Statistical Inference and Efficient Portfolio Investment Performance

By

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Abstract

Two main methods have been used in mutual funds evaluation. One is portfolio evaluation, and the other is data envelopment analysis (DEA). The history of portfolio evaluation dates from the 1960s with emphasis on both expected return and risk. However, there are many criticisms of traditional portfolio analysis which focus on their sensitivity to chosen benchmarks. Imperfections in portfolio analysis models have led to the exploration of other methodologies to evaluate fund performance, in particular data envelopment analysis (DEA). DEA is a non-parametric methodology for measuring relative performance based on mathematical programming.

Based on the unique characteristics of investment trusts, Morey and Morey (1999) developed a mutual funds efficiency measure in a traditional mean-variance model. It was based on Markowitz portfolio theory and related the non-parametric methodologies to the foundations of traditional performance measurement in mean-variance space. The first application in this thesis is to apply the non-linear programming calculation of the efficient frontier in mean variance space outlined in Morey and Morey (1999) to a new modern data set comprising a multi-year sample of investment funds. One limitation of DEA is the absence of sampling error from the methodology. Therefore the second innovation in this thesis extends Morey and Morey (1999) model by the application of bootstrapped probability density functions in order to develop confidence intervals for the relative performance indicators. This has not previously been achieved for the DEA frontier in mean variance space so that the DEA efficiency scores obtained through Morey and Morey (1999) model have not hitherto been tested for statistical significance. The third application in this thesis is to examine the efficiency of investment trusts in order to analyze the factors contributing to investment trusts' performance and detect the determinants of inefficiency. Robust-OLS regression, Tobit models and Papke-Wooldridge (PW) models are conducted and compared to evaluate contextual variables affecting the performance of investment funds.

From the thesis, new and original Matlab codes designed for Morey and Morey (1999) models are presented. With the Matlab codes, not only the results are obtained, but also how this quadratic model is programming could be very clearly seen, with all the details revealed.

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List of Abbreviations

CAPM – Capital Asset Pricing Model

CRS – Constant Returns to Scale

DEA – Data Envelopment Analysis

DRS – Decreasing Returns to Scale

ETF – Exchange Traded Funds

FTSE– Financial Times Stock Exchange

IRS – Increasing Returns to Scale

MLE – Maximum Likelihood Estimation

OEIC – Open Ended Investment Company

OLS – Ordinary Least Squares

RTS – Returns to Scale

Chapter 1 Introduction

1.1 Introduction and motivation

UK fund market is very large; according to Morningstar, there are more than 32,000 funds available in the UK market. The evaluation of mutual funds is of considerable importance. First, it is to see how well the mutual funds industry as a whole has performed in order to define their advantages as investment vehicles. Second, the evaluation results could help investors select better performing funds. Furthermore, the evaluation process motivates mutual fund companies to generate and report superior returns because investment dollars usually flow into top performing funds in response to industry publications and data. Evaluation could also help investors to understand what caused any superior performance.

Two main methods have been used in mutual funds evaluation. One is portfolio evaluation, and the other is data envelopment analysis (DEA) The history of portfolio evaluation dates from the 1960s (Sharp, 1966; Treynor, 1965 and Jensen, 1968), with emphasis on both expected return and risk. Mutual fund managers attempt to find efficient portfolios – those promising the greatest expected return for any given degree of risk, i.e. risk-adjusted return. However, there are many criticisms of traditional portfolio analysis which focus on their sensitivity to chosen benchmarks. For the CAPM, the market portfolio is an ideal portfolio that only exists in theory. In practice certain indexes are used as approximations, but this causes problems since different indexes are likely to give different results in empirical work. For multi-index models, the difficulties lie in justifying how many and which indexes should be included in the model and defining which category a particular equity belongs to, especially for some equities with properties that suit more than one category. These imperfections in portfolio analysis models have led to the exploration of other methodologies to evaluate fund performance.

Murthi et at. (1997) were the first to apply DEA methodology to fund performance evaluation. A large proportion of DEA models applied to mutual funds show pieceswise linear correspondence between multiple inputs and outputs. However, according to Markowitz portfolio theory, there is correlation between different assets which should not be ignored, and these co-movements between different securities affect the relationship between expected return and risk of the combined portfolio.

Based on the unique characteristics of investment trusts, Morey and Morey (1999) developed a mutual funds efficiency measure in a traditional mean-variance model. It was based on Markowitz portfolio theory and related the non-parametric methodologies to the foundations of traditional performance measurement in mean-variance space. The model is derived from the standard data envelopment analysis but differs from it in having non-linear constraints in the envelopment version of the model's structure. Although mean and variance are considered in Morey and Morey (1999) models, they distinguish their model from traditional portfolio analysis by the fact that there is no theoretical benchmark like the market portfolio of the Capital Asset Pricing Model. Instead, the benchmarking fund in Morey and Morey (1999) consists of certain funds in the group, each with a particular weight. So rather than being compared with an idealised fund that requires information about all the equities in the market, the Morey and Morey (1999) model benchmarks the funds under evaluation against themselves. This makes the Morey and Morey (1999) model practically feasible and easier to test. Therefore, the objective of the first chapter in this thesis is to apply the procedures in Morey and Morey (1999) to a new modern data set comprising a multi-year sample of investment funds.

The objective of the second chapter in this thesis is to extend Morey and Morey (1999) model by adding statistical significance tests. The purpose of the Monte Carlo bootstrapping analysis in the thesis is to treat the measured scores as statistical estimators and to construct the sampling distributions of these estimators. The motivation for this is that the DEA efficiency scores obtained through Morey and Morey (1999) model have not hitherto been tested for statistical significance. Banker (1993), Kneip et al. (1996), Korostelev et al. (1995a, 1995b), Gijbels et al (1999) have investigated the consistency and convergence properties of the DEA scores and found that DEA estimators have asymptotic sampling distributions, which means that the efficiency scores only converge when the sample size is large enough. They are also very sensitive to outliers and extreme values, for example, dropping one outlier can dramatically change the efficiency level for other decision making units. Thus the DEA estimators have been shown to be biased when using a finite number of observed units, so that significant tests are necessary to correct the bias. The confidence intervals obtained through the tests can give insights about whether the DEA scores obtained from the Morey and Morey (1999) quadratic models are just random results or statistically significant. Therefore, Simar Wilson (2008) bootstrapping algorithms are utilised to develop statistical inference and confidence intervals for the indexes of efficient investment fund performance.

The purpose of the third chapter in this thesis is to examine the efficiency of investment trusts, analyze the factors contributing to investment trusts performance and detect the determinants of inefficiency. The second stage DEA efficiency analyses are used to evaluate contextual variables affecting the fund performance. For the second stage analysis, robust-OLS regression, Tobit models and Papke-Wooldridge (PW) models are then conducted and compared to evaluate contextual variables affecting the performance of investment funds. The DEA efficiency scores are regressed on potential variables including Sharpe ratio, Jensen's alpha, expense ratio, P/E ratio, book to market ratio and market value of the investment funds to test the statistical significance of those factors.

1.2 Contributions to knowledge

In the first application, the Morey and Morey (1999) quadratic DEA model has been compared with traditional portfolio analysis and standard linear DEA models. Also, the Matlab codes for this model which have written especially for this thesis are reported as part of the contribution in the thesis. With the Matlab codes, one can not only obtain the results, but also see very clearly how this model is programmed, with all the details revealed. This might benefit later practitioners who are interested in the quadratic DEA models.

Another major contribution of this thesis is in its third application, because as far as I am aware, there is only one paper in the literature about the practical application of second stage DEA on investment trusts. Therefore, it is very meaningful to examine different potential factors affecting the fund performance. It is hoped that this application will draw the attention of other practitioners and promote the development of using second stage DEA models to explain the investment trusts efficiency.

1.3 Structure of this thesis

The thesis is organised as follows:

Chapter 2 provides a fairly inclusive literature review on portfolio analysis models and standard DEA models.

Chapter 3 illustrates Morey and Morey (1999) quadratic model, and explains the advantages of this quadratic DEA model compared with traditional portfolio analysis and standard linear DEA models. Then the procedures in Morey and Morey (1999) are applied to a new modern data set comprising a multi-year sample of investment funds.

Chapter 4 extends the Morey and Morey (1999) quadratic DEA model by utilizing Simar-Wilson (2008) bootstrapping algorithms to obtain statistical inference and confidence intervals for the indexes of efficient investment fund performance.

Chapter 5 constructs a second stage DEA model to evaluate contextual variables affecting the performance of investment trusts. The commonly used second stage DEA models are applied and compared. Also a recursive model is developed to compare the efficiency measures- DEA, Sharpe ratio and Jensen's alpha. Results and inferences are drawn from an extensive new dataset of investment funds.

Chapter 6 gives the conclusion.

Chapter 2 Literature Review

This literature review of the mutual fund evaluation covers two parts; one is portfolio evaluation, which is so far the mainstream focus of the analysis of the fund performance. The other is called Data Envelopment Analysis (DEA) of mutual funds evaluation, which has appeared in the operational research area fairly recently. The first half of this literature review is about portfolio evaluation, and DEA on funds evaluation literature review is covered in the second half.

Before the formal review of the academic mutual funds evaluation, it is necessary to make clear about the definition and characteristics of mutual funds as well as the purposes of the evaluation. A simple definition of a mutual fund is that a mutual fund is a portfolio of investments managed on behalf of a pool of investors by a fund manager. Funds exist for investment in many kinds of securities: stocks, bonds, money market instruments and commodities or mortgage-backed securities. Once the money is collected from the investors, it Is allocated into different types of investments and managed by the professionals on a regular basis.

There are several basic types of mutual fund. The earliest form of fund is a unit trust, which is an ‘open-ended’ fund – the size of the fund and the number of units can expand and contract with time according to demand. To liquidate, unit holders sell units back to the managers of the unit trusts. There is a spread between offer price and bid price here, unlike OEICs. An OEIC (open ended investment company) is a company that issues shares rather than units. In the UK many unit trust managers have converted to OEICs in recent years because OEICs have a simpler pricing system without spread, which means there is only one price for both buyers and sellers. Mutual funds can also be quoted on a stock exchange, like the stocks. ETFs and investment trusts are such kinds. ETFs (exchange traded funds) are index trackers. They follow a particular index like the FTSE 100 or a particular sector. ETFs are set up as companies issuing shares and the shareholders’ money is used to buy securities to form a sector-mimicking portfolio like pharmaceutical industry. They are also open-ended funds. As they are quoted companies, investors can buy and sell their shares in the secondary market like any stocks. The other type of quoted mutual funds is investment trusts. These are actually listed companies, and are therefore different from unit trust and OEICs. Unlike ETFs, they are close-ended funds; therefore the number of shares is fixed as with any other company that issues shares. An investment trust normally only invests in specific types of assets for example Chinese technology shares and is banned from switching to other segments.

Unit trusts, OEICs, ETFs and investment trusts differ in their type of organisation. However, mutual funds can also be differentiated by their size and the assets they invest in. For example, growth equity funds are funds that have most exposure to growth stocks; and a small stock fund is a fund that buys small stocks predominantly; others may be bond/income funds, money market funds etc.

Instead of buying shares directly, investors buy into a managed pool of investment funds. Through investing in a mutual fund, investors with a relatively small sum can gain access to a diversified portfolio. Furthermore, those who know little about investment can take advantage of professional management. However, all of these advantages are neither free nor without risk. There are many types of fees associated with the mutual funds investment, sales loads, redemption fees, exchange fees, management fees etc. Note that 12b-1 fees relate to a US SEC rule and are not relevant in the UK. Unit trusts may charge distribution fees but investment trusts do not. Therefore, the evaluation of the mutual funds is essential, to decide which are good and which are not; or whether investors could end up paying too much for poor management.

There is more than one purpose for the evaluation of mutual funds. First, it is to see how well the mutual funds industry as a whole has performed in order to define their advantages as investment vehicles. And also, the evaluation results could help investors select better performing funds. Furthermore, the evaluation process motivates mutual fund companies to generate therefore report superior returns because investment dollars usually flow into top performing funds based on industry publications and data; evaluation could also help them understand what caused the performance and where the superior comes from.

2.1 Portfolio Evaluation

Portfolio evaluation has evolved dramatically over the last two decades. Crude return calculations were the original idea which was soon replaced by the modern portfolio theory based on risk-return analysis. The key element in the portfolio analysis is the emphasis on both expected return and risk. Mutual fund managers attempt to find efficient portfolios those promising the greatest expected return for any given degree of risk, i.e. risk-adjusted return. The seminal models are those of Sharpe (1966), Treynor (1965) and Jensen (1968), and Merton and Henriksson (1981) and Treynor and Mazuy (1966).

Sharpe (1966) generated a reward-to-variability ratio (R/V) derived from the Tobin model (Tobin, 1958):

$$\gamma_t = \frac{1}{2\sigma_j^2(\tilde{\pi}_t)} \frac{dV(R_j)}{dE(R_j)} \left[E(\tilde{R}_m) + \tilde{\pi}_t^* \right] \quad (2.1.1)$$

The Sharpe ratio is simply $\frac{dV(R_j)}{dE(R_j)} = -\frac{\partial U}{\partial E(R_j)} / \frac{\partial U}{\partial V(R_j)}$. The numerator shows the difference between the funds' expected annual return and a risk-free interest rate or excess return; it is thus the reward provided the investor for bearing some risk. The denominator measures the standard deviation of the annual rate of return; it shows the amount of risk actually borne. The ratio is thus the reward per unit of variance. So mutual funds with lager Sharpe Ratios are assumed to have better performance than those with small ratios.

The Sharpe Ratio takes diversification into account: as the degree of diversification increases, the variance decreases therefore the ratio gets larger.

Treynor (1965) proposed the Treynor index $\beta_m = 1$, (also referred to as the reward-to-volatility ratio), which is an investment measure that, like the Sharpe ratio, evaluates the excess return to a risky investment per unit of risk. However, unlike Sharpe ratio, which takes diversification into account, risk in Treynor index is measured as non-diversifiable or systematic risk. Since the returns on all diversified portfolios move with the market, the extent to which changes in the market are reflected in changes in a fund's rate of return can stand as a good measure of the total volatility of the funds' return over time.

Treynor Index is obtained by simply substituting volatility (defined as the fund's beta) for variance in the formula for the R/V ratio:

$$\beta_{jt} = \gamma_t \qquad (2.2.2)$$

What should be noted is that the return calculated according to the Treynor index assumes that the portfolio is suitably diversified, as it only takes systematic risk into consideration. Unsystematic risk is not accounted for and therefore the results of a Treynor Index calculation for an undiversified portfolio are misleading. It is assumed that the idiosyncratic risk of a portfolio can be removed by further diversification, so that systematic risk is the valid measure. However, whether the use of total risk (Sharpe) or systematic risk (Treynor) is better in performance evaluation depends on the purpose of the evaluation.

Sharpe ratio and Treynor index are the simplest measurements yet they have still been used by academics. Another commonly used model is Jensen (1968). Jensen (1968) derived an ex-post CAPM from the market model and the CAPM:

$$b_{jt} = g_t = q_{jt}(E(\tilde{R}_m) + p_t^*) \tag{2.2.3}$$

From the market model

$$q_{jt} = \frac{1}{2s_j^2(\tilde{\beta}_t)} \frac{dV(R_j)}{dE(R_j)} \tag{2.2.4}$$

$$\frac{dV(R_j)}{dE(R_j)} \tag{2.2.5}$$

From the CAPM:

$$E(R_i) = R_f + \beta_i(E(R_m) - R_f) \tag{2.2.6}$$

If (2.2.5) and (2.2.6) hold, then

$$R_{it} = R_f + \beta_i(E(R_m) - R_f) + \beta_i(R_{mt} - E(R_m)) + \varepsilon_{it} \tag{2.2.7}$$

Rearranging (2.2.7) we get:

$$R_{jt} - R_f = a + bRMO(t) + sSMB(t) + hHML(t) \tag{2.2.8}$$

Add factor α_i to equation (2.2.8), it becomes:

$$R_{it} - R_f = \alpha_i + \beta_i(R_{mt} - R_{ft}) + \varepsilon_{it} \tag{2.2.9}$$

When β , (2.2.9) could be seen as it is derived from (2.2.8). R_{jt} can be positive or negative. Equation (2.2.8) holds when there is no managing of assets involved. However, if the fund manager has superior ability to find underpriced securities (perhaps because of special knowledge not available to others), there would be an excess return compared with the return without asset management. The excess return would then be captured by α_i in (2.2.9). And during this circumstance, there would be an $\alpha_i > 0$ in (2.2.9). Therefore α_i is called Jensen's alpha, which represents the average incremental rate of return on the portfolio per unit time which is due solely to the manager's stock-selection abilities.

The above three methods are common measures of the fund manager's overall performance. From Fama (1972) the fund manager's ability can also be decomposed into detailed factors that affect the overall performance: micro forecasting (forecast of price movements of individual stocks relative to stocks generally) and macro forecasting (forecast of price movements of the general stock market relative to fixed income securities). Micro forecast is frequently called 'security analysis' and macro forecasting is referred to as 'market timing'. Put another way, market timing ability is the fund's manager's talent to forecast whether the stock market as a whole will beat the bond market or vice versa. Selection ability is the ability of the manager to increase returns on the portfolio through successful prediction of future security prices given the level of the risk of his portfolio. The ability to time the ups and downs of the stock market has the potential to generate extraordinary returns as the same as selection ability, therefore testing the fund manager's market timing ability has been an important issue.

Merton and Henriksson (1981) and Henriksson (1984) describe a model that identifies market-timing ability separately from Jensen's α . In Henriksson and Merton model (HM), the market timer's forecasts take a simple form that the investment fund manager forecasts either that stocks will earn a higher return than bonds or that bonds will earn a higher return than stocks. Define $Z_p(t)$ as the realized return on the investment fund portfolio, $Z_m(t)$ as the return on the market portfolio, $x(t) \equiv Z_M(t) - R(t)$ is the realized excess return on the market, and let $y(t) \equiv \max[0, R(t) - Z_m(t)] = \max[0, -x(t)]$.

It assumes that the fund has two target risk levels: $\eta_1(t)$ for the forecast of a 'down market' ($Z_m(t) \le R(t)$) and $\eta_2(t)$ for the forecast of an 'up market' ($Z_m(t) > R(t)$)
Let $\theta(t)$ be a random variable such that $\theta(t) = 1$ if the forecast is correct and $\theta(t) = 0$ is the forecast is incorrect. The probability function for $\theta(t)$ conditional on the market return $Z_m(t) = Z$ is written as

$$\Pr ob\{\theta(t)=1|Z_m(t)=Z\}=p_1(t) \text{ for } 0\le Z_m(t)\le R(t) \qquad (2.2.10a)$$

$$=p_2(t) \text{ for } R(t)<Z_m(t)<\infty \qquad (2.2.10b)$$

Here $p_1(t)$ is the probability of correct forecast of a down market and $p_2(t)$ is the probability of correct forecast of an up market.

Now let $\gamma(t)=0$ if a 'down market' is forecasted at time t therefore ($0\le Z_m(t)\le R(t)$) and $\gamma(t)=1$ if an 'up market' is forecasted, in which case $R(t)<Z_m(t)<\infty$.

