.

In the last chapter we focus on a slightly different object than the local counting function. We discuss its Riesz means, i.e., the functions

$$R_k N(\lambda) = \int_{-\infty}^{\lambda} (1 - \tau\lambda^{-1})^k \, \mathrm{d}N(\tau), \quad k = 0, 1, 2, \ldots.$$

Note that for $k = 0$ we have the definition of the counting function. Because Riesz means of the counting function are more regular than the counting function, i.e. $R_k N \in C^{k-1}(\mathbb{R}_+)$, one can therefore expect that they admit a more precise expansion.

In 1987 Safarov in [43] has shown that

$$R_k N(\lambda) = \sum_{i=0}^{k} \frac{k!(n-i)!}{(n-i+k)!} a_i \lambda^{n-i} + O\left(\lambda^{n-k-1}\right), \quad \text{as } \lambda \to +\infty,$$

where exact formulas on a_i constants can be obtained, but, in general, for $i \geq 2$ are very complicated. Moreover for surfaces ($n = 2$) of constant negative curvature, Hejhal in [21] proved a logarithmic improvement in the error term for the Riesz mean of the counting function; more precisely

$$R_1 N(\lambda) = \frac{\mathrm{Vol}(M)}{12\pi} \lambda^2 + a_2 + O((\log \lambda)^{-2}). \tag{1.2}$$

The purpose of the last chapter is to generalise Hejhal's result to surfaces without conjugate points and to manifolds of non-positive curvature. So we want to extend his result in two ways: dimensionally and geometrically (assumption of constant curvature is not needed any more). Our results are contained in Theorem 5.0.4.

Chapter 2

Background

This chapter is devoted to some general background necessary for this thesis. It is a well known theory, so we give only definitions and theorem statements without proofs in general. Most of the concepts are taken from textbooks by Taylor [47] and Lieb and Loss [33] where the reader may find proofs of the stated theorems.

Here we introduce the basic objects that we will use later on. In particular we want to introduce the concept of the spectral calculus which will be widely used through the thesis. Our secondary objective is to fix a notation that will help avoid confusion.

2.1 Fourier analysis

Before we define the Fourier transform we define Schwartz space, $\mathcal{S}(\mathbb{R}^n)$, which is the natural domain of this transform. Moreover, using duality, we may then extend the Fourier transform to the dual space $\mathcal{S}'(\mathbb{R}^n)$, which for instance contains $L^2(\mathbb{R}^n)$.

Definition 2.1.1. A smooth function f on $\mathbb{R}^n$ is in *Schwartz space* $\mathcal{S}(\mathbb{R}^n)$ if for any two multi-indices α, β we have

$$p_{\alpha,\beta}(f) = \sup_{x\in\mathbb{R}^n} |x^\alpha \partial^\beta f(x)| < \infty.$$

The semi-norms $p_{\alpha,\beta}$ give $\mathcal{S}(\mathbb{R}^n)$ the structure of a Frechet space, which can be shown to be complete.
The space $\mathcal{S}'(\mathbb{R}^n)$ of *Schwartz distributions* is defined as the topological dual of $\mathcal{S}(\mathbb{R}^n)$. One can show that for a linear functional $\phi\colon \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$ to be in $\mathcal{S}'(\mathbb{R}^n)$, it is necessary and sufficient that there exist $c > 0$ and $N > 0$ such that

$$|\phi(f)| \leq c \sum_{|\alpha|+|\beta|\leq N} \sup_{x\in\mathbb{R}^n} |x^\alpha \partial^\beta f(x)|.$$

The space $\mathcal{S}'(\mathbb{R}^n)$ is equipped with the weak* topology, which means that $\phi_i \to \phi$ if $\phi_i(f) \to \phi(f)$ for every $f \in \mathcal{S}(\mathbb{R}^n)$.

One of the most important classes of Schwartz distributions is given by the natural injection

$$L^p(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$$

for any $p \in [1, \infty]$, given by

$$\phi(f) = \int_{\mathbb{R}^n} \phi(x) f(x) \, \mathrm{d}x, \quad f \in \mathcal{S}(\mathbb{R}^n), \quad \phi \in L^p(\mathbb{R}^n). \tag{2.1}$$

In particular, the image of $L^2(\mathbb{R}^n)$ in $\mathcal{S}'(\mathbb{R}^n)$ is sometimes called the space of *regular* distributions. Another important example is the Dirac-delta distribution defined by $\delta(f) = f(0)$.

Note that the delta distribution is not the image under the map (2.1) of any function. Thus, $\mathcal{S}'(\mathbb{R}^n)$ cannot be identified with any function space. However it can be shown that any element of $\mathcal{S}'(\mathbb{R}^n)$ is a sufficiently high distributional derivative of a function. In order to make this coherent we must define differentiation of distributions. This is done by duality.

Definition 2.1.2. Let ϕ be a distribution in $\mathcal{S}'(\mathbb{R}^n)$ and α be a multi index. Then the distributional derivative ∂^α is defined as

$$\partial^\alpha \phi(f) := \phi((-1)^{|\alpha|} \partial^\alpha f).$$

This definition works also for the larger space of distributions $\mathcal{D}'(\mathbb{R}^n)$, i.e. dual to the space of smooth compactly supported functions. One can show that under this definition, the derivative of the Heaviside function, which is discontinuous, is the Dirac-delta distribution.

Definition 2.1.3. Suppose $f \in L^1(\mathbb{R}^n)$. The Fourier transform of f is the function $\mathcal{F}f$ given by

$$\mathcal{F}f(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x) e^{-i\langle \xi, x \rangle} \, \mathrm{d}x.$$

It is clear that $\mathcal{F}\colon L^1(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)$. We will usually write $\widehat{f} = \mathcal{F}f$ for shorter notation. Bearing in mind that this is an abuse of notation we will also write $\mathcal{F}(f(x))(\xi)$, or $\widehat{f(x)}(\xi)$ for the Fourier transform of a function f, in order to simplify notation.

It turns out that the Fourier transform is invertible on $\mathcal{S}(\mathbb{R})$:

Theorem 2.1.4. *The Fourier transform is a continuous linear bijection from $\mathcal{S}(\mathbb{R}^n)$ onto $\mathcal{S}(\mathbb{R}^n)$. Its inverse is given by*

$$\mathcal{F}^{-1} f(\xi) = \check{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x) e^{i\langle \xi, x \rangle} \, \mathrm{d}x.$$

Now we define the Fourier transform for Schwartz distributions.

Definition 2.1.5. Let $F \in \mathcal{S}'(\mathbb{R}^n)$. Then the Fourier transform of F, denoted by $\widehat{F}$, is the Schwartz distribution defined by

$$\widehat{F}(f) = F(\widehat{f}).$$

Equation (2.2) shows that if we consider a Schwartz function as a distribution we indeed have

$$\widehat{f}(g) = \int_{\mathbb{R}^n} \widehat{f}(x)g(x)\,\mathrm{d}x = \int_{\mathbb{R}^n} f(x)\widehat{g}(x)\,\mathrm{d}x = f(\widehat{g}).$$

Therefore, the adjoint map $\mathcal{F}^*\colon \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ coincides with $\mathcal{F}$ on a dense subset $\mathcal{S}(\mathbb{R}^n)$. We can thus state following theorem:

Theorem 2.1.6. *The Fourier transform is the linear bijection from $\mathcal{S}'(\mathbb{R}^n)$ to itself which is the unique weakly continuous extension of the Fourier transform on $\mathcal{S}(\mathbb{R}^n)$. Additionally, for any $F \in \mathcal{S}'(\mathbb{R}^n)$ we have*

$$\partial^\alpha \widehat{F} = (-i)^{|\alpha|}\widehat{x^\alpha F}, \quad \widehat{\partial^\alpha F} = i^{|\alpha|}\xi^\alpha \widehat{F}.$$

Now that we have defined Fourier transforms of Schwartz distributions we can present some properties of the transform on various subsets of $\mathcal{S}'(\mathbb{R}^n)$. First we state Plancherel's theorem, which is extremely important in calculations.

Theorem 2.1.7 (Plancherel)**.** *The following statements hold:*

(a) for $f, g \in L^1(\mathbb{R}^n)$ we have

$$\int_{\mathbb{R}^n} \widehat{f}(x)g(x)\,\mathrm{d}x = \int_{\mathbb{R}^n} f(x)\widehat{g}(x)\,\mathrm{d}x, \tag{2.2}$$

(b) if $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ then $\widehat{f} \in L^2(\mathbb{R}^n)$ and $\|f\|_2 = \|\widehat{f}\|_2$.

Continuity of the Fourier transform implies the following corollary:

Corollary 2.1.8. *The Fourier transform extends uniquely to a unitary map of $L^2(\mathbb{R}^n)$ onto $L^2(\mathbb{R}^n)$. The inverse transform extends uniquely to its adjoint.*

Now we will introduce a new operation on functions, which has a nice characteristic from the Fourier transformation point of view.

Definition 2.1.9. The convolution of functions f and g on $\mathbb{R}^n$, denoted by $f * g$, is given by

$$f * g(x) = \int_{\mathbb{R}^n} f(x - t)g(t)\,\mathrm{d}t, \tag{2.3}$$

whenever this expression makes sense.

The problem of convergence of (2.3) is interesting in its own right. For example, one may require $f \in L^p(\mathbb{R}^n)$ and $g \in (L^p)'(\mathbb{R}^n)$; then by Hölder's inequality (2.3) is well defined. One can also show Young's inequality

$$\|f * g\|_r \leq \|f\|_p \|g\|_q,$$

for $f \in L^p(\mathbb{R}^n)$, $g \in L^q(\mathbb{R}^n)$ and $1/p + 1/q = 1 + 1/r$.

Convolution allows us to approximate functions in $L^p(\mathbb{R}^n)$ by smooth functions:

Theorem 2.1.10. *The space of compactly supported functions, C_0^∞, is dense in L^p for $1 \leq p < \infty$.*

The main advantage of the Fourier transform is that it turns both differentiation and convolution into multiplication operators.

Theorem 2.1.11. *Suppose that $f, g, h \in \mathcal{S}(\mathbb{R}^n)$. Then*

(a) $g \mapsto f * g$ is a continuous map of $\mathcal{S}(\mathbb{R}^n)$,

(b) convolution is associative and commutative i.e. $f * (g * h) = (f * g) * h$ and $f * g = g * f$,

(c) $\widehat{fg} = (2\pi)^{-n/2}\widehat{f} * \widehat{g}$ and $\widehat{f * g} = (2\pi)^{n/2}\widehat{f}\widehat{g}$.

Condition (c) shows that we could define convolution in the following way:

$$f * g(x) = (2\pi)^{n/2}\mathcal{F}(\widehat{f}\widehat{g})(x).$$

This also shows that convolution is a continuous operation. Because of its continuity, we can extend its definition to Schwartz distributions.

Definition 2.1.12. Suppose that $f \in \mathcal{S}(\mathbb{R}^n)$ and $F \in \mathcal{S}'(\mathbb{R}^n)$. Then the convolution of F and f, denoted $F * f$, is the Schwartz distribution given by

$$(F * f)(\phi) = F(\tilde{f} * \phi),$$

where $\tilde{f}(x) = f(-x)$.

Theorem 2.1.13. *Suppose that $f, g \in \mathcal{S}(\mathbb{R}^n)$ and $F \in \mathcal{S}'(\mathbb{R}^n)$. Then the map $F \mapsto F * f$ is a weakly* continuous map of $\mathcal{S}'(\mathbb{R}^n)$ into $\mathcal{S}'(\mathbb{R}^n)$ and*

(a) $F * f$ is a polynomially bounded smooth function such that

$$(F * f)(x) = F(\tilde{f}_x),$$

where $f_x(y) = f(y - x)$ and if $P = \sum_{|\alpha| \leq k} a_\alpha \partial^\alpha$ is a differential operator with constant coefficients, we have

$$P(f * g) = Pf * g = f * Pg;$$

(b) convolution is associative: $F * (f * g) = (F * f) * g$;

(c) $\widehat{F * f} = (2\pi)^{n/2}\widehat{f}\widehat{F}$.

Let us introduce the family of Hilbert spaces, that arises naturally in the analysis of the Laplace operator.

Definition 2.1.14. Let s be a non-negative real number. We define so called *the Sobolev space* of exponent s as

$$H^s(\mathbb{R}^n) := \left\{ f \in L^2(\mathbb{R}^n) : (1+|\xi|^2)^{s/2} \widehat{f}(\xi) \in L^2(\mathbb{R}^n) \right\},$$

equipped with the norm

$$\|f\|_{H^s} := \left(\int_{\mathbb{R}^n} (1+|\xi|^2)^s |\widehat{f}(\xi)|^2 \, \mathrm{d}\xi \right)^{1/2}.$$

Recall that the norm $\|\cdot\|_{H^s}$ is generated through the scalar product

$$\langle f, g \rangle := \int_{\mathbb{R}^n} (1+|\xi|^2)^s \widehat{f}(\xi) \overline{\widehat{g}(\xi)} \, \mathrm{d}\xi.$$

By Plancherel's theorem we have $H^0(\mathbb{R}^n) = L^2(\mathbb{R}^n)$. Because H^s is weighted L^2 space, it is a Hilbert space. Now we will present some theorems and definitions related to the question of embedding one space to another. Cited theorems can be found in the lecture notes of Kalantarov [29].

Definition 2.1.15. A Banach space B_1 is said to be *compactly embedded* in a Banach space B_2 if $B_1 \subset B_2$ and there is a constant $c > 0$ such that

$$\|u\|_{B_2} \leq c \|u\|_{B_1}$$

for every $u \in B_1$ and if every bounded set in B_1 is relatively compact in B_2, where $\|\cdot\|_{B_1}$, $\|\cdot\|_{B_2}$ are norms in B_1, B_2 respectively.

Proposition 2.1.16. *For $0 \leq s < t$, $H^t \subset H^s$ and $\|\cdot\|_{H^s} \leq \|\cdot\|_{H^t}$ on H^t.*

Let us define some other spaces which will be used in the next theorem.

Definition 2.1.17. Let $k \in \mathbb{N}$, then

$$\begin{aligned} C_\infty(\mathbb{R}^n) &:= \left\{ f \colon \mathbb{R}^n \to \mathbb{C} \text{ is continuous}, \forall_{\varepsilon>0} \exists_{K \subset \mathbb{R}^n \text{ compact}} : \sup_{x \in \mathbb{R}^n \setminus K} |f(x)| < \varepsilon \right\}, \\ C_\infty^k(\mathbb{R}^n) &:= \left\{ f \in C^k(\mathbb{R}^n) : f^{(\alpha)} \in C_\infty(\mathbb{R}^n) \, \forall_{\alpha \in \mathbb{N}_0^n}, |\alpha| < k \right\}, \end{aligned}$$

equipped with the norm

$$\|f\|_k := \sup\{|f^{(\alpha)}(x)| : x \in \mathbb{R}^n, \, |\alpha| \leq k\}.$$

Proposition 2.1.18. *The space of smooth and compactly supported functions, $C_0^\infty(\mathbb{R}^n)$, is dense in $H^s(\mathbb{R}^n)$ for $s \geq 0$.*

Theorem 2.1.19 (Sobolev embedding theorem)**.** *For $k \in \mathbb{N}_0$, let $s - k > \frac{n}{2}$. Then $H^s(\mathbb{R}^n)$ is contained in $C_\infty^k(\mathbb{R}^n)$.*

Recall that the theorem above implies that for $s > 1/2$ each function in $H^s(\mathbb{R})$ is continuous. The next proposition presents an equivalent definition for Sobolev spaces.

Proposition 2.1.20. *For each $k \in \mathbb{N}$, H^k is the set of L^2-functions f for which all weak derivatives $f^{(\alpha)}$ are L^2-functions and*

$$\|f\|_{H^k}^2 = \sum_{|\alpha|\leq k} B_{k,\alpha}\|f^{(\alpha)}\|_{L^2}^2, \text{ where } B_{k,\alpha} := \frac{k!}{(k-|\alpha|)!\alpha_1!\dots\alpha_n!}.$$

Although this proposition has the disadvantage that it works only for vectors α of natural numbers, it allow us to define Sobolev spaces on open subset Ω of $\mathbb{R}^n$. Additionally it shows that norm of $f \in H^s(\Omega)$ given by

$$\|f\|_{H^k}^2 = \sum_{|\alpha|\leq k} \|f^{(\alpha)}\|_{L^2}^2$$

is equivalent to the one in Definition 2.1.14. We will present some other Hilbert space.

Definition 2.1.21. Let Ω be an open subset of $\mathbb{R}^n$. For $s \geq 0$, define the space $H_0^s(\Omega)$ as the closure of the space of smooth compactly supported functions on Ω, $C_0^\infty(\Omega)$, in $H^s(\mathbb{R}^n)$.

Theorem 2.1.22 (F.Rellich)**.** *If $\Omega \subset \mathbb{R}^n$ is a bounded open domain, then $H^1(\Omega)$ is compactly embedded into $L^2(\Omega)$.*

Since the definition of the norm in Sobolev spaces implies that $H^2 \subset H^1$ continuously, this theorem shows that for a bounded open domain $\Omega \in \mathbb{R}^n$, the space $H^2(\Omega)$ is compactly embedded into $L^2(\Omega)$.

Theorem 2.1.23. *For each bounded open subset Ω of $\mathbb{R}^n$ with a C^1-boundary and $m < k - n/2$, we have the following inclusion*

$$H^k(\Omega) \subset C^m(\overline{\Omega}).$$

This theorem implies that $H^1((a,b))$ functions are continuous on $[a,b]$. This allow us to visualise how the local Sobolev space looks in the simplest setting.

Proposition 2.1.24. *Let $-\infty \leq a < b \leq \infty$, then*

$$H_0^1((a,b)) = \{f \in H^1((a,b)) : f(a) = f(b) = 0\}.$$

Theorem 2.1.25 (Rellich-Kondrashov)**.** *Assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain and the following inequalities hold*

$$1 \leq q \leq \frac{2n}{n-2}, \qquad n > 2.$$

Then $H_0^1(\Omega)$ is compactly embedded into $L^q(\Omega)$.

2.2 Homogeneous distributions

This section shows some basic facts about homogeneous distributions. For the detailed description we refer to the book of Hörmander [25].

Let us recall that the function $f\colon \mathbb{R}^n \to \mathbb{C}$ is homogeneous of degree m if for any $t > 0$ and $x \in \mathbb{R}^n$ we have $f(tx) = t^m f(x)$. Since we want to extend this definition to the space of distributions let us assume that $f \in L^p(\mathbb{R}^n)$ is a homogeneous function of degree m and $\varphi \in C_0^\infty(\mathbb{R}^n)$, then

$$t^m \int_{\mathbb{R}^n} f(x)\varphi(x)\,\mathrm{d}x = \int_{\mathbb{R}^n} f(tx)\varphi(x)\,\mathrm{d}x = t^{-n}\int_{\mathbb{R}^n} f(x)\varphi(t^{-1}x)\,\mathrm{d}x$$

This leads to the definition of the homogeneous distribution:

Definition 2.2.1. The distribution $F \in D'(\mathbb{R}^n)$ is homogeneous of degree $m \in \mathbb{C}$ if and only if for every $\varphi \in C_0^\infty(\mathbb{R}^n)$ we have

$$F(\varphi_t) = t^m F(\varphi)$$

where $\varphi_t(x) = t^{-n}\varphi(t^{-1}x)$.

The simplest examples of homogeneous distributions are homogeneous functions which are also distributions. Here we have some non-trivial examples

Example 2.2.2. The delta distribution on $\mathbb{R}^n$ is homogeneous of degree $-n$ because

$$\delta(\varphi_t) = t^{-n}\int_{\mathbb{R}^n} \delta(x)\varphi(t^{-1}x)\,\mathrm{d}x = t^{-n}\varphi(0) = t^{-n}\delta(\varphi).$$

Example 2.2.3. Suppose $F \in D'(\mathbb{R}^n)$ is homogeneous of degree m. Let α be a multi-index then $\partial^\alpha F$ is homogeneous of degree $m - |\alpha|$. By the definition of a distributional derivative and the change of coordinates we get

$$\partial^\alpha F(\varphi_t) = (-1)^{|\alpha|}F(\partial^\alpha \varphi_t) = (-1)^{|\alpha|}t^{m-|\alpha|}F(\partial^\alpha\varphi) = t^{m-|\alpha|}\partial^\alpha F(\varphi)$$

The aim of this chapter is to introduce some homogeneous distributions which appears, among the others, in the study of the kernel of the wave operator which is described later. Let a be a complex number such that $\operatorname{Re} a > -1$. Let us denote by 1_A a characteristic function of a set A, i.e.

$$1_A(x) = \begin{cases} 1 & x \in A, \\ 0 & x \notin A. \end{cases}$$

Then the function $\mathbb{R} \ni x \mapsto x^a 1_{\mathbb{R}_+}(x)$ is integrable and therefore it defines a homogeneous distribution of degree a, which we denote by x_+^a. It is obvious that

$$x x_+^a = x_+^{a+1} \quad \text{for } \operatorname{Re} a > -1, \tag{2.4}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}x}x_+^a = ax_+^{a-1} \quad \text{for Re } a > 0. \tag{2.5}$$

Our goal is to extend the definition of x_+^a to the whole complex plane, as a distribution, in a such a way that these properties are satisfied as far as possible.

Let φ be a test function on $\mathbb{R}$ and Re $a > -1$, then define the complex function as the action of x_+^a at φ i.e.

$$a \mapsto I_\varphi(a) = x_+^a(\varphi) = \int_0^\infty x^a \varphi(x)\,\mathrm{d}x. \tag{2.6}$$

This function is analytic for Re $a > -1$ and its derivative is given by

$$\frac{\mathrm{d}}{\mathrm{d}a}I_\varphi(a) = \int_0^\infty x^a \log x \varphi(x)\,\mathrm{d}x.$$

Note that for Re $a > -1$ we may rewrite the right hand side of (2.6) as

$$\int_0^\infty x^a\varphi(x)\,\mathrm{d}x = \int_0^1 x^a\left[\varphi(x) - \sum_{i=0}^{n-1}\frac{\varphi^{(i)}(0)}{i!}x^i\right] + \int_1^\infty x^a\varphi(x)\,\mathrm{d}x + \sum_{i=0}^{n-1}\frac{\varphi^{(i)}(0)}{i!(i+1+a)} \tag{2.7}$$

for any natural number n. Because the first term on the right hand side is well defined for Re $a > -n-1$, whereas the third makes sense for $a \neq -1, -2, \ldots, -n$ the expression above gives the formula for meromorphic continuation of I_φ to the half-plane Re $a > -n-1$ with simple poles at $-1, -2, \ldots, -n$. This defines x_+^a for $a \notin -\mathbb{N}$. Because

$$\sum_{i=0}^{n-1}\frac{\varphi^{(i)}(0)}{i!(i+1+a)} = \sum_{i=0}^{n-1}\frac{\varphi^{(i)}(0)}{i!}\int_1^\infty x^{a+i}\mathrm{d}x,$$

in the strip $-n-1 <$ Re $a < -n$ we can rewrite (2.7) in the form:

$$\langle x_+^a, \varphi\rangle = \int_0^\infty x^a\left[\varphi(x) - \sum_{i=0}^{n-1}\frac{\varphi^{(i)}(0)}{i!}x^i\right]\mathrm{d}x, \quad -n-1 < \text{Re } a < -n. \tag{2.8}$$

The formula (2.6) shows that function I_φ has simple poles at $a = -1, -2, \ldots$ and its residue at $a = -k$ is $\varphi^{(k-1)}(0)/(k-1)!$. Because of the definition of distributional derivative we can say that the functional x_+^a has a pole at $a = -k$, and the residue there is

$$\frac{(-1)^{k-1}}{(k-1)!}\delta^{(k-1)}(x), \quad \text{for } k = 1, 2, \ldots$$

For the sake of completeness let us define x_+^{-k} as its finite part, i.e.

$$x_+^{-k}(\varphi) :=$$
$$\lim_{a\to -k} I_a(\varphi) - \frac{\varphi^{(k-1)}(0)}{(k-1)!(a+k)} = -\frac{1}{(k-1)!}\left(\int_0^\infty \log x\varphi^{(k)}(x)\,\mathrm{d}x - \sum_{j=1}^{k}\frac{\varphi^{(k-1)}(0)}{j}\right).$$

Because the definition of x_+^a was given by the analytic continuation it defines a homogeneous distribution of degree a and satisfies (2.4) for $a \notin -\mathbb{N}$ and (2.5) for $a \notin -\mathbb{N}_0$. In order to get the formula analogous to (2.5) for $a = 0, -1, -2, \ldots$ let us consider the following limit

$$\begin{aligned}\lim_{a\to -k} \frac{\mathrm{d}}{\mathrm{d}x} x_+^a &= \lim_{a\to -k} a x_+^{a-1} = \lim_{a\to -k} \left(a x_+^{a-1} + k x_+^{a-1}\right) - k x_+^{-k-1} \\ &= \frac{(-1)^k}{k!}\delta^{(k)} - k x_+^{-k-1}.\end{aligned}$$

for $k \in \mathbb{N}_0$.

In order to have a complete picture one may also consider the functional corresponding to $x_-^a = 1_{\mathbb{R}_-}|x|^a$ for $\operatorname{Re} a > -1$. It is clear that

$$x_-^a(\varphi) = x_+^a(\widetilde{\varphi}),$$

for $\widetilde{\varphi}(x) = \varphi(-x)$, therefore we can easily transfer all results for x_+^a to x_-^a, by replacing $\varphi(x)$ by $\varphi(-x)$ and $\varphi^{(i)}(0)$ by $(-1)^i\varphi^{(i)}(0)$. In particular there is a meromorphic continuation of x_-^a to the whole complex plane with simple poles at negative integer points, $a = -k$, with residues $\delta^{(k-1)}(x)/(k-1)!$. The equation regularising the integral

$$\langle x_-^a, \varphi\rangle = \int_{-\infty}^0 |x|^a \varphi(x)\mathrm{d}x,$$

in the strip $-n-1 < \operatorname{Re} a < -n$ is given by

$$\langle x_-^a, \varphi\rangle = \int_0^\infty x^a \left[\varphi(-x) - \sum_{i=0}^{n-1} \frac{(-1)^i\varphi^{(i)}(0)}{i!} x^i\right] \mathrm{d}x.$$

With the help of already defined regularisation of x_+^a, x_-^a we can define two new distributions:

$$|x|^a := x_+^a + x_-^a, \tag{2.9}$$

$$|x|^a \operatorname{sign}(x) := x_+^a - x_-^a. \tag{2.10}$$

Of course regularisations of these functionals are given by regularisations of x_+^a, x_-^a individually. Recall that x_-^a, x_+^a have poles at $a = -k$, the first with residue $\delta^{(k-1)}(x)/(k-1)!$ and the second with residue $(-1)^{k-1}\delta^{(k-1)}(x)/(k-1)!$. Therefore sum of x_-^a, x_+^a is well defined for $a = 2m$. Additionally $x^{2m} = |x|^{2m}$, so the equality (2.9) defines x^{2m} for $m \in \mathbb{Z}$. By the same argument $x_+^a - x_-^a$ is well defined for $a = -2m-1$. Therefore it defines x^{-2m-1}.

Because we defined x^{-2m} and x^{-2m-1} in terms of x_+^a and x_-^a, equations (2.7) and (2.8) imply

$$\langle x^{-2m}, \varphi\rangle = \int_0^\infty x^{-2m} \left[\varphi(x) + \varphi(-x) - 2\sum_{i=0}^{m-1} \frac{\varphi^{(2i)}(0)}{(2i)!} x^{(2i)}\right] \mathrm{d}x, \tag{2.11}$$

$$\langle x^{-2m-1}, \varphi\rangle = \int_0^\infty x^{-2m-1} \left[\varphi(x) - \varphi(-x) - 2\sum_{i=0}^{m-1} \frac{\varphi^{(2i+1)}(0)}{(2i+1)!} x^{(2i+1)}\right] \mathrm{d}x. \tag{2.12}$$

Note that meromorphic continuations of x_+^a, x_-^a, $|x|^a$, $|x|^a\text{sign}(x)$ admit simple poles at negative integers. If we want to eliminate these poles we can divide each distribution by a function of a parameter a with simple poles at the same points. Let us present this idea in details for x_+^a. Fix the test function φ_0, then the complex function I_{φ_0}, given by

$$I_{\varphi_0}(a) := \frac{\langle x_+^a, \varphi\rangle}{\langle x_+^a, \varphi_0\rangle},$$

is analytic for any $\varphi \in C_0^\infty(\mathbb{R})$. Therefore, for fixed φ_0, $x_+^a/\langle x_+^a\varphi_0\rangle$ is a homogeneous distribution of degree a, with no poles. The question is: which test function, φ_0, is the most appropriate? We do not want to loose any data so $\langle x_+^a, \varphi_0\rangle$ has to have poles only at points where x_+^a has ones.