Then according to (2.2.11a) and (2.2.11b),

$$\Pr ob\{\gamma(t)=0|Z_m(t)\le R(t)\}=p_1(t) \qquad (2.2.11a)$$

$$\Pr ob\{\gamma(t)=1|Z_m(t)\le R(t)\}=1-p_1(t) \qquad (2.2.11b)$$

$$\Pr ob\{\gamma(t)=1|Z_m(t)>R(t)\}=p_2(t) \qquad (2.2.11c)$$

$$\Pr ob\{\gamma(t)=0|Z_m(t)>R(t)\}=1-p_2(t) \qquad (2.2.11d)$$

Here $p_1(t)$ is the probability of a correct forecast of a down market and $1-p_1(t)$ is the probability of an incorrect forecast of an up market; $p_2(t)$ is the probability of a correct forecast of an up market and $1-p_2(t)$ is the probability of an incorrect forecast of a down market. Merton (1981) showed that a necessary condition for a rational forecast to have a positive value is that the conditional probabilities satisfy $p_1+p_2(t)>1$

Henrikson and Merton market timing model is illustrated as follows:

$$Z_p(t)-R(t)=\alpha_p+\beta_1 x(t)+\beta_2 y(t)+\varepsilon(t) \qquad (2.2.12)$$

Henriksson and Merton (1981) showed that, ignoring the management fee for the fund, the large sample least squares estimates of β_1 and β_2 in Equation (2.2.12) can be written as:

$$\text{Plim } \hat{\beta}_1 = p_2\eta_2 + (1-p_2)\eta_1 \qquad (2.2.13a)$$

$$\text{Plim } \hat{\beta}_2 = (p_1 + p_2 - 1)(\eta_2 - \eta_1) \qquad (2.2.13b)$$

The market-timing ability of the forecaster is measured by β_2 and security analysis is identified by α_p. α_p and $\beta_2 y(t)$ together correspond to α_i in Jensen's model. Therefore, this model is used to estimate the separate contributions of selectivity and market timing. Merton and Henriksson (1981) claimed that the pattern of returns from successful market timing is similar to that from following certain option strategies. And they derive an equilibrium theory of value for market-timing forecasting skills based on this idea.

In an up market, $y(t) = 0$ so that $\hat{\beta}_1$ becomes the only risk factor and is the systematic risk given by β_i in Jenson's equation. However, when a down market happens, $y(t) = R(t) - Z_m(t) > 0$, the return of the portfolio will be protected by β_2 y(t), given the fund manager's forecasting ability.

Merton (1981) constructed a put options investment portfolio whose return was compared with those of the funds run by the market timer. His strategy was as follows:
For each dollar invested in the portfolio, allocate the fractions $\omega_1(t) \equiv p_2\eta_2 + (1-p_2)\eta_1$ to the market, $\omega_2(t) = (p_1 + p_2 - 1)(\eta_2 - \eta_1)$ to put options on the market, and $\omega_3(t) \equiv 1 - \omega_1(t) - \omega_2(t)$ to riskless bonds. The return per dollar on this portfolio, $Z_s(t)$ can be written as

$$Z_s(t) = \omega_1(t)Z_m(t) + \omega_2(t)\max\left[0, R(t) - Z_m(t)\right] + \omega_3(t)R(t) \qquad (2.2.14)$$

Merton (1981) showed that in equilibrium $Z_p(t)$ in (2.2.12) equals $Z_s(t)$ in (2.2.14). We can see that plim $\hat{\beta}_1 = \omega_1(t)$, plim $\hat{\beta}_2 = \omega_2(t)$ and $y(t)$ in (2.2.12) represents the return on one such option in (2.2.14). Therefore, from the comparison, the value of the market timing is reflected in the fact that the put options are obtained for free, under the assumption that the management fee is ignored. Market timing therefore gives investors exactly the same protection as a put option with an exercise price of $R(t)$.

With the management fees in consideration, let $A(t)$ denote the total value of investment in securities by the fund at time t and $F(t)$ the total fees paid at the beginning of period t for managing the fund between t and $t+1$, then the total (gross) dollar amount invested in the fund at time t, $I(t)$, satisfies $I(t) = A(t) + F(t)$ and the management fee denoted as a fraction of assets held by the fund is given by $m(t) \equiv \frac{F(t)}{A(t)}$. Let $g(t)$ denote the market price of a one-period put option on one share with an exercise price of $R(t)$. Merton (1981) showed that, in equilibrium, $m(t) = (p_1 + p_2 - 1)(\eta_2 - \eta_1)g(t)$, which means that the management fee should be equal to the cost of purchasing the number of $(p_1 + p_2 - 1)(\eta_2 - \eta_1)$ put options, otherwise there would be an arbitrage between the market timing fund and this specific portfolio including such a put option.

Another popular market timing model was proposed by Treynor and Mazuy (1966) which suggests that timing ability could be evaluated by including a quadratic term in a simple 'characteristic line' (market model) estimation. Based on this idea Jensen (1972) proposed a formal quadratic model regressing R_{jt} on $\tilde{\pi}_t$ and $\tilde{\pi}_t^{\,2}$, but Pfleiderer and Bhattacharya (1983) pointed out that $\tilde{\pi}_t = \tilde{R}_{mt} - E(\tilde{R}_m)$, and that a good estimate of $E(\tilde{R}_m)$ is very hard to obtain. They therefore used R_{mt} in the place of $\tilde{\pi}_t$. The final model is now generally written as:

$$\tilde{R}_{jt} = \eta_0^{'} + \eta_1^{'}\tilde{R}_{mt} + \eta_2^{'}R_{mt}^2 + \tilde{\omega}_t^{'} \tag{2.2.15}$$

Pleiderer and Bhattacharya (1983) proved that the probability limits of the coefficients obtained from traditional least square regression are

$$P\lim \hat{\eta}_0^{'} = \alpha^p \tag{2.2.16a}$$

$$P\lim \hat{\eta}_1^{'} = \theta E(\tilde{R}_m)(1-\varphi) \tag{2.2.16b}$$

$$P\lim \hat{\eta}_2^{'} = \rho^2\theta \qquad (2.2.16c)$$

The market-timing coefficient is $\eta_2^{'}$. The definition of ρ^2 and θ start from Jensen (1972), who defines $\tilde{\pi}_t$ as an unobservable 'market factor' which to some extent affects the returns on all securities, and assumes $E(\tilde{\pi}_t) = 0$. Then the excess returns on the market index can be expressed as: $\tilde{R}_{mt} \cong E(\tilde{R}_m) + \tilde{\pi}_t$

Define $\tilde{\pi}_t^*$ as the portfolio manager's forecast of the market factor $\tilde{\pi}_t$ and assume $\tilde{\pi}_t^* = E\left(\tilde{\pi}_t \middle| \Phi_{j,t-1}\right)$ where $\Phi_{j,t-1}$ is the information set available to the manager at time t-1. Given $\tilde{\pi}_t$ and $\tilde{\pi}_t^*$, define ρ as the correlation between the manager's forecast and the actual market returns. Thus

$$\rho = \frac{\operatorname{cov}(\tilde{\pi}^*, \tilde{\pi})}{\sigma_{\tilde{\pi}^*}, \sigma_{\tilde{\pi}}} \qquad (2.2.17)$$

So that the better the manager is informed, the closer ρ^2 is to unity.

Let γ_t be the fraction invested in the market portfolio at time *t*, and $1-\gamma_t$ the fraction invested in the riskless asset. Thus the expected excess return and variance of return on the portfolio are:

$$E(\tilde{R}_{jt}) = \gamma_t\left[E(\tilde{R}_m) + \tilde{\pi}_t^*\right] \qquad (2.2.18)$$

$$V(\tilde{R}_{jt}) = \gamma_t^2\sigma_t^2(\tilde{\pi}_t) \qquad (2.2.19)$$

The manager's problem is to

$$\max_{\gamma_T} U\left[E(\tilde{R}_{jt}), V(\tilde{R}_{jt})\right] = \max_{\gamma_T} U\left[\gamma_t E(\tilde{R}_m + \tilde{\pi}_t^*), \gamma_t^2\sigma_t^2(\tilde{\pi}_t^*)\right]$$

and the solution to this yields

$$\gamma_t = \frac{1}{2\sigma_j^2(\tilde{\pi}_t)} \frac{dV(R_j)}{dE(R_j)} \left[E(\tilde{R}_m) + \tilde{\pi}_t^* \right] \tag{2.2.20}$$

where

$$\frac{dV(R_j)}{dE(R_j)} = -\frac{\partial U}{\partial E(R_j)} / \frac{\partial U}{\partial V(R_j)} \tag{2.2.21}$$

(the slope of the indifference curve between variance and excess return) is a coefficient of risk aversion.

Since $\beta_m = 1$ and $\beta_{jt} = \gamma_t$ thus the manager's target risk can be given by $\beta_{jt} = \gamma_t = \theta_{jt}(E(\tilde{R}_m) + \pi_t^*)$

Where

$$\theta_{jt} = \frac{1}{2\sigma_j^2(\tilde{\pi}_t)} \frac{dV(R_j)}{dE(R_j)} \tag{2.2.22}$$

The more aggressive the fund manager is (even when he only has access to low quality information), the larger are both $\frac{dV(R_j)}{dE(R_j)}$ and θ_{jt}.

Therefore, we conclude that from Jensen(1972) and Pfleiderer and Bhattacharya (1983), the market timing returns of the mutual funds managers are decided by how well informed or how aggressive they are.

However, superior forecasting ability has not generally been found using these models. Jensen (1968) used a sample of 115 open end mutual funds for ten-year period 1955-1964, with annual data. He used returns both net and gross of expenses but found very few significantly positive alpha coefficients, suggesting that mutual funds showed very little ability collectively or individually to forecast security prices. Henriksson (1984) examined the performance of 116 open-end mutual funds using monthly data from February 1968 to

June 1980. Unfortunately the results show little evidence of market-timing ability and selectivity ability. (Only three of the 116 funds exhibited positive estimates for β_2 with 95% confidence and only one fund exhibited a significantly positive estimate of Alpha). Goetzmann and Zheng(2006) report the performance of a portfolio comprised of all equity mutual funds that existed in the CRSP database from the beginning of the data through 2004-5. 16 months of returns. The result of Alpha was negative, but the underperformance is less than normal expenses, so this evidence is consistent with the hypothesis that equity mutual funds have selection skill but probably do not have enough to cover expenses. Along with these two examples, almost all the early empirical work indicates that superior forecasting ability does not exist among mutual fund industry.

Further improvements were made years later by adding more factors which may have an influence on the mutual fund performance besides market factor. Banz (1981) found that stocks with small capitalization showed higher average returns than large stocks – an excess return or 'seize' effect that could not be explained by the CAPM. Other authors found that leverage (Bhandari, 1988), book to market ratio (Stattman, 1980; Rosenberg et al., 1985) and earnings-price ratio (Basu, 1983) all contributed to the explanation of cross-sectional stock returns.

Developing this work, Fama and French (1992) identified two factors other than the market factor which determined average stock portfolio return: size and Book-to-Market ratio, claiming that these absorb the effects of size, E/P, leverage, and book-to-market equity in the cross-sectional average returns on NYSE, AMEX, and NASDAQ stocks.

The Fama and French (1992) model is written as

$$R_{jt} - R_f = \alpha + bRMO(t) + sSMB(t) + hHML(t) \tag{2.2.23}$$

Here R_{jt} is the return of the portfolio j at time t, R_f is the risk-free rate, and RMO is the return on a market index. The three factor b is analogous to the classical β but not equal to it, since there are now two additional factors in the equation. SMB stands for 'small market capitalization minus big' and HML for 'high book-to-price ratio minus low'. They respectively measure the excess returns to size and of ‘value’ stocks over ‘growth’ stocks. SMB is often referred to as the ‘size factor’ and HML as ‘value factor’.

Fama and French found a negative relation between average stock return and firm size (stocks with the smallest market capitalization have the largest return) but that book-to-market, offers a significantly positive premium.

Fama and French (1993) extend their work by adding factors to explain returns government and corporate bonds in addition to those for common stocks. They identify two bond-market factors: a ‘term factor’, which captures the risk due to unexpected fluctuations in interest rates, and a ‘default’ factor, as follows:

$$R_{jt} - R_f = \alpha + mTERM(t) + dDEF + e(t) \tag{2.2.24}$$

TERM and DEF are respectively the excess return of the monthly long-term government bond return over the one-month Treasury bill rate measured and the excess return on a portfolio of long-term corporate bonds over long-term government bond.

Fama and French claim that these two bond-market factors capture the common variation in stocks as well as bonds returns. Thus they give a five factor model

$$R_{jt} - R_f = \alpha + bRMO(t) + sSMB(t) + hHML(t) + mTERM(t) + dDEF(t) + e(t) \tag{2.2.25}$$

To examine whether factors that are important in bond returns can help explain stock returns also, and vice versa. They used a time-series approach and the tests results show that TERM and DEF slopes for corporate bonds are 1 and that for stocks that around 0.8, which shows that these two risk factors have a common impact on both corporate bonds and stocks. But they interpret the results differently for stocks and bonds. The intercept from the time-series regression of the portfolio’s excess return on the five explanatory returns is the average abnormal return which can be used to judge a manager’s ability to beat the market (similar to Jensen’s α), that is whether he can use expertise to generate greater returns than the returns from those five indices.

Jegadeesh and Titman (1993) suggested that past performance of mutual funds might be able to predict the future, they call it momentum effect. Carhart (1997) proposed a four-factor model to test short- and long-term persistence in mutual funds, based on both the Fama-French model and momentum:

$$R_{jt} = \alpha_{iT} + b_{iT}RMRF_t + s_{iT}SMB_t + h_{iT}HML_t + p_{iT}PR1YR_t + e_{it} \quad (2.2.26)$$

RMRF is the market factor; SMB and HML are size and book-to-market factors from Fama and French model, PR1YR captures Jegadeesh and Titman's (1993) one-year momentum anomaly.

PR1YR in this model is calculated as equal-weighted average return of firms with the highest 30 percent eleven-month returns lagged one month minus the equal-weighted average of firms with the lowest 30 percent eleven-month returns lagged one month.

Carhart (1997) found that the excess returns from the 3-factor model were significantly negative for the previous year's loser stock portfolios but significantly positive for last year's winners. However, the 4-factor model containing eliminates these patterns in excess returns, indicating the great improvement to the 3-factor model.

To test short-term persistence, Carhart (1997) used lagged one-year returns. Mutual funds were sorted into decile portfolios according to their previous calendar year's return. Funds with the highest and lowest past one-year returns comprise deciles 1 and 10 respectively.

The 4-factor model explains most of the spread and pattern in these portfolios, with sensitivities to the size (SMB) and momentum (PR1YR) factors accounting for most of the explanation. The returns to the top decile funds have a significant and positive relationship with the one-year momentum factor, while the returns in the bottom decile are significantly negatively correlated with the factor. In addition, Carhart (1997) found that expenses, turnover, load fees and transaction costs were negatively related to fund performance. Decile 10 in particular suffers higher expenses, turnover, load fees and transaction costs. Overall, the results show robust short-run persistence and most of the persistence can be explained by the sensitivities to the four factors along with expenses, and transaction costs. However, when using two- to four year returns, the 4-factor model explains little of the excess return and nothing of the excess return in 5-year lagged portfolios. Thus evidence of long-term persistence was not found.

Carhart (1997) also sorted mutual funds by alphas instead of returns when ranking the portfolios into deciles, using their 4-factor alphas estimated over the prior 3 years. He then calculated the mean monthly excess returns on the funds in each decile portfolio for the first five years after ranking. The results show that the highest decile maintains a persistently high mean return for a full five years after the portfolio is initially formed, but that the mean returns on the lowest nine deciles converge after two years. However, the 4-factor model alphas on this portfolio over the five-year post-ranking period are not significantly different from zero. This suggests that these funds did not provide returns substantially beyond those predicted by the 4-factor model. Carhart (1997) therefore concluded that the persistence in mutual fund performance did not reflect the forecasting ability of fund managers.

The above models all assume constant risk coefficient, which was criticized by Ferson and Schadt (1996) who pointed out that variation in the expected returns and risks of stocks and bonds were likely to be due to some changes in dividend yields, interest rates or other variables. Ferson and Schadt(1996) claim that the traditional methods suffer from a number of biases, in particular that beta and the expected return cannot be constant as assumed in the traditional models. If expectations of future returns and risks fluctuate with this publicly available information, the measurement of the manager’s forecasting ability should accommodate the time variation too. Traditional market-timing models assume that any information related with future market returns is superior information. However, Ferson and Schadt (1996) consider this as a major drawback and claim that the skills of utilizing information readily available to the public should not be judged as superior forecasting ability. The fund manager’s forecasting ability should come from non-public information, so that, the essence of the conditional approach is to improve the traditional methods by accommodating common sources of variation from public information, using lagged instruments.

The betas of mutual funds will naturally change, for several reasons. First, the betas of the underlying assets may change over time. Second, fund managers may actively adjust the portfolio asset weights causing the fund beta to change. Last but not least, open-ended funds will experience net cash inflows and outflows from time to time, to allocate new cash or withdrawing underlying assets may lead to changes in fund betas. Betas will also vary as the percentage of cash held by the fund fluctuates. However, these three reasons can be reflected by two time-variation factors.

The lagged instruments used by Ferson and Schadt (1996) are the lagged level of the one-month Treasury bill yield and the lagged dividend yield. The conditional beta is illustrated as follows:

$$\beta(t) = b_0 + b_1(D/P_{t-1}) + b_2(TB_{t-1}) \tag{2.2.27}$$

Here $\beta(t)$ is some (undefined) function of D/P and TB. The linearization of equation (2.2.29) makes it operational. (D/P) is the lagged value of the market dividend yield and TB is the lagged value of a short-term Treasury yield. Thus the mutual fund's beta is conditional upon lagged dividend yield and short-term Treasury yield, which reflect the state of the stock market. b_0 is a 'target' or average beta, and remaining terms represent deviations of beta from target. Thus the manager's beta will change for period t in response to public information available at *t-1*.

Specifically, the fund betas will be higher if the two extra factors are positive, but this is a change in beta arising from the use of publically available information. So there should be no excess return to the fund in consequence.

The empirical results from Ferson and Schadt (1996) show that the coefficients on the dividend yield are positive, whereas those on the Treasury bill are negative, with both coefficients statistically significant for most of the data. This is reasonable since high dividend yields are a positive market indicator and high short-term interest rates predict low stock returns.

Using the modified β in (2.2.27), the conditional model has the following regression for the managed portfolio return:

$$RP_t = \alpha + b_0 RM_1 + b_1\left[RM_t(D/P)_{t-1}\right] + b_2\left[RM_t(TB)_{t-1}\right] + error \tag{2.2.28}$$

The conditional model adds two additional variables to the traditional regression, $[RM_t(D/P)_{t-1}]$ and $[RM_t(TB)_{t-1}]$. They are interaction terms between the market return and the lagged values of the market indicators. These interaction terms pick up the sources of movements in beta through time. The intercept, α is the conditional alpha, which measures the abnormal performance representing the fund manager's superior ability to earn returns not available by using public information.

In linearised form the conditional Jensen model is written as :

$$r_{pt+1} = \alpha_p + \delta_{1p} r_{mt+1} + \delta'_{2p}(z_t r_{mt+1}) + \varepsilon_{pt+1} \tag{2.2.29}$$

$z_t r_{mt+1}$ is the product of the market factor with the lagged information. This is both more and less general than equation (2.2.28). It is less general because (2.2.28) has been lineraised. More general because it allows you any vector of public information, not just D/P and TB.

The classic market timing regression is the quadratic regression of Treynor and Mazuy (1966):

$$r_{pt+1} = \alpha_p + b_p r_{mt+1} + \gamma_{tmn}[r_{m,t+1}]^2 + v_{pt+1} \tag{2.2.30}$$

A conditional version of the Treynor-Mazuy regression is

$$r_{pt+1} = \alpha_p + b_p r_{mt+1} + C'_p(z_t r_{mt+1}) + \gamma_{tmc}[r_{m,t+1}]^2 + v_{pt+1} \tag{2.2.31}$$

The term $C'_p(z_t r_{mt+1})$ controls for the public information effect.

To be consistent with Equation(2.2.28), (2.2.31) can also be written as

$$RP_t = \alpha + b_0 RM_1 + b_1[RM_t(D/P)_{t-1}] + b_2[RM_t(TB)_{t-1}] + \gamma RM_t^2 + error \tag{2.2.32}$$

The unconditional Merton and Henriksson model is written as

$$r_{pt+1} = \alpha_p + b_p r_{mt+1} + \gamma_u \left[r_{m,t+1} \right]^+ + v_{pt+1} \tag{2.2.33}$$

where $\left[r_{m,t+1} \right]^+$ is defined as $\text{Max}(0, r_{m,t+1})$

This is transformed to a conditional model by:

$$r_{pt+1} = b_p r_{mt+1} + B'_d (z_t r_{mt+1}) + \gamma_c r^*_{mt+1} + \Delta' \left[z_t r^*_{mt+1} \right] + u_{p,t+1} \tag{2.2.34}$$

where $\gamma_c = b_{up} - b_d$ and $\Delta = B_{up} - B_d$. Positive market timing ability is shown where $\gamma_c + \Delta' z_t > 0$

The empirical work of Ferson and Schadt (1996) shows that controlling for common variation using lagged instruments, yields an increase in the number of observed positive coefficients, compared with unconditional models.

All the above models use a general benchmark, an alternative, however, is to place the funds into different categories and adopt a set of benchmarks for each category. Roll (1978) has shown that single-index measures of performance are sensitive to the type of benchmark portfolio used. This is, the Beta when using the Standard and Poor's Index is not the same as the Beta calculated when using the Dow-Jones index as the benchmark portfolio. Ross (1976) developed arbitrage pricing theory (APT) which could overcome this problem.

Sharpe (1992) created a multi-index model under the assumption that a fund manager allocates investment among n asset classes. Bills, Intermediate-term Government Bonds, Long-term Government Bonds, Corporate Bonds, Mortgage-Related Securities, Large-Capitalization Value Stocks, Large-Capitalization Growth Stocks, Medium-Capitalization Stocks, Small-Capitalization Stocks, Non-U.S. Bonds, European and Asian Stocks, Japanese Stocks. Sharpe's (1992) model is illustrated as follows:

$$\tilde{R}_i = \left[b_{i1} \tilde{F}_{i1} + b_{i2} \tilde{F}_{i2} \ldots b_{in} \tilde{F}_{in} \right] + \tilde{e}_i \tag{2.2.35}$$

R_i represents the return on asset i, F_{i1} represents the value of the first factor, F_{i2} the value of the second factor, F_{in} the value of the nth factor. e_i the error term. Once the factors are determined, the exposure to these factors can then be measured using the above equation. Rearranging (2.2.35) gives the following form:

$$\tilde{e} = \tilde{R}_i - \left[b_{i1}\tilde{F}_1 + b_{i2}\tilde{F}_2 \ldots b_{in}\tilde{F}_n\right]_i \qquad (2.2.36)$$

The error term thus equals the difference between the return on the fund and that of the weighted factor indices. The goal of style analysis is to select the style (exposure to these asset classes) that minimizes the variance of this difference. The error term is thus called the fund's 'tracking error' and its variance the fund's 'tracking variance'. Shape uses quadratic programming to determine a fund's exposures to changes in the returns of these factors (calculating the b_{in}) which has the minimum tracking variance. This method is termed style analysis. The R^2 from this programming is defined as the contribution to the fund's style and the remainder $1 - R^2$ to the fund manager's selection.

Using this approach, Sharpe (1995) analyses the performance of the LS 100 funds – the 100 largest, seasoned U.S. funds that are chosen from bond funds, stock funds, balanced funds, global and international funds. Sharp uses style analysis to determine the sensitivities (betas) of these funds to 15 indexes. He defines a fund's selection return as the difference between its return and that of its style, as explained above. Thus selection returns equal to the return on LS100 minus that of the indicies. The statistics show that the average selection return is not significantly different from zero, suggesting that an actively-managed fund is not likely to beat a passive portfolio with the same type. This conclusion is consistent with other studies such as Elton, Gruber, Das, and Hlavka (1993), Brown and Goetzmann (1995) and Malkiel (1995).

Those are most important models in the portfolio evaluation, and one of the new research points is dynamic risk shifting modelling and how it can improve the accuracy of measurement of the funds performance. The following part of this literature review will shift to another methodology in operational research area. It is called data envelopment analysis. Data envelopment analysis gives completely different scenario from portfolio evaluation. It is a nonparametric analysis technique which was proposed by Charnes et al. (1978), and it was firstly used in measurement of the performance of educational institutions.

2.2 Data Envelopment Analysis

DEA is a linear programming formulation that defines a correspondence between multiple inputs and multiple outputs. It is a non-parametric analysis that does not require any theoretical models (CAPM or APT) as measurement benchmarks. Instead DEA measures how well a fund performs relative to the best set of funds within the category. DEA model is flexible and can evaluate performance on a number of outputs and inputs simultaneously.

The following paragraphs describe the DEA formulation as given in Charnes et al. (1978). The simple DEA program is formulated as a fractional programming problem and is then reduced to a linear programming problem that is easy to compute. In general, the program maximizes the ratio of the weighted average of the multiple outputs to the weighted average of the multiple inputs. Charnes et al (1978)'s idea is to define the efficiency measure by assigning to each unit the most favourable weights. This means that the weights will generally not be the same for the different units; therefore, the choice of the weights should not be responsible for the inefficiency of a fund.

In DEA, the organization under study is called a DMU (Decision Making Unit). The DMU is regarded as the entity responsible for converting inputs into outputs and whose performances are to be evaluated.

Define the input and output data for DMU_j

y_{rj} amount of output *r* for unit *j*

x_{rj} amount of input *i* for unit *j*

For each DMU_j, give the input and output as yet unknown weights

u_r weight assigned to output *r*

v_r weight assigned to input *i*

The DEA efficiency measure for a decision making unit *j* is given by a ratio of the weighted sum of outputs to the weighted sum of inputs:

$$h = \frac{\sum_{r=1}^{t} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij}} \qquad (2.2.37)$$

The weights in the ratio (2.2.37) are chosen in such a way that the efficiency measure h has an upper bound, usually chosen equal to 1, which will be reached only by the most efficient units. For each decision making unit the most favourable weights are chosen; they are computed by maximizing the efficiency ratio of the unit considered, subject to the constraints that the efficiency ratios of all units, computed with the same weights, have an upper bound of 1.

Formally, to compute the DEA efficiency measure for a target decision making unit separately identified by the index $j_o \in \{1,2,...,n\}$ we have to solve the following fractional linear programming problem:

$$\begin{aligned}
\max \quad & h_0 = \frac{\sum_{r=1}^{t} u_r y_{rj_0}}{\sum_{i=1}^{m} v_i x_{ij_0}} \\
s.t. \quad & \frac{\sum_{r=1}^{t} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij_0}} \le 1 \quad j = 1,...,n, \\
& u_r \ge 0, \quad r = 1 \\
& v_i \ge 0, \quad i = 1,...,m
\end{aligned}$$

$$(2.2.38)$$

The optimal objective function value (2.2.38) represents the efficiency measure assigned to the target unit j_o considered. To find the efficiency measures of other decision making units we have to solve similar problems, targeted on each unit in turn.

Mathematically, the nonnegativity constraint in (2.2.38) is not sufficient for the fractional terms $\dfrac{\sum_{r=1}^{t} u_r y_{rj_0}}{\sum_{i=1}^{m} v_i x_{ij_0}}$ to have a positive value. To solve this problem, the fractional problem (2.2.38) is conveniently replaced by an equivalent linear programming problem; by letting $\sum_{i=1}^{m} v_i x_{ij_0} = 1$ we obtain that so called input-oriented Charnes, Cooper and Rhodes (CCR) linear model.

$$
\begin{aligned}
\max \quad & \sum_{r=1}^{t} u_r y_{rj_0} \\
s.t. \quad & \sum_{i=1}^{m} v_i x_{ij_0} = 1, \\
& \sum_{i=1}^{m} v_i x_{ij} - \sum_{i=1}^{m} v_i x_{ij} \le 0, \quad j = 1,\dots,n \\
& u_r \ge 0, \quad r = 1,\dots,t, \\
& v_i \ge 0, \quad i = 1,\dots,m
\end{aligned}
\tag{2.2.39}
$$

Efficiency measures equal to 1 characterize the efficient units: at least with the most favourable weights, these units cannot be dominated by the other ones in the set, and therefore they lie on the efficient frontier.

The points either on the efficient frontier or within it are possible, therefore the set of feasible activities is called the production possibility set and is denoted by *P*. Arranging the data sets in matrices $X = (x_j)$ and $Y = (y_j)$; $x_j \in R^m$, $y_j \in R^s$.We can define the production possibility set *P* by

$$P = \left\{(x, y) \mid x \ge \sum_j \lambda_j x_j ; y \le \sum_j \lambda_j y_j ; \lambda_j \ge 0 \forall j\right\} \tag{2.2.40}$$

Where λ_j is a semi positive vector in R^n.

Based on the matrix (X_j, Y_J), (2.2.39) can be expressed in its dual form using a real variable θ and a non-negative vector $\lambda = (\lambda_1,\dots,\lambda_n)^T$ of variables as follows ([Envelopment form]):

$$
\begin{aligned}
&Min \quad \theta \\
&s.t. \quad \sum_{j=1}^{N} \lambda_j x_{ij} \le \theta x_{ij_0} \quad i = 1...m \\
&\qquad \sum_{j=1}^{N} \lambda_j y_{rj} \ge y_{rj_0} \quad r = 1...s, \\
&\qquad \lambda_j \ge 0, \quad j = 1...N
\end{aligned}
\tag{2.2.41}
$$

Where θ measures the technical input efficiency. It is known as the envelopment DEA model and it measures DEA efficiency with reference to a production possibility set boundary which 'envelops' the input and output levels observed at DMUs. To illustrate, see graph 2.2.1 for a simple case with two inputs x_1, x_2, the minimum θ gives the fraction each point needs to contract radially to arrive at the efficient frontier *BDGEF*. (e.g. point A contracts to point G)

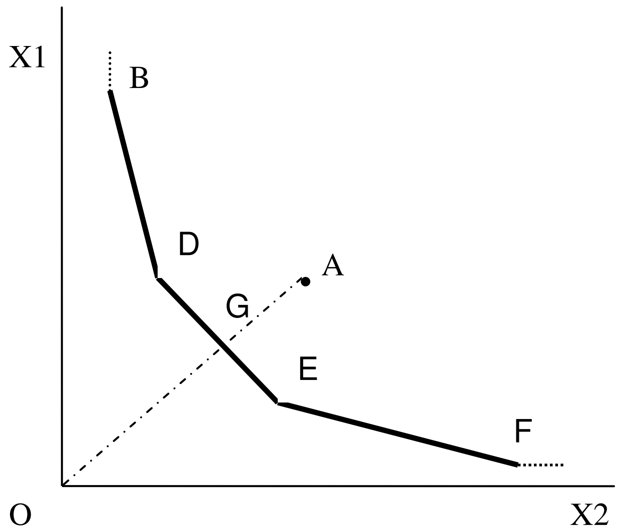

Figure 2.2.1 Input Space Representation

When $\theta_0 = 1$ at the optimal solution to (2.2.41), then DMU_0 lies on the efficient frontier, and it is deemed to be 100% efficient. However, the 100% efficiency in this sense is not necessarily Pareto-efficient because improvements to the individual levels of some inputs may still be possible. Such improvements are captured in the slacks. Therefore, in order to guarantee Pareto-efficiency, (2.2.41) can also be written in another form, in which inequalities are transformed into equalities using slacks.

$$
\begin{aligned}
&Min \quad && \theta - \varepsilon\left[\sum_{i=1}^{m} S_i^- + \sum_{r=1}^{s} S_r^+\right] \\
&s.t. && \sum_{j=1}^{N} \lambda_j x_{ij} = \theta x_{ij_0} - S_i^-, \quad i = 1...m \\
& && \sum_{j=1}^{N} \lambda_j y_{rj} = y_{rj_0} + S_r^+, \quad r = 1...s \\
& && \lambda_j \geq 0, \quad j = 1...N, \quad \lambda_j \geq 0, \quad \lambda_j \geq 0, \quad j = 1...N, \quad S_i^-, S^+ \geq 0
\end{aligned}
\tag{2.2.42}
$$

Where S_i^- and S_r^+ are slacks, ε is non-archimedean penalty term which has very small value such as 10^{-6}. The priority is given to the minimization of θ_0, once θ_0 has been minimized, the model seeks the maximum sum of the slack value S_i^- and S_r^+. If any one of these values is positive at the optimal solution to the model it means that the corresponding input of DMU j_0 can improve further, until it satisfies $\theta_0 = 1$ and all slacks are zero. Because slacks are multiplied by a very small value (identified by non-archimedean penalty term ε), the resulting objective junction is virtually equal to the optimal value of θ_0.

It can be seen that the CCR model gives a piecewise linear production surface which represents a production frontier: input oriented model aims to minimize inputs while satisfying at least the given output level; it gives the minimum amount of input required to achieve the given output levels. There is another type of model called the output-oriented model that attempts to maximize outputs without requiring more of any of the observed input values. The model shows as follows:

$$
\begin{aligned}
&Max \quad && \gamma \\
&s.t. && \sum_{j=1}^{N} \alpha_j x_{ij} \leq x_{ij_0}, \quad i = 1...m, \\
& && \sum_{j=1}^{N} \alpha_j y_{rj} \geq \gamma y_{rj_0}, \quad r = 1...s \\
& && \alpha_j \geq 0, \qquad j = 1...N
\end{aligned}
\tag{2.2.43}
$$

Where $1/\gamma$ measures the technical output efficiency. And the corresponding graph 2.2.2 of simple case with two outputs is as follows; in which maximum γ measures how many times each point must expand in order to get the efficient frontier denoted by *BDGEF*. (e.g. point A expand to G)

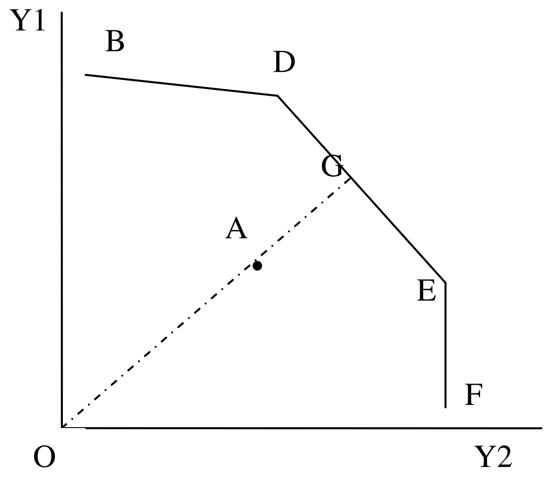

Figure 2.2.2 Output Space Representation

When $\gamma_0 = 1$ at the optimal solution to (2.2.43), which means DMU_0 lies on the efficient frontier. However, it is not necessarily Pareto-efficient because improvements to the individual levels of some outputs may still be possible. Such improvements are captured in the slacks. Therefore, (2.2.43) can also be written in the following form to achieve Pareto-efficiency.

$$
\begin{aligned}
Max \qquad & \gamma + \varepsilon\left[\sum_{i=1}^{m} I_i + \sum_{r=1}^{s} O_r\right] \\
& \sum_{j=1}^{N} \alpha_j x_{ij} = x_{ij_0} - I_i, \qquad i = 1...m, \\
& \sum_{j=1}^{N} \alpha_j y_{rj} = \gamma y_{rj_0} + O_r, \quad r = 1,...,s \\
& \alpha_j \geq 0, j = 1...N, \quad I_i, O_r \geq 0
\end{aligned}
\qquad (2.2.44)
$$

Where I_i and O_r are slacks, ε is non-archimedean penalty term. In (2.2.44), first, the priority is given to the maximisation of γ_0, and second, the maximization of slacks is sought. If any of the values of I_i or O_r is positive at the optimal solution to the model, it means that the corresponding output of DMU j_0 can improve further, after its output levels have been expanded by proportion γ_0. When $\gamma_0 = 1$ and all slacks are zero, Pareto-efficiency is guaranteed. Because slacks are multiplied by a very small value of ε, the resulting objective junction is virtually equal to the optimal value of γ_0

The above model is CCR model, which is built on the assumption of constant returns to scale in which case inputs and outputs are subject to change proportionally. For example, (x_1, x_2) change to $(\alpha x_1, \alpha x_2)$ (with $\alpha > 0$) In fact, since the very beginning of DEA studies, various extensions of the CCR model have been proposed, among which the BCC (Banker-Charnes-Cooper) model is representative. The BCC model is a variable returns to scale (VRS) version in which case inputs and outputs do not change proportionally.

Assessing the DEA under VRS compared with the one under CRS in example of single input and single output.

The CCR model assumes the constant returns-to-scale production possibility set, i.e., the radial expansion and reduction of all observed DMUs. On the other hand, the BCC model assumes that convex combinations of the observed DMUs form the production possibility set. If a DMU is fully efficient (100%) in both the CCR and BCC scores, it is operating in the most productive scale size. If a DMU has full BCC efficiency but a low CCR score, it is not scale efficient. Thus, the scale efficiency of a DMU is defined by the ratio of the two scores.

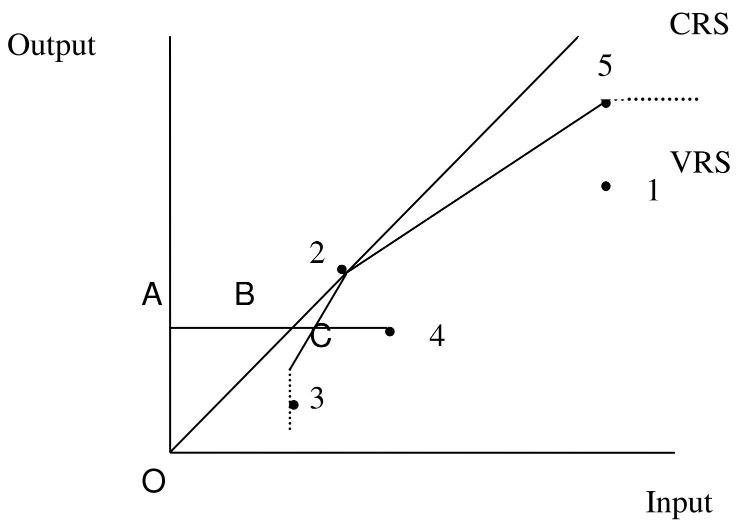

Figure 2.2.3 Graph Technology Representation

$$SE = \frac{\theta^*_{CCR}}{\theta^*_{BCC}} \tag{2.2.45}$$

Scale efficiency measures the impact of scale size on the productivity of a DMU. The Pure technical input efficiency will never be less than its technical input efficiency, so we have scale efficiency <=1.

Rearranged (2.2.45) we get $\theta^*_{CCR} = \theta^*_{BCC} * SE$ which means Technical Efficiency (TE) = Pure Technical Efficiency (PTE) * Scale Efficiency (SE). The CCR models estimates the overall efficiency. This efficiency comprises technical efficiency and scale efficiency. The BCC model measures pure Technical efficiency. Technical efficiency describes the efficiency in converting inputs to outputs, while scale efficiency recognizes that economies of scale cannot be attained at all scales of production, and therefore there is one most productive scale size, where the scale-efficiency is maximised at 100 per cent. This decomposition detects where the inefficiency lies. i.e. whether it is caused by technical problems (PTE) or by improper scale (SE) or by both.

In figure 2.2.3, under CRS, the efficient boundary is O2. However, under VRS, the efficient boundary O2 is no longer valid since we cannot scale up and down along the line O2. Instead, we have the piecewise efficient boundary 3-2-5. Therefore, the technical input efficiency of DMU 4 is AC/A4.

Formally, the BCC models are as follows:
Let us consider the *N* DMUs j=1…N using m inputs to secure s outputs. Let us denote x_{ij} and y_{rj} as the *i*th input and *r*th output of the DMU j.