Because x_+^a has poles at negative integers, the test function, φ_0, and all its derivatives must be non-zero at the origin in order to eliminate all poles. The most common choice is $\varphi_0(x) = e^{-x}$. We must say that e^{-x} is not a $C_0^\infty(\mathbb{R})$ function, but it decay fast enough for $x \to \infty$ to make the integral:

$$\int_0^\infty x^a e^{-x}\mathrm{d}x \tag{2.13}$$

converging for $\text{Re}(a) > -1$. We can use a trick given by (2.7) in order to get a meromorphic continuation of a function given by (2.13). Note that this procedure defines the gamma function and the expression (2.13) gives $\Gamma(a+1)$ for $\text{Re}(a) > -1$. Because the gamma function has poles at non-positive integers, they cancel with the poles of x_+^a. Therefore the following expression makes sense on the whole complex plane

$$\frac{x_+^a}{\Gamma(a+1)}$$

and defines the homogeneous distribution of degree a. Because x_-^a has poles at the same points as x_+^a we regularise it by a division by $\Gamma(a+1)$.

Note that $|x|^a$ has poles at negative and odd integers, so the regularising function must be chosen so that its derivatives of even order fail to vanish at $x = 0$. Let us take $\varphi_0(x) = e^{-x^2}$, then

$$\langle |x|^a, e^{-x^2}\rangle = \int_{\mathbb{R}} |x|^a e^{-x^2}\,\mathrm{d}x = 2\int_0^\infty x^a e^{-x^2}\,\mathrm{d}x = 2\int_0^\infty x^{\frac{a}{2}-\frac{1}{2}} e^{-x}\,\mathrm{d}x = \Gamma\left(\frac{a+1}{2}\right).$$

By the same reasoning we take $\varphi_0(x) = xe^{-x^2}$ for regularisation of $|x|^a\text{sign}(x)$.

Summing up, the contraction which we described gives us the following entire (in a) distributions

$$\frac{x_+^a}{\Gamma(a+1)}, \quad \frac{x_-^a}{\Gamma(a+1)}, \quad \frac{|x|^a}{\Gamma\left(\frac{a+1}{2}\right)}, \quad \frac{|x|^a}{\Gamma\left(\frac{a+2}{2}\right)}.$$

The value of these distribution at a zero of the denominator is given by the ratio of

the corresponding residues, which can be worked out from (2.7):

$$\left.\frac{x_+^a}{\Gamma(a+1)}\right|_{a=-n} = \delta^{(n-1)}(x), \quad n=1,2,3,\ldots, \tag{2.14}$$

$$\left.\frac{x_-^a}{\Gamma(a+1)}\right|_{a=-n} = (-1)^{n-1}\delta^{(n-1)}(x), \quad n=1,2,3,\ldots, \tag{2.15}$$

$$\left.\frac{|x|^a}{\Gamma\left(\frac{a+1}{2}\right)}\right|_{a=-2n-1} = \frac{(-1)^n n!}{(2n)!}\delta^{(2n)}(x), \quad n=0,1,2,\ldots, \tag{2.16}$$

$$\left.\frac{|x|^a}{\Gamma\left(\frac{a+2}{2}\right)}\right|_{a=-2n} = \frac{(-1)^n (n-1)!}{(2n-1)!}\delta^{(2n-1)}(x), \quad n=1,2,3,\ldots \tag{2.17}$$

Let us define two new distributions, $(x\pm i0)^a$. Define the function $\mathbb{C}\backslash\mathbb{R}_- \ni z \mapsto z^a$ by $e^{a\log z}$ where $\log z$ is understood as a principal value. The distribution we want to introduce is given by

$$(x\pm i0)^a := \lim_{\varepsilon\to 0^+}(x\pm i\varepsilon)^a.$$

It is clear that for Re $a>0$ we have

$$(x\pm i0)^a = x_+^a + e^{\pm\pi i a}x_-^a. \tag{2.18}$$

Since $\langle (x\pm i\varepsilon)^a,\varphi\rangle$ is an entire analytic function for $\varepsilon>0$ its limit is so too. Therefore (2.18)) is valid for $a\in\mathbb{C}\backslash -\mathbb{N}$. For $a\in -\mathbb{N}$ we can compare singularities of the right hand side of (2.18) to conclude that

$$(x\pm i0)^{-n} = x^{-n} \pm \frac{\pi i(-1)^k}{(n-1)!}\delta^{(n-1)}(x).$$

It turns out that distributions described above are also related by their Fourier transforms. Let us start with $\frac{x_+^a}{\Gamma(a+1)}$ with Re $a>-1$ and consider the family of functions $x\mapsto e^{-\varepsilon x}\frac{x_+^a}{\Gamma(a+1)}$, where $\varepsilon\in(0,1]$. Since this family is in $L^1(\mathbb{R})$ the Fourier transform is given by

$$\frac{1}{\Gamma(a+1)(2\pi)^{1/2}}\int_0^\infty x^a e^{-x(\varepsilon+i\xi)}\,\mathrm{d}x = \frac{(\varepsilon+i\xi)^{-a-1}}{(2\pi)^{1/2}}\frac{\int_0^\infty x^a e^{-x}}{\Gamma(a+1)}.$$

Notice that the numerator on the right is $\Gamma(a+1)$ and $(\varepsilon+i\xi)^{-a-1} = e^{-i\pi(a+1)/2}(\xi - i\varepsilon)^{-a-1}$. Since the Fourier transform is continuous in the space of tempered distributions we have that for for any Schwartz function φ

$$\left\langle \frac{x_+^a}{\Gamma(a+1)},\hat{\varphi}\right\rangle = \frac{e^{-i\pi(a+1)/2}}{(2\pi)^{1/2}}\langle(\xi-i0)^{-a-1},\varphi\rangle,$$

for Re $a>-1$. Since both sides of the equality are entire analytic functions of a so the equality holds for $a\in\mathbb{C}$. The similar reasoning shows that

$\widehat{\frac{x_-^a}{\Gamma(a+1)}}(\xi) = e^{i\pi(a+1)/2}/(2\pi)^{1/2}(\xi - i0)^{-a-1}$. Summarising

$$\widehat{\frac{x_\pm^a}{\Gamma(a+1)}}(\xi) = \frac{e^{\mp i\pi(a+1)/2}}{(2\pi)^{1/2}}(\xi \mp i0)^{-a-1} \tag{2.19}$$

Let us introduce one more distribution, namely the average of $(\xi + i0)^a$ and $(\xi - i0)^a$, we will denote it by $\underline{x}^a$. Let $k \in \mathbb{N}$ then

$$\sqrt{\frac{2}{\pi}}\mathcal{F}^{-1}(\underline{x}^{-k})(\xi) = \left(e^{-i\pi k}\frac{\xi_-^{2k-1}}{\Gamma(2k)} + e^{i\pi k}\frac{\xi_+^{2k-1}}{\Gamma(2k)}\right) = \frac{(-1)^k}{(2k-1)!}|\xi|^{2k-1}.$$

2.3 Fourier Tauberian theorems

This section summarises results obtained in [42] that we widely use in last two chapters. Although this results are available on author's website (`http://www.mth.kcl.ac.uk/~ysafarov/Publications/index.html`) we decided to place them here in order to make this thesis more self contained, we also adapted them to Our settings.

Suppose that F is a non-decreasing function and ρ is a function that satisfies certain conditions. Fourier Tauberian theorems are results that estimate the difference $F - \rho * F$. They are broadly used in estimates of the local counting function of elliptic differential operators, in our case we will apply them to the Laplace operator. Among the others, this concept goes back to [24], [13], [43].

Suppose that ρ is a continues function. Let us list some conditions to which we will refer in this section:

1. $|\rho(t)| \leq C(1+t^2)^{-m-1}$, where $m > -1/2$,
2. $\int \rho(t)\mathrm{d}t = 1$,
3. ρ is even,
4. ρ is non-negative,
5. $\mathrm{supp}\hat{\rho} \in [-1, 1]$.

Although there exist a function ρ which satisfies all of these condition, we will not assume that ρ meets all of them.

Following [42] let us define some modifications of function ρ:

$$\rho_T(\tau) = T\rho(T\tau), \quad \rho_{1,0}(\tau) = \int_\tau^\infty t\rho(t)\,\mathrm{d}t, \quad \rho_{T,0} = T\rho_{1,0}(T\tau).$$

Now one have the following Fourier Tauberian theorem:

Theorem 2.3.1. *Let ρ be a function that satisfies conditions 1-4 for some $m > 0$. Assume that $\rho_{T,0}(t-s)F(s) \to 0$ as $s \to \pm\infty$ and $\rho_{T,0} * F'(t) < \infty$ for some $T > 0$ and $t \in \mathbb{R}$ then*

$$|F(t) - \rho_T * F(t)| \leq \frac{\rho_{T,0} * F'(t)}{2T\rho_{1,0}(0)}.$$

Note that for fixed function ρ the values of $\rho_{T,0} * F'$ and $\rho_T * F$ depend only on values of $\hat{F}$ on the interval $(-T, T)$. Moreover we do not need to know the full Fourier transform but it is enough to know its even part.

Lemma 2.3.2. *If the cosine Fourier transforms of the derivatives F' and F_0' coincide on an interval $(-T, T)$ then the Fourier transforms of the functions $F(t) - F(-t)$ and $F_0(t) - F_0(-t)$ coincide on the same interval.*

Let us state one of the most useful lemma in this section:

Lemma 2.3.3. *Let ρ be a function satisfying (3), (5) and (1) with $m > n/2$. If* $\mathrm{supp}F \subset (0, +)$ *and the cosine transform of $F'(t)$ coincides on the interval (T, T) with the cosine transform of the function nt_+^{n-1} then*

$$\begin{aligned}
\rho_T * F(t) &\geq \int [P_n^+(t, \mu T^{-1}) - \sigma_n T^{-n}|\mu|^n]\rho(\mu)\mathrm{d}\mu, \\
\rho_T * F(t) &\leq \int P_n^+(t, \mu T^{-1})\rho(\mu)\mathrm{d}\mu, \\
\rho_{T,0} * F'(t) &\leq T^2 \int [P_n^-(t, \mu T^{-1}) + \sigma_n T^{-n-1}|\mu|^{n+1}]\rho(\mu)\mathrm{d}\mu
\end{aligned}$$

for all $t > 0$, where

$$\begin{aligned}
\sigma_n &:= \begin{cases} 0, & \text{if } n \text{ is odd,} \\ 1, & \text{if } n \text{ is even,} \end{cases} \\
P_n^+(t, \mu) &:= 1/2[(t+\mu)^n + (t-\mu)^n], \\
P_n^+(t, \mu) &:= 1/2\mu[(t+\mu)^n (t-\mu)^n].
\end{aligned}$$

In application we will use a particular test function ρ. We will define it in the following way: for the Dirichlet Laplacian on the interval $[-1/2, 1/2]$ let ν_m be the $2m$-th root of the first eigenvalue and ζ_m be the corresponding normalised eigenfunction extended by 0 to the real line. We define ρ as the square of the Fourier transform of ζ_m. Then ρ satisfies 1-5, moreover

$$\rho_{1,0}(0) \geq \frac{\pi}{4}$$

If we use this test function in Lemma 2.3.3 then we will get the following:

Corollary 2.3.4. *Under the assumptions of Lemma 2.3.3 one has:*

$$\begin{aligned}
F(t) &\leq t^n + \frac{n(2\nu_{m_n}^2 + \pi\nu_{m_n})}{\pi T}\left(t + \frac{\nu_{m_n}}{T}\right)^{n-1}, \\
F(t) &\geq t^n - \frac{2n\nu_{m_n}^2}{\pi T}\left(t + \frac{\nu_{m_n}}{T}\right)^{n-1}.
\end{aligned}$$

2.4 Spectral theory of self-adjoint operators

In this section we review some standard theory of unbounded linear operators on a Hilbert space, H, in particular, criteria for self-adjointness and the spectral theorem for self-adjoint operators that are not necessarily compact. It is worth mentioning that despite the misleading terminology, the class of bounded operators is contained in the class of unbounded operators.

Definition 2.4.1. An *unbounded operator* $T\colon H \to H$ is a linear map from a linear dense subspace $\mathcal{D}(T) \subset H$ called the *domain* of T into H.

It is important that an unbounded operator is not necessarily defined on all of H, but is allowed to be defined on a linear subspace only. For example, if we consider the Laplace operator acting on $L^2(\mathbb{R}^n)$, it is not well defined on all of $L^2(\mathbb{R}^n)$. We can take its domain to be either $C_0^\infty(\mathbb{R}^n)$ or the Sobolev space $H^2(\mathbb{R}^n)$.

Definition 2.4.2. The *graph* of T is the set of pairs

$$\Gamma(T) = \{(v, Tv) : v \in \mathcal{D}(T)\}.$$

Obviously $\Gamma(T)$ is a subset of $H \times H$, which is a Hilbert space with inner product defined in the natural way:

$$\langle (v_1, v_2), (u_1, u_2) \rangle = \langle v_1, u_1 \rangle + \langle v_2, u_2 \rangle.$$

Definition 2.4.3. Let $T, S\colon H \to H$ be unbounded operators, then we say that $T \subset S$ if $\Gamma(T) \subset \Gamma(S)$. We call operator *closed* if its graph is closed in $H \times H$. We call an operator T *closable* if there is a closed operator S such that $T \subset S$.

It is not obvious that the closure of the graph of T is the graph of some operator. The following proposition shows that taking the graph closure is indeed a natural way to obtain a closed extension of an operator.

Proposition 2.4.4. *If T is closable, then $\Gamma(\overline{T}) = \overline{\Gamma(T)}$.*

Definition 2.4.5. Let T be a densely defined linear operator on a Hilbert space H. Let $\mathcal{D}(T^*)$ be the set given by:

$$\mathcal{D}(T^*) = \{v \in H : \exists c(v) \geq 0 \text{ such that } |\langle Tu, v \rangle| \leq c(v)\|u\|, u \in \mathcal{D}(T)\}.$$

Then the adjoint of T is the operator $T^*\colon H \to H$ is given by

$$\langle u, T^* v \rangle := \langle Tu, v \rangle$$

for $u \in \mathcal{D}(T)$ and $v \in \mathcal{D}(T^*)$.

Definition 2.4.6. Let $T\colon H \to H$ be a densely defined unbounded operator. We say that T is *symmetric* if $T \subset T^*$ and *self-adjoint* if $T = T^*$.

Example 2.4.7. Let $V \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$ and $T = -\Delta + V$ with domain $\mathcal{D}(T) = C_0^\infty(\mathbb{R}^n)$. Then integration by parts shows that T is symmetric and Theorem 2.1.10 state that its domain is dense in $L^2(\mathbb{R}^n)$.

Theorem 2.4.8. *The graph of the Hilbert adjoint of a densely defined operator is closed.*

For the proof we refer to [39] Theorem VIII.1.

Example 2.4.9. Let $T, T_1 \colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be densely defined linear operators with domains $\mathcal{D}(T) = C_0^\infty(\mathbb{R})$ and $\mathcal{D}(T_1) = C_0^1(\mathbb{R})$ defined by

$$(Tf)(x) = if'(x),$$
$$(T_1 f)(x) = if'(x).$$

Of course $T \subset T_1$; we will show that $\Gamma(T_1) \subset \overline{\Gamma(T)}$. Simple integration by parts shows that T is symmetric. Therefore, because of Theorem 2.4.8 and Definition 2.4.6, T is closable. Since $C_0^1(\mathbb{R}) \subset L^2(\mathbb{R})$ then by Theorem 2.1.10, we have that for every function $\varphi \in C_0^1(\mathbb{R})$ there is a sequence $\varphi_\varepsilon \in C_0^\infty(\mathbb{R})$ such that

$$\varphi_\varepsilon = j_\varepsilon * \varphi \overset{L^2(\mathbb{R})}{\longrightarrow} \varphi,$$

where j_ε is smooth compactly supported function normalised in the $L^1(\mathbb{R})$ norm. Since $\varphi' \in C_0(\mathbb{R}) \subset L^2(\mathbb{R})$ and

$$i\frac{\mathrm{d}}{\mathrm{d}x}\varphi_\varepsilon = j_\varepsilon * i\frac{\mathrm{d}}{\mathrm{d}x}\varphi \overset{L^2(\mathbb{R})}{\longrightarrow} i\frac{\mathrm{d}}{\mathrm{d}x}\varphi.$$

We obtain $\Gamma(T_1) \subset \overline{\Gamma(T)}$.

Theorem 2.4.10. *Suppose T is a densely defined operator on a Hilbert space H. Then*

(a) T is closable if and only if $\mathcal{D}(T^*)$ is dense,

(b) if T is closable, then $\overline{T} = T^{**}$.

Recall that if T is densely defined and $T \subset S$ then $S^* \subset T^*$. A simple conclusion of Theorem 2.4.10 is that every self-adjoint operator is closed. One can also show something more general: symmetric operators are closable.

Definition 2.4.11. A symmetric operator T is called *essentially self-adjoint* if its closure is self-adjoint.

Because our considerations in the rest of this thesis will be focused on self-adjoint operators, we present some criteria for self-adjointness.

Theorem 2.4.12. *Let $T \colon H \to H$ be symmetric. Then the following statements are equivalent:*

(a) T is self-adjoint,

(b) T *is closed,* $\mathrm{Ker}(T^* + i) = \{0\}$ *and* $\mathrm{Ker}(T^* - i) = \{0\}$,

(c) $\mathrm{Im}(T \pm i) = H$ *for both signs,*

where $\mathrm{Ker}(T) = \{v \in \mathcal{D}(T) : Tv = 0\}$ *and* $\mathrm{Im}(T) = \{Tv : v \in \mathcal{D}(T)\}$.

Corollary 2.4.13. *Let* T *be a symmetric operator on a Hilbert space* H*. Then the following are equivalent:*

(a) T *is essentially self-adjoint,*

(b) $\mathrm{Ker}(T^* + i) = \{0\}$ *and* $\mathrm{Ker}(T^* - i) = \{0\}$,

(c) $\mathrm{Im}(T \pm i)$ *is dense for both signs.*

Example 2.4.14. $T = i\frac{\mathrm{d}}{\mathrm{d}x} \colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a densely defined linear operator with domain $\mathcal{D}(T) = H_0^1((0,1)) = \{f \in H^1((0,1)) : f(0) = f(1) = 0\}$. Integration by parts shows that T is symmetric. Moreover, T is also closed but not essentially self-adjoint. $T^* = i\,\mathrm{d}/\mathrm{d}x$ on $H^1((0,1))$, see [39], chapter VIII.

Theorem 2.4.15. *If a linear operator* $T \colon H \to H$ *is closed with dense domain in a Hilbert space* H*, then the operator*

$$B = (I + T^*T)^{-1}$$

is defined everywhere, bounded, symmetric and positive.

Theorem 2.4.16. *Suppose* T *is densely defined self-adjoint operator. Then* $(T^{-1})^* = T^{-1}$ *wherever* T^{-1} *exists, and for every real constant* c *we have* $(T + cI)^* = (T + cI)$.

Corollary 2.4.17. *Suppose* $T \colon H \to H$ *is a densely defined closed operator in* H*. Then* T^*T *with domain* $\mathcal{D}(T^*T) = \{f \in \mathcal{D}(T) : Tf \in \mathcal{D}(T^*)\}$ *is self-adjoint.*

Example 2.4.18. Since $T = i\frac{\mathrm{d}}{\mathrm{d}x}$ with domain $\mathcal{D}(T) = H_0^1((0,1))$ satisfies these conditions the operator $-\frac{\mathrm{d}^2}{\mathrm{d}x^2} \colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$ with the domain $\mathcal{D}(\frac{\mathrm{d}^2}{\mathrm{d}x^2}) = \{f : f \in H_0^1((0,1)) \wedge if' \in H^1((0,1))\}$ is self-adjoint. Since the condition $if' \in H^1((0,1))$ is equivalent to $f \in H^2((0,1))$, the domain of $-\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ is $\mathcal{D}(-\Delta) = H_0^1((0,1)) \cap H^2((0,1))$.

One of the most important notions in functional analysis is that of spectrum. It is a generalisation of the concept of eigenvalues for matrices, and has numerous applications in quantum mechanics. For instance, in quantum mechanics the energy operators, Hamiltonians, are self-adjoint unbounded operators on a Hilbert space. Their eigenvalues correspond to the energy levels of bound-states of the system.

Definition 2.4.19. Let V be a Banach space, $T \in \mathcal{L}(V) = \mathcal{L}(V, V)$ and I denote the identity operator on V. The complex number λ is in the *resolvent set* of T, denoted $\rho(T)$, if the inverse of $\lambda I - T$ exists and is a bounded linear operator defined on a dense subspace of V. The operator $R_T(\lambda) = (\lambda I - T)^{-1}$ is called the *resolvent* of T at λ. If $\lambda \notin \rho(T)$, then λ is said to be in *spectrum* of T, denoted $\sigma(T)$.

Definition 2.4.20. A non-zero vector, v, in a Banach space V which satisfies $Tv = \lambda v$ for some $\lambda \in \mathbb{C}$ is called *eigenvector* of T; λ is called the corresponding *eigenvalue*. If λ is an eigenvalue, then λ is also in the spectrum of T. The set of eigenvalues is called the *point spectrum* or *discrete spectrum* of T. For every λ in the point spectrum we can define the λ-eigenspace as follows: it is the linear subspace generated by all eigenvectors that correspond to the eigenvalue λ. The dimension of λ-eigenspace is called the *multiplicity* of the eigenvalue λ.

For an operator with purely discrete spectrum it is convenient to define a new function which describes its spectrum and multiplicities of eigenvalues are finite.

Definition 2.4.21. Let $T\colon H \to H$ be a densely defined self-adjoint operator with purely discrete spectrum. Suppose that each eigenvalue is of a finite multiplicity. Assume that there is a constant $M > 0$ such that $\langle Tu, u\rangle \geq -M\|u\|^2$ for every $u \in \mathcal{D}(T)$. Then

$$\mathrm{N}(\lambda) := \#\{\lambda_i\colon \lambda_i < \lambda\} = \sum_{i=1}^{\infty} 1_{(\lambda_i,\infty)}(\lambda) = \sum_{i=1}^{\infty} 1_{[-M,\lambda)}(\lambda_i),$$

is the *counting function* of T, where the λ_i are the eigenvalues of T, listed with multiplicity and 1_A is a characteristic function of the set A.

The following theorem gives a class of self-adjoint operators on any measure space.

Theorem 2.4.22. *Let $(M, \mathcal{M}, \mu)$ be a measure space with a σ-finite measure μ. Suppose that f is a measurable real-valued function on M. Then the operator $T_f\colon L^2(M,\mu) \to L^2(M,\mu)$ given by*

$$T_f\varphi(x) := f(x)\varphi(x),$$

with domain $\mathcal{D}(T_f) = \{\varphi \in L^2(M,\mu) : f\varphi \in L^2(M,\mu)\}$ is self-adjoint and $\sigma(T_f)$ is the essential range of f.

The next theorems show that the class of multiplication operators is indeed the only class of self-adjoint operators up to unitary maps.

Theorem 2.4.23 (Spectral theorem-operator form)**.** *Let T be a self-adjoint operator on a separable Hilbert space H. Then there are a measure space $(M, \mathcal{M}, \mu)$, where μ is a σ-finite measure, a measurable function $t\colon M \to \mathbb{R}$ and unitary operator $U\colon H \to L^2(M,\mu)$ such that:*

(a) a vector $v \in H$ belongs to $\mathcal{D}(T)$ if and only if $t \cdot U(v) \in L^2(M,\mu)$,

(b) If $v \in \mathcal{D}(T)$ then $U(Tv) = t \cdot U(v)$.

Let us denote the set of bounded Borel functions $h\colon \mathbb{R} \to \mathbb{C}$ by $\mathcal{B}_b(\mathbb{R})$. By the theorem above there is natural way to define functions of self-adjoint operators as follows. For a given function $h \in \mathcal{B}_b(\mathbb{R})$ we define

$$h(T) = U^{-1}(h \circ t)U.$$

Theorem 2.4.24 (Spectral theorem-functional calculus form)**.** *Let H be a separable Hilbert space, and let T be a self-adjoint operator on H. Then there exists a unique map:*

$$\mathcal{B}_b(\mathbb{R}) \ni h \mapsto h(T) \in \mathcal{L}(H),$$

with properties (a)-(d):

(a) $h(T)^* = \overline{h}(T)$, $h_1(T) + h_2(T) = (h_1 + h_2)(T)$ *and* $h_1(T) \circ h_2(T) = (h_1 \circ h_2)(T)$,

(b) $\|h(T)\| \leq \|h\|_\infty$

(c) if $h_n \in \mathcal{B}_b$, $\lim_{n\to\infty} h_n(x) = x$ *for each* x *and* $|h_n(x)| \leq |x|$ *for all* x *and* n. *Then* $\lim_{n\to\infty} h_n(T)v = Tv$ *for every* $v \in \mathcal{D}(T)$,

(d) if $h, h_n \in \mathcal{B}_b$, $\lim_{n\to\infty} h_n(x) = h(x)$ *point-wise and the sequence* $\|h_n\|_{L^\infty}$ *is bounded, then* $\lim_{n\to\infty} h_n(T)(v) = h(T)(v)$ *for every* $v \in \mathcal{D}(T)$.

In addition

(e) if $v \in \mathcal{D}(T)$ *and* $Tv = \lambda v$, *then* $h(T)v = h(\lambda)v$,

(f) if $h \geq 0$ *then* $\langle h(T)v, v\rangle \geq 0$ *for every* $v \in \mathcal{D}(T)$.

Because the class of bounded operators is contained in the class of unbounded operators, the functional calculus is valid also for bounded operators.

Example 2.4.25. The functional calculus is very useful. For example it allow us to define the bounded operator $\cos(\sqrt{-\Delta}t)\colon L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, which is a solution operator of the wave equation:

$$u_{tt} = \Delta u, \quad u(\cdot, 0) = f, \quad u_t(\cdot, 0) = 0. \tag{2.20}$$

By this we mean that the solution u is given by

$$u(x,t) = \left[\cos(\sqrt{-\Delta}t)f\right](x).$$

This can be seen as follows. The action of $\cos(\sqrt{-\Delta}t)$ is given by

$$\left[\cos(\sqrt{-\Delta}t)f\right](x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \cos(|\xi|t)e^{i\langle\xi, x-y\rangle} f(y)\,\mathrm{d}y\,\mathrm{d}\xi,$$

which clearly shows that $u(x,0) = f(x)$, because in this case the solution operator is simply $\mathcal{F}^{-1} \circ \mathcal{F}$. Differentiation under the integral sign gives:

$$\begin{aligned}
u_t(x,0) &= -\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |\xi|\sin(|\xi|t)e^{i\langle\xi, x-y\rangle} f(y)\,\mathrm{d}y\,\mathrm{d}\xi\Big|_{t=0} = 0,\\
u_{tt}(x,t) &= -\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |\xi|^2\cos(|\xi|t)e^{i\langle\xi, x-y\rangle} f(y)\,\mathrm{d}y\,\mathrm{d}\xi,\\
\Delta u(x,t) &= -\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |\xi|^2\cos(|\xi|t)e^{i\langle\xi, x-y\rangle} f(y)\,\mathrm{d}y\,\mathrm{d}\xi.
\end{aligned}$$

Function calculus has one more important advantage. Namely it helps us to state the spectral theorem in projection valued measure form. Let P_A be the operator $1_A(T)$, where 1_A is the characteristic function of the measurable set $A \subset \mathbb{R}$.

Definition 2.4.26. A sequence of bounded operators $A_n \colon V \to W$ converges to $A \colon V \to W$ strongly if and only if

$$\|A_n v - Av\| \to 0$$

for all $v \in V$. If A_n converges to A strongly we call A a strong limit of the sequence A_n and we denote it by: $A = \mathrm{s.lim} A_n$.

Then one can show that

(a) P_A is an orthogonal projection for any A,

(b) $P_\emptyset = 0$ and $P_{\mathbb{R}} = \mathrm{Id}$,

(c) $A = \bigcup_{n=1}^{\infty} A_n$, and $A_n \cap A_m = \emptyset$ for $m \neq m$ then $P_A = \mathrm{s.lim}_{N\to\infty} \sum_{n=1}^{N} P_{A_n}$,

(d) $P_{A_1} P_{A_2} = P_{A_1 \cap A_2}$.

The family of a such operators is called a projection-valued measure. Note that for any $\varphi \in H$, $\langle \varphi, P_A \varphi \rangle$ is a well defined Borel measure on $\mathbb{R}$, which we denote by $\mathrm{d}\langle \varphi, P_A \varphi \rangle$. Note that complex-valued measure may be obtained by the polarisation. If $A = (-\infty, \lambda)$, then we denote P_A by P_λ. Therefore the spectral theorem may be written in the following form:

Theorem 2.4.27. *There is a one to one correspondence between self-adjoint operator T and projection-valued measure P_λ on a Hilbert space H, given by*

$$\langle \phi, T\varphi \rangle = \int_{\mathbb{R}} \lambda \, \mathrm{d}\langle \phi, P_\lambda \varphi \rangle,$$

which, for simplicity, we denote by

$$T = \int_{\mathbb{R}} \lambda \, \mathrm{d}P_\lambda.$$

If f is a real-valued Borel function on $\mathbb{R}$, then the operator defined by $f(T) = \int f(\lambda) \mathrm{d}P_\lambda$, with a domain $D = \{\varphi \in H, \int |f(\lambda)|^2 \mathrm{d}\langle \varphi, P_\lambda \varphi \rangle \leq \infty\}$ is self-adjoint on H.

2.5 Spectrum of the Dirichlet Laplacian

As an example of unbounded operator with discrete spectrum we have chosen the Dirichlet Laplacian. Throughout this section we are going to define the Dirichlet Laplace operator and we will prove that it has a discrete spectrum. To achieve it we will firstly show that semi-bounded operators have a discrete spectrum if they have compact resolvent.

Secondly we will present a Rellich's theorem which gives us criterion for sets of being compact in $L^2(\mathbb{R}^n)$. Together, these facts give us desired results.

This section was based on the book of Michael Reed and Barry Simon [40]. We picked only few theorems that are necessary to show that the Dirichlet Laplacian has discrete spectrum but this book covers much more topics, for example it also discuss different Hamilton operators, like Neumann Laplacian or the Laplace operator with some potential.

Let U be an open and bounded subset of $\mathbb{R}^n$ with smooth boundary.

Let us consider the Laplace operator, $\Delta\colon L^2(U) \to L^2(U)$, given by

$$\Delta f(x) := -\sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}(x), \tag{2.21}$$

with the domain $\mathcal{D}(\Delta) = C_0^\infty(U)$. Since U has smooth boundary we can use the divergence theorem to the vectorfield $g(x)\nabla f(x)$ to get

$$-\sum_{i=1}^{n} \int_U \frac{\partial^2 \overline{f}}{\partial x_i^2}(x) g(x)\,\mathrm{d}x = -\int_{\partial U} \nabla \overline{f}(x) \cdot \vec{n} g(x)\,\mathrm{d}x + \int_U \nabla \overline{f}(x) \cdot \nabla g(x)\,\mathrm{d}x$$

where $\vec{n}$ is an outward normal vector to ∂U. Since $g|_{\partial U} = 0$ we have

$$\langle \Delta f, g\rangle = \int_U \nabla \overline{f}(x) \cdot \nabla g(x)\,\mathrm{d}x = \langle \nabla f, \nabla g\rangle. \tag{2.22}$$

Similarly $\langle f, \Delta g\rangle = \langle \nabla f, \nabla g\rangle$ which shows that the Laplacian is symmetric. Equations (2.22) shows that the Laplace operator is positive since

$$\langle \Delta f, f\rangle = \langle \nabla f, \nabla f\rangle = \sum_{i=1}^{n} \left\| \frac{\partial f}{\partial x_i^2} \right\|^2 \geq 0.$$

Definition 2.5.1. A *quadratic form* is a map $q\colon Q(q) \times Q(q) \to \mathbb{C}$, where $Q(q)$ is a linear subspace of $\mathcal{H}$ called the *form domain*, such that $q(\,\cdot\,, g)$ is conjugate linear and $q(f, \,\cdot\,)$ is linear for $f, g \in Q(q)$. If $q(f,g) = \overline{q(g,f)}$ we say that q is symmetric. If $q(f,f) \geq -c\|f\|^2$ for all $f \in Q(q)$, g is called *semi-bounded* or *bounded from below* by $-c$, in particular if $q(f,f) \geq 0$ for all $f \in Q(q)$ we say that q is *positive*.

Of course each densely defined operator, A, with the domain $\mathcal{D}(A)$ defines the quadratic form q_A with formal domain $Q(q) = \mathcal{D}(A)$ by

$$q_A(f,g) = \langle f, Ag\rangle, \quad f, g \in \mathcal{D}(A). \tag{2.23}$$

Additionally if A is symmetric then q_A is so, if A is positive then q_A is also positive, therefore the Laplace operator defines symmetric and positive quadratic form $q_\Delta\colon C_0^\infty(U) \times C_0^\infty(U) \to \mathbb{C}$ defined by

$$q_\Delta(f,g) = \langle \nabla f, \nabla g\rangle.$$

It is also worth to mention that the symmetric and bounded from below by $-c$ quadratic form defines a new inner product on $Q(q) \times Q(q)$ given by

$$\langle f, g\rangle_{+1} := q(f, g) + (c + 1)\langle f, g\rangle. \tag{2.24}$$

The inner product $\langle\cdot,\cdot\rangle_{+1}$ induces a norm $\|\cdot\|_{+1}$, which induces a metric $d_{+1}(f, g) = \|f - g\|_{+1}$. We obtain an inner product space $(Q(q), \langle\cdot,\cdot\rangle_{+1})$ which is not necessarily a Banach space, but we can always complete it by addition of certain elements. The completion of$(Q(q), d_{+1})$ we will denote by $\mathcal{H}_{+1}$.

It is also worth to mention that each element of $\mathcal{H}_{+1}$ is an element of $\mathcal{H}$ because $Q(q) \subset \mathcal{H}$ and $\|\cdot\| \leq \|\cdot\|_{+1}$. With this construction $(\mathcal{H}_{+1}, \langle\cdot,\cdot\rangle_{+1})$ becomes a Hilbert space and

$$Q(q) \subset \mathcal{H}_{+1} \subset \mathcal{H}.$$

Definition 2.5.2. Let q be a semi-bounded quadratic form, $q(f, f) \geq -c\|f\|^2$. Then q is called *closed* if $Q(q)$ is complete under the norm

$$\|f\|_{+1} = (q(f, f) + (c + 1)\langle f, f\rangle)^{\frac{1}{2}}.$$

Back to the Laplace operator, the above considerations hold in the case of quadratic form, q_Δ, associated with the Laplacian. Let us denote by $\overline{q_\Delta}$ the closed extension of q_Δ and by $H_0^1(U)$ the completion of $C_0^\infty(U)$ with respect to norm $\|\cdot\|_{+1} = (\|\cdot\|^2 + \|\nabla\cdot\|^2)^{1/2}$. We will call $H_0^1(U)$ the Sobolev space and to emphasise which norm acts on this space we will write $\|\cdot\|_{H^1}$ rather than $\|\cdot\|_{+1}$.

The Riesz lemma implies that the bounded quadratic form defines the bounded operator, for semi-bounded quadratic forms the situation is different which shows the theorem due to Friedrich.

Theorem 2.5.3 (Friedrich)**.** *Suppose that A be a semi-bounded symmetric operator. Let $q(\varphi, \psi) = \langle\varphi, A\phi\rangle$ for $\varphi, \psi \in D(A)$. Then q is a closable quadratic form and its closure $\tilde{q}$ is the quadratic form of a unique self-adjoint operator $\tilde{A}$. $\tilde{A}$ is a semi-bounded extension of A and the lower bound of its spectrum is the lower bound of q. Additionally, $\tilde{A}$ is the only self-adjoint extension of A whose domain is contained in the form domain of $\tilde{q}$.*

The self-adjoint extension of a semi-bounded symmetric operator given by Friedrich's theorem is called Friedrich's extension.

If q is a quadratic form of a self-adjoint operator A then we will write $Q(q) = Q(A)$ and we will call it by form domain of A and for $f, g \in Q(A)$ we will often write $q_A(f, g) = \langle f, Ag\rangle$ although Ag does not make sense for all $g \in Q(A)$.
Since we showed that $q_\Delta \colon C_0^\infty(U) \times C_0^\infty(U) \to \mathbb{C}$ has closed extension under the norm $\|\cdot\|_{H^1}$, the theorem above enables us to define a self-adjoint extension of the Laplacian on $L^2(U)$.

Definition 2.5.4. Let U be an open and bounded subset of $\mathbb{R}^n$. Then the Friedrich's extension of Δ on $C_0^\infty(U)$ is called the Dirichlet Laplacian and we denote it by Δ_U.

We will state some estimates of the norm of the resolvent. This information will be used in the proof of the next theorem.

Lemma 2.5.5. *Let A be a self-adjoint and positive operator. Then for $\lambda \notin [0, \infty)$*

$$\|R_\lambda(A)\| \leq \begin{cases} \dfrac{1}{|\mathrm{Im}(\lambda)|} & \mathrm{Re}(\lambda) \geq 0 \\ \dfrac{1}{|\lambda|} & \mathrm{Re}(\lambda) \leq 0 \end{cases}$$

Proof. For $\lambda = x + iy$ off the real line and $f \in \mathcal{D}(A)$ we have

$$\|(\lambda - A)f\|^2 = \|(x - A)f\|^2 + y^2\|f\|^2 \geq y^2\|f\|^2 = (\mathrm{Im}(\lambda))^2\|f\|^2.$$

Since by definition the resolvent is a bijection of $\mathcal{H}$ onto $\mathcal{D}(A)$ let us consider the element $g = (\lambda - A)f$ with $f \in \mathcal{D}(A)$. Then

$$\|g\| = \|(\lambda - A)f\| \geq |y|\|f\| = |y|\|(\lambda - A)^{-1}(\lambda - A)f\| = |y|\|(\lambda - A)^{-1}g\|$$

which gives

$$\|(\lambda - A)^{-1}\| \leq \frac{1}{|\mathrm{Im}(\lambda)|}.$$

For $\mathrm{Re}(\lambda) \leq 0$ we have

$$\|(\lambda - A)f\|^2 = \|Af\|^2 + 2|\mathrm{Re}(\lambda)|\langle f, Af\rangle + |\lambda|^2\|f\|^2 \geq |\lambda|^2\|f\|^2.$$

thus by the same argument as before

$$\|(\lambda - A)^{-1}\| \leq \frac{1}{|\lambda|}.$$

□

Lemma 2.5.6. *Let A be a self-adjoint and positive operator. Then for $\lambda \notin [0, \infty)$*

$$\|A(\lambda - A)^{-1}\| \leq 2$$

for $\mathrm{Re}(\lambda) \leq 0$.

Proof. Suppose that $f \in \mathcal{H}$ then

$$\|A(\lambda - A)^{-1}f\| = \|1 - (\lambda(\lambda - A)^{-1})f\| \leq \|1 - \lambda(\lambda - A)^{-1}\|\,\|f\| \leq (\|\lambda(\lambda - A)^{-1}\| + \|1\|)\|f\|$$

By the previous lemma

$$\|\lambda(\lambda - A)^{-1}\| \leq 1$$

Thus $\|A(\lambda - A)^{-1}\| \leq 2$. □

Although we know that this estimate is not optimal it is enough for our purposes.

Theorem 2.5.7. *Let A be a self-adjoint operator that is bounded from below. Then if the set*

$$F_b = \{f \in Q(A) : \|f\| \leq 1, \langle f, Af\rangle \leq b\}$$

is compact for all b then the resolvent of A, $R_\lambda(A)$, is compact for every $\lambda \in \rho(A)$.

Proof. Without loss of generality, we may assume that A is positive. Let $Im = \{(A+1)^{-1}f : \|f\| \leq 1\}$. Suppose that $g \in Im$ then there is an element $f \in \mathcal{H}$ such that $f = (A+1)g$ and $\|f\| \leq 1$. Then

$$\|g\| = \|(A+1)^{-1}f\| \leq \|(A+1)^{-1}\|\|f\| \leq \|f\| \leq 1$$

and

$$\langle g, Ag\rangle = \langle (A+1)^{-1}f, A(A+1)^{-1}f\rangle \leq \|(A+1)^{-1}\|\|f\|^2\|A(A+1)^{-1}\| \leq 2\|f\|^2 \leq 2.$$

Since the resolvent transforms $\mathcal{H}$ onto $\mathcal{D}(A)$ then $Im \subset F_2$. Since F_2 is closed $\overline{Im} \subset F_2$ so the image of a unit ball under the resolvent,$R_{-1}(A)$ is pre-compact. By a linearity of $R_{-1}(A)$ image of an arbitrary ball, $R_{-1}(A)(B_r)$ is pre-compact for $r < \infty$. Since every bounded set is contained in some ball of a finite radius the resolvent, $(A+1)^{-1}$, is compact. By the first resolvent formula

$$R_\lambda(A) = (-1-\lambda)R_\lambda(A)R_{-1}(A) + R_{-1}(A)$$

for every λ in the resolvent set. Since the resolvent is bounded and a class of compact operators is an ideal in $B(\mathcal{H})$ then $R_\lambda(A)$ is compact for every $\lambda \in \rho(A)$. □

Lemma 2.5.8. *Let $p < \infty$. Suppose $g \in L^p(\mathbb{R}^n)$ then*

$$\forall_{\varepsilon>0}\ \exists_{K-bounded} :\ \|g - \chi_K g\|_{L^p} < \varepsilon$$

where χ_K denotes the characteristic function of the set K.

Proof. This is a direct consequence of the density of C_0^∞ in L^p. □

Lemma 2.5.9. *Let $p \in [1, \infty)$. Then*

$$\forall_{g\in L^p(\mathbb{R}^n)}\ \forall_{\varepsilon>0}\ \exists_{\delta>0} : \|y\| < \delta \Rightarrow \|g - g(\cdot - y)\|_{L^p} < \varepsilon.$$

Proof. Suppose $g \in L^p$, since C_0^∞ space is dense in L^p then for each g in L^p there is a compact set K and a function $h \in C_0^\infty(K)$ such that $\|g - h\|_{L^p} \leq \varepsilon/3$. Let $K' = \{y : \text{dist}(K, y) < \delta\}$. Since h is continuous on a compact set it is uniformly continuous, so there is $\delta > 0$ such that for $\|y\| < \delta$

$$\|h - h(\cdot - y)\|_\infty < \frac{\varepsilon}{3\mu(K')^{\frac{1}{p}}}$$

where μ is the Lebesgue measure. By Jensen's inequality we have

$$\|h - h(\cdot - y)\|_{L^p} \leq \mu(K')^{\frac{1}{p}}\|h - h(\cdot - y)\|_\infty < \frac{\varepsilon}{3}.$$

Since $\|g\|_{L^p} = \|g(\cdot - y)\|_{L^p}$ then $\|g(\cdot - y) - h(\cdot - y)\|_{L^p} = \|g - h\|_{L^p}$. Thus we have that for $\|y\| < \delta$ and $g \in L^p(\mathbb{R}^n)$ we have

$$\|g(\cdot - y) - g\|_{L^p} \leq \|g(\cdot - y) - h(\cdot - y)\|_{L^p} + \|h(\cdot - y) - h\|_{L^p} + \|h - g\|_{L^p} < \varepsilon.$$

□

The following theorem gives us powerful tool to check if the subset of L^p space is compact.

Theorem 2.5.10 (Riesz criterion)**.** *Let $p < \infty$. Let $S \subset B(1)$, where $B(1)$ is a unit ball in $L^p(\mathbb{R}^n)$. The closure of S is compact in $L^p(\mathbb{R}^n)$ if and only if two following conditions hold*

1. *$f \to 0$ in L^p at infinity uniformly, i.e.*

$$\forall_{\varepsilon>0}\ \exists_{K\subset\mathbb{R}^n \text{ bounded}}\ \forall_{f\in S}\ \int_{\mathbb{R}^n\setminus K} |f(x)|^p \,\mathrm{d}x \leq \varepsilon^p, \tag{2.25}$$

2. *$f(\cdot - y) - f \to 0$ uniformly in S as $y \to 0$, i.e.*

$$\forall_{\varepsilon>0}\ \exists_{\delta>0}\ \forall f \in S\ \|y\| < \delta \Rightarrow \int_{\mathbb{R}^n} |f(x - y) - f(x)|^p \,\mathrm{d}x \leq \varepsilon^p. \tag{2.26}$$

Proof. In order to show that conditions 1. and 2. hold when $\overline{S}$ is compact we use a fact that $\overline{S}$ can be covered by finitely many balls of the same radius and a fact that the shift $L^p(\mathbb{R}^n) \ni g \mapsto g(\cdot - y)$ is a continuous map for $y \in \mathbb{R}^n$.

For a proof that 1. and 2. imply compactness of $\overline{S}$ one can use Ascoli's theorem, for details we refer to the book of Reed and Simon [40], Theorem XIII.66. □

Lemma 2.5.11. *Suppose that $\mathcal{H}$ is separable Hilbert space. Let $A\colon \mathcal{H} \to \mathcal{H}$ be a self-adjoint operator that is bounded from below. Then the set*

$$F_b = \{f \in Q(A) : \|f\| \leq 1, \langle f, Af\rangle \leq b\}$$

is closed in $\mathcal{H}$.

Proof. Let us take a sequence $\{f_n\}$ in F_b such that $f_n \to f$ in $\mathcal{H}$. In order to prove that $f \in F_b$ it is sufficient to show that $\langle f, Af\rangle \leq b$ and $\|f\| \leq 1$. Because $f = f - f_n + f_n$ the triangle inequality implies that $\|f\| \leq 1$. Note that semi-boundedness and definition of F_b imply that the operator A is bounded on F_b. Therefore

$$\langle f, Af\rangle = \int \lim_{n\to\infty} f_n(x) A f_n(x) \,\mathrm{d}x.$$

Fatou's lemma gives that

$$\int \lim_{n\to\infty} f_n(x) A f_n(x) \,\mathrm{d}x \leq \liminf_{n\to\infty} \int f_n(x) A f_n(x) \,\mathrm{d}x \leq b.$$

Summing up, $\langle f, Af\rangle \leq b$. □

Definition 2.5.12. Let F be a measurable function on $\mathbb{R}^n$ that is non-negative almost everywhere. We say that $F \overset{x\to\infty}{\longrightarrow} \infty$ if and only if

$$\forall_{N>0}\ \exists_{R_N>0}\ F(x) \geq N$$

for almost all x with $\|x\| \geq R_N$.

Corollary 2.5.13 (Rellich's criterion)**.** *Let F and G be two functions on $\mathbb{R}^n$ so that $F \overset{x\to\infty}{\longrightarrow} \infty$, $G \overset{x\to\infty}{\longrightarrow} \infty$. Then*

$$S = \left\{ f \in L^2(\mathbb{R}^n) : \|f\| \leq 1, \int G(x)|f(x)|^2 \mathrm{d}x \leq 1, \int F(\xi)|\widehat{f}(\xi)|^2 \mathrm{d}\xi \leq 1 \right\}$$

is a compact set of $L^2(\mathbb{R}^n)$.

Proof. See [40], Theorem XIII.65. □

Theorem 2.5.14. *Let U be a bounded open set in $\mathbb{R}^n$. The Dirichlet Laplacian Δ_U has compact resolvent.*

Proof. By the Theorem 2.5.7 it is enough to show that the set

$$F_b = \{f \in Q(\Delta_U) : \|f\|_{L^2} \leq 1, \langle f, \Delta_U f\rangle_{L^2} \leq b\}$$

is compact in $L^2(U)$.
Let F be any function that is 1 on U and $F \overset{x\to\infty}{\longrightarrow} \infty$. Then, the set

$$\mathbb{F}_b = \left\{ f \in L^2(\mathbb{R}^n) : \|f\|_{L^2} \leq 1, \langle f, \Delta f\rangle_{L^2} \leq b, \langle f, Ff\rangle \leq 1 \right\}$$

is compact in $L^2(\mathbb{R}^n)$ by Rellich's criterion; Δ is the operator with the domain $C_0^\infty(\mathbb{R}^n)$ defined by formula (2.21), and $Q(\Delta)$ is the form domain of a self-adjoint extension of Δ. Since $C_0^\infty(U) \subset C_0^\infty(\mathbb{R}^n)$, $Q(\Delta_U)$ and $Q(\Delta)$ are closures of $C_0^\infty(U)$, $C_0^\infty(\mathbb{R}^n)$ respectively, with respect to the norm $\|\cdot\|_{H^1}$, then $Q(\Delta_U) \subset Q(\Delta)$ and $\langle f, \Delta_U f\rangle = \langle f, \Delta f\rangle$ for $f \in Q(\Delta_U)$. Recall that for every $f \in Q(\Delta_U)$ we have $\langle f, Ff\rangle = \langle f, f\rangle$ thus $F_b \subset \mathbb{F}_b$.
F_b is closed in $L^2(U)$ by Lemma 2.5.9 and so is closed in $L^2(\mathbb{R}^n)$. Because it is a closed subset of a compact set it is compact in $L^2(\mathbb{R}^n)$. Since every cover of F_b in $L^2(U)$ is a cover in $L^2(\mathbb{R}^n)$, compactness of F_b in $L^2(\mathbb{R}^n)$ implies compactness of F_b in $L^2(U)$, which ends proof. □

Theorem 2.5.15. *Let A be a self-adjoint operator that is bounded from below. Then if the resolvent of A is compact then there is a complete orthonormal basis $\{f_n\}_{n=1}^\infty$ in $\mathcal{D}(A)$ so that $Af_n = \mu_n$ with $\mu_1 \leq \mu_2 \leq \ldots$ and $\mu_n \to \infty$.*

Proof. Suppose that $\langle f, Af\rangle < -a$, then $(a + A)^{-1}$ is a compact operator, therefore by the Spectral Theorem for bounded operators there is an orthonormal basis $\{f_n\}$ with $(a + A)^{-1} f_n = \lambda_n f_n$ and $\lambda_n \to 0$. Therefore

$$Af_n = (\lambda_n^{-1} - a) f_n.$$

Since $(A+a)^{-1}$ is a positive operator, then $\lambda_n \geq 0$, and we can arrange the λ's in descending order. Taking $\mu_n = \lambda_n^{-1} - a$ we obtain a non-decreasing sequence of eigenvalues of A and f_n are eigenfunctions of A. The fact that the spectrum of A is discrete is a result of the fact that if $\lambda \in \rho(R_{-a})$ and $\lambda \neq 0$ then $-a - \lambda^{-1} \in \rho(A)$. $\square$

Chapter 3

Hyperbolic geometry

3.1 Introduction

Let us denote by $\mathbb{H}^n$ $(n \geq 2)$ a complete simply connected, n-dimensional Riemannian manifold having constant sectional curvature equal to -1. We will call it hyperbolic space. There are several models of $\mathbb{H}^n$ but we will deal with two coordinate systems. First is the upper half space model i.e. $\{x \in \mathbb{R}^n : x_n > 0\}$ with the metric

$$\mathsf{g} = x_n^{-2}(\mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \ldots + \mathrm{d}x_n^2).$$

The Laplacian in this coordinate system is given by

$$\Delta \quad = \quad -x_n^2(\partial^2/\partial x_1^2 + \ldots + \partial^2/\partial x_n^2) + (n-2)x_n\partial/\partial x_n.$$

The Riemannian measure on $\mathbb{H}^n$ is expressed in terms of the Lebesgue measure simply by

$$\mathrm{d}\mu(x) = x_n^{-n}\mathrm{d}x_1\mathrm{d}x_2\ldots\mathrm{d}x_n.$$

Suppose that functions $f,\ g$ are such that $|fg|$ is integrable on $\mathbb{H}^n$, we define the inner product by

$$\langle f, g\rangle = \int_{\mathbb{H}^n} f(x)\overline{g(x)}\,\mathrm{d}\mu(x).$$

If $f,\ g \in C_0^\infty(\mathbb{H}^n)$, then integration by parts implies

$$\langle f, g\rangle = \int_{\mathbb{H}^n} x_n^{2-n}\left(\nabla f(x)\cdot\overline{\nabla g(x)}\right)\,\mathrm{d}x_1\ldots\mathrm{d}x_n,$$

where $\nabla f(x) = (\frac{\partial f}{\partial x_1}(x), \ldots, \frac{\partial f}{\partial x_n}(x))$. Thus, Δ is a symmetric and positive operator in the space of smooth and compactly supported functions on $\mathbb{H}^n$.

The second model is the Poincare unit ball, i.e. $\{x \in \mathbb{R}^n : \|x\| < 1\}$, where $\|\cdot\|$ is an Euclidean norm, endowed with the metric $\mathsf{g} = 4(1-\|x\|^2)^{-2}(\mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \ldots + \mathrm{d}x_n^2)$. These two models are related by a transformation:

$$\varphi(x) = \frac{2(x+e_n)}{\|x+e_n\|^2} - e_n,$$

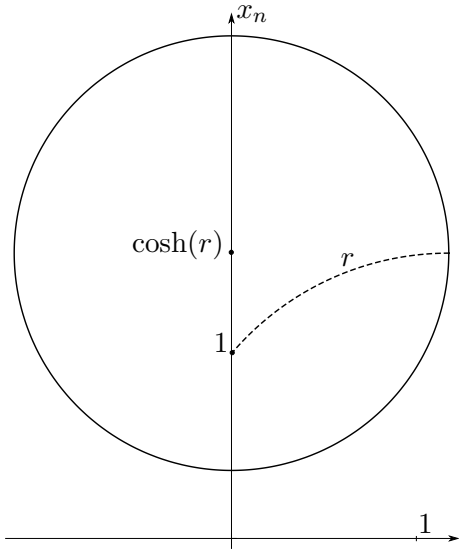

Figure 3.1: Hyperbolic ball of radius r

where $e_n = (0, 0, \ldots, 0, 1)$. One can easily recognise it as a reflection in a sphere of radius $\sqrt{2}$ with the centre in e_n. Since it is an involution the inverse function is given by the same formula. It is well known that reflection in a sphere sends Euclidean generalised circles into Euclidean generalised circles. By the radiality of the metric in the Poincare disc model it is easy to see that a Euclidean ball of radius $\tanh(r/2)$ with centre at the origin is a hyperbolic ball with the same centre but with radius r. Reflection by the sphere sends it to the Euclidean circle with centre at the point $(0, 0, \ldots, 0, \cosh(r))$ radius $\sinh(r)$. Because the hyperbolic space is homogeneous the hyperbolic spheres are Euclidean spheres with a different centre and a different radius. (See Figure 3.1.)

Let us introduce a function $u\colon \mathbb{H}^n \times \mathbb{H}^n \to \mathbb{R}_+$ defined by

$$u(x, y) = \frac{\|x - y\|^2}{4x_n y_n}. \tag{3.1}$$

Again, $\| \cdot \|$ denotes the Euclidean norm. The hyperbolic distance $\rho : \mathbb{H}^n \times \mathbb{H}^n \to \mathbb{R}_+$ is given by the relation

$$1 + 2u = \cosh(\rho). \tag{3.2}$$

Note that we can give the explicit form of the distance function, which is: $\rho(x, y) = \log \frac{\|x-\overline{y}\|+\|x-y\|}{\|x-\overline{y}\|-\|x-y\|}$, where $\overline{(x_1, x_2, \ldots, x_{n-1}, x_n)} = (x_1, x_2, \ldots, x_{n-1}, -x_n)$, however it is rather complicated formula and we will prefer to deal with the function u.

The rotational symmetry of the ball model is motivation to introduce another coordinate system, namely *geodesic spherical coordinates* i.e vector $(\rho, \varphi_1, \varphi_2, \ldots, \varphi_{n-1})$,

these are related to the rectangular coordinates by

$$\begin{aligned} x_n &= (\cosh(\rho) + \sinh(\rho)\sin(\varphi_1)\sin(\varphi_2)\dots\sin(\varphi_{n-1}))^{-1}, \\ x_j &= x_n(\sinh(\rho)\sin(\varphi_1)\sin(\varphi_2)\dots\sin(\varphi_{j-1})\cos(\varphi_j)) \quad \text{for } j = 2, 3, \dots, n-1, \\ x_1 &= x_n \sinh(\rho)\cos(\varphi_1), \end{aligned}$$

where ρ is a distance of a point x to e_n, $\varphi_1, \dots, \varphi_{n-2}$ ranges over $[0, \pi)$, $\varphi_{n-1} \in [0, 2\pi)$. For simplicity let us denote this map by $\phi(\rho, \varphi_1, \varphi_2, \dots, \varphi_{n-1}) = x$. The volume element and the metric in this case are given by

$$\begin{aligned} \mathrm{d}\mu(x) &= (\sinh(\rho))^{n-1}\sin(\varphi_1)^{n-2}\sin(\varphi_2)^{n-3}\dots\sin(\varphi_{n-2})\mathrm{d}r\mathrm{d}\varphi_1\dots\mathrm{d}\varphi_{n-1} \\ g &= \mathrm{d}r^2 + \sinh(r)^2(\mathrm{d}\varphi_1^2 + \sin^2(\varphi_1)\mathrm{d}\varphi_2^2 + \\ &\quad \sin^2(\varphi_1)\sin^2(\varphi_2)\mathrm{d}\varphi_3^2 + \dots + \sin^2(\varphi_1)\dots\sin^2(\varphi_{n-2})\mathrm{d}\varphi_{n-1}^2), \end{aligned}$$

So the Laplace operator in geodesic polar coordinates is given by

$$\Delta = -\partial^2/\partial\rho^2 - (n-1)\coth(\rho)\partial/\partial\rho + \Delta_{S(e_n,\rho)},$$

where $\Delta_{S(e_n,r)}$ is the Laplacian on the geodesic sphere of radius ρ about e_n. Suppose that γ is an isometry such that for some point p in $\mathbb{H}^n$, $\gamma p = e_n$. This allow us to introduce a geodesic polar coordinates about p, they are given by $\phi^{-1}(\gamma x)$. The Laplace operator in geodesic polar coordinates about p is given by:

$$\Delta = -\partial^2/\partial\rho^2 - (n-1)\coth(\rho)\partial/\partial\rho + \Delta_{S(p,\rho)},$$

3.2 Pre-trace formula

Let us define an integral operator by the formula

$$(Lf)(x) = \int_{\mathbb{H}^n} k(x,y)f(y)\,\mathrm{d}\mu(y),$$

where $\mathrm{d}\mu$ is a Riemannian volume form and $k\colon \mathbb{H}^n \times \mathbb{H}^n \to \mathbb{C}$ is a given function, called the *kernel* of L. We will always assume that the function k and f are that the integral converge absolutely. Let us define the operator T_γ acting on functions on $\mathbb{H}^n$ by

$$[T_\gamma f](x) = f(\gamma z),$$

where $\gamma \in \mathrm{Isom}^+(\mathbb{H}^n)$. A linear operator L is said to be *invariant* under the action of the group G if it commutes with T_γ for $\gamma \in G$. An example of an invariant operator is the Laplacian; it is invariant under the action of isometric diffeomorphisms.