Input-orientation

$$
\begin{aligned}
&Min \quad && \theta - \varepsilon\left[\sum_{i=1}^{m} S_i^- + \sum_{r=1}^{s} S_r^+\right] \\
&s.t. \quad && \sum_{j=1}^{N} \lambda_j x_{ij} = \theta x_{ij_0} - S_i^-, \quad i = 1...m, \\
& && \sum_{j=1}^{N} \lambda_j y_{rj} = y_{rj_0} + S_r^+, \quad r = 1...s \\
& && \sum_{j=1}^{N} \lambda_j = 1, \\
& && \lambda_j \geq 0, \quad j = 1...N \quad S_i^-, S_r^+ \geq 0
\end{aligned}
\tag{2.2.46}
$$

It differs from the formula under CRS only in that it includes the so-called convexity constraint $\sum_{j=1}^{N} \lambda_j = 1$. This constraint does not allow any free scaling up or down to form a referent point for efficiency measurement. The convexity constraint $\sum_{j=1}^{N} \lambda_j = 1$ essentially ensures that an inefficient firm is only ‘benchmarked’ against firms of a similar size. This convexity restriction is not imposed in the CRS case. Hence, in a CRS DEA, a firm may be benchmarked against firms that are substantially larger (smaller) than it. In this instance the λ -weights sum to a value less than (greater than) one. $\sum_{j=1}^{N} \lambda_j \leq 1$ ensures that the *j*-th firm is not ‘benchmarked’ against firms that are substantially larger than it, but maybe compared with firms smaller than it. The optimal level of input efficiency θ^* is termed as *pure* technical input efficiency of DMU j_0 and they are ‘net’ of any scale effects.

Output Orientation

$$
\begin{aligned}
Max \quad & \gamma + \varepsilon\left[\sum_{i=1}^{m} I_i + \sum_{r=1}^{s} O_r\right] \\
s.t. \quad & \sum_{j=1}^{N} \alpha_j x_{ij} = x_{ij_0} - I_i, \quad i = 1...m, \\
& \sum_{j=1}^{N} \alpha_j y_{rj} = \gamma y_{rj_0} + O_r, \quad r = 1...s, \\
& \sum_{j=1}^{N} \alpha_j y_{rj} = \gamma y_{rj_0} + O_r, r = 1...s \quad r = 1...s, \\
& \sum_{j=1}^{N} \alpha_j = 1 \quad \alpha_j \geq 0, \quad j = 1...N, \quad I_i, O_r \geq 0
\end{aligned}
\tag{2.2.47}
$$

where ε is a non-Archimedean infinitesimal. Therefore $\frac{1}{\gamma^*}$ is a measure of output efficiency of DMU j_0. Under VRS, $\frac{1}{\gamma^*}$ is the pure technical output efficiency of DMU j_0.

The above DEA models assume that the units which are combined by the observed units in any way can exist, in other words, convexity is assumed. So the underlying units are compared with the efficient frontier, which are hypothetical but potentially efficient combinations of the actual observations. However, there is some debate in the literature concerning the validity of this assumption. For example, in situations where commodities are not continuously divisible, the assumption of convexity does not apply. The best-known non convex technological set is the free disposal hull (FDH). It requires input and output disposability (i.e. there are slacks in the inputs and outputs which can be reduced without using up other additional resources). The efficiency frontier in the FDH is composed only of observed production units.

Free disposal hull (FDH) analysis is an alternative method to the conventional DEA methodology which was introduced by Deprins, Simar, and Tulkens (1984) and further developed by Tulkens (1993) and Vanden Eckaut, Jamar and Tulkens (1993).

Graph 2.2.4 illustrates what a FDH is like. In this example, there are two inputs X_1 and X_2. The DEA isoquant is the piecewise linear frontier connecting B, D, E and F, with C and G as inefficient points. The FDH isoquant is the stepped line connecting B, D, C, E and F. Each of these points is then regarded as efficient. It is only observed point G in the FDH counts as inefficient.

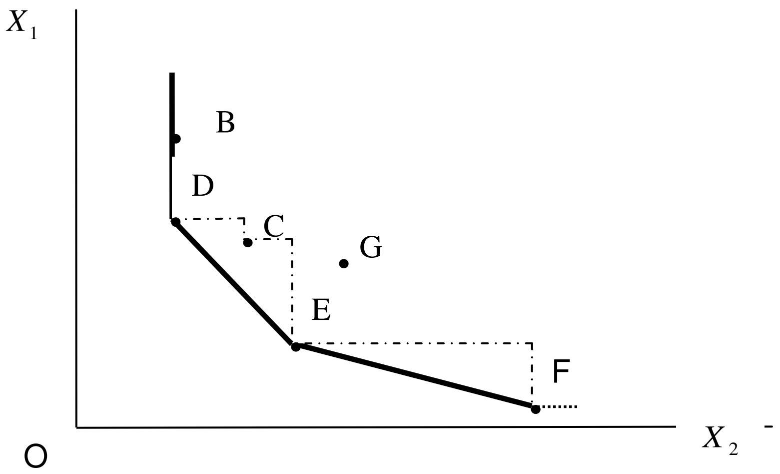

Figure 2.2.4 Free disposal hull (FDH) input space representation

Formally, the free disposal hull (FDH) efficient frontier suggested by Tulkens (1993) is as follows:

$$T_D^* = \{(x, y) : x \geq \sum_{j=1}^{n} \lambda_j x_j , y \leq \sum_{j=1}^{n} \lambda_j y_j , \lambda_j \in \{0,1\}, \sum_{j=1}^{n} \lambda_j = 1\} \quad (2.2.48)$$

This is a mixed-integer programming problem as the weights λ_j can take only 0 or 1 as values.

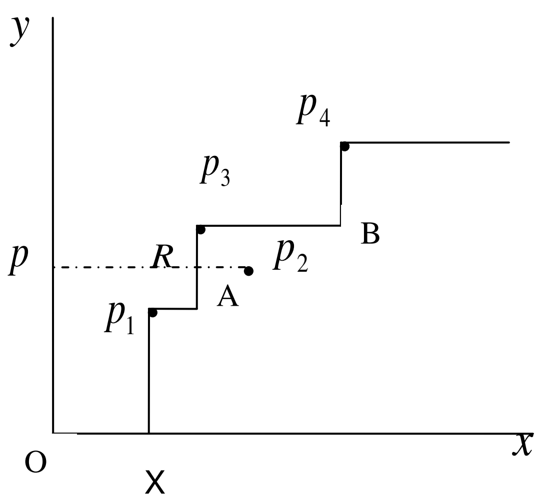

Figure 2.2.5 FDH Production Possibilities Set

Figure 2.2.5 is an example of FDH using a single input and a single output. The efficient frontier is $XP_1AP_3BP_4$ and the production possibility set is estimated by the area on and to the right of this frontier. P_1, P_3, P_4 are on the frontier while P_2 lies inside the frontier. Therefore, P_1, P_3, P_4 are efficient and P_2 is inefficient. The efficiency score of P_2 can be measured by the radial contraction in input needed to reach the frontier. This is the ratio: PR / PP_2.

The input orientated FDH model is as follows:

$$\begin{aligned} &\theta^* = Min \quad \theta \\ &s.t. \quad \sum_{j=1}^{N} \lambda_j x_{ij} \le \theta x_{ij_0}, \quad i = 1...m, \\ &\qquad \sum_{j=1}^{N} \lambda_j y_{rj} \ge y_{rj_0}, \quad r = 1...s, \\ &\qquad \sum_{j=1}^{N} \lambda_j = 1 \\ &\qquad \lambda_j \in \{0,1\}, \quad j = 2,3,...,N \end{aligned} \tag{2.2.49}$$

Similarly, the following formula gives the output-oriented FDH:

$$
\begin{aligned}
&\gamma^* = Max \quad \gamma \\
&s.t. \quad \sum_{j=1}^{N} \alpha_j x_{ij} \le x_{ij_0}, \quad i = 1...m \\
&\qquad \sum_{j=1}^{N} \alpha_j y_{rj} \ge \gamma y_{rj_0}, \quad r = 1...s, \\
&\qquad \sum_{j=1}^{N} \alpha_j y_{rj} \ge \gamma y_{rj_0}, \quad r = 1...s, \\
&\qquad \alpha_j \ge 0, \qquad j = 1...N, \\
&\qquad \sum_{j=1}^{N} \alpha_j = 1; \quad \alpha_j \in \{0,1\} \quad j = 1,2,...,N
\end{aligned} \tag{2.2.50}
$$

Where $1/\gamma^*$ measures the technical output efficiency

Murthi et at. (1997) were the first to apply the DEA method to fund performance evaluation. They presented a non-parametric benefit and cost analysis in the original CCR ratio form with standard deviation of returns, expense ratio, load and turnover as inputs and mean gross return as output. They developed a new measure that avoids the benchmark problem that exists in the traditional portfolio analysis. They employed data for a sample of 2083 US equity mutual funds for the third quarter of 1993. They detected a significant positive relation between their efficiency index and Jensen's alpha for all categories of funds, which indicated that the DEA measure of performance is consistent with traditional indices while offering more advantages over the traditional methods.

Basso & Funari (2001) tested the DEA performance indexes for 47 Italian investment funds that were classified as equity, bond and balanced funds in the period 01/01/1997 to 30/06/1999. They used several risk measures such as standard deviation, the square root of the half-variance and beta coefficient as inputs, and other inputs include subscription and redemption costs. The expected return and the stochastic dominance indicator defined using the DARA criterion were used as outputs. Also they considered subscription fees and redemption fees when calculating the costs. The results indicate that it is important to deduct the subscription and redemption costs when determine the fund ranking.

Galagedera and Silvapulle (2002) conducted DEA models to assess the relative performance of 257 Australian investment funds for the period 1995 to 1999. They used four output variables to capture the short-,the medium- and the long-term gross performances and seven inputs including four standard deviations of the 1-,2-,3- and 5-year gross performance, sales charges, operating expenses and minimum initial investment. Their results suggest that scale efficiency is the main source of overall technical efficiency, and that risk-averse funds with high positive net asset flows tend to have higher overall technical efficiency as well as the scale efficiency, while structure, classification, size and the age of funds have little impact on the level of relative efficiency.

Daraio & Simar (2006) proposed a robust non-parametric performance measure based on the concept of order-m frontier and on a probabilistic approach. They compared the performance of six categories of funds: asset allocation, aggressive growth, balanced, equity income, growth and growth income, and examined more than 3000 US mutual funds for the period June 2001-May 2002. They used standard deviation, expense ratio, and turnover and fund size as inputs and mean return as output. Also, economies of scale, slacks and market risks are investigated. The results show that for some categories including asset allocation, aggressive growth and equity income, the investment funds did not lie on the mean-variance efficiency frontier due to the slacks. In addition, the economies of scale deriving from portfolio management and shareholder service have no impact on most investment funds in terms of efficiency.

Lozano & Gutierez (2008) combined DEA with stochastic dominance criteria and performed an efficiency analysis for a sample of 108 Spanish funds in a four-year period from January 2002 to December 2005. They presented six distinct DEA-like linear programming (LP) models that incorporate second-order stochastic dominance under the assumption that investors are rational, risk-averse. They conducted four return-risk DEA models which use return as input and risk as output and two return-safety DEA models which are pure output DEA models with both return and safety measures as outputs. Similar results were obtained from five of the proposed LP models.

One of the main advantages of these non-parametric frontiers is that they can handle multiple dimensions simultaneously and that these yield a single real number performance. However, there is no evident rule for the selection among various candidates of input-like and output-like variables. Therefore it is not always clear whether a certain variable should be included in the model calculating the efficiency measure, or rather should be used to explain the observed variations in the efficiency measures in the second stage analysis.

Recently, some researchers gave a new angle to the mutual fund evaluation. They followed the Markowitz portfolio theory and related the non-parametric methodologies to the foundations of traditional performance measurement in the mean-variance space. Markowitz portfolio theory postulates that an investor chooses unobserved subjective weights to maximise the utility of a portfolio subject to constraints on the mean (M) and variance (V) of the sum of the weighted returns. Therefore the Markowitz model establishes a tangency point between an unobserved indifference curve in MV space and the efficient portfolio frontier. Morey and Morey (1999) developed a mutual funds efficiency measure in a traditional mean-variance (MV) model. They presented two basic quadratic programming approaches to identify those funds that are efficient. The purpose of the Morey & Morey model is to compare the relative improvement that could be achieved by a sample portfolio when compared to other candidate portfolios. It uses the Markowitz model as a template for specifying the theoretical optimizing behaviour of the fund managers. The Morey & Morey model solves for objective weights that establish the relative distance from the efficient portfolio frontier of each sample portfolio. The efficient frontier simulates the unobserved Markowitz frontier and compares the achieved realizations of the sample portfolios. Therefore the Morey & Morey model measures the relative success of different fund managers assuming that their subjective optimizing behaviour can be described by the Markowitz theory of behaviour. It measures the relative success of different fund managers in behaving like Markowitz optimizers, and therefore differs in purpose and implementation from the Markowitz model. The first quadratic program is mean return augmentation, it is as follows:

Determine $w_j \geq 0 (j = 1,2,... j_0,..., N)$ and $\theta \geq 1$ so that:

$$
\begin{aligned}
&Max \quad \theta \\
&s.t. \quad \sum_{j=1}^{N} w_j = 1 \\
&\qquad \sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j Cov(R_{i,t}, R_{j,t}) \le \sigma_{j_0}^2 \\
&\qquad \sum_{j=1}^{N} w_j E(R_{j,t}) \ge \vartheta E(R_{j_0,t}) \\
&\qquad (t = 1,2,...T)
\end{aligned} \tag{2.2.51}
$$

Where $\theta \ge 1$ and higher θ indicates poorer performance. So the frontier funds are those with theta value of one.

The second quadratic program is risk contraction, and the formula is as follows:

$$
\begin{aligned}
&Min \quad Z \\
&s.t. \quad \sum_{j=1}^{N} w_j = 1 \\
&\qquad \sum_{j=1}^{N} w_j E(R_{j,t}) \ge E(R_{j_0,t}) \\
&\qquad \sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j Cov(R_{i,t}, R_{J,t}) \le Z\sigma_{j_{0,t}}^2 \\
&\qquad (t = 1,2,...T)
\end{aligned} \tag{2.2.52}
$$

Where $Z \le 1$ and the efficient frontier is composed by funds with Z equal to one.

Figure 2.2.6 illustrates these two quadratic models. They give different but similar rankings of different mutual funds. Further, Morey and Morey (1999) claimed that even a fund with θ or Z equal to one could still have further slacks possible. Therefore, in addition to the quadratic optimization, they apply a standard device used in DEA which is a lexicographic, pre-emptive programming to help identify the maximum increases possible in mean returns or the reduction in risks for a given fund.

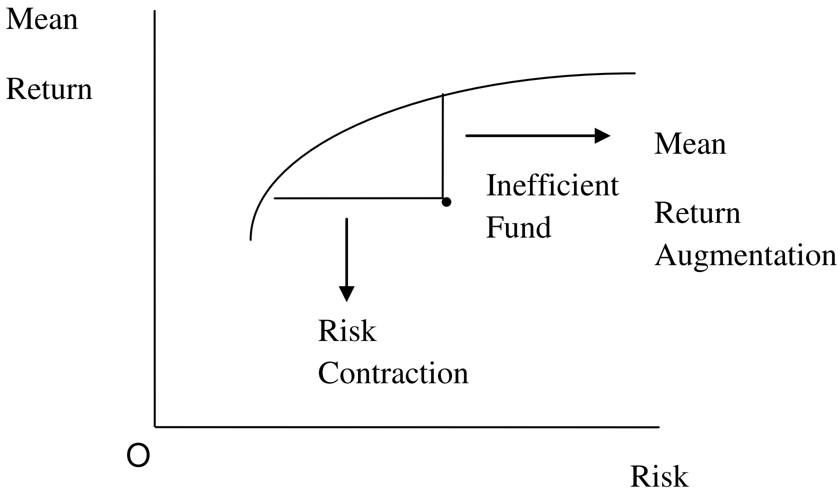

Figure 2.2.6 Different paths to the efficiency frontier

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Later, W. Briec, K. Kerstens, And J. B. Lesourd (2004) applied the directional distance function and its properties into the mutual fund evaluation and introduced an efficiency improvement possibility (EIP) function. It is as follows:

$$S_g(x) = \sup\{\delta; (V(R(x)) - \delta_{gV}, E(R(x)) + \delta_{gE}) \in R\} \tag{2.2.53}$$

Where $S_g(x)$ is the EIP function for the portfolio x in the direction of vector $g = (-g_V, g_E)$. It allows simultaneous changes in the direction of reducing inputs *x* and expanding outputs *y*.

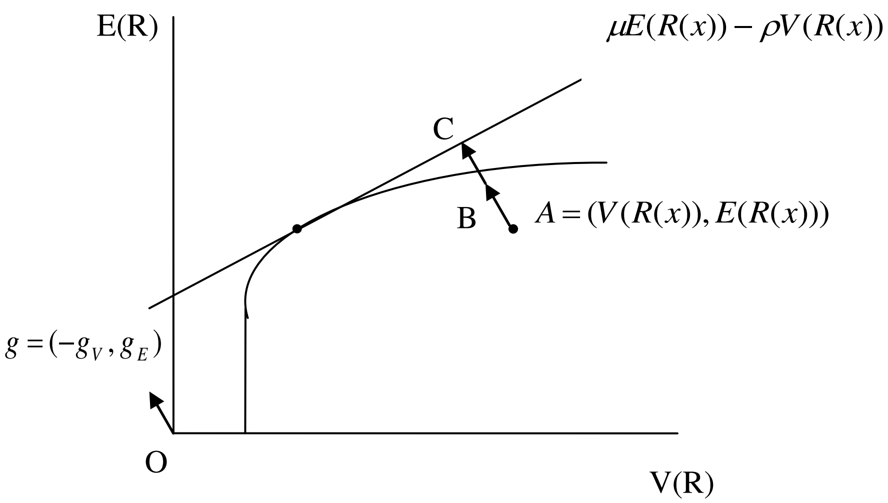

Figure 2.2.7 Efficiency Improvement Possibility Function & Decomposition

Figure 2.2.7 illustrates the principle of the EIP function, where the inefficient portfolio A is projected onto the efficient frontier at point B.

Also, W. Briec, K. Kerstens, And J. B. Lesourd (2004) defined an indirect mean-variance utility function for given parameters (ρ,μ), where ρ is the weight in the utility function on expected return, and μ is the coefficient of risk.

$$\begin{aligned} U^*(\mu,\rho) = \sup\ & \mu E(R(x)) - \rho V(R(x)), \\ & Ax \le b \\ & \sum_{i=1,\dots,n} x_i = 1, \qquad x \ge 0 \end{aligned} \tag{2.2.54}$$

It determines the maximum value function for the decision maker for a given set of parameters (ρ,μ) which represents the investor's return preference and risk aversion. The overall efficiency (OE), allocative efficiency (AE), and portfolio efficiency (PE) are distinguished as follows:

$$\begin{aligned} & OE(x,\rho,\mu) = \sup\{\delta;\mu(E(R(x)) + \delta g_E) - \rho(V(R(x)) - \delta g_V) \le U^*(\rho,\mu)\} \\ & AE(x,\rho,\mu) = OE(x,\rho,\mu) - S_g(x) \\ & PE(x) = S_g(x) \end{aligned} \tag{2.2.55}$$

Portfolio efficiency (PE) measures the distance needed for the point in evaluation to reach the portfolio frontier. Allocative efficiency (AE), however, measures the portfolio reallocation along the portfolio frontier, in order to achieve the maximum of the indirect mean-variance utility function. And the following relationship holds:

$$OE(x,\rho,\mu) = AE(x,\rho,\mu) + PE(x) \tag{2.2.56}$$

According to the above definitions, a standard quadratic optimization method is computed:

$$
\begin{aligned}
\max \quad & \delta, \\
s.t. \quad & E(R(y^k)) + \delta g_E \le \sum_{i=1,..n} x_i E(R_i), \\
& V(R(y^k)) - \delta g_V \ge \sum_{i,j} \Omega_{i,j} x_i x_j, \\
& Ax \le b, \\
& \sum_{i=1,...,n} x_i \quad x_i \ge 0, \quad i = 1,...,n.
\end{aligned}
\tag{2.2.57}
$$

W. Briec, K. Kerstens and O. Jokung (2007) extended the W. Briec et al (2004) into a mean-variance-skewness space using cubic programming. They claimed that portfolio returns are generally not normally distributed as investors prefer positive skewness so that the probability of obtaining a negative return is low. This idea can be related to the Prospect Theory model of Kahneman and Tversky (1979) which underlies many of the recent developments in behavioural finance. The cubic utility function in the MVS space is as follows, where SK is a measure of skewness:

$$
U_{k,\mu,\rho}(x) = \mu E(R(x)) - \rho V(R(x)) + kSK(R(x)) \tag{2.2.58}
$$

And the indirect utility function is accordingly rewritten as:

$$
\begin{aligned}
& \max E(R(x)) - \varphi V(R(x)) + \Psi SK(R(x)) \\
& s.t. \quad \sum_{i=1,..n} x_i = 1, \quad x \ge 0
\end{aligned}
\tag{2.2.59}
$$

The cubic program is computed as follows:

\

$$
\begin{aligned}
\max \quad & \delta, \\
s.t. \quad & E(R(y^k)) + \delta g_E \le \sum_{i=1,..n} x_i E(R_i) \\
& V(R(y^k)) - \delta g_V \ge \sum_{i,j} \Omega_{i,j} x_i x_j, \\
& Sk[R(y^k)] + \delta g_S \le \sum_{i.j.k} CSK_{i,j,k} x_i x_j x_k \\
& \sum_{i=1,...,n} x_i \quad x_i \ge 0, \quad i = 1,...,n.
\end{aligned}
\tag{2.2.60}
$$

Similar to the model in MV space, this cubic program divides the overall efficiency into portfolio efficiency, allocative efficiency, and convexity efficiency. The portfolio efficiency is the distance from the point in evaluation to the boundary of the efficient frontier, and allocative efficiency is the necessary move along the efficient frontier in order to get the portfolio most preferred. *Convexity efficiency measures the difference between the shortage functions computed on both the convex representation set CR and the initial non-convex representation sex DR*. (W. Briec et al (2007))

K Kerstens, A Mounir, and I V Woestyne (2010) examined different returns to scale, convexity problems and higher order moments in the quadratic and cubic optimization programming and argued that various return to scale (VRS), Free Disposal Hull and higher moments are desirable methodologies for the mutual funds evaluation.

Chapter 3 A quadratic DEA model

3.1 Introduction and motivation

The key element in portfolio analysis is the emphasis on both expected return and risk. Thus investment fund managers attempt to find efficient portfolios –those promising the greatest expected return for any given degree of risk, i.e. risk-adjusted return. Consequently there is considerable interest in comparing the performance of investment fund management companies.

Morey and Morey (1999) developed a mutual funds efficiency measure in a traditional mean-variance model. It was based on Markowitz portfolio theory and related the non-parametric methodologies to the foundations of traditional performance measurement in mean-variance space. The model is derived from the standard data envelopment analysis but differs from it in having non-linear constraints in the envelopment version of the model's structure. These constraints give rise to dual multipliers with economically important interpretations. Morey and Morey presented two basic programming approaches which have radial efficiency measures with both linear and quadratic constraints: mean return augmentation and risk contraction to identify those funds that are relatively efficient in the data envelopment analysis sense. Briec et al. (2009) and other authors further developed the model into a mean-variance-skewness space using cubic programming, as I showed in the preceding literature review.

As stated in the literature, two main methods have been used in mutual funds evaluation. One is portfolio evaluation, and the other is data envelopment analysis. The history of portfolio evaluation dates from the 1960s (Sharp, 1966; Treynor, 1965 and Jensen, 1968), with emphasis on both expected return and risk. Mutual fund managers attempt to find efficient portfolios – those promising the greatest expected return for any given degree of risk, i.e. risk-adjusted return. Murthi et at. (1997) were the first to apply DEA methodology to fund performance evaluation. A large proportion of DEA models applied to mutual funds show pieceswise linear correspondence between multiple inputs and outputs. Murthi et al. (1997) used standard deviation of returns, expense ratio, load and turnover as inputs, and mean gross return as output. Basso and Funari (2001) used several risk measures (standard deviation, standard semi-deviation and beta) and subscription and redemption costs as inputs, and the mean return and the percentage of periods in which the fund was non-dominated as outputs. Those linear DEA programs are good at handling multiple dimensions simultaneously and then yield a single real number performance. However, there is no evident rule for choosing between the various candidates of input and output variables and it is not always clear whether any given variable should be included in the model calculating the efficiency measure, or rather should be used to explain the observed variations in the efficiency measures in the second stage analysis. Also, those linear models show piecewise linear representation of inputs and outputs. However, according to Markowitz portfolio theory, there is correlation between different assets which should not be ignored, and these co-movements between different securities affect the relationship between expected return and risk of the combined portfolio.

Departing from linear models, Morey and Morey (1999) constructed a quadratic one-input, one-output DEA model. They used the fund's risks as outputs and mean returns as inputs. It was a quadratic model because it did not only contain linear constraints in the model, but also quadratic constraints. It utilized the insights from the traditional Markowitz portfolio theory that imperfect correlation between different assets leads to diversification of risk, and thus exploits the quadratic relationship between expected return and risk of the combined portfolio. Instead of having a piecewise frontier, as in linear DEA models, the efficient frontiers for Morey and Morey (1999) quadratic models are smooth concave curves in mean-variance space.

In the essence of data envelopment analysis, Morey and Morey (1999) quadratic models used the idea of 'funds of funds': for each fund there is a corresponding composite benchmarking fund, which lies on the efficient frontier. These are hypothetical but potentially efficient combinations of the actual observations. DEA scores are obtained by measuring the direct distance from the position of the fund in question in mean-variance space to that of the efficient composite benchmarking fund.

Although only mean and variance are considered in Morey and Morey (1999) models, they distinguish their model from traditional portfolio analysis by the fact that there is no theoretical benchmark like the market portfolio of the Capital Asset Pricing Model. Instead, the benchmarking fund in Morey and Morey (1999) consists of certain funds in the group, each with a particular weight. So rather than being compared with an idealised fund that requires information about all the equities in the market, Morey and Morey (1999) model benchmarks the funds under evaluation again themselves. This makes Morey and Morey (1999) model practically feasible and easier to test.

Chapter 3 is organized as follows: Section 2 provides a brief literature review of DEA models on fund performance evaluation; Section 3 describes in detail the Morey and Morey (1999) quadratic model, Section 4 shows the data collection; and Section 5 presents the results.

3.2 Literature review

Data envelopment analysis is a methodology in operational research which gives completely different scenario from portfolio evaluation. It is a nonparametric analysis technique which was proposed by Charnes et al. (1978), and it was firstly used in measurement of the performance of educational institutions. Murthi et at. (1997) were the first to apply this method to fund performance evaluation. As one of the linear models; they used standard deviation of returns, expense ratio, load and turnover as inputs, and mean gross return as output. Basso & Funari (2001) used several risk measures (standard deviation, standard semi-deviation and beta) and subscription and redemption costs as inputs, and the mean return and the percentage of periods in which the fund was non-dominated as outputs. They also incorporate a stochastic dominance criterion as one of the outputs in their module. Sengupta (2003) employed loads, expenses, turnover, risk (standard deviation or beta) and skewness of returns as inputs and raw returns as output in his model. Daraio & Simar (2006) used standard deviation, expense ratio, turnover and fund size as inputs and mean raw return as output. And their model was based on an order-m frontier. Lozano & Gutierez (2008) incorporated second-order stochastic dominance in their models and used mean return as input and various measures of risk as outputs. Besides the variables of return, risk and transaction costs, Galagedera and Silvapulle (2002) include the minimum initial investment as an additional variable. Haslem and Scheraga (2006) include the percentage of stocks; Premachandra, Powell, and Shi (1998) add a variable indicating the total amount that is invested risk-free, etc.

Considering the drawbacks of the linear models, Morey and Morey (1999) developed quadratic data envelopment analysis models which followed the Markowitz portfolio theory and related the non-parametric methodologies to the foundations of traditional performance measurement in the mean-variance space. It is also constrained to avoid short sales. They presented two basic quadratic programming approaches to identify those funds that are efficient. Under this quadratic programming, only technical efficiency is being evaluated.

3.3 Methodology

This paper first of all applied Morey and Morey (1999) models to a recent dataset. Morey and Morey (1999) presented two basic quadratic programming approaches to identify those funds that are efficient. These two approaches are mean return augmentation and risk contraction. Figure 3.3.1 illustrates these two quadratic models.

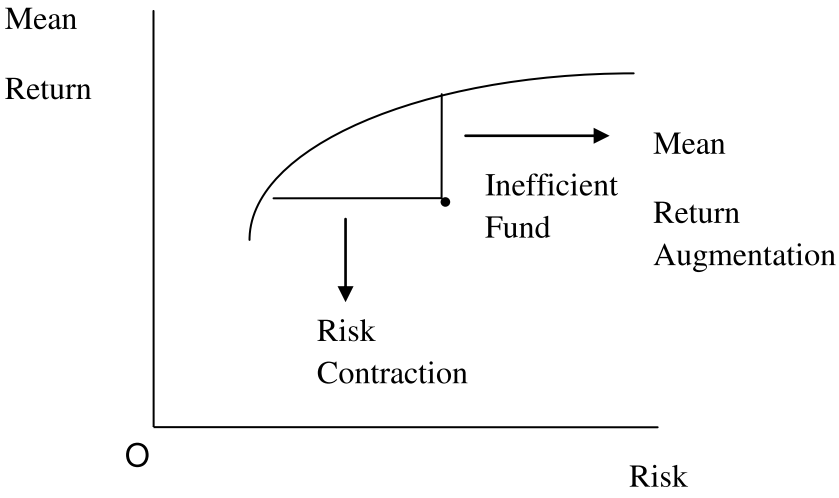

Figure 3.3.1 Quadratic DEA efficiency frontier

As illustrated in Figure 3.3.1, it is in the mean-variance space with risk as input and mean return as output. These two approaches show different paths to the efficiency frontier. Mean return augmentation method could be seen as output oriented DEA, and it represents a vertical path towards the efficient frontier, while the risk contraction model is input-oriented DEA which follows the horizontal path.

Consider N mutual funds to be evaluated, indexed j=1, 2,...,N, where j_0 is the fund in evaluation for each run. $j_0 = 1,2,...N,$ and there are N runs totally. Let T denote the number of different time horizons, where t=1, 2,…,T. Denote $E(R_{j,t})$ as the mean return for fund j, and σ_j^2 as its the variance as well as $Cov(R_{i,t}, R_{j,t})$ as its covariance. Denote w_j as the weight allocated to each fund to form the benchmarking fund in each run. The formula for mean return augmentation is as follows:

Determine $w_j \geq 0 (j = 1,2,...j_0,..., N)$ so that:

$$
\begin{aligned}
&Max \quad \theta \\
&s.t. \quad \sum_{j=1}^{N} w_j = 1 \\
&\qquad \sum w_j^2 \sigma_{j,t}^2 + \sum_{\substack{i=1 \\ i+j}}^{N} \sum_{j=1}^{N} w_i w_j Cov(R_{i,t}, R_{j,t}) \le \sigma_{j_0}^2 \qquad (3.3.1) \\
&\qquad \sum_{j=1}^{N} w_j E(R_{j,t}) \ge \vartheta E(R_{j_0,t}) \\
&\qquad (t = 1,2,...T)
\end{aligned}
$$

Where θ is the efficiency score and we have $\theta \ge 1$. It can be seen that this is an implementation of Markowitz portfolio analysis.

θ is calculated by running the above programming problem once for each fund. Efficient funds will have a value of one, while inefficient ones will get a value greater than one which shows how much the actual return should be expanded for the fund to be considered technically efficient. The efficiency score measured here is called 'technical efficiency' since it treats fund risk characteristics and costs as inputs in a simulation of the production of 'return' as an output. The measured efficiency scores relate to sampled funds and are relative in the sense of measuring the relative distance of each sample point to the efficient frontier of sample funds. They do not directly measure an abstract or theoretical efficiency. However, the purpose of the Monte Carlo bootstrapping analysis later in the thesis is to treat the measured scores as statistical estimators and to construct the sampling distributions of these estimators.

This model is a nonparametric technique, as weights w_j are produced by the programming itself rather than set up beforehand. The first constraint is a convexity constraint; the second constraint maintains the risk level of the resulting composite fund obtained from this programming the same as that of the fund being evaluated, if not smaller; In the third constraint, because theta could only be equal or larger than 1, this constraint guarantees that mean return of the resulting benchmarking fund is larger than or at least equal to that of the target fund. Note that constraints 2 and 3 hold for the all T time horizons. So this programming has 2T+1constraints totally. And the objective of this programming is to simultaneously maximize the increases in the mean returns over all these periods, without occurrence of any increase in the risks. θ measures how many times the target fund must vertically expand in order to get the efficient frontier in this mean-variance space. Note that Figure 3.3.1 represents one of the $2T+1$ periods, while for T periods programming it has T figures like this, and the resulting θ must satisfy the conditions in all T periods. This paper involves three periods, a 3-year period, a 5-year period and a 10-year period. So T=3.

The second quadratic program is risk contraction, and the formula is as follows:

$$
\begin{aligned}
& Min \quad Z \\
& s.t. \quad \sum_{j=1}^{N} w_j = 1 \\
& \qquad \sum_{j=1}^{N} w_j E(R_{j,t}) \geq E(R_{j_0,t}) \\
& \qquad \sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_j Cov(R_{i,t}, R_{J,t}) \leq Z\sigma_{j_0,t}^2 \\
& \qquad (t = 1,2,...T)
\end{aligned}
\tag{3.3.2}
$$

Where $Z \leq 1$ and the efficient frontier is composed by funds with Z equal to one.

Similarly to the first approach, the first constraint is a convexity constraint; the second constraint then promises that the mean return of the efficient benchmarking fund maintains the same level as that of the fund being evaluated, if not larger; For the third constraint, since Z is equal to or less than 1, it requires that the efficient composite fund has smaller or at least the same risk level as that of the target fund. Again, constraints 2 and 3 hold for the all T time horizons. The objective of this programming is to simultaneously minimize the contraction in the risk level over T periods, without any decrease in the mean returns. As shown in Figure 3.3.1, Z measures the minimum contraction the target fund needs in order to reach the efficient frontiers, considering all the conditions over T periods.

3.4 Data collection

The database used is MorningStar Direct. The funds chosen were ‘Acc’ open ended funds. The ‘Acc’ distribution status means that dividends generated from these funds are automatically reinvested back to the funds. We examine one specific type of the open ended funds: those classified by Morningstar as ‘UK mid-cap equity’ as of July 1, 2011. This is because it has a fairly small number of funds which makes the analysis of the entire group easier. The choice of sample was based on considerations of homogeneity of the business of the underlying trusts. Another important criterion is that the funds we select must have at least 10 years of monthly return data available, because the models require mean returns for three periods: 3 years, 5 years and 10 years. Our sample period was from July 1 2011 to June 30, 2011, so each fund selected had an inception date at or before July 1, 2001.

32 funds were found that satisfied the above criteria. For each fund, funds with negative mean monthly returns in any period were deleted, leaving 29 funds in the data set.

For each of the 29 mutual funds, the following figures were calculated for each of the 3, 5 and 10-year time periods: (i) Mean monthly returns; (ii) Covariances; (iii) Variances. These values were calculated using monthly return data from Morningstar Direct database. Expressed in percentage terms, Morningstar's calculation of monthly return is determined by taking the change in monthly net asset value, reinvesting all income and capital-gains distributions during that month, and dividing by the starting net asset value. The total returns account for management, administrative and other costs taken out of fund assets. Note that Morningstar does not adjust total returns for sales charges (such as front-end loads, deferred loads and redemption fees), preferring to give a clearer picture of a fund's performance.

The selected twenty-nine funds with their 3-year, 5-year and 10-year mean monthly returns are presented in Table 3.4.1

Table 3.4.1 Funds analysed and their mean monthly returns

Fund No. and name	3 Year mean monthly return	5 Year mean monthly return	10 Year mean monthly return
(1) Aberdeen UK Mid Cap A Acc	1.33	0.53	0.60
(2) AEGON Ethical Equity A	0.56	0.59	0.60
(3) AEGON Ethical Equity B	0.63	0.66	0.67
(4) Allianz RCM UK Mid Cap A	1.02	0.72	0.77
(5) Artemis UK Special Situations	0.84	0.57	0.82
(6) Aviva Investors SF UK Growth SC1	0.47	0.37	0.36
(7) Aviva Investors SF UK Growth SC2	0.54	0.43	0.42
(8) Aviva Investors UK Ethical SC1	0.42	0.41	0.44
(9) Aviva Investors UK Ethical SC2	0.44	0.43	0.46
(10) BlackRock UK Special Situations A Acc	1.03	0.82	0.83
(11) Ecclesiastical Amity UK C	0.86	0.44	0.48
(12) F&C Stewardship Growth 1 Acc	0.56	0.22	0.36
(13) F&C Stewardship Income 1 Acc	0.46	0.28	0.56
(14) GAM Exempt Trust UK Opportunities	0.84	0.44	0.65
(15) GAM UK Diversified Acc	0.89	0.45	0.70
(16) Henderson UK Equity Income I Acc	1.14	0.63	0.63
(17) HSBC FTSE 250 Index Retail Acc	1.18	0.74	0.83
(18) Marlborough UK Primary Opps A Acc	0.59	0.44	0.63
(19) Marlborough UK Primary Opps B Acc	0.56	0.44	0.64
(20) MFM Bowland	1.22	0.63	0.84
(21) Saracen Growth Alpha	0.11	0.24	0.67
(22) Saracen Growth Beta	0.16	0.29	0.72
(23) Schroder UK Mid 250 Acc	0.84	0.44	0.92
(24) Standard Life UK Eq High Alpha Inst Acc	1.66	1.10	0.80
(25) Standard Life UK Eq High Alpha R Acc	1.60	1.04	0.83
(26) Standard Life UK Ethical Inst	0.80	0.55	0.60
(27) Standard Life UK Ethical R	0.73	0.48	0.55
(28) SVM UK Opportunities Instl	1.42	0.90	0.96
(29) SVM UK Opportunities Retail	1.35	0.83	0.90

3.5 Results analysis

Efficiency scores are calculated only with respect to the sample units. Note that Morey-Morey routines were validated on Morey-Morey's own data. For each of the 29 funds, we ran both mean return augmentation and risk contraction programming, and the results are listed as in table 3.5.1. Recall that the return augmentation programme records efficiency of performance as ≥ 1, with fully efficient performance shown as a score of 1, while the risk reduction programme records efficiency of performance as ≤ 1, with fully efficient performance shown as a score of 1.