Lemma 3.2.1. *The integral operator is invariant under the action of the group G if and only if*

$$k(\gamma x, \gamma y) = k(x,y) \quad \textit{for all } \gamma \in G.$$

A function k satisfying this relation is called a point-pair invariant of group G.

Because we are interested only in analysis of the Laplace operator we restrict our considerations to point -pair invariants of group of orientation preserving, isometric diffeomorphisms.

In this setting k depends purely on the distance. Therefore we will often write $k(\rho(x,y))$ or shortly $k(q)$ rather than $k(x,y)$. Recall that $\rho(x,y)$ denotes the hyperbolic distance between x and y. We will also use the function u defined by (3.1) and write $k(u)$ for the point-pair invariant $k(x,y)$.

Lemma 3.2.2. *Suppose $k(x,y)$ is a smooth point pair invariant. Then $\Delta_x k(x,y) = \Delta_y k(x,y)$.*

Let us introduce another notation. We will call functions $f\colon \mathbb{H}^n \times \mathbb{H}^n \to \mathbb{R}$ *radial* about x if they depend on x and the distance from x to y, i.e. $f(x,y) = \tilde{f}(x,\rho(x,y))$. The simplest example of a radial function at every point is a point pair invariant since it depends only on the distance.

Throughout this thesis we will call a function which satisfies an eigenvalue equation *generalised eigenfunction*. This notation appears because we want to distinguish the eigenfunction which are in the domain of the operator from these function which satisfy the eigenvalue problem but are not in the domain.

Let us assume that ψ is a radial generalised eigenfunction about x of the Laplacian. The eigenvalue equation, $\Delta\psi = \lambda^2\psi$, in geodesic polar coordinates about a point x is

$$\psi''(\rho) + (n-1)\coth(\rho)\psi'(\rho) + \lambda^2\psi(\rho) = 0.$$

It is a second order linear differential equation, so its general solution is a linear combination of two independent solutions, f_1, f_2 such that $f_1(0) = 1$ and f_1 is an entire function of ρ and f_2 is well defined for $\rho > 0$ and has a singularity for $\rho = 0$ of the logarithmic type for $n = 2$ and of type ρ^{2-n} for $n > 2$.
After a substitutions

$$u = 1/2(\cosh(\rho) - 1), \tag{3.3}$$
$$\lambda^2 = s(n-1-s) \tag{3.4}$$

we obtain

$$u(u+1)\psi(u) + n(n+1/2)\psi'(u) + s(n-1-s)\psi(u) = 0. \tag{3.5}$$

Recall, that the hypergeometric function ${}_2\mathrm{F}_1(\lambda,\beta,\gamma,z)$ is defined to be a solution of the differential equation

$$z(1-z)F''(z) - ((\alpha+\beta+1)-\gamma)F'(z) - \alpha\beta F(z) = 0,$$

with an initial condition $F(0) = 1$. Therefore a function

$$F_{s,n}(u) = {}_2\mathrm{F}_1(s, n-1-s, n/2, -u)$$

solves (3.5), moreover $F_{s,n}(0) = 1$.

Suppose now that f is a function on $\mathbb{H}^n$, $x \in \mathbb{H}^n$. We want to produce the rotational symmetrisation of f about x. Define

$$f_x^\circ(\rho) := \frac{1}{A(\rho)} \int_{S_\rho(x)} f(s)\,\mathrm{d}\omega(s), \quad \rho > 0,$$

where $S_\rho(x)$ is the geodesic sphere in $\mathbb{H}^n$ with the centre at x and the radius ρ, $A(\rho)$ is its area, $\mathrm{d}\omega$ is the area element of $S_\rho(x)$, wherever this expression makes sense. By definition, $f_x^\circ(0) = f(x)$. We call f_x° the radicalisation of f about x. By the definition it is obvious that $f_x^\circ(x) = f(x)$ and f_x° is radial at x.

Lemma 3.2.3. *Suppose that k is a point-pair invariant. Then for every function $f\colon \mathbb{H}^n \to \mathbb{C}$ with a well-defined radicalisation we have*

$$\int_{\mathbb{H}^n} k(x,y) f(y) \mathrm{d}\mu(y) = \int_{\mathbb{H}^n} k(x,y) f_x^\circ(y) \mathrm{d}\mu(y).$$

Lemma 3.2.4. *Let $\lambda \in \mathbb{C}$ and $y \in \mathbb{H}^n$. There is a unique function $\omega_\lambda(\,\cdot\,, y)$ which is radial at y such that*

$$\begin{aligned} \omega_\lambda(y,y) &= 1 \\ (\Delta_x - \lambda^2)\omega_\lambda(x,y) &= 0. \end{aligned}$$

This is given by

$$\omega_\lambda(x,y) = F_{s,n}(u(x,y)), \quad \lambda^2 = s(n-1-s),$$

where $F_{s,n}(u)$ is the Gauss hypergeometric function ${}_2\mathrm{F}_1(s, n-1-s, n/2, -u)$.

Corollary 3.2.5. *If f is an generalised eigenfunction of Laplacian with eigenvalue $\lambda^2 = s(n-1-s)$, then*

$$f_x^\circ(y) = F_{s,n}(u(x,y)) f(x)$$

Theorem 3.2.6. *Suppose ψ is an generalised eigenfunction of the Laplace operator, with $\Delta\psi = \lambda^2\psi$. Then ψ is an generalised eigenfunction for the integral operator corresponding to any point-pair invariant, and its eigenvalue depends only on λ and k, i.e there is a function Λ defined on the spectrum, such that*

$$\int_{\mathbb{H}^n} k(x,y)\psi(y)\,\mathrm{d}\mu(y) = \Lambda(\lambda)\psi(x).$$

Proof. Note that

$$\int_{\mathbb{H}^n} k(x,y)\psi(y)\,\mathrm{d}\mu(y) = \int_{\mathbb{H}^n} k(x,y)\psi_x^\circ(y)\,\mathrm{d}\mu(y) = \psi(x) \int_{\mathbb{H}^n} k(u(x,y)) F_{s,n}(u(x,y))\,\mathrm{d}\mu(y).$$

Neither k nor $F_{s,n}$ depends on x therefore we have

$$\Lambda(\lambda) = \int_{\mathbb{H}^n} k(u(x,y)) F_{s,n}(u(x,y))\,\mathrm{d}\mu(y) \tag{3.6}$$

which ends the proof. □

The theorem above shows that any point-pair invariant defines the function on the spectrum on the spectrum of the Laplace operator. On the other hand the spectral theorem implies that this point-pair invariant is the integral kernel of $\Lambda(\sqrt{\Delta})$. Because it is more convenient in the hyperbolic space to deal with functions of $\sqrt{\Delta-(n-1)^2/4}$ let us show some change of coordinates that will make our calculations easier.

First, substitute

$$s = (n-1)/2 + it. \tag{3.7}$$

In this notation $\lambda^2 = \left(\frac{n-1}{2}\right)^2 + t^2$, so

$$t(\lambda) = \left(\lambda^2 - \left(\frac{n-1}{2}\right)^2\right)^{1/2}, \quad \arg(t) \in [-\pi/2, \pi/2).$$

Define a function h by the following relation

$$(h \circ t)(\lambda) = \Lambda(\lambda).$$

Now the substitution (3.7), the formula (3.6) and the relation above imply that

$$h(t) = \int_{\mathbb{H}^n} k(u(x,y))\, {}_2\mathrm{F}_1\left(\frac{n-1}{2} + it, \frac{n-1}{2} - it, \frac{n}{2}, -u(x,y)\right) \mathrm{d}\mu(y).$$

Since the right hand side does not depend on x let us take $x = (0, \ldots, 0, 1)$ and introduce geodesic polar coordinates. Then $h(t)$ is equal to:

$$\int_0^\pi \ldots \int_0^\pi \int_0^{2\pi} \sin^{n-2}(\varphi_1)\sin^{n-3}(\varphi_2)\ldots\sin(\varphi_{n-2})\,\mathrm{d}\varphi_1\ldots\mathrm{d}\varphi_{n-1} \times$$
$$\int_0^\infty k\left(\frac{\cosh(r)-1}{2}\right) {}_2\mathrm{F}_1\left(\frac{n-1}{2} + it, \frac{n-1}{2} - it, \frac{n}{2}, -\frac{\cosh(r)-1}{2}\right) \sinh^{n-1}(r)\,\mathrm{d}r.$$

After a substitution $u = \sinh^2(r/2)$ we get

$$h(t) = \Omega_n 2^{n-1} \int_0^\infty k\,(u)\, {}_2F_1\left(\frac{n-1}{2} + it, \frac{n-1}{2} - it, \frac{n}{2}, -u\right) (u^2+u)^{(n-2)/2}\,\mathrm{d}u, \tag{3.8}$$

where Ω_n denotes the area of an Euclidean unit ball in $\mathbb{R}^n$. We showed that any point-pair invariant defines a function $h(\sqrt{\Delta-(n-1)^2/4})$. The spectral theorem suggests that the inverse relation holds, i.e. the function h give a rise to the point-pair invariant k. In [27] we can find that the inverse formula is given by

$$k(u) = \frac{1}{4\pi}\int_{\mathbb{R}} h(t) {}_2F_1\left(\frac{1}{2} + it, \frac{1}{2} - it, 1, -u\right) \tanh(\pi t)\, t\,\mathrm{d}t, \quad \text{for } n = 2. \tag{3.9}$$

Unfortunately the reverse formula is not known for $n \neq 2$. We will present it in the next section for even n in Theorem 4.1.4.

Let us consider a closed, connected hyperbolic n-dimensional manifold M^n. Then group theory implies that M^n can be regarded as $\Gamma\backslash\mathbb{H}^n$, where Γ is a discrete subgroup

of isometries acting on $\mathbb{H}^n$ discontinuously and has no fixed points. The space of square integrable functions on $\Gamma\backslash\mathbb{H}^n$ may be identified with the space of measurable functions $f\colon \mathbb{H}^n \to \mathbb{C}$ satisfying the properties

$$f(\gamma^{-1}z) = f(z) \quad \forall_{\gamma\in\Gamma} \tag{3.10}$$

and

$$\|f\|^2 := \int_{\mathcal{F}_\Gamma} |f|^2 \,\mathrm{d}\mu < \infty,$$

where $\mathcal{F}_\Gamma$ is any fundamental domain of Γ in $\mathbb{H}^n$ and $\mathrm{d}\mu$ is a volume element. Obviously we will denote this space by $L^2(\Gamma\backslash\mathbb{H}^n)$, it will not be surprising if we say that the inner product on this space is given by

$$\langle f_1, f_2\rangle = \int_{\mathcal{F}_\Gamma} \overline{f_1} f_2 \,\mathrm{d}\mu.$$

By similar arguments we may define the space of smooth functions on $\Gamma\backslash\mathbb{H}^n$ as the space of functions $f \in C^\infty(\mathbb{H}^n)$ satisfying (3.10). One more time consider the operator

$$(Lf)(x) = \int_{\mathbb{H}^n} k(x,y) f(y) \,\mathrm{d}\mu y,$$

where $f \in L^2(\Gamma\backslash\mathbb{H}^n)$. The images of $\mathcal{F}_\Gamma$ under Γ tessellate $\mathbb{H}^n$, therefore

$$\begin{aligned}(Lf)(x) &= \sum_{\gamma\in\Gamma}\int_{\gamma\mathcal{F}_\Gamma} k(x,y)f(y)\mathrm{d}\mu(y) = \sum_{\gamma\in\Gamma}\int_{\mathcal{F}_\Gamma} k(x,\gamma y)f(\gamma y)\mathrm{d}\mu(y)\\ &= \sum_{\gamma\in\Gamma}\int_{\mathcal{F}_\Gamma} k(x,\gamma y)f(y)\mathrm{d}\mu(y) = \int_{\mathcal{F}_\Gamma}\sum_{\gamma\in\Gamma} k(x,\gamma y)f(y)\mathrm{d}\mu(y),\end{aligned}$$

where we assume that the change of the order of the integration and summation is justified by appropriate decay conditions of k. The equalities above give a rise to a new integral operator $L\colon M^n \to M^n$

$$Lf(x) = \int_{M^n} k_\Gamma(x,y) f(y)\mathrm{d}\mu(y),$$

with a kernel

$$k_\Gamma(x,y) = \sum_{\gamma\in\Gamma} k(x,\gamma y). \tag{3.11}$$

It is an easy exercise to check that k_Γ is a point pair invariant, in addition the appropriate decay conditions provide convergence of the sum. For details we refer reader to [35].

Chapter 4

Explicit estimates of spectral functions

Let (M, g) be an n-dimensional Riemannian manifold and as usual denote by $C_0^\infty(M)$ the space of compactly supported smooth functions and by $L^2(M)$ the space of square integrable functions, i.e. the completion of $C_0^\infty(M)$ in the L^2-inner product $\langle f_1, f_2 \rangle = \int_M f_1(x)\overline{f_2(x)}\,\mathrm{dvol}_{\mathsf{g}}(x)$, where $\mathrm{dvol}_{\mathsf{g}}$ is the Riemannian volume form. Suppose that Δ is a non-negative (in the sense of operator theory) self-adjoint extension of the Laplace operator acting on $C_0^\infty(M)$. Note that such self-adjoint extensions always exist and are unique in case the manifold is complete. We do not however assume completeness here.

By the spectral theorem for unbounded self-adjoint operators, there exists a spectral family P_λ, described in the first chapter, with associated spectral measure such that

$$\Delta = \int_{\mathbb{R}} \lambda^2 \,\mathrm{d}P_\lambda.$$

For each $\lambda \in \mathbb{R}$ the operator P_λ is a projection and since $\Delta^N P_\lambda$ is L^2-bounded for any $N > 0$ it follows from elliptic regularity that P_λ has a smooth integral kernel $e_\lambda \in C^\infty(M \times M)$, i.e.

$$(P_\lambda f)(x) = \int_M e_\lambda(x, y) f(y)\,\mathrm{dvol}_g(y).$$

The restriction of e_λ to the diagonal in $M \times M$ is called the local counting function, and is denoted by

$$N_x(\lambda) := e_\lambda(x, x). \tag{4.1}$$

Note that in the case that Δ has compact resolvent the spectrum is purely discrete and there exists an orthonormal basis $(\varphi_j)_{j\in\mathbb{N}_0}$ in $L^2(M)$ consisting of eigenfunctions with eigenvalues λ_j^2 for Δ:

$$\Delta\varphi_j = \lambda_j^2 \varphi_j.$$

The local counting function is then given by

$$N_x(\lambda) = \sum_{\lambda_j < \lambda} |\varphi_j(x)|^2,$$

and integration over M gives the usual counting function of the Laplacian

$$N(\lambda) = \int_M N_x(\lambda)\,\mathrm{dvol}_{\mathbf{g}}(x) = \#\{\lambda_j < \lambda\}.$$

The local counting function is well defined independently of any assumptions on the discreteness of the spectrum.

This chapter is organised as follows: first, we present the classical formula for the solution to the shifted wave equation (4.2) in hyperbolic space and relate it to the integral kernel of $\cos(\sqrt{\Delta - (n-1)^2/4}\,t)$; through this dissertation we call this the shifted wave kernel. Second, through the analysis of the aforementioned wave kernel we present the generalisation of the Mehler-Fock formula (Theorem 4.1.4), which gives us a way to compute the value of the integral kernel of $\int \cos(\sqrt{\Delta - (n-1)^2/4}\,t)g(t)\mathrm{d}t$ on the diagonal, where g is a test function. Next, by finite propagation speed and the Fourier-Tauberian argument given in [42] we are able to show our main result: estimates on the local counting function of the Laplace operator (Theorem 4.2.1). As consequences of this result we show bounds on the local heat trace and estimate the growth of eigenfunctions of the Laplacian as well as growth of their first derivative. At the end we show how to find estimates of higher derivatives of the eigenfunctions for the hyperbolic surfaces.

4.1 The integral kernel of the solution operator of the shifted wave equation

In order to study the local counting function of the Laplace operator at a point $x \in M^n$ near which M^n is hyperbolic, we first need to study the shifted wave kernel on $\mathbb{H}^n$. By way of motivation, Safarov in [42] showed that for the suitable test functions ϱ_1, ϱ_2 one has:

$$|N_x(\lambda) - N_x * \varrho_1(\lambda)| \leq N_x' * \varrho_2(\lambda)$$

For detailed discussion on this estimate see Section 2.3, Theorem 2.3.1.

The Weyl law implies that the local counting function has polynomial growth, therefore the right hand side of the above inequality is of lower order and we may treat it as correction term in the expansion. In addition if the Fourier transforms of functions ϱ_1 and ϱ_2 are compactly supported, then $N_x * \varrho_1$ and $N_x' * \varrho_2$ depend only on values of the Fourier transforms of N_x and N_x' on the on the support of ϱ_1 and ϱ_2 respectively. Moreover by a simple modification of the counting function, one does not need the full information about the Fourier transform of N_x and N_x' but it is enough to know just the cosine transform of N_x'. By the lemma below we will show how it is connected to the wave kernel.

Lemma 4.1.1. *The cosine transform of the derivative of the local counting function of the Laplacian on the manifold M is the diagonal of the wave kernel on M.*

Proof. By the spectral theorem, for any λ, the projection E_λ may be expressed in terms of its smooth integral kernel. By the definition of functional calculus for any L^2 function

f we have

$$\langle f, \cos(\sqrt{\Delta}t)f\rangle = \int_{\mathbb{R}} \cos(\lambda t)\,\mathrm{d}\int_{M\times M} e_\lambda(x,y)f(y)\overline{f(x)}\,\mathrm{dVol_g}(y)\mathrm{dVol_g}(x).$$

Of course we need to understand this equality in the distributional sense since E_λ is a distribution in λ. When we consider the diagonal of e_λ the proof is complete. □

Denote by $L_{\mathbb{H}^n}$ the shifted Laplace operator $\Delta_{\mathbb{H}^n} - \left(\frac{n-1}{2}\right)^2$. Consider the Cauchy problem for the shifted wave equation in hyperbolic n-space:

$$\left\{\begin{array}{rcl} -L_{\mathbb{H}^n}v &=& \frac{\partial^2 v}{\partial t^2},\\ v|_{t=0} &=& f,\\ \frac{\partial v}{\partial t}|_{t=0} &=& 0,\end{array}\right. \tag{4.2}$$

for a given $f \in L^2(\mathbb{H}^n)$. This problem may be solved using functional calculus:

$$v(x,t) = \int_{\mathbb{R}} \cos(\sqrt{\lambda^2 - (n-1)^2/4})\,\mathrm{d}P_\lambda f(x) =: \cos\left(\sqrt{L_{\mathbb{H}^n}}t\right) f(x). \tag{4.3}$$

Because the family of operators $\cos(\sqrt{L_{\mathbb{H}^n}}t)$ is bounded on $L^2(\mathbb{H}^n)$ for all $t \in \mathbb{R}$, the Schwartz kernel theorem implies that it has an associated distributional kernel $C(x,y,t)$ which we call the shifted wave kernel. On the other hand, there are well known formulae for the solution to (4.2) (see e.g. [16]):

$$v(x,t)=\begin{cases} \dfrac{\partial_t}{(2m-1)!!}\left(\left(\dfrac{1}{\sinh t}\partial_t\right)^{m-1}\left(\sinh^{2m-1}(t)f_x^\circ(t)\right)\right), & n=2m+1,\\ \dfrac{\partial_t}{2^{m+\frac{1}{2}}m!}\displaystyle\int_0^t \frac{\sinh\rho}{\sqrt{\cosh t-\cosh\rho}}\left(\frac{1}{\sinh\rho}\partial_\rho\right)^m\left(\sinh^{2m}(\rho)f_x^\circ(\rho)\right)\mathrm{d}\rho, & n=2m+2.\end{cases}$$

Since we do not need the exact formula for the shifted wave kernel but just local (in time) data, let us pair the solution to the wave equation with a test function $g \in C_0^\infty(\mathbb{R})$:

$$\int_{\mathbb{R}} v(x,t)g(t)\,\mathrm{d}t = \begin{cases} \dfrac{2(-1)^m\omega_{2m+1}^{-1}}{(2m+1)!!}\displaystyle\int_{\mathbb{H}^n} f(y)\left(\frac{1}{\sinh\rho}\partial_\rho\right)^m g(\rho)\,\mathrm{d}\mu(y), & n=2m+1,\\ \dfrac{(-1)^{m+1}\omega_{2m+2}^{-1}}{2^{m+\frac{1}{2}}(m+1)!}\displaystyle\int_{\mathbb{H}^n} f(y)\left(\frac{1}{\sinh\rho}\partial_\rho\right)^m\int_\rho^\infty \frac{g'(t)\,\mathrm{d}t}{\sqrt{\cosh t-\cosh\rho}}\,\mathrm{d}\mu(y), & \\ & n=2m+2,\end{cases}$$

where ρ denotes the hyperbolic distance from x to y.

From the formula above and (4.3), one can easily extract the formula for the integral kernel of the operator $\int_{\mathbb{R}} \cos\left(\sqrt{L_{\mathbb{H}^n}}t\right) g(t)\,\mathrm{d}t$. Let $k_{n,g}$ be the integral kernel of $\int_{\mathbb{R}} \cos\left(\sqrt{L_{\mathbb{H}^n}}t\right) g(t)\,\mathrm{d}t$, i.e.

$$k_{n,g} = \int_{\mathbb{R}} C(x,y,t)g(t)\,\mathrm{d}t,$$

where we recall that $C(x,y,t)$ denotes the shifted wave kernel on $\mathbb{H}^n$.

Lemma 4.1.2. *The integral kernel, $k_{n,g}$, of the operator $\int_{\mathbb{R}} \cos\left(\sqrt{L_{\mathbb{H}^n}}t\right) g(t)\,\mathrm{d}t$ is given by*

$$k_{2m+1,g}(x,y) = (-2\pi)^{-m}\left(\frac{1}{\sinh\rho}\partial_\rho\right)^m g(\rho), \tag{4.4}$$

$$k_{2m+2,g}(x,y) = \frac{\sqrt{2}}{(-2\pi)^{m+1}}\left(\frac{1}{\sinh\rho}\partial_\rho\right)^m \int_\rho^\infty \frac{g'(t)\,\mathrm{d}t}{\sqrt{\cosh t-\cosh\rho}} \tag{4.5}$$

where $\rho = \mathrm{dist(x,y)}$.

4.1.1 The shifted wave kernel in hyperbolic space on the diagonal

The question we want to answer is: how does the integral kernel $k_{n,g}$ look on the diagonal, where $\rho = 0$? As we mentioned in the introduction, the local counting function is smooth, therefore the wave kernel paired with some test function is also smooth. Thus it makes sense to talk about the value of $k_{n,g}$ at a point. Since $\mathbb{H}^n$ is a homogeneous space, this value does not depend on x. We cannot examine this directly from (4.4), (4.5), because the operator $\frac{1}{\sinh\rho}\partial_\rho$ is singular for $\rho = 0$. Moreover the integral does not satisfy the assumptions of the fundamental theorem of calculus there.

Instead, we manipulate $k_{n,g}$. The case of $n = 2$ it is well known (see e.g. (70) and (114) in [35] and [27]):

$$k_{2,g}(x,x) = \frac{1}{4\pi}\int_{\mathbb{R}} h(t)\tanh(\pi t)t\,\mathrm{d}t,$$

where h is the cosine transform of $g \in C_0^\infty(\mathbb{R})$, i.e.,

$$h(t) := \int_{\mathbb{R}} g(x)\cos(tx)\,\mathrm{d}x. \tag{4.6}$$

It is easier to express $k_{n,g}$ in terms of the cosine transform of g, that we denote h, rather than in terms of g. In addition, it turns out that the change of coordinates

$$u = \sinh^2(\rho/2)$$

makes the calculations much simpler. Bearing in mind that it is an abuse of notation let us write $k_{n,g}(u)$ for $k_{n,g}(x,y)$, where u is the function defined by (3.1). Now, using the point pair invariant u we can rewrite formulas (4.5), (4.4) in a recursive form:

$$\begin{cases} k_{1,g}(u) = g\left(\operatorname{arcosh}(1+2u)\right), \\ k_{2,g}(u) = -\dfrac{1}{2\sqrt{2}}\displaystyle\int_{\rho(u)}^\infty \frac{g'(t)\,\mathrm{d}t}{\sqrt{\cosh t-1-2u}}, \\ k_{2m+1,g}(u) = (-4\pi)^{-m}\,\partial_u^m k_1(u), & \text{for } m\in\mathbb{N}_+, \\ k_{2m+2,g}(u) = (-4\pi)^{-m}\,\partial_u^m k_2(u), & \text{for } m\in\mathbb{N}_+ \end{cases} \tag{4.7}$$

The explicit formula for $k_{2,g}$ is well known (see e.g. [27]). It is expressed in terms of the function h, of course, and the main tool to recover h from k for $n = 2$ is the

Mehler-Fock formula. The following theorem accomplishes this for general n. This result may also be proved using Harish-Chandra's Plancherel formula for spherical functions, see [10], p. 292. For the sake of completeness and since we will derive through the proof a generalisation of the Mehler-Fock formula, which is interesting in its own right, we give the proof here. Moreover, the transform which we developed allows us to give the estimate of the derivatives of the eigenfunctions.

Lemma 4.1.3. *Suppose $g \in C_0^\infty(\mathbb{R})$. The integral kernel, $k_{n,g}$, on the diagonal is given by*

$$k_{n,g}(x,x) = \begin{cases} \dfrac{1}{(4\pi)^{m+1} m!} \displaystyle\int_{\mathbb{R}} h(t) \tanh(\pi t) t \left(\frac{1}{2} + it\right)_m \left(\frac{1}{2} - it\right)_m \mathrm{d}t, & n = 2m+2 \\ \dfrac{1}{(2\pi)^{m+1}(2m-1)!!} \displaystyle\int_{\mathbb{R}} h(t) \, (it)_m \, (-it)_m \, \mathrm{d}t, & n = 2m+1, \end{cases}$$

where h is the cosine transform of g and $(x)_n$ is the Pochhammer symbol, i.e. $(x)_n = \Gamma(x+n)/\Gamma(x) = x(x+1)\dots(x+n-1)$.