Table 3.5.1 (a) Frontier values from the mean return augmentation programming

Fund No. and name	Frontier mean returns		
	3 Year	5 Year	10 Year
(1) Aberdeen UK Mid Cap A Acc	1.35	0.6468	0.6991
(2) AEGON Ethical Equity A	0.8471	0.6788	0.7826
(3) AEGON Ethical Equity B	0.8489	0.6803	0.7831
(4) Allianz RCM UK Mid Cap A	1.1586	0.8178	0.8746
(5) Artemis UK Special Situations	0.84	0.57	0.82
(6) Aviva Investors SF UK Growth SC1	0.8752	0.689	0.7943
(7) Aviva Investors SF UK Growth SC2	0.8735	0.6843	0.7868
(8) Aviva Investors UK Ethical SC1	0.9188	0.7375	0.7999
(9) Aviva Investors UK Ethical SC2	0.9182	0.737	0.7998
(10) BlackRock UK Special Situations A Acc	1.03	0.82	0.83
(11) Ecclesiastical Amity UK C	0.956	0.4942	0.6265
(12) F&C Stewardship Growth 1 Acc	1.0825	0.5582	0.7086
(13) F&C Stewardship Income 1 Acc	0.46	0.28	0.56
(14) GAM Exempt Trust UK Opportunities	0.8702	0.4735	0.6892
(15) GAM UK Diversified Acc	1.0173	0.5797	0.8001
(16) Henderson UK Equity Income I Acc	1.4287	0.8368	0.804
(17) HSBC FTSE 250 Index Retail Acc	1.2191	0.7645	0.8575
(18) Marlborough UK Primary Opps A Acc	1.1007	0.7242	0.8313
(19) Marlborough UK Primary Opps B Acc	1.0204	0.6148	0.8943
(20) MFM Bowland	1.2238	0.6884	0.8426
(21) Saracen Growth Alpha	0.8628	0.4858	0.8888
(22) Saracen Growth Beta	0.8628	0.4858	0.8887
(23) Schroder UK Mid 250 Acc	0.84	0.44	0.92
(24) Standard Life UK Eq High Alpha Inst Acc	1.66	1.1	0.8
(25) Standard Life UK Eq High Alpha R Acc	1.601	1.0462	0.8305
(26) Standard Life UK Ethical Inst	1.151	0.7913	0.8632
(27) Standard Life UK Ethical R	1.1495	0.7559	0.8661
(28) SVM UK Opportunities Instl	1.42	0.9	0.96
(29) SVM UK Opportunities Retail	1.43	0.9083	0.9533

Table 3.5.1 (b) DEA scores from the mean return augmentation programming

Fund No. and name	DEA scores
(1) Aberdeen UK Mid Cap A Acc	1.015
(2) AEGON Ethical Equity A	1.1505
(3) AEGON Ethical Equity B	1.0308
(4) Allianz RCM UK Mid Cap A	1.1359
(5) Artemis UK Special Situations	1
(6) Aviva Investors SF UK Growth SC1	1.8621
(7) Aviva Investors SF UK Growth SC2	1.5914
(8) Aviva Investors UK Ethical SC1	1.7987
(9) Aviva Investors UK Ethical SC2	1.714
(10) BlackRock UK Special Situations A Acc	1
(11) Ecclesiastical Amity UK C	1.1116
(12) F&C Stewardship Growth 1 Acc	1.9331
(13) F&C Stewardship Income 1 Acc	1
(14) GAM Exempt Trust UK Opportunities	1.036
(15) GAM UK Diversified Acc	1.143
(16) Henderson UK Equity Income I Acc	1.2532
(17) HSBC FTSE 250 Index Retail Acc	1.0331
(18) Marlborough UK Primary Opps A Acc	1.3196
(19) Marlborough UK Primary Opps B Acc	1.3974
(20) MFM Bowland	1.0031
(21) Saracen Growth Alpha	1.3265
(22) Saracen Growth Beta	1.2344
(23) Schroder UK Mid 250 Acc	1
(24) Standard Life UK Eq High Alpha Inst Acc	1
(25) Standard Life UK Eq High Alpha R Acc	1.0006
(26) Standard Life UK Ethical Inst	1.4387
(27) Standard Life UK Ethical R	1.5747
(28) SVM UK Opportunities Instl	1
(29) SVM UK Opportunities Retail	1.0593

Table 3.5.2 (a) Frontier values from the risk contraction programming

Fund No. and name	Frontier variance		
	3 Year	5 Year	10 Year
(1) Aberdeen UK Mid Cap A Acc	41.7244	32.5916	26.8937
(2) AEGON Ethical Equity A	26.2296	20.6322	20.9228
(3) AEGON Ethical Equity B	28.5375	22.1238	22.108
(4) Allianz RCM UK Mid Cap A	31.9833	24.7411	21.5681
(5) Artemis UK Special Situations	25.4702	20.234	22.9294
(6) Aviva Investors SF UK Growth SC1	23.2782	18.7204	18.1069
(7) Aviva Investors SF UK Growth SC2	23.6344	18.9101	18.2147
(8) Aviva Investors UK Ethical SC1	23.4662	18.7967	17.9295
(9) Aviva Investors UK Ethical SC2	23.742	18.9806	18.0692
(10) BlackRock UK Special Situations A Acc	36.6318	27.8395	23.7489
(11) Ecclesiastical Amity UK C	26.3676	21.0007	18.0637
(12) F&C Stewardship Growth 1 Acc	23.7016	18.9708	16.2692
(13) F&C Stewardship Income 1 Acc	24.8915	19.832	14.9057
(14) GAM Exempt Trust UK Opportunities	23.2325	18.8981	21.5468
(15) GAM UK Diversified Acc	24.0626	19.5407	20.1135
(16) Henderson UK Equity Income I Acc	36.1744	28.0543	21.7694
(17) HSBC FTSE 250 Index Retail Acc	37.9648	29.4226	23.0613
(18) Marlborough UK Primary Opps A Acc	26.9454	20.9756	16.6189
(19) Marlborough UK Primary Opps B Acc	27.0885	21.0694	16.5898
(20) MFM Bowland	40.5012	32.2574	24.7067
(21) Saracen Growth Alpha	25.1772	19.9353	16.7109
(22) Saracen Growth Beta	26.8616	21.1923	17.8542
(23) Schroder UK Mid 250 Acc	58.5356	43.2719	33.049
(24) Standard Life UK Eq High Alpha Inst Acc	79.6227	55.9455	37.1673
(25) Standard Life UK Eq High Alpha R Acc	79.1058	55.7109	36.8609
(26) Standard Life UK Ethical Inst	29.5848	22.7898	18.0845
(27) Standard Life UK Ethical R	28.229	21.8734	17.0729
(28) SVM UK Opportunities Instl	103.6475	72.8428	48.6614
(29) SVM UK Opportunities Retail	68.9601	49.9436	34.5464

Table 3.5.2 (b) DEA scores from the risk contraction programming

Fund No. and name	DEA scores
(1) Aberdeen UK Mid Cap A Acc	0.9596
(2) AEGON Ethical Equity A	0.9005
(3) AEGON Ethical Equity B	0.9698
(4) Allianz RCM UK Mid Cap A	0.6503
(5) Artemis UK Special Situations	1
(6) Aviva Investors SF UK Growth SC1	0.8141
(7) Aviva Investors SF UK Growth SC2	0.8266
(8) Aviva Investors UK Ethical SC1	0.7536
(9) Aviva Investors UK Ethical SC2	0.7614
(10) BlackRock UK Special Situations A Acc	1
(11) Ecclesiastical Amity UK C	0.9467
(12) F&C Stewardship Growth 1 Acc	0.7693
(13) F&C Stewardship Income 1 Acc	1
(14) GAM Exempt Trust UK Opportunities	0.9926
(15) GAM UK Diversified Acc	0.8532
(16) Henderson UK Equity Income I Acc	0.7876
(17) HSBC FTSE 250 Index Retail Acc	0.8456
(18) Marlborough UK Primary Opps A Acc	0.5565
(19) Marlborough UK Primary Opps B Acc	0.55
(20) MFM Bowland	0.9813
(21) Saracen Growth Alpha	0.586
(22) Saracen Growth Beta	0.6242
(23) Schroder UK Mid 250 Acc	1
(24) Standard Life UK Eq High Alpha Inst Acc	1
(25) Standard Life UK Eq High Alpha R Acc	0.996
(26) Standard Life UK Ethical Inst	0.6736
(27) Standard Life UK Ethical R	0.6365
(28) SVM UK Opportunities Instl	1
(29) SVM UK Opportunities Retail	0.7167

Table 3.5.1 and table 3.5.2 show that for both mean return augmentation and risk contraction, the 5th fund Artemis UK Special Situations, the 10^{th} fund BlackRock UK Special Situations A Acc, the 13th fund F&C Stewardship Income 1 Acc, the 23^{rd} fund Schroder UK Mid 250

Acc, the 24th fund Standard Life UK Eq High Alpha Inst Acc and the 28th fund SVM UK Opportunities Instl have the efficiency score of 1, which means they are most efficient funds among these 28 funds in both approaches. To illustrate more, look at one fund, the 4th fund, particularly. The efficiency score is $\theta = 1.1359$ in the mean return augmentation approach, and in the risk contraction approach, the efficiency score is $z = 0.6503$.Table 4 shows the actual mean monthly returns and frontier mean returns for the 4th fund Allianz RCM UK Mid Cap A, as long as the actual monthly variance and frontier variance. We could see that the frontier mean returns (the mean returns for the composite benchmarking fund) expand the actual mean returns by 13.59% for each period. And the frontier variance (the variance for the composite benchmarking variance) contracted the actual monthly variance by more than one third. Note that the actual monthly variances times efficiency scores are much larger than the frontier variance for 3-year period and slightly larger for 5-year period 10-year period, which means that the third constraint in (3.1.2) is not strictly binding. In this case the frontier variances contract even more than the efficiency score indicates. Note that there is duplication among the sample of unit trusts; eg. Marlborough A and B units are claims on the same fund but one is available at lower charge to a minimum investment of £25,000 and the other for a minimum of £1000. There are several other example of institutional and retail units in the same fund. Therefore, it is not surprisingly the correlation coefficient between each pairs of such funds=1.

Mean return augmentation method could be seen as output oriented DEA, and it represents a vertical path towards the efficient frontier, and the frontier mean returns from augmentation approach are the mean returns of the hypothetical funds lying on the efficient frontier at the end of the vertical path from the inefficient funds. The relationship between the frontier mean returns and portmanteau DEA scores is that frontier mean returns equal to actual mean monthly returns times efficiency scores. Risk contraction model is input-oriented DEA which follows the horizontal path to the efficient frotier. And the frontier variances from risk contraction approach are the variances of the hypothetical funds lying on the efficient frontier at the end of the horizontal path from the inefficient funds. The relationship between the frontier variances and portmanteau DEA scores is that frontier variances equal to actual monthly variances times efficiency scores.

Table 3.5.3 Comparison of frontier and actual levels of returns and risks for Allianz RCM UK Mid Cap A fund

	3-year	5-year	10-year
Actual mean monthly returns	1.02	0.72	0.77
Frontier mean returns	1.1586	0.8178	0.8746
Actual mean monthly returns *efficiency scores($\theta = 1.1359$)	1.1586	0.8178	0.8746
Actual monthly variances	53.5373	38.0468	33.1930
Frontier variances	31.9833	24.7411	21.5681
Actual monthly variances *efficiency scores($z = 0.6503$)	34.8151	24.7418	21.5854

For the 4^{th} fund, in the mean return augmentation approach, the composite benchmarking fund consists of five other funds, with each having a particular weight: w5=0.0709 (5th fund Artemis UK Special Situations); w10 =0.5308 (10th fund BlackRock UK Special Situations A Acc); w20=0.052(20th fund MFM Bowland); w23=0.0049 (23rd fund Schroder UK Mid 250 Acc) and w28=0.3414(28th fund SVM UK Opportunities Instl). So the weights of all other funds equal to the zero. The second approach, risk contraction results a different set of weights for the corresponding composite benchmarking fund w5=0.1106 (5th fund Artemis UK Special Situations); w10 =0.6162 (10th fund BlackRock UK Special Situations A Acc); w14=0.1076 (14th fund GAM Exempt Trust UK Opportunities) and w20=0.1656 (20th fund MFM Bowland) with all other w's at the zero level.

Morey and Morey (1999) also described an approach to further discriminate the 6 most efficient funds. The idea is that for those funds with $\theta = 1$ there could still be a 'slack' or possible improvement in the mean returns for at least one of its horizons. Because the efficient frontier illustrated in Figure 3.3.1 represents the situation in one period, for three periods there are three frontiers. The two quadratic programming problems described in (3.1.1) and (3.1.2) simultaneously consider the constraints over all three periods, however for those funds with $\theta = 1$, if the fund was not on the frontiers for all periods, a 'slack' would be detected. In this approach the new objective is to maximize the mean return for only the most important period, that is:

$$Max \quad \sum_{j=1}^{N} w_j E(R_j, \tau) \tag{3.5.1}$$

With all the constraints remaining the same.

Here τ is the time period in consideration.

According to the order of importance in the mutual fund industry 10-year period is the most meaningful time horizon, followed by 5-year and 3-year period. So let $\tau = t_{10}$ first, then if the return of the benchmarking fund cannot be further maximised, then let $\tau = t_5$ and $\tau = t_3$ successively.

For the risk contraction approach, the new objective is:

$$Min \quad \sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_j Cov(R_{i,\tau}, R_{J,\tau}) \tag{3.5.2}$$

Executing the above procedures no further improvement in either expected return or risk was found. Also a further look at the results from programming (3.3.1) and (3.3.2) we found that for those funds the frontier mean returns and variances are equal to their actual mean returns and variances, which mean that they are on the frontier for all three periods therefore no further improvements are possible. So these six efficient funds are ‘tied’.

Table 3.5.4 Rankings of funds from two approaches

Fund No. And Name	Ranking from mean return	Ranking from risk
(1) Aberdeen UK Mid Cap A Acc	9	11
(2) AEGON Ethical Equity A	17	13
(3) AEGON Ethical Equity B	10	10
(4) Allianz RCM UK Mid Cap A	15	24
(5) Artemis UK Special Situations	1	1
(6) Aviva Investors SF UK Growth SC1	28	17
(7) Aviva Investors SF UK Growth SC2	25	16
(8) Aviva Investors UK Ethical SC1	27	21
(9) Aviva Investors UK Ethical SC2	26	20
(10) BlackRock UK Special Situations A	1	1
(11) Ecclesiastical Amity UK C	14	12
(12) F&C Stewardship Growth 1 Acc	29	19
(13) F&C Stewardship Income 1 Acc	1	1
(14) GAM Exempt Trust UK	12	8
(15) GAM UK Diversified Acc	16	14
(16) Henderson UK Equity Income I Acc	19	18
(17) HSBC FTSE 250 Index Retail Acc	11	15
(18) Marlborough UK Primary Opps A	20	28
(19) Marlborough UK Primary Opps B	22	29
(20) MFM Bowland	8	9
(21) Saracen Growth Alpha	21	27
(22) Saracen Growth Beta	18	26
(23) Schroder UK Mid 250 Acc	1	1
(24) Standard Life UK Eq High Alpha Inst	1	1
(25) Standard Life UK Eq High Alpha R	7	7
(26) Standard Life UK Ethical Inst	23	23
(27) Standard Life UK Ethical R	24	25
(28) SVM UK Opportunities Instl	1	1
(29) SVM UK Opportunities Retail	13	22

Table 3.5.4 lists the rankings of funds from two approaches. We could see that except the most efficient funds, only fund 3, 25 and 26 have the same ranking, with other funds rank either slightly or dramatically differently for different approaches. The correlation between

two rankings is 0.8299, which is very high. This means that although mean return augmentation approach and risk contraction approach emphasize different aspects and have different benchmarking fund on the efficient frontier, a fund could get similar ranking based on the two approaches. Also, the correlation between the DEA score from the mean return augmentation approach and Sharpe measure is 0.8535 and the correlation between the DEA score from risk contraction approach is 0.8156. This indicates that DEA scores share similar results with Sharpe ratio.

The non-linear programming Eqs. (3.3.1) can also be solved by maximizing the following function, which is the lagrangean function for this problem:

$$\Phi = \theta + \sum_{t=1}^{t=T} \lambda_t \left[\sum_{j=1}^{j=n} \left(w_{jt} ER_{jt} \right) - \theta ER_{0t} \right] + \sum_{t=1}^{t=T} \alpha_t \left[\sigma_{0t}^2 - \sum_{j=1}^{j=n} \sum_{k=1}^{k=n} w_j w_k \left(\mathrm{cov}\left(R_{jt} R_{jt} \right) \right) \right] + \lambda \left[\sum_{j=1}^{j=n} w_j - 1 \right] \tag{3.5.3}$$

Take the derivative, $\partial\Phi / \partial\theta = 0$, the following relationship could be obtained:

$$\sum \lambda_t^* E(R_{j_{0,}t}) = 1 \tag{3.5.4}$$

Similarly, formulation of Eqs. (3.3.2) could be solved by maximising the following function:

$$-\Phi' = -Z + \sum_{t=1}^{t=T} \mu_t \left[\sum_{j=1}^{j=n} \left(w_{jt} ER_{jt} \right) - ER_{0t} \right] + \sum_{t=1}^{t=T} u_t \left[Z\sigma_{0t}^2 - \sum_{j=1}^{j=n} \sum_{k=1}^{k=n} w_j w_k \left(\mathrm{cov}\left(R_{jt} R_{jt} \right) \right) \right] + v \left[\sum_{j=1}^{j=n} w_j - 1 \right] \tag{3.5.5}$$

Take the derivative, $\partial\Phi / \partial Z = 0$, one could get:

$$\sum u_t^* \sigma_{j_0,t} = 1 \tag{3.5.6}$$

λ_t^* and u_t^*, are called ‘virtual weights’. They are useful in exploring the marginal contribution of the mean reuturn and variance in each period to the fund’s efficiency. Table 3.5.5 presents the Lagrangians for the 29 funds.