Proof. In the previous chapter we worked out a relation between functions $k_{n,g}$ and h, see (3.8). Recall that it is given by:

$$h(t) = \frac{\omega_n}{n} 2^{n-1} \int_0^\infty k_{n,g}(u) \, {}_2\mathrm{F}_1\left(\frac{n-1}{2} + it, \frac{n-1}{2} - it, \frac{n}{2}, -u\right) (u^2+u)^{(n-2)/2} \, \mathrm{d}u, \quad (4.8)$$

where ω_n denotes the volume of an Euclidean unit ball in $\mathbb{R}^n$ and h is as before the cosine transform of $g \in C_0^\infty(\mathbb{R})$. The inverse transform for $n = 2$ is as follows (see e.g.[27]):

$$k_2(x,y) = \frac{1}{4\pi} \int_{\mathbb{R}} h(t) \, {}_2\mathrm{F}_1\left(\frac{1}{2} + it, \frac{1}{2} - it, 1, -u(x,y)\right) \tanh(\pi t) t \, \mathrm{d}t, \quad \text{for } n = 2. \quad (4.9)$$

Note that the hypergeometric function ${}_2\mathrm{F}_1(\alpha, \beta, \gamma, z)$ has an analytic continuation over the plane $\mathbb{C}$ cut along $[1, +\infty)$, given by the integral representation:

$${}_2\mathrm{F}_1(\alpha, \beta, \gamma, z) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-tz)^{-\alpha} \mathrm{d}t.$$

Moreover the m-th derivative of ${}_2\mathrm{F}_1$ is

$$\partial_z^m \, {}_2\mathrm{F}_1(\alpha, \beta, \gamma, z) = \frac{\prod_{j=0}^{m-1}(\alpha+j) \prod_{j=0}^{m-1}(\beta+j)}{\prod_{j=0}^{m-1}(\gamma+j)} \, {}_2\mathrm{F}_1(\alpha+im, \beta+m, \gamma+m, z). \quad (4.10)$$

Since the hypergeometric function satisfies the initial condition $F_{t,n}(0) = 1$, the recursion formula (4.7) and (4.9) complete the proof for even n. For odd n note that

$${}_2\mathrm{F}_1(it, -it, 1/2, -u) = \cos(2t \operatorname{arcsinh} \sqrt{u}). \quad (4.11)$$

Substitute $\rho = 2\operatorname{arccosh}\sqrt{u}$ into the integrals (4.8) and (4.4) for $m = 0$. Then one can easily recognise that these formulas simplify to the cosine transform and its inverse. Now use identities (4.8) and (4.11) to get that

$$k_{1,g}(x,y) = \frac{1}{2\pi}\int_{\mathbb{R}} h(t)\,{}_2\mathrm{F}_1(it,-it,1/2,-u(x,y))\,\mathrm{d}t.$$

The recursion formula (4.7) and the differential identity (4.10) imply

$$k_{2m+1,g}(x,y) = \frac{(2\pi)^{-m-1}}{(2m-1)!!}\int_{\mathbb{R}} h(t)(it)_m(-it)_m\,{}_2\mathrm{F}_1(m+it,m-it,m+1/2,-u(x,y))\,\mathrm{d}t.$$

When we take into account the initial condition $F_{t,n}(0) = 1$ the proof is complete. □

The proof of Theorem 4.1.3 gives us a class of transformations which contains the cosine transform as well as the Mehler-Fock transform. Further, we know that the class of functions of interest to us are in the domain of these transforms.

Theorem 4.1.4 (Generalised Mehler-Fock formula)**.** *Suppose $g \in C_0^\infty(\mathbb{R})$ and g is even. Define f_e, f_o by*

$$f_e(u) = -\frac{1}{2\sqrt{2}}\int_{\rho(u)}^\infty \frac{g'(t)\,\mathrm{d}t}{\sqrt{\cosh t - 1 - 2u}}, \quad f_o(u) = g(\rho(u))$$

where $\rho(u) = 2\operatorname{arccosh}\sqrt{u}$. Then the following inverse formulae hold:

$$\begin{aligned} f_e(u) = &\frac{1}{\Gamma(m+1)^2}\int_{\mathbb{R}}\int_0^\infty f_e(v)\,{}_2\mathrm{F}_1\left(\frac{1}{2}+m+it,\frac{1}{2}+m-it,m+1,-v\right)(v^2+v)^m\,\mathrm{d}v\times \\ &{}_2\mathrm{F}_1\left(\frac{1}{2}+m+it,\frac{1}{2}+m-it,m+1,-u\right)\tanh(\pi t)t\left(\frac{1}{2}+it\right)_m\left(\frac{1}{2}-it\right)_m\mathrm{d}t, \quad (4.12) \\ f_o(u) = &\Gamma\left(m+\frac{1}{2}\right)^{-2}\int_{\mathbb{R}}\int_0^\infty f_o(v)\,{}_2\mathrm{F}_1\left(m+it,m-it,m+\frac{1}{2},-v\right)(v^2+v)^{m-\frac{1}{2}}\,\mathrm{d}v\times \\ &{}_2\mathrm{F}_1\left(m+it,m-it,m+\frac{1}{2},-u\right)(it)_m(-it)_m\,\mathrm{d}t, \quad (4.13) \end{aligned}$$

for $m \in \mathbb{N}_0$, where $(x)_n$ is the Pochhammer symbol, i.e.

$$(x)_n = \Gamma(x+n)/\Gamma(x) = x(x+1)\dots(x+n-1).$$

4.1.2 The wave kernel near a point where the manifold is hyperbolic

As we mentioned at the beginning of this section, in order to estimate the local counting function on the manifold M we need to know the diagonal of the wave kernel on M. So far we obtained just the shifted wave kernel on $\mathbb{H}^n$. In this section we show how one may move from $\mathbb{H}^n$ to M and change the shifted wave kernel to the wave kernel.

Let (M^n, g) be an n-dimensional Riemannian manifold, we say that M^n is hyperbolic around $x \in M$ if there exists a positive radius ρ such that the open metric ball, $B_\rho(x)$,

centred at x is isometric to the ball in hyperbolic n-space. Let us denote by $d(x)$ twice the maximal radius of that ball.

In order to distinguish the integral kernel of $\int_{\mathbb{R}} \cos\left(\sqrt{L_{M^n}}t\right) g(t)\,\mathrm{d}t$ from the integral kernel of $\int_{\mathbb{R}} \cos\left(\sqrt{L_{\mathbb{H}^n}}t\right) g(t)\,\mathrm{d}t$ we will denote it by $k_{n,g}(x,y)$. The unit wave propagation speed implies that the solution of the shifted wave equation (4.2) in $\mathbb{H}^n$ coincides with the solution in $B_\rho(x) \subset M^n$ for $t \in (-d(x), d(x))$, whenever the initial function f coincides on isometric balls and vanishes outside. In other words $k_n(x,y) = k_n(x,y)$, whenever

$$\operatorname{supp}(g) \subset (-d(x), d(x)). \tag{4.14}$$

Therefore we can move from $\mathbb{H}^n$ to M^n for function g that has a support of an appropriate size. In addition, if M^n is compact, κ_j^2 is the shifted eigenvalue of Δ_{M^n}, satisfying $\kappa_j^2 = \lambda_j^2 - (n-1)^2/4$, $\arg(\kappa_j) \in [-\pi/2, \pi/2)$, where $\Delta_{M^n}\varphi_j = \lambda_j^2\varphi_j$. Then the integral kernel k_n is just

$$\sum_{j=0}^{\infty} h(\kappa_j)\varphi_j(x)\overline{\varphi_j(y)},$$

which converges absolutely, uniformly in $x, y \in M^n$ for $g \in C_0^\infty(\mathbb{R})$.

Let us assume from this point that $g \in C_0^\infty(-d(x), d(x))$. By Lemma 4.1.1, the cosine transform of the derivative of $N_{n,x}$ tested against a function g coincides with the diagonal of the wave kernel on M^n tested against g. Introducing functions $\tilde{h}, \tilde{g}$ given by

$$\tilde{h}(t) = h(\sqrt{t^2 + (n-1)^2/4}), \tag{4.15}$$

$$\tilde{g}(t) := \frac{1}{2\pi}\int_{\mathbb{R}} \tilde{h}(x)\cos(xt)\,\mathrm{d}x, \tag{4.16}$$

we have

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \cos(\lambda t) N'_{n,x}(\lambda)\,\mathrm{d}\lambda\, g(t)\,\mathrm{d}t = \int_{\mathbb{R}} k_n(t,x,x)\tilde{g}(t)\,\mathrm{d}t. \tag{4.17}$$

It is surprising, if we look at formulas (4.15), (4.16), but it turns out that $\tilde{g}$ is also smooth and compactly supported as well as g, as we show in the technical lemma below. Therefore we can use Lemma 4.1.3 applied to $k_{n,\tilde{g}}$.

Lemma 4.1.5. *Suppose that g is an even, smooth and compactly supported function, such that* $\operatorname{supp}(g) \subset (-a,a)$ *then $\tilde{g}$ defined by (4.15), (4.16) is also smooth, compactly supported and even function such that* $\operatorname{supp}(\tilde{g}) \subset (-a,a)$.

Proof. The Paley-Wiener theorem implies that the function h is entire. For the sake of concreteness let us define this square root as a square root on the Riemann surface and if $t \in i(0,(n-1)/2)$ then $\arg(t^2 + (n-1)^2/4) = 0$, if $t \in i(-(n-1)/2, 0)$ then $\arg(t^2 + (n-1)^2/4) = -2\pi$.

Let us define the auxiliary function $\gamma\colon \mathbb{C} \to \mathbb{R}$

$$\gamma(t) = \frac{1 + (t^2 + (n-1)^2/4)^{\frac{1}{2}}}{1+|t|}.$$

It turns out that this function is continuous and has a maximal value $(n+1)/2$ at 0, a minimal value at $t = i(n-1)/2$ and

$$\inf_{t\in\mathbb{C}} \gamma(t) = \frac{2}{n+1}. \tag{4.18}$$

Our choice of branch of the square root implies that the function $|\Im(t-(t+(n-1)^2/4)^{1/2})|$ has symmetries along the real and an imaginary axis. Moreover for $t = a + ib$, where $a, b \geq 0$ we have

$$|\Im(t-(t+(n-1)^2/4)^{1/2})| = \\ b - \sqrt{\tfrac{1}{2}\sqrt{\left(a^2-b^2+\left(\tfrac{n-1}{2}\right)^2\right)^2+4a^2b^2} - \tfrac{1}{2}\left(a^2-b^2+\left(\tfrac{n-1}{2}\right)^2\right)}.$$

This implies the following estimate

$$|\Im(t-(t+(n-1)^2/4)^{1/2})| \leq \frac{n-1}{2},$$

where the maximal value is reached for $t = \pm i(n-1)/2$. The Paley-Wiener theorem, triangle inequality and the estimate (4.18) imply

$$|\tilde{h}(t)| \leq \frac{c_\nu e^{a|\Im(\sqrt{t^2+(n-1)^2/4})|}}{(1+\sqrt{t^2+(n-1)^2/4})^\nu} \leq \frac{c_\nu 2^\nu e^{\frac{a(n-1)}{2}}}{(n+1)^\nu}\frac{e^{a|\Im(t)|}}{(1+|t|)^\nu}.$$

Therefore $\tilde{g} \in C_0^\infty(-a,a)$. □

This allows us to use what we already know about the shifted wave kernel on $\mathbb{H}^n$ to control the local counting function on M^n.

Lemma 4.1.6. *The cosine transform of the derivative of $N_{n,x}$ coincides in the interval $(-d(x), d(x))$ with the cosine transform of the derivative of F_n for $n \in \mathbb{N}_+$, $n \geq 2$ where*

$$F_2'(\tau) = \frac{1}{2\pi}H\left(\tau-\frac{1}{2}\right)\tanh\left(\pi\sqrt{\tau^2-\frac{1}{4}}\right)\tau,$$

$$F_3'(\tau) = \frac{H(\tau-1)}{2\pi^2}\tau\sqrt{\tau^2-1},$$

$$F_{2m+2}'(\tau) = \frac{2H\left(\tau-m-\frac{1}{2}\right)}{(4\pi)^{m+1}m!}\tanh\left(\pi\sqrt{\tau^2-\left(m+\frac{1}{2}\right)^2}\right)\tau\prod_{l=0}^{m-1}\left(\tau^2-m^2+l^2-m+l\right),$$

$$F_{2m+1}'(\tau) = \frac{2H(\tau-m)}{(2\pi)^{m+1}(2m-1)!!}\tau\sqrt{\tau^2-m^2}\prod_{l=1}^{m-1}\left(\tau^2-m^2+l^2\right),$$

$$F_n(0) = 0$$

and H is the Heaviside step function, i.e.

$$H(\tau) = \begin{cases} 1 & \tau > 0, \\ 1/2 & \tau = 0, \\ 0 & \tau < 0. \end{cases}$$

Proof. By Lemma 4.1.5 we have that the function $\tilde{g}$ is smooth for a given smooth and compactly supported function g, moreover its support is contained in the support of g. therefore by (4.17) and Lemma 4.1.3 we have

$$\int_{\mathbb{R}} N'_{n,x}(\lambda) h(\lambda)\,\mathrm{d}\lambda = \begin{cases} \dfrac{1}{(4\pi)^{m+1} m!} \displaystyle\int_{\mathbb{R}} \tilde{h}(t) \tanh(\pi t) t \left(\frac{1}{2} + it\right)_m \left(\frac{1}{2} - it\right)_m \mathrm{d}t, & \\ & n = 2m+2, \\ \dfrac{1}{(2\pi)^{m+1}(2m-1)!!} \displaystyle\int_{\mathbb{R}} \tilde{h}(t)\,(it)_m\,(-it)_m\,\mathrm{d}t, & \\ & n = 2m+1, \end{cases}$$

The statement is just a result of a coordinate change. □

4.2 Estimates

The goal of this section is to present explicit estimates of the local counting function on the manifold M^n. We will use explicit Fourier-Tauberian theorems due to Safarov which were used previously to estimate the local counting function for domains in $\mathbb{R}^n$. We have already presented them in Section 2.3.

Lemma 4.1.6 shows that unfortunately the cosine transform of the derivative of the local counting function on M^n is not a polynomial of order $n-1$ but it behaves like a polynomial as $\lambda \to \infty$. The idea is to compare it to this asymptotic polynomial and estimate the error which we make by this comparison. Let us define a function G_n as the error when we compare F_n to a polynomial:

$$\begin{cases} G'_{2m+2}(\tau) := & \\ \dfrac{2H\left(\tau - m - \frac{1}{2}\right)}{(4\pi)^{m+1} m!} \left[\tanh\left(\pi\sqrt{\tau^2 - \left(m+\frac{1}{2}\right)^2}\right) - 1\right] \tau \displaystyle\prod_{l=0}^{m-1} \left(\tau^2 - m^2 + l^2 - m + l\right), & \\ G_{2m+2}(0) = 0. & \end{cases}$$

By definition G'_{2m+2} is clearly an integrable and negative function. Note that the function $\sqrt{t^2 - m^2}$ has the following expansion

$$\sqrt{t^2 - m^2} = t - \frac{1}{2} \sum_{j=1}^{\infty} \frac{m^{2j} \left(\frac{1}{2}\right)_{j-1}}{t^{2j-1} j!},$$

where $(x)_n = \Gamma(x+n)/\Gamma(x)$ and the series converges for $|t| > m$. Let us define $G'_{2m+1}(\tau)$ as the part of the Laurent series of $F'_{2m+1}(\tau)$ with negative powers of τ multiplied by

the characteristic function of the set $[m,\infty)$. Let us set $G_{2m+1}(0)=0$. The function G_n satisfies the following bounds

$$\|G_n\|_\infty \leq \begin{cases} \dfrac{1}{48\pi}, & n=2,\\ \dfrac{1}{12\pi^2}, & n=3,\\ \min\left\{\dfrac{2\left(\left(m-\frac{1}{2}\right)^2+\frac{1}{4\pi^2}\right)^m(2m+1)!}{\pi(4\pi)^{m+2}m!}, \dfrac{2(2m+1)!e^{\pi(2m-1)}}{(16\pi^3)^{m+1}m!}\right\}, & n=2m+2\\ \dfrac{m^{2m+1}(1-m^4)+(2m-1)!m^2(1+m^{2m-2})}{(2\pi)^{m+1}(2m-1)!!(1-m^4)}, & n=2m+1 \text{ and } m \text{ odd},\\ \dfrac{m^{2m+1}(1-m^4)+(2m-1)!m^4(1+m^{2m-4})}{(2\pi)^{m+1}(2m-1)!!}, & n=2m+1 \text{ and } m \text{ even}. \end{cases} \tag{4.19}$$

Theorem 4.2.1. *Let (M^n,g) be a Riemannian manifold and let $x\in M$ be a point such that the metric ball of radius $d(x)/2$ is isometric to a ball in $\mathbb{H}^n$. Then the local counting function on M^n satisfies the following estimates*

$$N_{n,x}(\tau) \leq \frac{\omega_n}{(2\pi)^n}\left[\tau^n+\frac{n}{d(x)}\left(\frac{2}{\pi}\nu^2_{\lfloor\frac{n+2}{2}\rfloor}+\nu_{\lfloor\frac{n+2}{2}\rfloor}\right)\left(\tau+\frac{\nu_{\lfloor\frac{n+2}{2}\rfloor}}{d(x)}\right)^{n-1}\right] \tag{4.20}$$

For $n=2m+2$ we have

$$N_{2m+2,x}(\tau) \geq -\|G_{2m+2}\|_\infty+\frac{\omega_{2m+2}}{(2\pi)^{2m+2}}\left\{\left(\tau-m-\frac{1}{2}\right)^{2m+2}-\left(m+\frac{1}{2}\right)^{2m+2}-\frac{(2m+2)\nu_{m+2}}{d(x)}\left[\left(m+\frac{1}{2}+\frac{\nu_{m+2}}{d(x)}\right)^{2m+1}+\frac{2\nu_{m+2}}{\pi}\left(\tau+\frac{\nu_{m+2}}{d(x)}\right)^{2m+1}\right]\right\},$$

while for $n=2m+1$ we have

$$N_{2m+1,x}(\tau)\geq \begin{cases} \frac{\omega_{2m+1}}{(2\pi)^{2m+1}}\left(\tau^{2m+1}-\frac{(4m+2)\nu^2_{m+1}}{d(x)\pi}\left(\tau+\frac{\nu_{m+1}}{d(x)}\right)^{2m}\right)-\|G_{2m+1}\|_\infty,\ \tau\in[0,m],\\ \frac{\omega_{2m+1}}{(2\pi)^{2m+1}}\left((\tau-m)^{2m+1}-\frac{(4m+2)\nu^2_{m+1}}{d(x)\pi}\left(\tau+\frac{\nu_{m+1}}{d(x)}\right)^{2m}\right)-\|G_{2m+1}\|_\infty,\ \tau>m. \end{cases}$$

As before ν_m denotes $2m$-th root of the first eigenvalue of Δ^m on the interval $[-1/2,1/2]$ subject to Dirichlet boundary conditions, ω_n is the volume of n-dimensional Euclidean ball and constants $\|G_n\|_\infty$ are given by (4.19).

The proof of this theorem is constructive and thus also is of interest itself as it gives an algorithm for computing estimates of the local counting function as well as estimates of the derivatives (in a space variable) of the local counting function. Some examples will be given in Example 4.2.4 and we will also derive an estimate of the counting function on compact hyperbolic manifolds with an explicit remainder estimate (Corollary 4.2.5).

Let us recall some notation given in Section 2.3:

Notation 4.2.2. For the Dirichlet extension of Δ^m on the interval $[-1/2, 1/2]$ let as before ν_m be the $2m$-th root of the first eigenvalue and ζ_m be the corresponding normalised eigenfunction extended by 0 to the real line. We define ρ as the square of the Fourier transform of ζ_m. Following [42] let

$$\rho_\delta(\tau) = \delta\rho(\delta\tau), \quad \rho_{\delta,0} = \delta\rho_{1,0}(\delta\tau), \quad \rho_{1,0}(\tau) = \int_\tau^\infty t\rho(t)\,\mathrm{d}t. \tag{4.21}$$

Plancherel theorem implies that $\|\rho_\delta\|_1 = 1$. The function G'_{2m+1} is non-positive, therefore

$$\rho_\delta * G_n(\tau) \le 0, \quad \rho_{\delta,0} * G'_n \le 0 \tag{4.22}$$

for $\delta > 0$. By Young's inequality we have $|\rho_\delta * G_n(\tau)| \le \|G_n\|_\infty$. Note, that

$$\frac{n\omega_n}{(2\pi)^n} H\left(\tau - \frac{n-1}{2}\right)\left(\tau - \frac{n-1}{2}\right)^{n-1} \le F'_n(\tau) - G'_n(\tau) \le \frac{n\omega_n}{(2\pi)^n} H(\tau)\tau^{n-1}, \tag{4.23}$$

where ω_n is a volume of n-dimensional Euclidean unit ball. In order to give bounds on the local counting function we will need the following Fourier Tauberian theorem.

Lemma 4.2.3. *Let ρ be an even function such that* $\operatorname{supp}(\hat\rho) \subset [-1,1]$ *and suppose* $|\rho(t)| \le \text{const.}(1+t^2)^{-m-1}$ *for $m > n/2$. Define ρ_δ as in (4.21). Suppose that* $\operatorname{supp} E_n \in (0,+\infty)$ *and the cosine transform of $E'_n(t)$ coincides on the interval $(-\delta,\delta)$ with the cosine transform of the function $nH(t-\frac{n-1}{2})(t-\frac{n-1}{2})^{n-1}$, then*

$$\rho_\delta * \widetilde{E}_{2m+2}(\tau) \ge \left(\tau - m - \frac{1}{2}\right)^{2m+2} - \left(m+\frac{1}{2}\right)^{2m+2} - (2m+2)\int \left|\frac{\nu}{\delta}\right| \left(m+\frac{1}{2}+\frac{\nu}{\delta}\right)^{2m+1} \rho(\nu)\mathrm{d}\nu,$$
$$\text{for } \tau > 0,$$
$$\rho_\delta * \widetilde{E}_{2m+1}(\tau) \ge \begin{cases} \tau^{2m+1} & \text{for } \tau \in [0,m], \\ (\tau-m)^{2m+1} & \text{for } \tau > m, \end{cases}$$

where $\tilde E(t) = E(t) - E(-t)$ and H is the Heaviside step function.

Proof. The assumption implies that the Fourier transform of $\tilde E_n$ coincides on the interval $(-\delta,\delta)$ with the Fourier transform of $H(t-\frac{n-1}{2})(t-\frac{n-1}{2})^n - H(-t-\frac{n-1}{2})(-t-\frac{n-1}{2})^n$, while the compactness of the support of the Fourier transform of ρ implies that for

$n = 2m + 2$ we have

$$\begin{aligned}
\rho_\delta * \widetilde{E}_{2m+2}(\tau) &= \int_{-\infty}^{\delta(\tau-m-\frac{1}{2})} \left(\tau - \frac{\nu}{\delta} - m - \frac{1}{2}\right)^{2m+2} \rho(\nu)\mathrm{d}\nu - \int_{\delta(\tau+m+\frac{1}{2})}^{\infty} \left(\tau - \frac{\nu}{\delta} + m + \frac{1}{2}\right)^{2m+2} \rho(\nu)\mathrm{d}\nu \\
&= \int_{\mathbb{R}} \left(\tau - \frac{\nu}{\delta} - m - \frac{1}{2}\right)^{2m+2} \rho(\nu)\mathrm{d}\nu - \int_{\delta(\tau-m-\frac{1}{2})}^{\infty} \left(\tau - \frac{\nu}{\delta} - m - \frac{1}{2}\right)^{2m+2} \rho(\nu)\mathrm{d}\nu - \int_{\delta(\tau+m+\frac{1}{2})}^{\infty} \left(\tau - \frac{\nu}{\delta} + m + \frac{1}{2}\right)^{2m+2} \rho(\nu)\mathrm{d}\nu \\
&\geq \int_{\mathbb{R}} \left(\tau - \frac{\nu}{\delta} - m - \frac{1}{2}\right)^{2m+2} \rho(\nu)\mathrm{d}\nu - \int_{\mathbb{R}} \left(\frac{\nu}{\delta} - m - \frac{1}{2}\right)^{2m+2} \rho(\nu)\mathrm{d}\nu,
\end{aligned}$$

where the inequality is satisfied just for positive τ. Let us define $P_n(\tau, \nu) := 1/2[(\tau + \nu)^n + (\tau - \nu)^n]$. Then P_n is a polynomial in τ and ν which contains only even powers of τ. The fact that ρ is an even function results in the inequality

$$\rho_\delta * \widetilde{E}(\tau) \geq \int_{\mathbb{R}} \left[P_{2m+2}(\tau - m - 1/2, \nu/\delta) - P_{2m+2}(m + 1/2, \nu/\delta)\right] \rho(\nu)\,\mathrm{d}\nu$$

The basic estimates

$$\tau^{2m+2} \leq P_{2m+2}(\tau, \nu) \leq \tau^{2m+2} + (2m + 2)|\nu|(|\tau| + |\nu|)^{2m+1}$$

complete the proof in the case of $n = 2m + 2$. For $n = 2m + 1$ we have

$$\begin{aligned}
\rho_\delta * \widetilde{E}_{2m+1}(\tau) &= \int_{-\infty}^{\delta(\tau-m)} \left(\tau - \frac{\nu}{\delta} - m\right)^{2m+1} \rho(\nu)\mathrm{d}\nu + \int_{\delta(\tau+m)}^{\infty} \left(\tau - \frac{\nu}{\delta} + m\right)^{2m+1} \rho(\nu)\mathrm{d}\nu \\
&= \int_{\mathbb{R}} \left(\tau - \frac{\nu}{\delta} - m\right)^{2m+1} \rho(\nu)\mathrm{d}\nu + \int_{\delta(\tau-m)}^{\infty} \left(\frac{\nu}{\delta} + m - \tau\right)^{2m+1} \rho(\nu)\mathrm{d}\nu + \int_{\delta(\tau+m)}^{\infty} \left(\tau - \frac{\nu}{\delta} + m\right)^{2m+1} \rho(\nu)\mathrm{d}\nu \\
&\geq \int_{\mathbb{R}} \left(\tau - \frac{\nu}{\delta} - m\right)^{2m+1} \rho(\nu)\mathrm{d}\nu + \int_{\mathbb{R}} \left(m - \frac{\nu}{\delta}\right)^{2m+1} \rho(\nu)\mathrm{d}\nu,
\end{aligned}$$

for $\tau > 0$. This implies the following estimates

$$\rho_\delta * \widetilde{E}_{2m+1}(\tau) \geq \begin{cases} \tau^{2m+1} & \text{for } \tau \in [0, m], \\ (\tau - m)^{2m+1} & \text{for } \tau > m. \end{cases}$$

□

At this point we have all tools needed to give the proof of Theorem 4.2.1:

Proof. Safarov shoved that the test function defined in Notation 4.2.2 satisfies the assumptions of the Theorem 2.3.1 and Lemma 2.3.3. Moreover Theorem 2.3.1, our Theorem 4.1.6 and support properties of the function ρ imply

$$\rho_{d(x)}*F_n(\tau)-\frac{d(x)^{-1}}{\int|\nu|\rho(\nu)\mathrm{d}\nu}\rho_{d(x),0}*F_n'(\tau)\leq N_{n,x}\leq\rho_{d(x)}*F_n(\tau)+\frac{d(x)^{-1}}{\int|\nu|\rho(\nu)\mathrm{d}\nu}\rho_{d(x),0}*F_n'(\tau) \tag{4.24}$$

Let us define the asymptotic polynomial of F_n as

$$p_a(n,\tau)=F_n(\tau)-G_n(\tau).$$

Notice that the asymptotic polynomial is just the Taylor part of a Laurent series of F_n multiplied by the characteristic function of the interval $\left[\frac{n-1}{2},\infty\right)$. Safarov in [42] has shown that $\int|\nu|\rho(\nu)\mathrm{d}\nu\geq\pi/2$. Inequalities (4.19) admit the estimate

$$N_{n,x}\leq\rho_{d(x)}*p_a(n,\tau)+\frac{2}{\pi d(x)}\rho_{d(x),0}*\partial_\tau p_a(n,\tau) \tag{4.25}$$

The monotonicity property of the Riemann integral and (4.23) lead to the conclusion that

$$N_{n,x}(\tau)\quad\leq\quad\frac{\omega_n}{(2\pi)^n}\left[\tau^n+\frac{n}{d(x)}\left(\frac{2}{\pi}\nu^2_{\lfloor\frac{n+2}{2}\rfloor}+\nu_{\lfloor\frac{n+2}{2}\rfloor}\right)\left(\tau+\frac{\nu_{\lfloor\frac{n+2}{2}\rfloor}}{d(x)}\right)^{n-1}\right]$$

Using one more time (4.19) one may show that

$$\rho_{d(x)}*p_a(n,\tau)-\frac{2}{\pi d(x)}\rho_{d(x),0}*\partial_\tau p_a(n,\tau)+\rho_{d(x)}*G_n(\tau)\leq N_{n,x}(\tau). \tag{4.26}$$

When we take into account inequality (4.22) we arrive at

$$N_{n,x}(\tau)\geq\rho_{d(x)}*p_a(n,\tau)-\frac{2}{\pi d(x)}\rho_{d(x),0}*\partial_\tau p_a(n,\tau)-\|G_n\|_\infty.$$

Inequality (4.23) and Lemma 4.2.3 complete the proof. □

In the special cases $n=2,3,4$ slightly stronger estimates can be obtained.