Table 3.5.5 Lagrangians for the 29 funds

Fund number	Approaches	Type of return	3 years	5 years	10 years
1	Mean return augmentation	λ_t^*	0.7519	0	0
		α_t^*	0.0084	0	0
	Risk contraction	μ_t^*	1.9807	0	0
		u_t^*	0.0230	0	0
2	Mean return augmentation	λ_t^*	0.0000	1.6949	0
		α_t^*	0.0381	0	0
	Risk contraction	μ_t^*	0	0.6758	0
		u_t^*	0	0.0265	0.0179
3	Mean return augmentation	λ_t^*	0	1.5152	0
		α_t^*	0.0339	0	0
	Risk contraction	μ_t^*	0	1.4503	0
		u_t^*	0.0340	0	0
4	Mean return augmentation	λ_t^*	0.0165	0.1206	1.1590
		α_t^*	0	0.0043	0
	Risk contraction	μ_t^*	0.4200	0.6142	0
		u_t^*	0	0.0263	0
5	Mean return augmentation	λ_t^*	0.6519	0	0
		α_t^*	0.0073	0	0
	Risk contraction	μ_t^*	1.8803	0	0
		u_t^*	0	0.0230	0
6	Mean return augmentation	λ_t^*	0.0000	1.6643	0
		α_t^*	0.0169	0	0
	Risk contraction	μ_t^*	0.4758	0	1.965
		u_t^*	0	0.0265	0
7	Mean return augmentation	λ_t^*	0	1.5152	0
		α_t^*	0.0375	0	0
	Risk contraction	μ_t^*	0	1.6503	0
		u_t^*	0.0390	0	0
8	Mean return augmentation	λ_t^*	0.0165	0.1206	1.1640
		α_t^*	0	0.0043	0
	Risk contraction	μ_t^*	0.4200	0.6142	0
		u_t^*	0	0.0263	0

9	Mean return augmentation	λ_t^*	0	0	0.7553
		α_t^*	0.0035	0	0
	Risk contraction	μ_t^*	0	1.6574	0
		u_t^*	0.0350	0	0
10	Mean return augmentation	λ_t^*	0.0000	0	1.6593
		α_t^*	0.0381	0	0
	Risk contraction	μ_t^*	0	0.6758	0
		u_t^*	0	0.0265	0.0169
11	Mean return augmentation	λ_t^*	0	1.5152	0
		α_t^*	0.0339	0	0
	Risk contraction	μ_t^*	0	1.4503	0
		u_t^*	0.0340	0	0
12	Mean return augmentation	λ_t^*	0.0165	0.1206	1.065
		α_t^*	0	0.0043	0
	Risk contraction	μ_t^*	0.4200	0.6142	0
		u_t^*	0	0.0263	0
13	Mean return augmentation	λ_t^*	0.0339	0	0
		α_t^*	0	1.4503	0
	Risk contraction	μ_t^*	0.0340	0	0
		u_t^*	0.0165	0.8759	1.007
14	Mean return augmentation	λ_t^*	0	0.0043	0
		α_t^*	0.4200	0.6142	0
	Risk contraction	μ_t^*	0	0.0263	0
		u_t^*	0.0340	0	0
15	Mean return augmentation	λ_t^*	0.0753	0.1706	1.1640
		α_t^*	0	0.0043	0
	Risk contraction	μ_t^*	0.4200	0.6142	0
		u_t^*	0	0.0263	0
16	Mean return augmentation	λ_t^*	0.0381	0	0
		α_t^*	0	0.6758	0
	Risk contraction	μ_t^*	0	0.0265	0.0169
		u_t^*	0	1.5152	0
17	Mean return augmentation	λ_t^*	0.0339	0	0
		α_t^*	0	1.4503	0
	Risk contraction	μ_t^*	0.0340	0	0

		u_t^*	0.0165	0.1204	1.785
18	Mean return augmentation	λ_t^*	0	0.0043	0
		α_t^*	0.4200	0.6142	0
	Risk contraction	μ_t^*	0	0.0263	0
		u_t^*	0.0339	0	0
19	Mean return augmentation	λ_t^*	0	1.4503	0
		α_t^*	0.0340	0	0
	Risk contraction	μ_t^*	0.0165	0.1206	1.4670
		u_t^*	0	0.0043	0
20	Mean return augmentation	λ_t^*	0.4200	0	0.6142
		α_t^*	0	0.0263	0
	Risk contraction	μ_t^*	0.0340	0	0
		u_t^*	0.0165	0.1206	1.356
21	Mean return augmentation	λ_t^*	0	0.0043	0
		α_t^*	0.4200	0.6142	0
	Risk contraction	μ_t^*	0	0.0263	0
		u_t^*	0.0245	0.1206	1.4530
22	Mean return augmentation	λ_t^*	0	0.0043	0
		α_t^*	0.4200	0.6142	0
	Risk contraction	μ_t^*	0	0.0263	0
		u_t^*	0.0340	0	0
23	Mean return augmentation	λ_t^*	0.0456	0.1206	1.4576
		α_t^*	0	0.0043	0
	Risk contraction	μ_t^*	0.6785	0.6142	0
		u_t^*	0	0.0263	0
24	Mean return augmentation	λ_t^*	0.4563	0	0
		α_t^*	0	1.3452	0
	Risk contraction	μ_t^*	0.0340	0	0
		u_t^*	0	0.1206	1.3456
25	Mean return augmentation	λ_t^*	0	0.0043	0
		α_t^*	0.3462	0.7895	0
	Risk contraction	μ_t^*	0	0.0843	0
		u_t^*	0.2345	0	0
26	Mean return augmentation	λ_t^*	0.0165	0	1.5673
		α_t^*	0	0.0043	0

	Risk contraction	μ_t^*	0.4467	0	0
		u_t^*	0.4563	0	0
27	Mean return augmentation	λ_t^*	0.0165	0.7854	1.2570
		α_t^*	0	0.4768	0
	Risk contraction	μ_t^*	0.3452	0.8432	0
		u_t^*	0	0.0974	0
28	Mean return augmentation	λ_t^*	0.4568	0	0
		α_t^*	0.0343	0.7543	0
	Risk contraction	μ_t^*	0	0.0234	0
		u_t^*	0.0234	0.0123	0
29	Mean return augmentation	λ_t^*	0	0.0456	0
		α_t^*	0	0.0239	0
	Risk contraction	μ_t^*	0	0	0.6943
		u_t^*	0	0.5427	0

For the first fund, the Lagrangian on the 3 year mean return is $\lambda_1^* = 0.7519$, which indicates that if the fund's actual 3 year mean return were decreased 0.1 unit to 1.43 (from 1.33), the DEA score of the fund would be worsened by 0.7519 units, which becomes 1.015+0.7519=1.7669. To test (5.4), 0.7519(1.33)+0(0.53)+0(0.60)=1. And for (5.5), there is 0.0230(43.47992)+0(34.91566)+0(31.23086)=1.

These 'virtual weights' could also be used to look at substitution possibilities. For example, for the fourth fund, taking the ratio of the Lagrangians for the 3 year mean return and the 5 year mean return (devided 0.1206 by 0.0165, or 7.3), means that one could exchange a 0.1 increase in the 5 year return (from 0.72 to 0.82) with a 0.73 decrease in the fund's 3 year mean return (i.e. from 1.02 to 0.29), all with no change in the fund's overall rating of 1.1359. This is also true when analysing the substitution of risks.

3.6 Conclusion

The quadratic DEA model presented in Morey and Morey (1999) departs from the traditional DEA models and utilises insights from Markowitz portfolio theory which reveals the quadratic relationship between fund's return and risk. This application applies the procedures to a new modern data set comprising a multi-year sample of investment funds and identifies six efficient funds among 29 funds.

Morey and Morey (1999) laid a foundation for further development of quadratic and cubic DEA models. Briec et al. (2004) applied a directional distance function which allowed simultaneous changes in the direction of reducing inputs and expanding outputs. They also defined an indirect mean-variance utility function, and divided overall efficiency (OE) into allocative efficiency (AE), and portfolio efficiency (PE). Briec et al. (2007) claimed that portfolio returns are generally not normally distributed, with investors preferring positive skewness so that the probability of obtaining a negative return is low. They extended the work of Briec et al. (2004) into mean-variance-skewness space using cubic programming and divided overall efficiency into portfolio efficiency, allocative efficiency, and convexity efficiency. Kerstens et al. (2011) examined different returns to scale, convexity problems and higher order moments in both quadratic and cubic optimization programming and decided that various return to scale (VRS), free disposal hull and higher moments are essential methodologies for mutual funds evaluation. Relevant empirical papers applying these methods are very few, and none of these papers discuss the statistical properties of DEA estimators. However, as our empirical example shows, ignoring the uncertainty surrounding DEA estimates can lead to erroneous conclusions.

Chapter 4 Bootstrapping of the DEA scores

4.1 Introduction and motivation

Data Envelopment Analysis has been proved to be a powerful frontier methodology to estimate production efficiency in a nonparametric framework. However, this methodology has to be used carefully; one major reason is that DEA estimators have unknown asymptotic sampling distributions. Banker (1993), Kneip et al. (1996), Korostelev et al. (1995a, 1995b), Gijbels et al (1999) have investigated the consistency and convergence properties of the DEA scores and found that the efficiency scores only converge when the sample size is large enough. They are also very sensitive to outliers and extreme values, for example, dropping one outlier can dramatically change the efficiency level for other decision making units. Thus the DEA estimators have shown to be biased when using a finite number of observed units.

Simar and Wilson (1992) claimed that the DEA results by themselves are not true efficiency scores, and the true efficiency scores cannot be known. The only way to obtain some information of the unknown true levels of efficiency is through the analysis of the distributions of the samples. For example, the confidence intervals can give insights about how reliable the DEA scores obtained from Morey and Morey (1999) quadratic models are, whether they're just random results or statistically significant. And it's also necessary to correct the bias of the DEA scores to give more accurate estimation of the investment fund's performance. A very effective way to investigate sampling properties of DEA estimators is to use a methodology called bootstrapping, i.e. sampling with replacement in order to simulate sampling distributions. In addition, the DEA estimators can be improved using bias correction in the bootstrap.

The second application of this thesis is to extend the Morey and Morey (1999) paper by utilizing Simar-Wilson (2008) bootstrapping algorithms to develop statistical inference and confidence intervals for the indexes of efficient investment fund performance.

Chapter 4 is organized as follows: Section 2 provides a literature review and methodology of bootstrap; Section 3 describes the algorithm of smoothed bootstrap for this quadratic DEA model; Section 4 shows the data collection; and Section 5 presents the results.

4.2 Literature review and methodology

The bootstrap was introduced by Efron (1979) and it has been widely used to analyze the distribution of a statistic, for instance, mean and variance, without using normal theory such as z-statistic and t-statistic. This is convenient when the distribution of a certain statistic is uncertain. The first use of the bootstrap in frontier models was Simar (1992) and it was later developed by Simar and Wilson (1998a) etc.

The essence of the bootstrap idea (Efron, 1979, 1982; Efron and Tibshirani, 1993) is that distribution of the true efficiency scores which is unknown could approximate to the sampling distribution, given that a proper data generating process (DGP) is used to create resamples. Therefore, in bootstrap, through obtaining the sampling distribution which can be calculated following a certain procedure, the information of the true efficiency scores is revealed. The crucial step is to create a DGP to simulate, or mimic, the real unknown DGP from which the true DEA estimators are generated.

There are two general ways to resample. The first one is called Monte Carlo resampling. The first step in this resampling is to draw a new set of data independently, uniformly, and with replacement from the set of original observations. Under the Monte Carlo resampling, all the resamples are drawn only from the original data, and the size of the resample is equal to the size of the original data set. Therefore there could be some duplicates since the replacement comes from random picking from the original data. In the second step, a new efficiency score is computed from the resample in the first step. The first two steps then are repeated as many times as needed to get a precise estimate of the distribution of the true DEA scores. A more complicated way to resample is the 'exact' version of resampling. The procedure is similar, but all possible resample of the data sets are enumerated exhaustively. In a case where data size is n, there are $\binom{2n-1}{n}$ different resamples totally.

The DGP described in this application is based on Monte Carlo resampling. To illustrate the DGP and resampling procedure (in an input-oriented DEA case):

In a DEA application, the true attainable set Ψ, therefore the true efficiency score θ are unknown, and we use the observations of input and output to capture some characteristics of these unknown variables.

Denote X as the observed dataset including both inputs and outputs,

$$\mathrm{X} = \{(x_i, y_i), i = 1,\ldots,n\} \tag{4.2.1}$$

Denote P (which is unknown) as the DGP which generates X.

$$P = P(\Psi, f(x, y)) \tag{4.2.2}$$

where Ψ is unknown , and $f(x, y)$ is the probability density function of the random variables (x, y) in X.

Let $\hat{\mathrm{P}}$ be a consistent estimator of P:

$$\hat{\mathrm{P}} = \mathrm{P}\left(\hat{\Psi}, \hat{f}(x, y)\right) \tag{4.2.3}$$

where

$$\hat{\Psi} = \left\{(x, y) \in R^{p+q} \middle| y \leq \sum_{i=1}^{n} \gamma_i y_i; x \geq \sum_{i=1}^{n} \gamma_i x_i; \gamma_i \geq 0; i = 1,...,n \right\} \tag{4.2.4}$$

Therefore $\widehat{\Psi}$ is the piecewise linear representation of the technology or attainable set. The observed X are used to construct estimates $\hat{\theta}_i$, and $\hat{\theta}_i$ are the efficiency scores of the production units (x_i, y_i) obtained by following DEA procedures. In a simple case of constant return to scale and input oriented DEA, there is:

$$\hat{\theta}_i = \min\left\{\theta_i \middle| y_i \leq \sum_{j=1}^{n} \gamma_j y_j; \theta x_i \geq \sum_{j=1}^{n} \gamma_j x_j; \sum_{i=1}^{n} \gamma_i = 1; \theta > 0;_i \gamma_i \geq 0; j = 1,...,n \right\}, i = 1,...,n \tag{4.2.5}$$

Then a new dataset, which includes all the resamples, $\mathrm{X}^* = \{(x_i^*, y_i^*), i = 1,...,n\}$needs to be generated from $\hat{\mathrm{P}}$. In the bootstrap of input-oriented DEA, the resamples are composed of new inputs and the original outputs in which case $\mathrm{X}^* = \{(x_i^*, y_i), i = 1,...,n\}$ while in the bootstrap of output-oriented DEA the resamples include the original inputs and new outputs where $\mathrm{X}^* = \{(x_i, y_i^*), i = 1,...,n\}$And the procedures to create $\mathrm{X}^* = \{(x_i^*, y_i), i = 1,...,n\}$ in this input-oriented case are as follows:

First of all, select $\theta_i^* i = 1,...,n$ from $\hat{\theta}_1,...,\hat{\theta}_n$ randomly.

Then for a given output level y_i, the efficient level of input is determined by:

$$x_i^{\partial}(y_i) = \hat{\theta}_i(x_i, y_i)x_i.\ i = 1,...n. \qquad (4.2.6)$$

Which lies on the efficient boundary $\partial\hat{\Psi}$, along the ray x and orthogonal to y.

And for each replicate θ_i^*, there must be a corresponding input which could be projected on the same point on the efficient frontier by θ_i^* given the same y_i. This is illustrated by the following formula:

$$x_i^{\partial}(y_i) = \theta_i^*(x_i^*, y_i)x_i^*. \qquad (4.2.7)$$

From (4.2.6) and (4.2.7), the bootstrap inputs are obtained by the following formula:

$$x_i^* = \frac{x_i^{\partial}(y_i)}{\theta_i^*} = \frac{\hat{\theta}_i}{\theta_i^*}x_i.\ i = 1,...n. \qquad (4.2.8)$$

Once $\mathrm{X}^* = \{(x_i^*, y_i), i = 1,...,n\}$ is obtained, $\hat{\theta}_i^*$ is then computed by solving the following linear program, in this simple case of constant return to scale.

$$\hat{\theta}_i^* = \min\left\{\theta \middle| y_{\mathrm{i}} \le \sum_{j=1}^{n}\gamma_j y_j; \theta x_{\mathrm{i}} \ge \sum_{j=1}^{n}\gamma_j x_j^*; \sum_{j=1}^{n}\gamma_j = 1; \theta > 0;\ \ \gamma_j \ge 0, j = 1,...,n\right\}, \quad i = 1,...,n \qquad (4.2.9)$$

Where $\hat{\theta}_i^*$ is an estimator of $\hat{\theta}_{\mathrm{i}}$ under DGP $\hat{\mathrm{P}}$. It is in the same way as $\hat{\theta}_{\mathrm{i}}$ is an estimator of the true efficiency score ϑ_{i}, given the unknown true DGP P.

The above procedures show how the computed sampling distributions mimic the original unknown distributions of efficiency estimators, and it could be written as:

$$(\hat{\theta}_i^* - \hat{\theta}_i) \Big| \hat{P} \sim (\hat{\theta}_i - \theta_i) \Big| P \tag{4.2.10}$$

To summarize, the bootstrapping is based on the idea of repeatedly simulating the data generating process (DGP) to obtain resamples, and then calculate the distribution of the resamples which mimic the distribution of the unknown true efficiency estimator. It involves mainly three steps:

1. Apply the DGP $\hat{P}$ to generate resamples $X^* = \{(x_i^*, y_i^*), i = 1,...,n\}$ (i.e., simulate).
2. Use the resamples $X^* = \{(x_i^*, y_i^*), i = 1,...,n\}$ to compute $\hat{\theta}_i^*$, which is the estimator of the efficiency score $\hat{\theta}_i$
3. Repeat the first two steps.

The distribution of the estimates obtained in the end approximates the distribution of the true efficiency estimator.

In the bootstrapping, the quality of the approximation relies partly on the number of times the simulation repeated. (indicate it as B). It is proved that the larger value of B the better. Simar and Wilson (2000) claimed that when $B \to \infty$, the error of this approximation due to DGP $\hat{p}$ tends to be zero. To illustrate, DGP $\hat{\bar{P}}$ generates B samples $X_b^*, b = 1,...B$. In particular, for a given unit (X_i, Y_i), there is $\left\{\hat{\theta}_{ib}^*\right\}_{b=1}^{B}$; therefore the distribution of $\left\{\hat{\theta}_{ib}^*\right\}_{b=1}^{B}$ is the approximation of the distribution of the true efficiency scores.

Bootstrap is applied to correct the bias of the efficiency estimator and construct corresponding hypothesis tests.

(i) Correcting the bias

The bias of $\hat{\theta}_i$ as the estimator of true efficiency score θ is given by:

$$bias_i = E(\hat{\theta}_i) - \theta_i, \tag{4.2.11}$$

The bias of $\hat{\theta}_i^*$ as the estimator of $\hat{\theta}_i$ is:

$$bias_i = E(\hat{\theta}_i^*) - \hat{\theta}_i. \tag{4.2.12}$$

The latter can also be written as the following given B replications in the bootstrap,

$$bias_i = \frac{1}{\mathrm{B}}\sum_{\mathrm{b=1}}^{\mathrm{B}} \hat{\theta}_{\mathrm{i,b}}^* - \hat{\theta}_{\mathrm{k}}. = \overline{\theta}_{\mathrm{i}}^* - \hat{\theta}_{\mathrm{i}}. \tag{4.2.13}$$

Due to (4.2.8), the following relationship holds,

$$bias_i = bias_i = \hat{\mathrm{bias}}_{\mathrm{i}} = \overline{\theta}_{\mathrm{i}}^* - \hat{\theta}_{\mathrm{i}}. \tag{4.2.14}$$

Define $\tilde{\theta}_{\mathrm{i}}$ as a bias-corrected estimator of θ_{i}, there is:

$$\tilde{\theta}_{\mathrm{i}} = \hat{\theta}_{\mathrm{i}} - \hat{\mathrm{bias}}_{\mathrm{i}}. = 2\hat{\theta}_{\mathrm{i}} - \overline{\theta}_{\mathrm{i}}^*. \tag{4.2.15}$$

Define $\tilde{\theta}_{\mathrm{i,b}}^*$ as a bias-corrected estimator of $\hat{\theta}_{\mathrm{i,b}}^*$, there is,

$$\tilde{\theta}_{\mathrm{i,b}}^* = \hat{\theta}_{\mathrm{i,b}}^* - 2\,\hat{\mathrm{bias}}_i. \tag{4.2.16}$$

Note that $\hat{\theta}_{\mathrm{i,b}}^*$ has to be shifted by $2\,\hat{\mathrm{bias}}_{\mathrm{i}}$ to the left in order to centre on bias-corrected estimator of θ_{i}, $\tilde{\theta}_{\mathrm{i}}$.

Finally, the sample variance of the bootstrap values $\hat{\theta}_i^*$ could be estimated by:

$$\hat{\sigma}_i^2 = \frac{1}{\mathrm{B}}\sum_{b=1}^{B}\left[\left(\hat{\theta}_{\mathrm{i,b}}^{*} - \overline{\theta}_{\mathrm{i}}^{*}\right)\right]^2 \tag{4.2.17}$$

However, this bias correction has shown to introduce additional noise (Efron and Tibshirani, 1993); the mean square error of the bias-corrected estimator $\tilde{\theta}_{\mathrm{i}}$ maybe greater than the mean square error of $\hat{\theta}_i$ .The value of variance of $\tilde{\theta}_{\mathrm{i}}$ is approximately $4\hat{\sigma}_i^2$. Therefore, the bias correction should not be used unless $4\hat{\sigma}_i^2$ is well less than $[bias_i]^2$; otherwise, $\tilde{\theta}_{\mathrm{i}}$ is likely to have mean square error larger than the mean square error of $\hat{\theta}_i$. Efron and Tibshirani (1993) proved that the bias correction in (4.2.13) should be avoided unless

$$\frac{|bias_i|}{\hat{\sigma}_i} > \frac{1}{4} \tag{4.2.18}$$

(ii) Confidence interval

Bootstrapping allows one to calculate the confidence intervals for the true efficiency score θ_{i} . The formula for confidence interval of $\hat{\theta}_{i,b}^{*}, b = 1,...,B,$ at α significance level is as follows:

$$\Pr\left(-\hat{\mathrm{b}}_\alpha \le \hat{\theta}_{DEA}^{*}(x_i, y_i) - \hat{\theta}_{DEA}(x_i, y_i) \le -\hat{\alpha}_\alpha \,\middle|\, \hat{P}(x_n)\right) = 1 - \alpha. \tag{4.2.19}$$

Similar to (4.2.17), the formula for confidence interval of $\hat{\theta}_{i,b}^{*}, b = 1,...,B,$ at α significance level is given by:

$$\Pr\left(-\mathrm{b}_\alpha \le \hat{\theta}_{DEA}(x_i, y_i) - \theta(x_i, y_i) \le -\alpha_\alpha\right) = 1 - \alpha. \tag{4.2.20}$$

Due to (4.2.8), (4.2.17) could be rewritten as:

$$\Pr\left(-\hat{\mathrm{b}}_\alpha \le \hat{\theta}_{DEA}(x_{\mathrm{i}}, y_{\mathrm{i}}) - \theta(x_{\mathrm{i}}, y_{\mathrm{i}}) \le -\hat{\alpha}_\alpha\right) = 1 - \alpha. \tag{4.2.21}$$

For $(1-\alpha)\%$ confidence interval, $\hat{a}_\alpha$ and $\hat{b}_\alpha$ can be found by sorting the values $(\hat{\theta}_i^* - \hat{\theta}_i), b = 1,..., B$ in increasing order and then delete $\left(\frac{\alpha}{2} \times 100\right)$-percent of the elements at either end of the sorted list. Then set $-\hat{b}_\alpha$ and $-\hat{a}_\alpha$ equal to the endpoints of the truncated, making sure $\hat{a}_\alpha \leq \hat{b}_\alpha$.