Example 4.2.4. Recall that $p_a(2,\tau)=H\left(\tau-\frac{1}{2}\right)\frac{\tau}{2\pi}$, $p_a(3,\tau)=H(\tau-1)\left(\frac{\tau^3}{6\pi^2}-\frac{\tau}{4\pi^2}\right)$, $p_a(4,\tau):=H(\tau-\frac{3}{2})\left(\frac{\tau^4}{32\pi^2}-\frac{\tau^2}{8\pi^2}\right)$. Inequality (4.25) with Corollary 2.3 from [42] imply

$$N_{2,x}(\tau)\quad\leq\quad\frac{1}{4\pi}\left(\tau^2+\frac{4\nu_2^2+2\nu_2\pi}{\pi d(x)}\left(\tau+\frac{\nu_2}{d(x)}\right)\right), \tag{4.27}$$

$$N_{3,x}(\tau)\quad\leq\quad\frac{1}{6\pi^2}\left(\tau^3+\left(\frac{6\nu_2^2+3\pi\nu_3}{\pi d(x)}\right)\left(\tau+\frac{\nu_2}{d(x)}\right)^2\right)-\frac{1}{4\pi^2}\left(\tau-\frac{2\nu_1^2}{\pi d(x)}\right), \tag{4.28}$$

$$N_{4,x}(\tau)\quad\leq\quad\frac{1}{32\pi^2}\left(\tau^4+\left(\frac{8\nu_3^2+4\pi\nu_3}{\pi d(x)}\right)\left(\tau+\frac{\nu_3}{d(x)}\right)^3\right)-$$
$$\frac{1}{8\pi^2}\left(\tau^2-\frac{4\nu_2^2}{\pi d(x)}\left(\tau+\frac{\nu_2}{d(x)}\right)\right) \tag{4.29}$$

Moreover, one may compute the exact value for the supremum norm of G_n,

$$\|G_2\|_\infty = \frac{1}{48\pi}, \quad \|G_3\|_\infty = \frac{1}{12\pi^2}, \quad \|G_4\|_\infty = \frac{17}{7680\pi^2}$$

These estimates combined with inequality (4.26) and Corollary 2.3 from [42] give

$$N_{2,x}(\tau) \geq \frac{1}{4\pi}\left(\tau^2 - \frac{4\nu_2^2}{\pi d(x)}\left(\tau + \frac{\nu_2}{d(x)}\right) - \frac{1}{12}\right), \tag{4.30}$$

$$N_{3,x}(\tau) \geq \frac{1}{6\pi^2}\left(\tau^3 - \frac{6\nu_2^2}{\pi d(x)}\left(\tau + \frac{\nu_2}{d(x)}\right)^2\right) - \frac{1}{4\pi^2}\left(\tau + \frac{2\nu_1^2 + \pi\nu_1}{\pi d(x)}\right) - \frac{1}{12\pi^2}, \tag{4.31}$$

$$N_{4,x}(\tau) \geq \frac{1}{32\pi^2}\left(\tau^4 - \frac{8\nu_3^2}{\pi d(x)}\left(\tau + \frac{\nu_3}{d(x)}\right)^3\right) - \frac{1}{8\pi^2}\left(\tau^2 + \frac{4\nu_2^2 + 2\pi\nu_2}{\pi d(x)}\left(\tau + \frac{\nu_2}{d(x)}\right)\right) - \frac{17}{7680\pi^2}, \tag{4.32}$$

where the numerical values of ν_n are

$$\nu_2 = 4.73004074\ldots,$$
$$\nu_3 = 6.28318530\ldots,$$
$$\nu_4 = 7.81870734\ldots.$$

Notice that simple integration of the presented estimates gives bounds on the counting function of the Laplacian for hyperbolic manifolds. Denote by l_n the length of the shortest closed geodesic on M^n. The following theorem gives a proof of the Weyl's law, moreover it gives the estimate on the remainder term.

Corollary 4.2.5. *The counting function of Laplacian on a compact hyperbolic manifold M^n satisfies the following estimates*

$$N_n(\tau) \leq \frac{\omega_n |M^n|}{(2\pi)^n}\left[\tau^n + \frac{n}{l_n}\left(\frac{2}{\pi}\nu^2_{\lfloor\frac{n+2}{2}\rfloor} + \nu_{\lfloor\frac{n+2}{2}\rfloor}\right)\left(\tau + \frac{\nu_{\lfloor\frac{n+2}{2}\rfloor}}{l_n}\right)^{n-1}\right] \tag{4.33}$$

For $n = 2m+2$ we have

$$\frac{N_{2m+2}(\tau)}{|M^{2m+2}|} \geq -\|G_{2m+2}\|_\infty + \frac{\omega_{2m+2}}{(2\pi)^{2m+2}}\left\{\left(\tau - m - \frac{1}{2}\right)^{2m+2} - \left(m + \frac{1}{2}\right)^{2m+2} - \frac{(2m+2)\nu_{m+2}}{l_{2m+2}}\left[\left(m + \frac{1}{2} + \frac{\nu_{m+2}}{l_{2m+2}}\right)^{2m+1} + \frac{2\nu_{m+2}}{\pi}\left(\tau + \frac{\nu_{m+2}}{l_{2m+2}}\right)^{2m+1}\right]\right\}, \tag{4.34}$$

while for $n = 2m+1$ we have

$$\frac{N_{2m+1}(\tau)}{|M^{2m+1}|} \geq \begin{cases} \frac{\omega_{2m+1}}{(2\pi)^{2m+1}}\left(\tau^{2m+1} - \frac{(4m+2)\nu^2_{m+1}}{l_{2m+1}\pi}\left(\tau + \frac{\nu_{m+1}}{l_{2m+1}}\right)^{2m}\right) - \|G_{2m+1}\|_\infty, \tau \in [0, m], \\ \frac{\omega_{2m+1}}{(2\pi)^{2m+1}}\left((\tau - m)^{2m+1} - \frac{(4m+2)\nu^2_{m+1}}{l_{2m+1}\pi}\left(\tau + \frac{\nu_{m+1}}{l_{2m+1}}\right)^{2m}\right) - \|G_{2m+1}\|_\infty, \tau > m. \end{cases} \tag{4.35}$$

4.2.1 Estimates of eigenfunctions and heat trace for the Laplacian on compact manifolds that are hyperbolic near a point

The above estimates can be used to obtain information about the eigenfunctions. Let us assume that λ^2 is an eigenvalue of the Laplace operator on a closed manifold M^n that is hyperbolic near x. Suppose φ is a corresponding normalised eigenfunction, then by the definition of a local counting function we have

$$|\varphi(x)|^2 \leq \limsup_{\tau\to\lambda^+} N_{n,x}(\tau) - \liminf_{\tau\to\lambda^-} N_{n,x}(\tau) \tag{4.36}$$

where the equality holds only for single eigenvalues. Let us use inequalities (4.25), (4.26) one more time to show

$$|\varphi(x)|^2 \leq \frac{4}{\pi d(x)}\rho_{d(x),0} * \partial_\lambda p_a(n,\lambda) + \|G_n\|_\infty. \tag{4.37}$$

Let us summarise these considerations.

Corollary 4.2.6. *Let (M^n, g) be a Riemannian manifold and let $x \in M$ be a point such that the metric ball of radius $d(x)/2$ is isometric to a ball in $\mathbb{H}^n$. Let φ be an eigenfunction of the Laplacian on M^n with eigenvalue λ^2, then*

$$|\varphi(x)|^2 \leq \frac{8n\ \nu^2_{\lfloor\frac{n+2}{2}\rfloor}\omega_n}{d(x)(2\pi)^{n+1}}\left(\lambda + \frac{\nu_{\lfloor\frac{n+2}{2}\rfloor}}{d(x)}\right)^{n-1} + \|G_n\|_\infty,$$

where the supremum norm of G_n is given by (4.19).

Proof. Notice that $\rho_{d(x),0} \geq 0$ and by (4.24) we have $\partial_\lambda p_a(n,\lambda) \leq \frac{n\omega_n}{(2\pi)^n}\lambda_+^{n-1}$. Therefore (4.37) implies that

$$|\varphi(x)|^2 \leq \frac{4\omega_n n}{\pi d(x)(2\pi)^n}\rho_{d(x),0} * \lambda_+^{n-1} + \|G_n\|_\infty.$$

now Lemma 2.3.3 and Corollary 2.3.4 finish the proof. □

Similarly bounds for the local heat trace may be obtained. For $t > 0$ the local heat trace $k_t(x)$ is defined as the diagonal of the integral kernel of the heat operator and can be expressed in terms of the eigenfunctions as

$$k_t(x) = \sum_{\lambda_j \geq 0} e^{-\lambda_j^2 t}|\varphi_j(x)|^2.$$

Following [46] let us denote by R_t^c the remainder of a truncated series at $c > 0$:

$$R_t^c(x) = \sum_{\lambda_j^2 \geq c} e^{-\lambda_j^2 t}|\varphi_j(x)|^2.$$

It is often necessary to estimate this quantity in numerical computations if only finitely many eigenvalues are available. The quantity $R_t^c(x)$ represents the error made when the expansion series for the local heat trace is truncated. Let us introduce a rescaled counting function $\tilde{N}_x(\tau) := N_x(\sqrt{\tau})$. Then, the remainder, R_t^c, in terms of rescaled counting function is given by $R_t^c(x) = \int_c^\infty \tilde{N}_x'(\tau)e^{-t\tau}\,\mathrm{d}\tau$. By integration by parts we obtain

$$R_t^c(x) = -\lim_{\varepsilon\to 0^-} \tilde{N}_x(c+\varepsilon)e^{-ct} + t\int_c^\infty \tilde{N}_x(\tau)e^{-t\tau}\,\mathrm{d}\tau. \tag{4.38}$$

As usual the incomplete gamma function, $\Gamma : \mathbb{C}\times\mathbb{R}_+ \to \mathbb{C}$, is defined by

$$\Gamma(z,r) := \int_r^\infty e^{-t}t^{z-1}\,\mathrm{d}t.$$

Of course $\Gamma(z,0)$ is just the usual gamma function. Equation (4.38) together with our estimate (4.20) imply the following bound on the remainder

$$\begin{aligned} R_t^c(x) \;\leq\; & -\lim_{\varepsilon\to 0^-} \tilde{N}_x(c+\varepsilon)e^{-ct} + \\ & c_1 t^{-\frac{n}{2}}\Gamma\left(\frac{n}{2}+1, tc\right) + c_1c_2\sum_{l=0}^{n-1}\binom{n-1}{l}c_3^{n-1-l}t^{-\frac{l}{2}}\Gamma\left(\frac{l}{2}+1,tc\right), \end{aligned}$$

where $c_1 = \omega_n/(2\pi)^n$, $c_2 = n(2\nu^2_{\lfloor\frac{n+2}{2}\rfloor} + \pi\nu_{\lfloor\frac{n+2}{2}\rfloor})/(d(x)\pi)$, $c_3 = \nu_{\lfloor\frac{n+2}{2}\rfloor}/d(x)$.

Because $N_x(0^-) = 0$ and $k_t(x) = R_t^0(x)$ the inequality above implies the following theorem:

Theorem 4.2.7. *Let (M^n, g) be a Riemannian manifold and let $x \in M$ be a point such that the metric ball of radius $d(x)/2$ is isometric to a ball in $\mathbb{H}^n$. Then the local heat trace satisfies the estimate*

$$k_t(x) \leq \frac{\omega_n}{(2\pi)^n}\left[\Gamma\left(\frac{n+2}{2}\right)t^{-\frac{n}{2}} + \frac{n\Gamma\left(\frac{n+1}{2}\right)(2\nu^2_{\lfloor\frac{n+2}{2}\rfloor}+\pi\nu_{\lfloor\frac{n+2}{2}\rfloor})}{d(x)\pi}\left(\frac{1}{\sqrt{t}}+\frac{\nu_{\lfloor\frac{n+2}{2}\rfloor}}{d(x)}\right)^{n-1}\right],$$

for x in the ball locally isometric to the hyperbolic n-space, $d(x)$ is a twice of the maximal radius of this ball.

For hyperbolic manifolds integration over x gives a bound on the heat trace.

Corollary 4.2.8. *Let M^n be a closed connected hyperbolic manifold of dimension n. Then the heat trace $k_t = \mathrm{tr}(e^{-t\Delta})$ satisfies the estimate*

$$k_t \leq \frac{\omega_n|M^n|}{(2\pi)^n}\left[\Gamma\left(\frac{n+2}{2}\right)t^{-\frac{n}{2}} + \frac{n\Gamma\left(\frac{n+1}{2}\right)(2\nu^2_{\lfloor\frac{n+2}{2}\rfloor}+\pi\nu_{\lfloor\frac{n+2}{2}\rfloor})}{l_n\pi}\left(\frac{1}{\sqrt{t}}+\frac{\nu_{\lfloor\frac{n+2}{2}\rfloor}}{l_n}\right)^{n-1}\right],$$

where l_n is the length of the shortest geodesic in M^n.

4.2.2 Estimates of the derivatives of eigenfunctions

In the previous subsection obtained bounds for the eigenfunctions of the Laplace operator. A similar technique may be used to obtain bounds for the derivatives of the eigenfunctions. Assuming that M^n is hyperbolic near x notice that

$$\sum_{j=0}^{\infty} h(r_j)|\vec{v}_x\varphi_j|^2 = -\frac{1}{2}\partial_u\, k_n(x,y)|_{u=0}\,, \tag{4.39}$$

where $\vec{v}_x$ is a tangent vector at a point x.

Definition 4.2.9. Let $\nabla^l\varphi$ denotes l-th covariant derivative of $\varphi \in C^\infty(M^n)$. Define the absolute value of l-th covariant derivative of φ by

$$|\nabla^l\varphi|^2 := \mathsf{g}^{i_1j_1}\mathsf{g}^{i_2j_2}\dots\mathsf{g}^{i_lj_l}(\nabla^l\varphi)_{i_1i_2\dots i_l}\overline{(\nabla^l\varphi)_{j_1j_2\dots j_l}}$$

for $l \in \mathbb{N}^+$.

Define

$$N^l_{n,x}(\tau) := \sum_{j=0}^{\infty} H(\tau^2-\lambda_j^2)|\nabla^l\varphi_j(x)|^2.$$

Equation (4.39) implies the following theorem about the function N^1_x.

Lemma 4.2.10. *The cosine transform of the derivative of $N^1_{n,x}$ coincides in the interval $(-d(x), d(x))$ with the cosine transform of the derivative of F^1_n for $n \in \mathbb{N}_+$, $n \geq 2$ where*

$$\begin{aligned}
{F^1_{2m+2}}'(\tau) &= \\
&\frac{2H\left(\tau-m-\frac{1}{2}\right)}{(4\pi)^{m+1}m!}\tanh\left(\pi\sqrt{\tau^2-\left(m+\frac{1}{2}\right)^2}\right)\tau^3\prod_{l=0}^{m-1}\left(\tau^2-m^2+l^2-m+l\right),\\
{F^1_{2m+1}}'(\tau) &= \frac{2H(\tau-m)}{(2\pi)^{m+1}(2m-1)!!}\tau^3\sqrt{\tau^2-m^2}\prod_{l=1}^{m-1}(\tau^2+l^2-m^2),\\
F_n(0) &= 0.
\end{aligned}$$

This Lemma gives us a tool to estimate the first derivative of an eigenfunction; an application of this result may be found in [46]. Let us define ${G^1_n}'(\tau)$ as the part of the Laurent series of ${F^1_n}'(\tau)$ with negative powers of τ multiplied by a characteristic function of the set $[(n-1)/2, \infty)$. Set $G^1_{2m+1}(0) = 0$. The function ${G^1_n}'$ is non-positive, therefore

$$\rho_\delta * G^1_n(\tau) \leq 0, \quad \rho_{\delta,0} * {G^1_n}' \leq 0 \tag{4.40}$$

for $\delta > 0$. By Young's inequality we obtain

$$|\rho_\delta * G^1_n(\tau)| \leq \|G^1_n\|_\infty. \tag{4.41}$$

Moreover, one can show that G_n^1 is bounded and

$$\|G_n^1\|_\infty \leq \begin{cases} \frac{17}{1920\pi}, & n=2, \\ \frac{11}{240\pi^2}, & n=3, \\ \min\left\{ \frac{2\left(\left(m-\frac{1}{2}\right)^2+\frac{1}{4\pi^2}\right)^m (2m+3)!}{\pi(4\pi)^{m+4}m!}, \frac{e^{2\pi(m+1/2)}(2m+3)!}{m!2^{4m+5}}\pi^{2m+5} \right\}, & n=2m+2, \\ \frac{11}{60}\frac{m^{2m+3}(1-m^4)+(2m-1)!m^4(1+m^{2m-2})}{(2\pi)^{m+1}(2m-1)!!(1-m^4)}, & n=2m+1 \text{ and } m \text{ odd}, \\ \frac{11}{60}\frac{m^{2m+3}(1-m^4)+(2m-1)!m^6(1+m^{2m-4})}{(2\pi)^{m+1}(2m-1)!!(1-m^4)}, & n=2m+1 \text{ and } m \text{ even}. \end{cases}$$

By the definition of $N_{n,x}^1$ we have

$$|\nabla\varphi(x)|^2 \leq \overline{\lim_{\tau\to\lambda^+}} N_{n,x}^1(\tau) - \underline{\lim_{\tau\to\lambda^-}} N_{n,x}^1(\tau),$$

where equality holds only for simple eigenvalues. By a similar argument as before

$$|\nabla\varphi(x)|^2 \leq \frac{8(n+2)\ \nu^2_{\lfloor\frac{n+4}{2}\rfloor}\omega_n}{d(x)(2\pi)^{n+1}}\left(\lambda+\frac{\nu_{\lfloor\frac{n+4}{2}\rfloor}}{d(x)}\right)^{n+1} + \|G_n^1\|_\infty.$$

Example 4.2.11. Suppose that φ is an eigenfunction of the Laplacian wit eigenvalue λ^2 on a compact Riemannian manifold M^n hyperbolic near $x \in M^n$ and let $d(x)$ be as before. Then

$$\begin{aligned} |\nabla\varphi(x)|^2 &\leq \frac{4\ \nu_3^2}{d(x)\pi^2}\left(\lambda+\frac{\nu_3}{d(x)}\right)^3 + \frac{17}{1920\pi} \quad \text{for } n=2 \\ |\nabla\varphi(x)|^2 &\leq \frac{10\ \nu_3^2}{3d(x)\pi^3}\left(\lambda+\frac{\nu_3}{d(x)}\right)^4 + \frac{11}{240\pi^2} \quad \text{for } n=3 \\ |\nabla\varphi(x)|^2 &\leq \frac{3\nu_4^2}{4d(x)\pi^3}\left(\lambda+\frac{\nu_4}{d(x)}\right)^5 + \frac{367}{64512\pi^2} \quad \text{for } n=4 \end{aligned}$$

4.2.3 Estimates of higher derivatives of eigenfunctions on hyperbolic surfaces

Adopting the method from [46], we will derive bounds on the higher derivatives of the eigenfunctions on hyperbolic surfaces. By the Selberg pre-trace formula and finite

propagation speed, the cosine transform of $\partial_\tau N^2_{2,x}(\tau)$ coincides with the cosine transform of the function $\partial_\tau F^2_2(\tau)$ on the interval $(-d(z), d(z))$, where

$$F^2_2(0) = 0, \quad F^{2\prime}_2(\tau) = \frac{1}{2\pi} H(\tau^2 - 1/4)(|\tau|^3 + |\tau|^5) \tanh(\pi\sqrt{\tau^2 - 1/4}).$$

Taking into account the facts that

$$\begin{aligned} \int_{1/2}^{\infty} \tau^3 (\tanh(\pi\sqrt{\tau^2 - 1/4}) - 1)\, \mathrm{d}\tau &= -\frac{17}{960}, \\ \int_{1/2}^{\infty} \tau^5 (\tanh(\pi\sqrt{\tau^2 - 1/4}) - 1)\, \mathrm{d}\tau &= -\frac{407}{40320}, \end{aligned}$$

we get the following estimate:

$$-\frac{29}{1260\pi} \leq \operatorname{sign}(\tau) G^2_2(\tau) \leq 0, \tag{4.42}$$

where $G^2_2(\tau) := F^2_2(\tau) - \operatorname{sign}(\tau)\frac{1}{8\pi}\tau^4 - \operatorname{sign}(\tau)\frac{1}{12\pi}\tau^6$. The Fourier Tauberian Theorem 1.3 in [42] and estimate (4.42) implies the following estimates for $N^2_{2,x}$.

Theorem 4.2.12. *The function $N^2_{2,x}$ satisfies*

$$\begin{aligned} N^2_{2,x}(\tau) &\leq \frac{1}{12\pi}\left(\tau^6 + \frac{12\nu_4^2 + 6\pi\nu_4}{\pi d(x)}\left(\tau + \frac{\nu_4}{d(x)}\right)^5\right) \\ &\quad + \frac{1}{8\pi}\left(\tau^4 + \frac{8\nu_3^2 + 4\pi\nu_3}{d(x)\pi}\left(\tau + \frac{\nu_3}{d(x)}\right)^3\right) + \frac{29}{1260\pi}, \\ N^2_{2,x}(\tau) &\geq \frac{1}{12\pi}\left(\tau^6 - \frac{12\nu_4^2}{\pi d(x)}\left(\tau + \frac{\nu_4}{d(x)}\right)^5\right) + \frac{1}{8\pi}\left(\tau^4 - \frac{8\nu_3^2}{d(x)\pi}\left(\tau + \frac{\nu_3}{d(x)}\right)^3\right) - \frac{29}{1260\pi}. \end{aligned}$$

The theorem above implies the following estimate on the derivatives of eigenfunctions.

Corollary 4.2.13. *Let φ be a normalised eigenfunction of the Laplace operator on a compact manifold M^2 with eigenvalue λ. Then*

$$\begin{aligned} |\nabla^2\varphi(x)|^2 &\leq \frac{1}{12\pi}\left(\frac{24\nu_4^2 + 6\pi\nu_4}{\pi d(z)}\left(\lambda + \frac{\nu_4}{d(x)}\right)^5\right) \\ &\quad + \frac{1}{8\pi}\left(\frac{16\nu_3^2 + 4\pi\nu_3}{d(x)\pi}\left(\lambda + \frac{\nu_3}{d(x)}\right)^3\right) + \frac{29}{630\pi}. \end{aligned}$$

The same method can be used to get bounds for higher derivatives of eigenfunctions. Again by Selberg's pre-trace formula and finite propagation speed one gets that the cosine transform of $\partial_\tau N^3_{2,x}(\tau)$ coincides with the cosine transform of the function $\partial_\tau F^3_2(\tau)$ on the interval $(-d(x), d(x))$, where

$$F_{2,3}(0) = 0, \quad F'_{2,3}(\tau) = \frac{1}{2\pi} H(\tau^2 - 1/4)(4|\tau|^3 + 3|\tau|^5 + |\tau|^7) \tanh(\pi\sqrt{\tau^2 - 1/4}).$$

Note that

$$\int_{1/2}^{\infty} \tau^7(\tanh(\pi\sqrt{\tau^2-1/4})-1)\,\mathrm{d}\tau = -\frac{1943}{215040},$$

and so

$$\|G_2^3\|_\infty = \frac{2467}{26880\pi}, \tag{4.43}$$

where $G_2^3(\tau) = F_2^3(\tau) - \mathrm{sign}(\tau)\frac{1}{16\pi}\tau^8 - \mathrm{sign}(\tau)\frac{1}{4\pi}\tau^6 - \mathrm{sign}(\tau)\frac{1}{2\pi}\tau^4$. Taking into account Fourier Tauberian Theorem 2.3.1 and estimate (4.43) we obtain the following:

Theorem 4.2.14. *The function $N_{2,x}^3$ satisfies*

$$\begin{aligned}
N_{2,x}^3(\tau) &\le \frac{1}{16\pi}\left(\tau^8+\frac{16\nu_5^2+8\pi\nu_5}{\pi d(x)}\left(\tau+\frac{\nu_5}{d(x)}\right)^7\right)+\frac{1}{4\pi}\left(\tau^6+\frac{12\nu_4^2+6\pi\nu_4}{\pi d(x)}\left(\tau+\frac{\nu_4}{d(x)}\right)^5\right)\\
&\quad+\frac{1}{2\pi}\left(\tau^4+\frac{8\nu_3^2+4\pi\nu_3}{d(x)\pi}\left(\tau+\frac{\nu_3}{d(x)}\right)^3\right)+\frac{2467}{26880\pi},\\
N_{2,x}^3(\tau) &\ge \frac{1}{16\pi}\left(\tau^8-\frac{16\nu_5^2}{\pi d(x)}\left(\tau+\frac{\nu_5}{d(x)}\right)^7\right)+\frac{1}{4\pi}\left(\tau^6-\frac{12\nu_4^2}{\pi d(x)}\left(\tau+\frac{\nu_4}{d(x)}\right)^5\right)\\
&\quad+\frac{1}{2\pi}\left(\tau^4-\frac{8\nu_3^2}{d(x)\pi}\left(\tau+\frac{\nu_3}{d(x)}\right)^3\right)-\frac{2467}{26880\pi}.
\end{aligned}$$

Corollary 4.2.15. *Let φ be a normalised eigenfunction of the Laplace operator with eigenvalue λ^2. Then*

$$\begin{aligned}
|\nabla^3\varphi(x)|^2 &\le \frac{1}{2\pi^2 d(x)}\left((4\nu_5^2+\pi\nu_5)\left(\lambda+\frac{\nu_5}{d(x)}\right)^7+(12\nu_4^2+3\pi\nu_4)\left(\lambda+\frac{\nu_4}{d(x)}\right)^5+\right.\\
&\quad\left.(16\nu_3^2+4\pi\nu_3)\left(\lambda+\frac{\nu_3}{d(x)}\right)^3\right)+\frac{2467}{13440\pi}.
\end{aligned}$$

Similarly the cosine transform of $\partial_\tau N_z^l(\tau)$ coincides with the cosine transform of the

function $\partial_\tau F_2^l(\tau)$ on the interval $(-d(z), d(z))$, where

$$
\begin{aligned}
F_2^l(0) &= 0, \quad \text{for } l = 4,5,6,7,8\\
\partial_\tau F_2^4(\tau) &= \frac{1}{2\pi} H(\tau^2 - 1/4)\tanh(\pi\sqrt{\tau^2-1/4})(32|\tau|^3 + 23|\tau|^5 + 6|\tau|^7 + |\tau|^9),\\
\partial_\tau F_2^5(\tau) &= \frac{1}{2\pi} H(\tau^2 - 1/4)\tanh(\pi\sqrt{\tau^2-1/4})(328|\tau|^3 + 280|\tau|^5 + 75|\tau|^7 + 10|\tau|^9 + |\tau|^{11}),\\
\partial_\tau F_2^6(\tau) &= \frac{1}{2\pi} H(\tau^2 - 1/4)\tanh(\pi\sqrt{\tau^2-1/4})\left(5752|\tau|^3 + 5040|\tau|^5 + 1399|\tau|^7 + 185|\tau|^9 + \right.\\
&\qquad \left. 15|\tau|^{11} + |\tau|^{13}\right),\\
\partial_\tau F_2^7(\tau) &= \frac{1}{2\pi} H(\tau^2 - 1/4)\tanh(\pi\sqrt{\tau^2-1/4})\left(140944|\tau|^3 + 125864|\tau|^5 + 36096|\tau|^7 + \right.\\
&\qquad \left. 4893|\tau|^9 + 385|\tau|^{11} + 21|\tau|^{13} + |\tau|^{15}\right)\\
\partial_\tau F_2^8(\tau) &= \frac{1}{2\pi} H(\tau^2 - 1/4)\tanh(\pi\sqrt{\tau^2-1/4})\left(4883472|\tau|^3 + 4419704|\tau|^5 + 1299288|\tau|^7 + \right.\\
&\qquad \left. 181275|\tau|^9 + 7231|\tau|^{11} + 189|\tau|^{13} + 15|\tau|^{15} + |\tau|^{17}\right).
\end{aligned}
$$

This information might be used to give bounds in up to the C^8 norm for the eigenfunction on a hyperbolic manifold of dimension 2.

4.3 Applications

As an example of an application, we show how the contribution to the length spectrum in Selberg's trace formula can be bounded. For a compact hyperbolic surface M Selberg's trace formula states (see e.g. [35])

$$
\operatorname{tr}(e^{-\Delta t}) = \frac{|M|e^{-t/4}}{4\pi t}\int_0^\infty \frac{\pi e^{-r^2 t}}{\cosh^2(\pi r)}\mathrm{d}r + \frac{e^{-t/4}}{2\sqrt{\pi t}}\sum_{n=1}^{\infty}\sum_{\gamma}\frac{l(\gamma)e^{\frac{-n^2 l(\gamma)^2}{4t}}}{2\sinh\frac{nl(\gamma)}{2}}, \tag{4.44}
$$

where the sum is over the set of primitive closed geodesics γ. The first term can be computed and does not depend on the geometry of the manifold. Let us denote by l the length of the shortest closed geodesic. Then because each term in (4.44) is positive, the second term is bounded for $t < T < \sqrt{l^2+1} - 1$ by

$$
F_T(t) = \sqrt{\frac{T}{t}}\operatorname{tr}(e^{-\Delta T})e^{\frac{T}{4}+\frac{l^2}{4T}}e^{\frac{-l^2}{4t}},
$$

and rapidly decreasing in t as $t \to 0^+$. Our estimates (4.30), (4.27) of the local counting function and the equality (4.38) imply that we have the following bound on a function R_t^c:

$$
R_t^c(x) \le \frac{1}{4\pi}\left[\frac{\Gamma(2,tc)}{t} + \frac{c_1\Gamma(3/2,tc)}{t^{1/2}} + e^{-ct}\left(-c + \frac{c^{1/2}4\nu_2^2}{\pi d(x)} + \frac{8\nu_2^3 + 2\nu_2^2\pi}{\pi d(x)^2} + \frac{1}{12}\right)\right], \tag{4.45}
$$

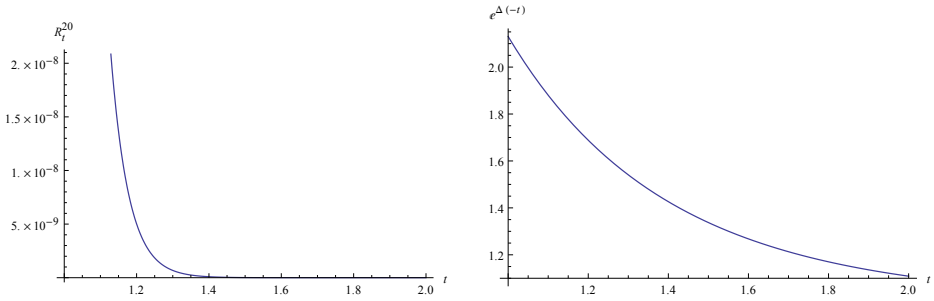

Figure 4.1: The function R_t^{20} on the left and the approximation of the heat trace on the right.

where Γ is the incomplete gamma function, $c_1 = \frac{(4\nu_2^2+2\nu_2\pi)}{\pi d(x)}$, $d(x)$ is twice the injectivity radius and ν_2 is the first non-zero solution to the equation $\cos(\lambda)\cosh(\lambda) = 1$.

Example 4.3.1. The Bolza surface is a compact hyperbolic surface of genus 2 which maximises the order of the symmetry group in this genus. The shortest simple closed geodesic has length $2\cosh(1+\sqrt{2})$, whereas the volume of the manifold is 4π. On can see that the integral over the manifold of $R_t^c(x)$ gives us the error which we make computing the heat trace when we know only finitely many eigenvalues. The estimate for R_t^c blows up near the origin but it gives a good approximation for large t, which may been seen on the plot of R_t^c when only 20 first eigenvalues are known. We used the list of eigenvalues obtained by Strohmaier and Uski [46], which may be found at: `http://homepages.lboro.ac.uk/~maas3/publications/eigdata/eig-bolza-refined0-1000.dat`

Because we obtained the estimate of the local heat trace, we have for a hyperbolic surface of genus $\mathtt{g}$

$$F_T(t) \leq \frac{\mathtt{g}-1}{\sqrt{t}} e^{\frac{T}{4}+\frac{l^2}{4T}-\frac{l^2}{4t}} \left(\frac{1}{\sqrt{T}} + \frac{2\nu_2^2+\nu_2\pi}{\sqrt{\pi}l} + \sqrt{T}\left(\frac{4\nu_2^3+2\nu_2^2\pi}{\pi l^2} \right) \right).$$

This shows that for small $t > 0$ the main contribution to the heat trace is from the first term of the expansion (4.44), which does not depend on the geometry of the manifold. By our heat trace estimates we can actually estimate this quantity.

Estimates may also be obtained for the spectral determinant of Δ. Let us consider the spectral zeta function, $\zeta_\Delta(s)$, defined as the meromorphic continuation of the function

$$\zeta_\Delta(s) = \sum_{\lambda_i \neq 0}^{\infty} \lambda_i^{-2s},$$

where λ_i^2 are the eigenvalues of the Laplace operator. Then the zeta-regularised determinant $\det_\zeta(\Delta)$ of the Laplacian is defined by

$$\log(\det_\zeta(\Delta)) = -\zeta'_\Delta(0).$$

Because 0 is not a pole of ζ_Δ the zeta-regularised determinant is well defined. Strohmaier and Uski [46] showed that

$$\zeta'_\Delta(0) = L_1^\epsilon + L_2^\epsilon + L_3^\epsilon,$$

where

$$\begin{aligned}
L_1^\epsilon &= \sum_{i=1}^{\infty} \Gamma(0, \epsilon\lambda_i^2), \\
L_2^\epsilon &= -\frac{|M|}{4\pi\epsilon} - \left(\frac{|M|}{12\pi} + 1\right)(\tilde{\gamma} + \log(\epsilon)) + \\
&\frac{|M|}{4}\int_0^\infty \operatorname{sech}^2(\pi r)\left(\frac{1 - \mathrm{E}_2(\epsilon(r^2+1/4))}{\epsilon} + \left(r^2 + \frac{1}{4}\right)(\tilde{\gamma} - 1 + \log(\epsilon(r^2+1/4)))\right)\mathrm{d}r, \\
L_3^\epsilon &= \sum_{n=1}^{\infty}\sum_{\gamma}\int_0^\epsilon e^{-\frac{t}{4}} \frac{l(\gamma)e^{-\frac{n^2 l(\gamma)^2}{4t}}}{4\sqrt{\pi}t^{3/2}\sinh(\frac{1}{2}nl(\gamma))}\mathrm{d}t.
\end{aligned}$$

Here $\tilde{\gamma}$ is the Euler constant, $\mathrm{E}_2(x)$ is the generalised exponential integral which is given by the formula

$$\mathrm{E}_n(x) = \int_1^\infty e^{-xt}t^{-n}\,\mathrm{d}t.$$

Let us assume that we have the list of eigenvalues up to c, i.e., we know $\{\lambda_i^2 | \lambda_i^2 \leq c\}$ and we know the constant l. Then we can give a good estimate for $\log\det_\zeta(\Delta)$. That integral L_2^ϵ can be evaluated with high accuracy with numerical integration. We can split L_1 into

$$L_1^\epsilon = \sum_{0<\lambda_i^2\leq c} \Gamma(0, \epsilon\lambda_i^2) + \int_\epsilon^\infty t^{-1}R_t^c\,\mathrm{d}t.$$

By the (4.45) we know that the integral in the formula above is bounded by

$$(g-1)\left[\left(-c + \frac{\sqrt{c}4\nu_2^2}{\pi l} + \frac{8\nu_2^3 + 2\nu_2^2\pi}{\pi l^2} + \frac{1}{12}\right)\Gamma(0, c\epsilon) + \frac{e^{-c\epsilon}}{\epsilon} + \frac{\Gamma(\frac{1}{2}, c\epsilon)(4\nu_2^2 + 2\nu_2\pi)}{\sqrt{\epsilon}\pi l}\right].$$

Note that for $\epsilon \leq T$ we have

$$|L_3^\epsilon| \leq \int_0^\epsilon \frac{F_T(t)}{2t}\mathrm{d}t,$$

therefore L_3^ϵ is bounded by

$$\frac{\mathrm{g}-1}{l}e^{\frac{T}{4}+\frac{l^2}{4T}}\left(\frac{1}{\sqrt{T}} + \frac{2\nu_2^2 + \nu_2\pi}{\sqrt{\pi}l} + \sqrt{T}\left(\frac{4\nu_2^3 + 2\nu_2^2\pi}{\pi l^2}\right)\right)\Gamma\left(\frac{1}{2}, \frac{l^2}{4\epsilon}\right).$$

Summarising we have found:

Corollary 4.3.2. *The spectral determinant of the Laplace operator on a compact, connected hyperbolic manifold of dimension 2 and genus* g *satisfies the lower estimate:*

$$
\begin{aligned}
&-\log(\det{}_\zeta(\Delta)) \leq \\
&\pi(\mathsf{g}-1)\int_0^\infty \operatorname{sech}^2(\pi r)\left(\frac{1-\mathrm{E}_2(\epsilon(r^2+\frac{1}{4}))}{\epsilon}+\left(r^2+\frac{1}{4}\right)\left(\gamma-1+\log\left(\epsilon\left(r^2+\frac{1}{4}\right)\right)\right)\right)\mathrm{d}r- \\
&\frac{(\mathsf{g}-1)}{\epsilon}-\left(\frac{g+2}{3}\right)(\gamma+\log(\epsilon))+\sum_{0<\lambda_i^2\leq c}\Gamma(0,\epsilon\lambda_i^2)+ \\
&(\mathsf{g}-1)\left[\left(-c+\frac{\sqrt{c}4\nu_2^2}{\pi l}+\frac{e^{-c\epsilon}}{\epsilon}+\frac{8\nu_2^3+2\nu_2^2\pi}{\pi l^2}+\frac{1}{12}\right)\Gamma(0,c\epsilon)+\frac{\Gamma(\frac{1}{2},c\epsilon)(4\nu_2^2+2\nu_2\pi)}{\sqrt{\epsilon}\pi l}\right]+ \\
&\frac{\mathsf{g}-1}{l}e^{\frac{T}{4}+\frac{l^2}{4T}}\left(\frac{1}{\sqrt{T}}+\frac{2\nu_2^2+\nu_2\pi}{\sqrt{\pi}l}+\sqrt{T}\left(\frac{4\nu_2^3+2\nu_2^2\pi}{\pi l^2}\right)\right)\Gamma\left(\frac{1}{2},\frac{l^2}{4\epsilon}\right).
\end{aligned}
$$

An upper bound is given by:

$$
\begin{aligned}
&-\log(\det{}_\zeta(\Delta)) \geq \\
&\pi(\mathsf{g}-1)\int_0^\infty \operatorname{sech}^2(\pi r)\left(\frac{1-\mathrm{E}_2(\epsilon(r^2+\frac{1}{4}))}{\epsilon}+\left(r^2+\frac{1}{4}\right)\left(\gamma-1+\log\left(\epsilon\left(r^2+\frac{1}{4}\right)\right)\right)\right)\mathrm{d}r- \\
&\frac{(\mathsf{g}-1)}{\epsilon}-\left(\frac{g+2}{3}\right)(\gamma+\log(\epsilon))+\sum_{0<\lambda_i^2\leq c}\Gamma(0,\epsilon\lambda_i^2)- \\
&\frac{\mathsf{g}-1}{l}e^{\frac{T}{4}+\frac{l^2}{4T}}\left(\frac{1}{\sqrt{T}}+\frac{2\nu_2^2+\nu_2\pi}{\sqrt{\pi}l}+\sqrt{T}\left(\frac{4\nu_2^3+2\nu_2^2\pi}{\pi l^2}\right)\right)\Gamma\left(\frac{1}{2},\frac{l^2}{4\epsilon}\right),
\end{aligned}
$$

for $0<\epsilon\leq T<\sqrt{l^2+1}-1$ *and* $c>0$.

Example 4.3.3. In the example of the Bolza surface the inverse exponent of the sum of the first term in L_1^ϵ and L_2^ϵ yields the numerical value 4.73115. With the same $c=20$ as before and $\epsilon=0.3524$, $T=2.2165$ one obtains upper and the lower bounds 4.88303 and 4.51591 respectively. For $c=50$, $\epsilon=0.22161$, $T=2.2165$ upper and lower estimates are 4.71927 and 4.7253 respectively. The known value is

$$\det{}_\zeta(\Delta)\approx 4.72273280444557.$$

Chapter 5

Riesz means

Let (M, g) denote again a smooth compact n-dimensional Riemannian manifold. We have already mentioned that the Weyl law states that there is a constant $c > 0$ such that the following estimate is satisfied

$$\left| N(\lambda) - \frac{\mathrm{Vol}(B^n)\mathrm{Vol}(M)}{(2\pi)^n}\lambda^n \right| \leq c\lambda^{n-1}, \tag{5.1}$$

where $\mathrm{Vol}(B^n)$ is the volume of the unit ball in $\mathbb{R}^n$. Moreover, in the previous chapter of the dissertation we gave the explicit constant for manifolds of constant negative curvature.

This chapter is devoted to the analysis of some regularisation of the counting function rather than the counting function itself. We will consider the Riesz means i.e.

$$R_k N(\lambda) = \int_{-\infty}^{\lambda} (1 - \tau\lambda^{-1})^k \, \mathrm{d}N(\tau), \quad k = 0, 1, 2, \ldots.$$

As we mentioned in the introduction one can improve the estimate of the remainder term in (5.1) by the factor of $(\log \lambda)^{-1}$ in the case of non-positively curved manifolds. The fact that the Riesz means are more regular than the counting function itself leads to results of Safarov in [43], where he showed that for $k < n$

$$R_k N(\lambda) = \sum_{i=0}^{k} \frac{k!(n-i)!}{(n-i+k)!} a_i \lambda^{n-i} + O\left(\lambda^{n-k-1}\right), \quad \text{as } \lambda \to +\infty,$$

where a_i are constants directly related to the heat coefficients. The logarithmic improvement of the error estimate and Safarov's result motivated us to prove the following:

Theorem 5.0.4. *Let (M, g) be a compact n-dimensional smooth Riemannian manifold. Assume that either $n = 2$ and M has no conjugate points or that M has non-positive sectional curvature. Then for $k \in \{0, 1, 2, \ldots, n-1\}$*

$$R_k N(\lambda) = \sum_{i=0}^{k+1} \frac{k!(n-i)!}{(n-i+k)!} a_i \lambda^{n-i} + O\left(\frac{\lambda^{n-k-1}}{(\log \lambda)^{k+1}}\right), \quad \textit{as } \lambda \to +\infty,$$

where constants a_i may be calculated explicitly. For $k \geq n$ one has

$$R_k N(\lambda) = \sum_{i=0}^{n} \frac{k!(n-i)!}{(n-i+k)!} a_i \lambda^{n-i} + O\left(\lambda^{-1+\epsilon}\right), \quad \text{as } \lambda \to +\infty,$$

for any $\epsilon > 0$.

Note that this result contains Bérard's asymptotic formula, moreover it generalises Hejhal's results from [21] to the case of non-positively curved manifolds.

The Weyl asymptotic formula implies that

$$a_0 = \frac{\mathrm{Vol}(B^n)\mathrm{Vol}(M)}{(2\pi)^n}.$$

Moreover, according to the work of Duistermaat and Guillemin [13] we have that the only coefficients which appear in the expansion are those which have even indices.

Before we will prove the main theorem let us show something much simpler:

Proposition 5.0.5. *Let (M, g) be a compact n-dimensional smooth Riemannian manifold. Assume that either $n = 2$ and M has no conjugate points or that M has non-positive sectional curvature. Then for $k \in \mathbb{N}_0$, $k < n$*

$$R_k N(\lambda) = \sum_{i=0}^{k+1} \frac{k!(n-i)!}{(n-i+k)!} a_i \lambda^{n-i} + O\left(\frac{\lambda^{n-k-1}}{\log \lambda}\right), \quad \text{as } \lambda \to +\infty,$$

where constants a_i may be calculated explicitly and are directly related to the heat invariants.

Let us explore the relation of a_i to the heat trace coefficients, but first let us define what they are. First, the heat equation is given by

$$\begin{cases} \frac{\partial}{\partial t} u(t,x) = -\Delta u, & (t,x) \in \mathbb{R}_+ \times M, \\ u(0,x) = u_0(x), & x \in M. \end{cases}$$

By the functional calculus the solution to this problem in $L^2(M)$ is given by the solution operator $e^{-\Delta t}$, i.e. $u(t,x) = e^{-\Delta t} u_0(x)$. While the heat kernel is an integral kernel of this operator, which for compact manifolds can be expressed in terms of the eigendata:

$$k(t,x,y) = \sum_{i=0}^{\infty} e^{-\lambda_i^2 t} \varphi_i(x) \overline{\varphi_i(y)},$$

where λ_i^2 are the eigenvalues of the Laplace operator and φ_i are the corresponding eigenfunctions. Because the solution operator is trace class for $t > 0$ the following quantity is finite and it is called the heat trace

$$\mathrm{Tr}\left(e^{-\Delta t}\right) := \sum_{i=0}^{\infty} e^{-\lambda_i^2 t}.$$

In [28] Kac showed that for small t the heat trace has a full asymptotic expansion

$$\mathrm{Tr}\left(e^{-\Delta t}\right) \sim \sum_{i=0}^{\infty} c_{\frac{-n+i}{2}} t^{\frac{-n+i}{2}}, \quad \text{as } t \to 0^+. \tag{5.2}$$

Constants $c_{\frac{-n+i}{2}}$ that appear in the expansion are precisely heat trace coefficients. Let us show their relation to constants a_i from Theorem 5.0.4. Note that

$$\mathrm{Tr}\left(e^{-\Delta t}\right) = \int_{\mathbb{R}_+} e^{-st} \tilde{N}'(s)\,\mathrm{d}s,$$

where $\tilde{N}$ is a rescaled counting function of the Laplacian: $\tilde{N}(s^2) = N(s)$. Now, $k+1$-fold integration by parts gives

$$\mathrm{Tr}\left(e^{-\Delta t}\right) = t^{k+1} \int_0^{\infty} e^{-st} \int_0^{s} \int_0^{s_1} \dots \int_0^{s_{k-1}} \tilde{N}(s_k)\mathrm{d}s_k\mathrm{d}s_{k-1} \dots \mathrm{d}s.$$

Because the Riesz means can be understood as a repeated integral divided by an appropriate power of an independent variable one has

$$\mathrm{Tr}\left(e^{-\Delta t}\right) = t^{k+1} \sum_{i=0}^{k+1} \frac{\Gamma(\frac{n-i}{2}+1)}{\Gamma(\frac{n-i}{2}+k+1)} a_i \int_0^{\infty} e^{-st} s^{\frac{n-i}{2}+k}\mathrm{d}s + O\left(t^{\frac{-n+1}{2}+k}\right).$$

Let us change the variable $s \to s/t$, use the definition of the gamma function and compare coefficients of (5.2) with the expansion above in order to note that

$$c_{\frac{-n+i}{2}} = a_i \Gamma\left(\frac{n-i}{2}+1\right)$$

In particular, a_2 on a hyperbolic surface of genus g is given by

$$a_2 = -\frac{g-1}{3}$$

where we used an expansion of a heat kernel from [7] (eq. 9.22 and 9.23). Proposition 5.0.5 shows that for manifolds of non-positive curvature one can improve the Safarov estimate quite easily: Let $\rho \in S(\mathbb{R})$, $\hat{\rho} \in C_0^{\infty}(\mathbb{R})$, $\rho = 1$ in the neighbourhood of zero, then according to the work of Duistermaat and Guillemin [13] we have that the smoothing of the counting function has a full asymptotic expansion, i.e.

$$\rho * N(\lambda) = \sum_{i=0}^{n-1} a_i \lambda^{n-i}. \tag{5.3}$$

whenever the support of ρ is sufficiently small. Moreover, the only coefficients which appear in the expansion are that which have an even indices.

Let us prove the following Tauberian type statement.

Lemma 5.0.6. *Suppose that $\rho \in S(\mathbb{R})$ such that $\hat{\rho} \in C_0^\infty(\mathbb{R})$, $\hat{\rho} = 1/\sqrt{2\pi}$ in the neighbourhood of 0. Let N be a function of a local bounded variation supported on the positive semi-axis, $N \in S'(\mathbb{R})$. If $\rho * N$ has a full asymptotic expansion and $N(\lambda) - \rho * N(\lambda)$ is $O(\lambda^p/\log(\lambda))$ for some $p \in \mathbb{N}$. Then there are constants $c > 0$, $\lambda_0 > e$ such that*

$$\left| \int_0^\lambda N(\mu)\,\mathrm{d}\mu - \int_0^\lambda \rho * N(\mu)\,\mathrm{d}\mu \right| \leq c\left(\frac{\lambda^p}{\log(\lambda)} + 1\right),$$

for $\lambda \geq \lambda_0$.

Proof. Because $\int_{\mathbb{R}} \rho(t)\mathrm{d}t = 1$ one has

$$\int_0^\lambda N(\mu)\,\mathrm{d}\mu - \int_0^\lambda \rho * N(\mu)\,\mathrm{d}\mu = \int_0^\lambda \int_{\mathbb{R}} (N(\mu) - N(\mu - t))\rho(t)\,\mathrm{d}t\,\mathrm{d}\mu.$$

Now, the change of the order of the integration and a fact that N is supported on the positive semi-axis imply that

$$\int_0^\lambda N(\mu)\,\mathrm{d}\mu - \int_0^\lambda \rho * N(\mu)\,\mathrm{d}\mu = \int_{\mathbb{R}} \rho(t) \int_{\lambda - t}^\lambda N(\mu)\,\mathrm{d}\mu\mathrm{d}t. \tag{5.4}$$

Since $\hat{\rho}$ is constant near the origin, we have $\int_{\mathbb{R}} t^k \rho(t)\mathrm{d}t = 0$ for $k = 1, 2, 3, \ldots$ and so $\int_{\mathbb{R}} \rho(t) \int_{\lambda-t}^\lambda \rho * N(\mu)\,\mathrm{d}\mu\mathrm{d}t$ vanishes.

Note that the basic estimate $\nu^p/\log(\nu) \leq \nu^p$ for $\nu \geq e$ implies that

$$(1 + |t|)^{-p-1} \int_{\lambda-t}^\lambda \frac{\nu^p}{\log(\nu)}\mathrm{d}\nu \leq c\lambda^p$$

Since $\int_{\lambda-t}^\lambda \frac{\nu^p}{\log(\nu)}\mathrm{d}\nu$ is a continuous function of t and λ for $\lambda - t \geq e$ and

$$\lim_{\lambda\to\infty} \frac{\int_{\lambda-t}^\lambda \frac{\nu^p}{\log(\nu)}\mathrm{d}\nu - \frac{\lambda^p}{\log(\lambda)}t}{\lambda^{p-1}} = 0$$

we showed that

$$\left| \int_{\lambda-t}^\lambda \frac{\nu^p}{\log(\nu)}\mathrm{d}\nu - \frac{\lambda^p}{\log(\lambda)}t \right| \leq c\left(\lambda^{p-1}(1 + |t|)^{p+1}\right). \tag{5.5}$$

Denote by N_2 the function $N - N * \rho$ and split the integral (5.4) into 2 pieces:

$$\int_{\mathbb{R}} \rho(t) \int_{\lambda-t}^\lambda N_2(\mu)\,\mathrm{d}\mu\mathrm{d}t = \int_{-\infty}^{\lambda-\lambda_0} \rho(t) \int_{\lambda-t}^\lambda \frac{\nu^p}{\log(\nu)}\,\mathrm{d}\mu\mathrm{d}t + \int_{\lambda-\lambda_0}^\infty \rho(t) \int_{\lambda-t}^\lambda N_2(\mu)\,\mathrm{d}\mu\mathrm{d}t$$

Because $\rho \in S(\mathbb{R})$ we have $\|\rho\|_{L^1} = c$. Therefore the estimate (5.5) implies that the first integral on the right hand side of the above equation is bounded by $c\lambda^p/\log(\lambda)$, whereas the second integral is $O(\lambda^{-\infty})$ since $\rho \in S(\mathbb{R})$ and N_2 is a function of local bounded variation, which ends the proof. □

Duistermaat and Guillemin [13] showed that for any $\rho \in S(\mathbb{R})$, $\hat{\rho} \in C_0^\infty(\mathbb{R})$ the convolution of ρ with the counting function of the Laplace operator, N, has a full asymptotic expansion. Moreover Bérard proved that for a compact n-dimensional smooth Riemannian manifold that either $n = 2$ and M has no conjugate points or that M has non-positive sectional curvature the difference $N(\lambda) - \rho * N(\lambda)$ is of order $\lambda^{n-1}/\log(\lambda)$. Therefore the counting function of the Laplace operator on non-positively curved manifolds satisfies the assumptions of Lemma 5.0.6 and that proves Proposition 5.0.5

5.1 Main estimates

Our goal is to estimate the local counting function and its Riesz means for fixed $x \in M$. The bounds will be obtained in the following way: first we will estimate the smoother version of the counting function, i.e. $N_x * \rho$ for some smooth and rapidly decreasing function, and afterwards we will prove that $N_x(\lambda) - N_x * \rho(\lambda)$ is of the same order as $N_x' * \rho$. We will repeat the analogous splitting for the Riesz means.

This is done as usual by exploiting the properties of a suitable parametrix for the operator $e^{it\sqrt{\Delta}}$. In our case we are will use the parametrix for $\cos(t\sqrt{\Delta})$ that was constructed by Bérard in [5]. Let $\tilde{M}$ be the universal covering of M and Γ be the group of automorphisms of covering $\pi : \tilde{M} \to M$. Denote by $C(t, x, y)$ the integral kernel of $\cos(\sqrt{\Delta}t)$ on $\tilde{M}$, then

$$C(t,x,y) = C_0 \sum_{l=0}^{N} (-4)^{-l} u_l(x,y) |t| \left. \frac{\left(d(x,y)^2 - t^2\right)_-^{l-\alpha}}{\Gamma(l+1-\alpha)} \right|_{\alpha=\frac{n+1}{2}} + \tilde{\epsilon}_N(t,x,y), \tag{5.6}$$

where $d(x, y)$ denotes the distance between x and y on $\tilde{M}$, note that regularisations of $x^\alpha/\Gamma((\alpha+1)/2)$, $x^\alpha/\Gamma(\alpha)$, x^α are described in Section 2.2. Moreover the functions u_l and $\Delta_y^m u_l$ have all at most an exponential growth as $d(x, y)$ tends to infinity, i.e. for all l, m exists c such that the following estimate is satisfied:

$$|\Delta_y^m u_l(x,y)| \leq c\, e^{c\, d(x,y)}. \tag{5.7}$$

Moreover for $N \geq \lfloor n/2 \rfloor + 3$, the error term is continuous and bounded:

$$|\tilde{\epsilon}_N(t,x,y)| \leq c\, e^{c|t|}. \tag{5.8}$$

Note that the distributional kernel of $\cos(\sqrt{\Delta})$ on M is given by:

$$C_\Gamma(t,x,y) = \sum_{\gamma \in \Gamma} C(t,x,\gamma y). \tag{5.9}$$

The integral kernel of $\cos(\sqrt{\Delta}t)$ has a finite propagation speed property, i.e. it is supported where $d(x, y) \leq t$. This property implies that for fixed x and y the number of terms in (5.9) is finite for every t. In fact non-positivity of the curvature implies that the number of terms in (5.9) is $O(e^{c|t|})$ for some c which depends only on the curvature

and the dimension of the manifold. Since we are interested in estimates on the local counting function, restrict our considerations to the diagonal, i.e. we fix $x \in M$.

Let us recall that we have already defined the local counting function of the Laplace operator on M as the integral kernel of $1_{\mathbb{R}_+}(\lambda - \sqrt{\Delta})$ on the diagonal, where 1_A denotes the characteristic function of the set A. By functional calculus we have that the operator of our interest may be expressed formally in the following way

$$1_{\mathbb{R}_+}(\lambda - \sqrt{\Delta}) = (2\pi)^{-1/2} \int_{\mathbb{R}} \hat{1}_{\mathbb{R}_+}(t) e^{-it\sqrt{\Delta}+it\lambda} \, \mathrm{d}t$$

Because $1_{(0,+\infty)}(t) = t_+^0$ in the distributional sense, the Fourier transform of $1_{(0,+\infty)}$ is the distribution

$$\frac{-i}{\sqrt{2\pi}(t - i0)} = \frac{-i}{\sqrt{2\pi}t} + \frac{\sqrt{\pi}}{\sqrt{2}}\delta(t),$$

where a precise description of these distribution is given in the first chapter. Therefore the local counting function is the integral kernel of the following operator

$$(2\pi)^{-1} \int_{\mathbb{R}} \frac{e^{it\lambda}}{i(t - i0)} e^{-it\sqrt{\Delta}} \, \mathrm{d}t \tag{5.10}$$

on the diagonal, i.e. where $x = y$.

Since Bérard's paper provides an approximate integral kernel of the even part of $e^{-it\sqrt{\Delta}}$ for t in a finite interval it is reasonable to split this operator into 2 parts by a cut-off function $\hat{\rho}$. From this point assume that $\hat{\rho}$ satisfies the following conditions

1. $\hat{\rho} \in C_0^\infty(\mathbb{R})$,
2. $\hat{\rho}$ is even, i.e. $\hat{\rho}(t) = \hat{\rho}(-t)$,
3. $\hat{\rho}(t) = 1$ for $|t| \leq 1/2$,
4. $\mathrm{supp}(\hat{\rho}) \in [-1, 1]$.

In order to control the support of a test function $\hat{\rho}$ let us introduce its modification:

$$\hat{\rho}_T(t) := \hat{\rho}(t/T), \tag{5.11}$$

then $\mathrm{supp}(\hat{\rho}_T) \in [-T, T]$. Now (5.10) splits into

$$\frac{1}{2\pi} \int_{\mathbb{R}} \frac{\hat{\rho}_T(t)}{i(t - i0)} e^{it\lambda} e^{-it\sqrt{\Delta}} \, \mathrm{d}t + \frac{1}{2\pi} \int_{\mathbb{R}} \frac{1 - \hat{\rho}_T(t)}{it} e^{it\lambda} e^{-it\sqrt{\Delta}} \, \mathrm{d}t \tag{5.12}$$

If we want to translate this splitting in terms of the local counting function, then it equivalent to say that the local counting function splits into

$$N_x(\lambda) = N_x * \rho_T(\lambda) + [N_x * (\delta - \rho_T)](\lambda),$$

where $\rho_T(t) = T\rho(Tt)$. We will split further the first term:

$$N_x(\lambda) = N_x * \rho_\epsilon(\lambda) + [N_x * (\rho_T - \rho_\epsilon)](\lambda) + [N_x * (\delta - \rho_T)](\lambda), \tag{5.13}$$

where $\epsilon > 0$ is a small parameter, which is smaller than the injectivity radius at x. The first term has full asymptotic expansion i.e

$$N_x * \rho_\epsilon(\mu) \sim \sum_{i=0}^{\infty} a_i(x)\mu^{n-i} \tag{5.14}$$

for $\mu \to \infty$ as it was proved in [13] and [26] e.g.. Duistermaat and Guillemin proved that the only coefficients which appear in the expansion are these with an even indices. Because N_x is supported on the positive semi-axis and ρ_ϵ is the Schwartz function $N_x * \rho_\epsilon(\mu)$ is rapidly decreasing as $\nu \to -\infty$.

Let us recall the definition of Riesz means:

$$\begin{aligned} R_k N_x(\lambda) &= \int_{-\infty}^{\lambda} (1 - \tau\lambda^{-1})^k \, \mathrm{d}N_x(\tau) = \lambda^{-k} \int_{-\infty}^{\lambda} (\lambda - \tau)^k \, \mathrm{d}N(\tau) \\ &= \lambda^{-k} \, \tau_+^k * N_x'(\lambda) = \lambda^{-k} k \, \tau_+^{k-1} * N_x(\lambda), \end{aligned}$$

where Γ denotes the gamma function and τ_+^k was defined in the second chapter (for $k > -1$ equation (2.7)). Let us apply Riesz means to (5.13):

$$\begin{aligned} (R_k N_x)(\lambda) &= \lambda^{-k} k \left[\tau_+^{k-1} * (N_x * \rho_\epsilon)\right](\lambda) + \lambda^{-k} k \left[\tau_+^{k-1} * (N_x * (\rho_T - \rho_\epsilon))\right](\lambda) + \\ &\quad \lambda^{-k} k \left[\tau_+^{k-1} * (N_x * (\delta - \rho_T))\right](\lambda). \end{aligned} \tag{5.15}$$

Note that the convolution with the distribution τ_+^k may be understood as a repeated integral from $-\infty$ to λ multiplied by a constant factor, therefore for $k \geq 1$

$$\left[\tau_+^{k-1} * (N_x * \rho_\epsilon)\right](\lambda) = (k-1)! \int_{-\infty}^{\lambda} \int_{-\infty}^{\lambda_1} \dots \int_{-\infty}^{\lambda_{k-1}} (N_x * \rho_\epsilon)(\lambda_k) \, \mathrm{d}\lambda_k \dots \mathrm{d}\lambda_2 \mathrm{d}\lambda_1.$$

Taking it into account together with the full asymptotic expansion (5.14) we get that

$$\lambda^{-k} k \left[\tau_+^{k-1} * (N_x * \rho_\epsilon)\right](\lambda) = \sum_{i=0}^{n} \frac{(n-i)! a_i(x)}{(n-i+k)!} \lambda^{n-i} + O(\lambda^{-1+\epsilon}) \tag{5.16}$$

for any $\epsilon > 0$ as $\lambda \to \infty$.

Note that Bérard's paper provides the formula for the integral kernel of $\cos(\sqrt{\Delta}t)$ for t in a finite interval whereas we need the kernel of $e^{-it\sqrt{\Delta}}$. We can avoid this difficulty in the following way: the Laplace operator is non-negative so the local counting function is supported for $\lambda \geq 0$, let us define

$$N_x^{odd}(\lambda) := N_x(\lambda) - N_x(-\lambda), \quad N_x^{neg}(\lambda) := N_x(-\lambda).$$

It is clear that the functions just defined sum up to N_x. Clearly the convolution, $[(N_x^{odd})*(\rho_T-\rho_\epsilon)](\lambda)$, admits the same asymptotics as $[(N_x)*(\rho_T-\rho_\epsilon)](\lambda)$ as $\lambda\to\infty$. Moreover, the similar considerations as at the beginning of this section show that $[(N_x^{odd})*(\rho_T-\rho_\epsilon)](\lambda)$ is the integral kernel of

$$(2\pi)^{-1}\int_{\mathbb{R}}(\hat\rho_T(t)-\hat\rho_\epsilon(t))t^{-1}\sin(t\lambda)\cos(t\sqrt{\Delta})\,\mathrm{d}t \tag{5.17}$$

on the diagonal, therefore we can make use of Bérard's parametrix (5.6). A similar argument works for the Riesz mean of the counting function. Note that the Fubini theorem implies

$$\tau_+^{k-1}*(N_x*(\rho_T-\rho_\epsilon))(\lambda)=N_x*(\tau_+^{k-1}*(\rho_T-\rho_\epsilon))(\lambda) \tag{5.18}$$

Let as usual $p_{\alpha,\beta}$ be the Schwartz space semi-norms defined by $p_{\alpha,\beta}(f)=\sup_x|x^\alpha\partial_x^\beta f|$. Then it easy to check that for all $T>1$ we have

$$p_{\alpha,\beta}(\hat\rho_T-\hat\rho_\epsilon)\le C_{\alpha,\beta}T^\alpha.$$

Thus, since N_x is supported in the half line, $N_x*(\tau_+^{k-1}*(\rho_T-\rho_\epsilon))(\lambda)=O(T^L\lambda^{-\infty})$ as $\lambda\to-\infty$ for some $L\ge0$. Hence, $N_x*(\tau_+^{k-1}*(\rho_T-\rho_\epsilon))(\lambda)$ and $N_x^{odd}*(\tau_+^{k-1}*(\rho_T-\rho_\epsilon))(\lambda)$ differ by a function of order $O(T^L\lambda^{-\infty})$ as $\lambda\to\infty$ for some $L>0$. Because

$$\widehat{\tau_+^{k-1}}(\xi)=\frac{(-i)^k(k-1)!}{\sqrt{2\pi}}(\xi-i0)^{-k},$$

the function $N_x^{odd}*(\tau_+^{k-1}*(\rho_T-\rho_\epsilon))(\lambda)$ coincides with the integral kernel of

$$\frac{(k-1)!}{2\pi}\mathrm{Re}\int_{\mathbb{R}}\frac{\hat\rho_T(t)-\hat\rho_\epsilon(t)}{(it)^{k+1}}e^{it\lambda}\cos(t\sqrt{\Delta})\,\mathrm{d}t \tag{5.19}$$

restricted to the diagonal.

We will make use of the Berard parametrix to estimate this integral. The error term, ϵ_N contributes by $O[\exp(cT)]$, due to (5.8) and a fact that $\hat\rho_T$ is supported in $[-T,T]$.

Let us focus on the single term in the sum (5.9). The estimate (5.7) implies that there is a constant c_l independent on x such that

$$|u_l(x,\gamma x)|\le c_l\exp(c_lT) \tag{5.20}$$

on the support of $\widehat{\rho_T}$.

Note that the integral kernel of $\cos(t\sqrt{\Delta})$, $(\rho_T-\rho_\epsilon)(t)$ and $\mathrm{Re}((it)^{-k-1}e^{it\lambda})$ are even functions with respect to the independent variable t. Therefore we may restrict our integration to the positive semi-axis and in order to get a bound on (5.19) we need to estimate terms of the following form:

$$\mathrm{Re}\int_0^\infty\frac{\hat\rho_T(t)-\hat\rho_\epsilon(t)}{(it)^k}\left.\frac{\left(d(x,\gamma x)^2-t^2\right)_-^{l-\alpha}}{\Gamma(l+1-\alpha)}\right|_{\alpha=\frac{n+1}{2}}e^{it\lambda}\,\mathrm{d}t \tag{5.21}$$

Note that for $\gamma = \mathrm{id}$ and odd dimension $n = 2m - 1$ the distribution $t^{l-m}/\Gamma(l-m+1)$ is supported at the origin for $l < m$, therefore in this case the paring above is 0, since $\hat{\rho}_T - \hat{\rho}_\epsilon$ is supported for $|t| > \epsilon$ otherwise for $\gamma = \mathrm{id}$ we have the Fourier transform of $C_0^\infty(-T, T)$ function. In the following we use the notation $O_T(g(x))$ for $O(g(x))$ in case the implied constant can be chosen independent of T, i.e. $f(x,T) = O_T(g(x))$ if $|f(x,T)| \leq C|g(x)|$ with C not dependent on T.

Lemma 5.1.1. *Let ρ and ρ_T are as in our considerations. Suppose further that $\epsilon < T$ then*

$$\left| \int_0^\infty (\hat{\rho}_T(t) - \hat{\rho}_\epsilon(t)) t^k e^{it\lambda} \, \mathrm{d}t \right| = O_T(\lambda^{-\infty})$$

as $\lambda \to \infty$ for $T \geq 1$.

Proof. Recall that $\hat{\rho}$ is smooth and compactly supported thus the change of coordinates $t \to Tt$ and η-fold integration by parts implies that the absolute value of $\int (\hat{\rho}_T(t) - \hat{\rho}_\epsilon(t)) t^k e^{it\lambda} \, \mathrm{d}t$ is bounded by

$$\frac{T^{k+1-\eta}}{\lambda^\eta} \int_0^1 \left| \frac{\partial^\eta}{\partial t^\eta} \left[(\hat{\rho}(t) - \hat{\rho}(tT/\epsilon)) t^k \right] \right| \mathrm{d}t.$$

Because $\hat{\rho}(t) = 1$ for $|t| < 1/2$ the integral above splits to

$$\int_{\frac{\epsilon}{T}}^{\frac{2\epsilon}{T}} \left| \frac{\partial^\eta}{\partial t^\eta} \left[\hat{\rho}(tT/\epsilon) t^k \right] \right| \mathrm{d}t + \int_{\frac{2\epsilon}{T}}^{\frac{1}{2}} \left| \frac{\partial^\eta}{\partial t^\eta} t^k \right| \mathrm{d}t + \int_{\frac{1}{2}}^1 \left| \frac{\partial^\eta}{\partial t^\eta} \left[\hat{\rho}(t) t^k \right] \right| \mathrm{d}t.$$

The third term depends neither on λ nor T so is bounded by some positive constant c. For non-negative k, the second term vanishes if $\eta < k$. For negative k the second term is clearly bounded by $c(1 + T^{k-\eta+1})$. Finally, the product rule implies that the first term is bounded by the sum of the following terms:

$$c_i \|\hat{\rho}^{(\eta-i)}\|_\infty T^{\eta-k-1},$$

where i varies from 0 to $\min\{\eta, k\}$ when $k \in \mathbb{N}_0$, and from 0 to η if k is negative. Let us sum it up and multiply by $\lambda^{-\eta}$, then we have that the integral from the argument is bounded by

$$\frac{T^{k+1-\eta}}{\lambda^\eta} \int_0^1 \left| \frac{\partial^\eta}{\partial t^\eta} \left[(\hat{\rho}(t) - \hat{\rho}(tT/\epsilon)) t^k \right] \right| \mathrm{d}t \leq \frac{cT^{k+1-\eta}}{\lambda^\eta} \left(T^{\eta-k-1} + 1 \right), \quad \eta > k.$$

In order to finish the proof it is enough to note that $T^{k+1-\eta} \leq 1$ for $\eta > k$. □

Now, the only terms which left to estimate in the sum (5.9) are such that $d(x, \gamma x) > \epsilon$. For a shorter notation denote by d_γ the distance between x and γx.

Lemma 5.1.2. *For any $T \geq 1 > \epsilon > 0$, $k \geq 0$, and $m \in \mathbb{R}$ there exists an $L > 0$ such that*

$$T^{-L} \int_0^\infty \hat{\rho}_T \, t^{-k} \, \frac{(d^2 - t^2)_-^m}{\Gamma(m+1)} \, e^{i\lambda t} \, dt = O_{T,d}\left(1 + |\lambda|^{-m-1}\right)$$

as $|\lambda| \to \infty$ for all $T > 1$ and d with $\epsilon < d < T$.

Proof. For $m \geq 0$ the estimate follows for $L = 2m+1-k$ immediately from the support properties of $\hat{\rho}_T$ and the fact that $\hat{\rho}_T$ is bounded. It remains to show the estimate in case $m < 0$. Therefore, assume $m < 0$. Let $\phi_T \in C_0^\infty(\mathbb{R}_0)$ such that $\phi_T = 0$ in a neighbourhood of zero, $\phi_T = 1$ on $[d, T]$ and $\phi_T = 0$ on $[2T, \infty)$, such that $\phi_T^{(\beta)} \leq C_\beta$ uniformly in T and d for $T > 1$ and $T > d > \epsilon$. Then one shows easily that the Schwartz semi-norms of $\phi_T \hat{\rho}_T t^{-k}(t+d)^m$ satisfy

$$p_{\alpha,\beta}\left(\phi_T \hat{\rho}_T t^{-k}(t+d)^m\right) \leq C_{\alpha,\beta} T^{\alpha-k}$$

for $T > 1$ and $d > \epsilon$, where $C_{\alpha,\beta}$ is independent of T and d. If ψ_T is the Fourier transform of the function $\phi_T \hat{\rho}_T t^{-k}(t+d)^m$ we therefore have

$$p_{\alpha,\beta}\left(\psi_T\right) \leq \tilde{C}_{\alpha,\beta} T^{\beta-k}$$

On the other hand the Fourier transform of

$$\hat{\rho}_T t^{-k} \frac{(d^2 - t^2)_-^m}{\Gamma(m+1)}$$

is the convolution of ψ_T with the Fourier transform of $\frac{(d-t)_-^m}{\Gamma(m+1)}$. The Fourier transform of the distribution $\frac{(d-t)_-^m}{\Gamma(m+1)}$ can be computed explicitly and is a locally integrable function of order $O_d(\lambda^{-m-1})$. In order to estimate this convolution we use

$$(1 + |\mu - \lambda|)^{-m-1} \leq (1 + |\mu|)^{|m+1|}(1 + |\lambda|)^{-m-1},$$

which is a simple consequence of the triangle inequality. Together with the fact that

$$\int \psi_T(\mu)(1 + |\mu|)^{|m+1|} \mathrm{d}x$$

can be bounded by a multiple of $\sup_\mu (1+\mu^2)^q \psi_T(x)$ for all integers q with $2q > (|m+1|+2)$. Now simply choose $L = -k$ to conclude that the convolution is of order $O_{T,d}(T^L \lambda^{-m-1})$. □

Summing up, the growth rate (5.20) and Lemma 5.1.2 imply that

$$N_x^{odd} * (\chi_+^{k-1} * (\rho_T - \rho_\epsilon))(\lambda) \leq c e^{cT} \lambda^{\frac{n-1}{2}} \tag{5.22}$$

and consequently

$$N_x * (\chi_+^{k-1} * (\rho_T - \rho_\epsilon))(\lambda) \leq c e^{cT} \lambda^{\frac{n-1}{2}}$$

In order to finish the proof of the main theorem we need to estimate $N_x * (\delta - \rho_T)$. This can be done by the following Tauberian argument which was inspired by Theorem B.2.1 in [31]:

Theorem 5.1.3. *Let N be a real-valued, non-decreasing function on $\mathbb{R}$ supported on the positive semi-axis such that $\lambda^p N(\lambda) \to 0$ for some positive $p \in \mathbb{N}_0$. Let ρ_T and $\hat{\rho}_T$ be functions as used throughout this chapter. Then if there exists a non-negative Schwartz function $\tilde{\rho}$, such that $\operatorname{supp}(\hat{\tilde{\rho}}) \subset [-1,1]$ and its modification $\tilde{\rho}_T(t) := T\tilde{\rho}(Tt)$ satisfies*

$$N' * \tilde{\rho}_T(\lambda) \leq c_1(T)\lambda^\nu, \quad \forall \lambda \geq 1,\ T \geq 1,$$

the following estimate is satisfied

$$|N(\lambda) - N * \rho_T| \leq c_1(T)c^{-1}T^{-1}\lambda^\nu, \quad \forall \lambda \geq 1,\ T \geq 1$$

for some constant c independent of λ and T.

Proof. Note that the positivity of the Schwartz function $\tilde{\rho}_T$ implies the following estimate

$$c_1(T)\lambda^\nu \geq (N' * \tilde{\rho}_T)(\lambda) = T\int_{\mathbb{R}} \tilde{\rho}(T(\lambda-\mu))\mathrm{d}N(\mu) \geq T\int_{\lambda-1/T}^{\lambda+1/T} \tilde{\rho}(T\mu)\mathrm{d}N(\mu) \geq$$
$$Tc\int_{\lambda-1/T}^{\lambda+1/T} \mathrm{d}N(\mu) = Tc(N(\lambda+1/T) - N(\lambda-1/T)).$$

Because the function N is non-decreasing it satisfies the following conditions

$$\begin{aligned} N(\lambda+1/T) - N(\lambda) &\leq \tilde{c}(T)\lambda^\nu T^{-1}, \quad \lambda \geq 1, T \geq 1, \\ N(\lambda) - N(\lambda-1/T) &\leq \tilde{c}(T)\lambda^\nu T^{-1}, \quad \lambda \geq 1, T \geq 1, \end{aligned}$$

where $\tilde{c}(T) = c_1(T)/c$. Suppose that μ is a positive real number, then inequalities above imply that

$$\begin{aligned} N\left(\lambda+\frac{\mu}{T}\right) - N(\lambda) &\leq N\left(\lambda+\frac{\lceil\mu\rceil}{T}\right) - N(\lambda) \leq \frac{\tilde{c}(T)}{T}\sum_{i=0}^{\lceil\mu\rceil-1}\left(\lambda+\frac{i}{T}\right)^\nu \leq \\ &\frac{\tilde{c(T)}\lceil\mu\rceil}{T}\left(\lambda+\frac{\lceil\mu\rceil-1}{T}\right)^\nu \leq \frac{\tilde{c}(T)}{T}(\mu+1)(\lambda+\mu)^\nu \leq \frac{\tilde{c}(T)}{T}(\mu+1)^{\nu+1}\lambda^\nu, \\ N(\lambda) - N\left(\lambda-\frac{\mu}{T}\right) &\leq N(\lambda) - N\left(\lambda-\frac{\lceil\mu\rceil}{T}\right) \leq \frac{\tilde{c}(T)}{T}\sum_{i=0}^{\lceil\mu\rceil-1}\left(\lambda-\frac{i}{T}\right)^\nu \leq \\ &\frac{\tilde{c}(T)}{T}\lceil\mu\rceil\lambda^\nu \leq \tilde{c}(T)T^{-1}(\mu+1)\lambda^\nu. \end{aligned}$$

Therefore for $\mu \in \mathbb{R}$ we have

$$\left|N\left(\lambda-\frac{\mu}{T}\right) - N(\lambda)\right| \leq \frac{\tilde{c}(T)\lambda^\nu}{T}(1+|\mu|)^{\nu+1}, \quad \lambda - \frac{\mu-1}{T} \geq 1,\ T \geq 1.$$

In order to prove the estimate from the argument note that

$$\begin{aligned} |N(\lambda) - N * \rho_T| &= \left| \int_{\mathbb{R}} (N(\lambda) - N(\lambda - \mu)) \rho_T(\mu) \mathrm{d}\mu \right| = \\ & \left| \int_{\mathbb{R}} \left(N(\lambda) - N\left(\lambda - \frac{\mu}{T}\right)\right) \rho(\mu) \mathrm{d}\mu \right| \leq \\ & \frac{\tilde{c}(T)\lambda^{\nu}}{T} \left(\int_{-\infty}^{(\lambda-1)T+1} (1+|\mu|)^{\nu+1} |\rho(\mu)| \, \mathrm{d}\mu + N(\lambda) \int_{(\lambda-1)T+1}^{\infty} |\rho(\mu)| \mathrm{d}\mu \right). \end{aligned}$$

Because ρ is a Schwartz function $\int (1+\mu)^{1+\nu} |\rho(\mu)| \mathrm{d}\mu$ is bounded for any $\nu > 0$, moreover it also implies that $\int_{(\lambda-1)T+1}^{\infty} |\rho(\mu)| \mathrm{d}\mu$ is $O(\lambda^{-\infty})$ uniformly in T, which one had to prove. □

Let us define $\hat{\tilde{\rho}} = \zeta * \zeta$, where $\zeta \in C_0^{\infty}(-1/2, 1/2)$. Then of course $\mathrm{supp}(\hat{\tilde{\rho}}) \subset [-1, 1]$ and $\tilde{\rho}$ is a non-negative Schwartz function. As before define $\tilde{\rho}_T(t) := T\tilde{\rho}(Tt)$. Lemma 5.1.2 also holds with the function ρ_T replaced by $\tilde{\rho}_T$ and in the same way as before, using Berard's parametrix one obtains

$$N'_x * \tilde{\rho}_T(\lambda) \leq c\lambda^{n-1} + ce^{cT}\lambda^{\frac{n-1}{2}}.$$

In this estimate the term $c\lambda^{n-1}$ is derived from the contribution of the identity element in Γ. Since this computation is standard we omit the details here. Theorem 5.1.3 then implies that

$$[N_x * (\delta - \rho_T)] \leq \frac{c\lambda^{n-1}}{T} + \frac{ce^{cT}}{T}\lambda^{\frac{n-1}{2}} \tag{5.23}$$

Combining this with equation (5.24) in the case $k = 0$ gives

$$|N_x(\lambda) - \sum_{i=0}^{n-1} a_i(x)\lambda^n| \leq \frac{c\lambda^{n-1}}{T} + ce^{cT}\lambda^{\frac{n-1}{2}} \tag{5.24}$$

What remains in order to prove Theorem 5.0.4 is to estimate the third term in (5.15). In a general case we will use the proposition derived by Safarov in [43], which we adapt to our situation.

Proposition 5.1.4. *There exists a constant $C > 0$ depending only on ρ such that the following statement holds. Suppose N is a function of local bounded variation, $N \in S'(\mathbb{R})$ and assume that there exist $\nu_1, \nu_2 \in \mathbb{R}_+$ such that*

$$|N(\lambda) - \sum_{i=0}^{\infty} a_i \lambda^{n-i}| \leq C_1\lambda^{\nu_1} + C_2\lambda^{\nu_2},$$

where a_i are constants. Then,

$$\left| \int_{-\infty}^{\lambda} N(\mu) - N * \rho_T(\mu) \, \mathrm{d}\mu \right| \leq \frac{C}{T}(C_1\lambda^{\nu_1} + C_2\lambda^{\nu_2}) + C, \quad \lambda \geq 0.$$

Proposition 5.1.4 applied to the estimate (5.24) gives

$$\left[\chi_+^{k-1} * (N_x * (\delta - \rho_T))\right](\lambda) \leq c\left(\frac{\lambda^{n-1}}{T^{k+1}} + 1\right) + ce^{cT}\left(\frac{\lambda^{\frac{n-1}{2}}}{T^k} + 1\right)$$

When we substitute this estimate into (5.15), and use also estimates (5.16), (5.24) we get that

$$\left|R_k N_x(\lambda) - \sum_{i=0}^{n-1} \frac{k!(n-i)!}{(n-i+k)!} a_i \lambda^{n-i}\right| \leq c\left(\frac{\lambda^{n-1-k}}{T^{k+1}} + 1\right) + ce^{cT}\left(\lambda^{\frac{n-1}{2}-k} + 1\right)$$

Since this estimate is valid for $T \geq 1$ and $\lambda \geq \lambda_0$, let us take $T = \alpha \log \lambda$ for some small $\alpha > 0$ to finish the proof of Theorem 5.0.4.

Note that Theorem 5.0.4 is very useful in numerics. It helps to check if the list of computed eigenvalues is complete. Whereas the Weyl type estimates are useless for this purpose, our theorem is very handy. In order to visualise it, let us plot the Riesz mean of a counting function of the Laplace operator on the Bolza surface with the list of eigenvalues derived by Strohmaier [46] minus its asymptotic expansion. The theorem which we derived implies that the function obtained in this example will be of order $(\log \lambda)^{-2}$.

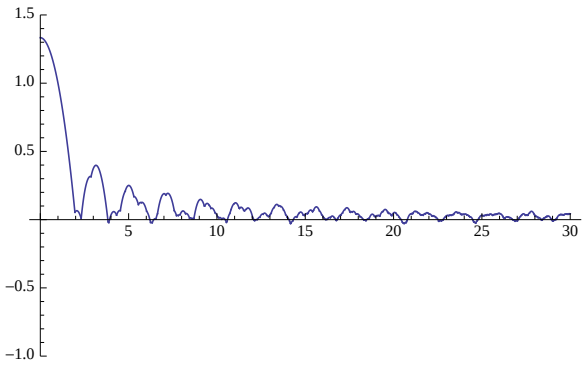

Figure 5.1: $R_1 N(\lambda) - \lambda^2 + 1/6$

The second plot presents the same function but from the list of eigenvalues we removed one number.

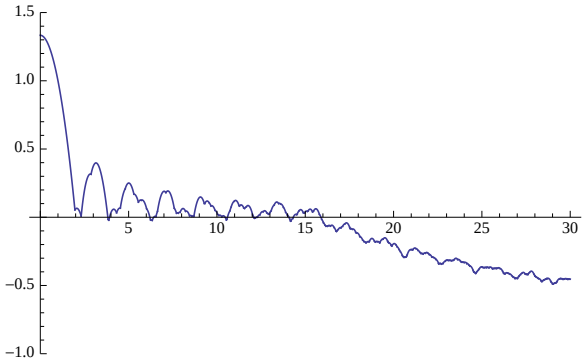

Figure 5.2: $R_1 N^{\text{incorrect}}(\lambda) - \lambda^2 + 1/6$

Simple calculations shows that the missed eigenvalue causes an error $1 - \lambda_i/\lambda$. One can easily see, that the missed eigenvalue is around 15^2. The simpler detectable the error is, the earlier it appears in the available list of eigenvalues, for example if we remove fifth eigenvalue from the list we get the following plot:

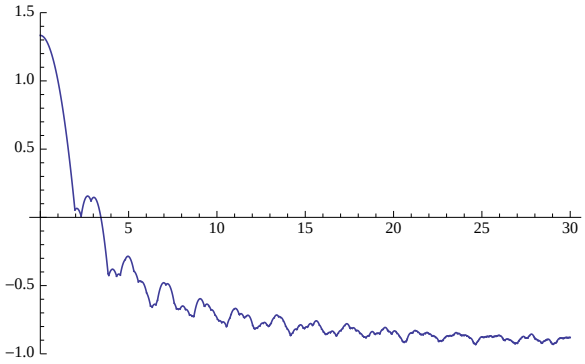

Figure 5.3: $R_1 N^{\text{incorrect}}(\lambda) - \lambda^2 + 1/6$

Chapter 6

Conclusions

Throughout this thesis we have proven many results related to the counting function of the Laplace operator. We gave a lot of explicit estimates on the spectral functions, see: Theorem 4.2.1, Corollary 4.2.6, Theorem 4.2.7, Corollary 4.3.2. These results are very useful for numerics, which was the main motivation for this work. We have also managed to extend existing results about Riesz means to the more general setting (Theorem 5.0.4). The asymptotic, which we obtained, has also an application in numerics. It gives a criterion if our list of eigenvalues is complete.

This work was mainly inspired by three papers: the fourth chapter was inspired by the paper of Safarov [42], where the author gives explicit estimates on the local counting function of the Laplace operator for domains in $\mathbb{R}^n$. We used his result to give analogous estimates in hyperbolic spaces. Moreover we presented how one can use these estimates in order to get an estimate of the heat kernel and a numerical value of the determinant of Δ.

The last chapter was inspired by papers of Safarov [43] and Bérard [5]. The former author sketched the idea of the main proof in the last chapter, the latter provided necessary tools for improving existing results.

The direction of a further study may be finding an explicit constant in our estimates of Riesz means in a case of hyperbolic manifolds. Of course it will not be a straightforward computation but much is known in this setting. One can use an explicit expression for a wave kernel, which will remove an error made by using Bérard's parametrix. Moreover the number of terms in the sum $\sum_\gamma C(t, x, \gamma x)$ was estimated by the size of a ball, which is also known for manifolds of constant curvature.

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