Then the $(1-\alpha)$-percent confidence interval for the true efficiency score is:

$$\hat{\theta}(x_i, y_i) + \hat{\alpha}_\alpha \leq \theta(x_i, y_i) \leq \hat{\theta}(x_i, y_i) + \hat{b}_\alpha \qquad (4.2.22)$$

If bias-corrected estimators are considered, then $\hat{\theta}(x_i, y_i)$ would be replaced by $\tilde{\theta}(x_i, y_i)$ in (4.2.22).

The above is standard bootstrap, which is also called 'naïve' bootstrap. Silverman and Young (1987) and Efron and Tibshirani (1993) claimed that standard bootstrap has some problems. In the standard bootstrap, the true DEA scores are subject to an unknown distribution, i.e.

$$(\theta_1,...,\theta_n) \sim i.i.d.F \qquad (4.2.23)$$

where F is an unknown density function on $(0,1]$. Define $\hat{F}$ as the estimator of F, there is

$$(\hat{\theta}_1,...,\hat{\theta}_n) \sim i.i.d.\hat{F} \qquad (4.2.24)$$

Because with limited data, $\hat{F}$ is actually a discrete distribution, therefore samples constructed from $\hat{F}$ may have some peculiar properties. Most seriously, it's not consistent under some circumstances which mean (4.2.8) doesn't hold all the time. In other words, the distribution of the $\hat{\theta}_i^*$ will not approximate the sampling distribution of $\hat{\theta}_i$. Because $\hat{\mathrm{F}}$ provides a poor estimate of F near the upper bound for θ (when $\theta = 1$). Also, it is difficult to estimate F from $\hat{\mathrm{F}}$ in the extreme tails.

There are two techniques to deal with this problem, one is subsampling technique (in which the sample size equals $m = n^k$, for $0 < k < 1$), and the other is smoothing technique. Kneip et al. (2003) proved that both ideas provide consistent results in the simulation. The following part of this sector describes the smoothing technique.

Smoothed bootstrap is introduced and developed by Efron (1979, 1982). The essential idea of the smoothed bootstrap is to resample not assuming $\hat{F}$, but a smoothed version of $\hat{F}$ which is a joint density function. Defined it as $\hat{F}_h(t)$, (4.2.21) in the standard bootstrap is then replaced by

$$(\theta_1,...,\theta_n) \sim i.i.d \hat{F}_h(t) \tag{4.2.25}$$

In the smoothed bootstrap, kernel density estimation is chosen as the joint density function $\hat{F}_h(t)$. Kernel density estimation is a non-parametric way of estimating the probability density function of a random variable. It provides a smoothing function through a parameter, particularly when data sample is finite.

To illustrate, the kernel density estimator is,

$$\hat{f}_h(x) = \frac{1}{n}\sum_{i=1}^{n} K_h(x - x_i) = \frac{1}{nh}\sum_{i=1}^{n} K\left(\frac{x - x_i}{h}\right) \tag{4.2.26}$$

where $K(\cdot)$ is a symmetric but not necessarily positive function that integrates to one, which means $K(\cdot)$ satisfies $K(t) = K(-t)$, $\int_{-\infty}^{\infty} K(t)dt = 1$, and $\int_{-\infty}^{\infty} tK(t)dt = 0$. Any symmetric probability function with mean zero satisfies these conditions. $\hat{f}_h(x)$ could be understood as the average of *n* different probability densities $K(\cdot)$ with the parameter *h* controlling the dispersion of the *n* densities.

In the smoothed bootstrap, the normal kernel is used, in which case $K(x) = \phi(x)$, and $\phi(x)$ is the standard normal density function. Therefore, in the smoothed bootstrap,

$$\hat{F}_h(t) = \frac{1}{nh}\sum_{i=1}^{n}\phi\left(\frac{t-\hat{\theta}_i}{h}\right) \tag{4.2.27}$$

here h is the smoothing parameter, which is also called the window width or bandwidth. The smoothing parameter h determines to what extent the data are smoothed in the DGP. Larger values of h provide more smoothing than smaller values of h because when h is small, only a few observations closest to the point where the density is estimated influence the value of $\hat{F}_h(t)$. As h gets larger, further observations are included to determine $\hat{F}_h(t)$. Consequently, in two extreme situations when $h \to 0$ and when $h \to \infty$ the density will become the discrete empirical density function and a flat horizontal line respectively.

Also, it is proved that with h, the bias of $\hat{F}_h(t)$ increases while the variance decreases. So when choosing the optimal h, there is always a trade-off between the bias of the estimator and its variance.

Silverman (1986) provides a formula for the optimal value of h, when the underlying density function is Gaussian and $K(\cdot)$ is standard normal:

$$h_{NR} = 1.06\hat{\sigma}n^{-1/5} \tag{4.2.28}$$

This is referred to as the 'normal reference rule' or Silverman's rule of thumb.

A more robust choice is given by,

$$h_R = 1.06\min\left(\hat{\sigma}, \hat{R}/1.34\right)n^{-1/5} \tag{4.2.29}$$

where $\hat{R}$ is the interquartile range.

Another commonly used criteria for choosing the bandwidth h in kernel density estimation is data-driven criterion. Because, with discrete data, especially when the sample size is not very large, the density of the efficiency scores is likely to be not normally distributed, therefore, this density may have moments different from a normal distribution. Data-driven methods provide ad hoc rules for choosing *h*. The approach is to minimize an estimate of either mean-integrated square error (MISE) or its asymptotic version (AMISE) which are called unbiased and biased cross-validation respectively. Silverman (1986) described a least-squares cross-validation method which is a form of unbiased cross-validation; Simar and Wilson (2002) adapts it to the DEA context.

The MISE of kernel density function is given by

$$MISE(h) = E[\int_{-\infty}^{\infty} (\hat{F}_h(x) - F(x))^2 dx]. \tag{4.2.30}$$

This is computed by the Leave-one-out cross-validation least-square (LOOCV) and the function is as follows:

$$CV(h) = \int_{-\infty}^{\infty} \hat{F}_h^2(x)dx - \frac{1}{2m}\sum_{i=1}^{2m} \hat{F}_{h,(i)}^2(\tilde{x}_i), \tag{4.2.31}$$

Where $\hat{F}_{h,(i)}$ is the leave-one-out estimator of $F(x)$ based on the original observations (the m values $\hat{x}_j \neq 1$), except $\hat{x}_i$, with bandwidth *h*. And the optimal *h* could be obtained by minimizing (4.2.29).

The above estimation has one problem; however, that is it does not take into account the boundary condition that $t < 1$. To overcome this problem the reflection method has been used, which was described by Silverman (1986). In the reflection method, the points $\hat{\theta}_i \leq 1$ are reflected by its symmetric image $2 - \hat{\theta}_i \geq 1,\ i = 1,\ldots,n,$ and then the kernel density estimator are modified from the resulting set of 2n points to be,

$$\hat{G}_h(t) = \frac{1}{2nh}\sum_{i=1}^{n}\left[\phi\left(\frac{t-\hat{\theta}_i}{h}\right) + \phi\left(\frac{t-2+\hat{\theta}_i}{h}\right)\right] \tag{4.2.32}$$

Define

$$\hat{F}_{s,h}(t) = \begin{cases} 2\hat{G}_h(t) & \text{if } t \le 1 \\ 0 & \text{otherwise} \end{cases} \qquad (4.2.33)$$

It can be proved that $\hat{F}_{s,h}(t)$ is consistent for all $t \le 1$.

Under this reflection method, a certain procedures should be followed to generate samples $\theta_1^*,...,\theta_n^*$ from $\hat{F}_{s,h}(t)$. First of all, let $\beta_1^*,...,\beta_n^*$ be a set of bootstrap resample from $\hat{\theta}_1,...,\hat{\theta}_n$. According to the convolution theorem in Efron and Tibshirani (1993), there is,

$$t_i = \beta_i^* + h\varepsilon_i^* \sim \hat{G}_{1,h}(t) = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{h}\phi\left(\frac{t-\hat{\theta}_i}{h}\right) \qquad (4.2.34)$$

where ε_i^* is an error term from the standard normal distribution. Similarly, let t_i^R be the reflection of t_i then

$$t_i^R = 2 - \beta_i^* - h\varepsilon_i^* \sim \hat{G}_{2,h}(t) = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{h}\phi\left(\frac{t-2+\hat{\theta}_i}{h}\right) \qquad (4.2.35)$$

And $\hat{G}_h(t)$ in (4.2.28) may be written as:

$$\hat{G}_h(t) = \frac{1}{2}\hat{G}_{1,h}(t) + \frac{1}{2}\hat{G}_{2,h}(t) \qquad (4.2.36)$$

Now the estimator of $\hat{\theta}_i$ is given by:

$$\tilde{\theta}_i^* = \begin{cases} \beta_i^* + h\varepsilon_i^* & \text{if } \beta_i^* + h\varepsilon_i^* \le 1 \\ 2 - \beta_i^* - h\varepsilon_i^* & \text{otherwise} \end{cases} \qquad (4.2.37)$$

It has been proved that

$$\tilde{\theta}_i^* \sim \hat{F}_{s,h}(t) \tag{4.2.38}$$

and $\tilde{\theta}_i^*$ has the following properties:

$$E\left(\tilde{\theta}_i^* \middle| \hat{\theta}_1, ..., \hat{\theta}_n\right) = \hat{\mu}, \tag{4.2.39}$$

$$V\left(\tilde{\theta}_i^* \middle| \hat{\theta}_1, ..., \hat{\theta}_n\right) = \hat{\sigma}_\theta^2 + h^2 \tag{4.2.40}$$

where $\hat{\sigma}_i^2$ is the sample variance of $\hat{\theta}_1, ..., \hat{\theta}_n$, i.e.,

$$\hat{\sigma}_i^2 = \frac{1}{n}\sum\left(\hat{\theta}_i^2 - \hat{\bar{\theta}}\right)^2 \tag{4.2.41}$$

And $\hat{\mu}$ is the sample mean of the $\hat{\theta}_1, ..., \hat{\theta}_n$.

When kernel estimators are used, the variance of the generated bootstrap sequence must be corrected by computing

$$\theta_i^* = \bar{\beta}^* + \frac{1}{\sqrt{1 + h^2 / \hat{\sigma}_\theta^2}}\left(\tilde{\theta}_i^* - \bar{\beta}^*\right) \tag{4.2.42}$$

Where $\bar{\beta}^* = (1/n)\sum_{i=1}^n \beta_i^*$,

(4.2.32) and (4.2.33) become,

$$E\left(\tilde{\theta}_i^* \middle| \hat{\theta}_1, ..., \hat{\theta}_n\right) = \hat{\mu}, \text{ and} \tag{4.2.43}$$

$$\mathrm{V}\left(\tilde{\theta}_{\mathrm{i}}^{*}\middle|\hat{\theta}_1,...,\hat{\theta}_n\right)=\hat{\sigma}_{\theta}^{2}\left(1+\frac{h^2}{n\left(\hat{\sigma}_{\theta}^{2}+h^2\right)}\right) \tag{4.2.44}$$

θ_i^* has better properties than $\tilde{\theta}_i^*$ as variance of θ_i^* is asymptotically correct.

In the smoothed bootstrap, the problems that appear in the standard bootstrap are avoided. The empirical results to be reported in the thesis were all derived from MATLAB codes written by the researcher and these MATLAB codes are included as appendices to the thesis**.**

4.3 Algorithm of smoothed bootstrap for this quadratic DEA model

First of all, DEA estimators are obtained from Morey and Morey (1999) quadratic model mean augmentation approach using the following program:

$$\begin{aligned}
&Max \quad \theta \\
&s.t. \quad \sum_{j=1}^{N} w_j = 1 \\
&\qquad \sum w_j^2\sigma_{j,t}^2 + \sum_{\substack{i=1\\ i+j}}^{N}\sum_{j=1}^{N} w_i w_j Cov(R_{i,t}, R_{j,t}) \le \sigma_{j_0}^2 \\
&\qquad \sum_{j=1}^{N} w_j E(R_{j,t}) \ge \vartheta E(R_{j_0,t}) \\
&\qquad (t = 1, 2, ...T)
\end{aligned} \tag{4.3.1}$$

Secondly, to select replicates $\{\theta_{ib}^*\}_{\mathrm{b}=1}^{\mathrm{B}}$ $i = 1,...\mathrm{n}.\ \mathrm{B} = 2000$ from kernel density function using reflection method based on the DEA estimators obtained from Morey and Morey (1999) quadratic model mean augmentation approach, with the optimal bootstrap smooth parameter h automatically chosen in the program utilising cross-validation approach described by Silverman (1986). The corresponding formulas are from (4.2.30) to (4.2.44). In this application, the minimum and maximum replicates among $\{\theta_{ib}^*\}_{\mathrm{b}=1}^{\mathrm{B}}$ $i = 1,...\mathrm{n}.\ \mathrm{B} = 2000$ are set to be not below the minimum orginal DEA score and no above the maximum orginal DEA score. This way it could make sure all the replicates have the value equal or larger than one, and all the replicates produced are reasonable in this quadratic DEA case.

The third step is to obtain the bootstrap inputs using the following formula:

$$E(R_{ib},t)^* = \frac{\hat{\theta}_i}{\theta^*_{ib}} E(R_{ib},t).\ i = 1,...n.\ \mathrm{b} = 1,...B,\ \mathrm{B} = 2000,\ \mathrm{t} = 1,2,...\mathrm{T} \qquad (4.3.2)$$

Finally $\hat{\theta}^*_{ib}$ which is convenient to be expressed as $\hat{\theta}^*_{j_0 b}$ in the following formula is computed by solving the following program.

$$\begin{aligned}
&Max \quad \hat{\theta}^*_{j_0 b} \\
&s.t. \quad \sum_{j=1}^{N} w_j = 1 \\
&\qquad \sum w_j^2 \sigma^2_{j,t} + \sum_{\substack{i=1 \\ i+j}}^{N} \sum_{j=1}^{N} w_i w_j Cov(R_{i,t}, R_{j,t}) \le \sigma^2_{j_0} \\
&\qquad \sum_{j=1}^{N} w_j E(R_{jb,t})^* \ge \hat{\vartheta}^*_{ib} E(R_{j_0,t}) \\
&\qquad (t = 1,2,...T)
\end{aligned} \qquad (4.3.3)$$

The distribution of the 2000 estimates $\hat{\theta}_{ib}{}^*$ obtained from (4.3.3) approximates the distribution of the true efficiency estimator.

The bias of $\hat{\theta}_i$ as the estimator of true efficiency score θ is given by:

$$bias_i = \frac{1}{\mathrm{B}} \sum_{\mathrm{b}=1}^{\mathrm{B}} \hat{\theta}^*_{\mathrm{ib}} - \hat{\theta}_{\mathrm{i}}. = \overline{\theta}_{\mathrm{i}}^* - \hat{\theta}_{\mathrm{i}}. \qquad (4.3.4)$$

Bias corrected estimator $\tilde{\theta}_{\mathrm{i}} = \hat{\theta}_{\mathrm{i}} - \widehat{\mathrm{bias}}_{\mathrm{i}}. = 2\hat{\theta}_{\mathrm{i}} - \overline{\theta}_{\mathrm{i}}^*.$ (4.3.5)

$(1-\alpha)$-percent confidence interval for the true efficiency score is calculated by:

$$\hat{\theta}(x_i, y_i) + \hat{\alpha}_\alpha \le \theta(x_i, y_i) \le \hat{\theta}(x_i, y_i) + \hat{b}_\alpha \qquad (4.3.6)$$

For 95% confidence interval, $\hat{a}_\alpha$ and $\hat{b}_\alpha$ can be found by sorting the values $\left(\hat{\theta}_i^* - \hat{\theta}_i\right), b = 1,\ldots,B, B = 2000$ in increasing order and then delete 50 elements at either end of the sorted list. Then set $-\hat{b}_\alpha$ and $-\hat{a}_\alpha$ equal to the endpoints of the truncated, making sure $\hat{a}_\alpha \leq \hat{b}_\alpha$.

If bias-corrected estimators are considered, then replace $\tilde{\theta}(x_i, y_i)$ by $\hat{\theta}(x_i, y_i)$.
The consistency of the bootstrap depends on the resampling procedure (avoidance of naive bootstrapping) and this has been addressed by the design of the Simar-Wilson algorithm.

4.4 Data collection

The data used in this application is the DEA scores obtained from the Morey and Morey (1999) return augmentation approach. And the results in the following table are from Table 3.5.1 in the third chapter.

Table 4.4.1 Frontier values and scores from the mean return augmentation programming

Fund No. and name	DEA scores
(1) Aberdeen UK Mid Cap A Acc	1.015
(2) AEGON Ethical Equity A	1.1505
(3) AEGON Ethical Equity B	1.0308
(4) Allianz RCM UK Mid Cap A	1.1359
(5) Artemis UK Special Situations	1
(6) Aviva Investors SF UK Growth SC1	1.8621
(7) Aviva Investors SF UK Growth SC2	1.5914
(8) Aviva Investors UK Ethical SC1	1.7987
(9) Aviva Investors UK Ethical SC2	1.714
(10) BlackRock UK Special Situations A Acc	1
(11) Ecclesiastical Amity UK C	1.1116
(12) F&C Stewardship Growth 1 Acc	1.9331
(13) F&C Stewardship Income 1 Acc	1
(14) GAM Exempt Trust UK Opportunities	1.036
(15) GAM UK Diversified Acc	1.143
(16) Henderson UK Equity Income I Acc	1.2532
(17) HSBC FTSE 250 Index Retail Acc	1.0331
(18) Marlborough UK Primary Opps A Acc	1.3196
(19) Marlborough UK Primary Opps B Acc	1.3974
(20) MFM Bowland	1.0031
(21) Saracen Growth Alpha	1.3265
(22) Saracen Growth Beta	1.2344
(23) Schroder UK Mid 250 Acc	1
(24) Standard Life UK Eq High Alpha Inst Acc	1
(25) Standard Life UK Eq High Alpha R Acc	1.0006
(26) Standard Life UK Ethical Inst	1.4387
(27) Standard Life UK Ethical R	1.5747
(28) SVM UK Opportunities Instl	1
(29) SVM UK Opportunities Retail	1.0593

4.5 Results analysis

The Bandwidth h selected is 0.2186, calculated by least-squares cross-validation method described by Silverman (1986). The corresponding formulas are (4.2.28) and (4.2.29).

The results of bootstrapping DEA scores are as follows:

Table 4.5.1 Bootstrap results with Bandwidth h=0.2186;

N	estimates	bias	bias-corrected estimator	Standard deviation	$\frac{\lvert bias_i \rvert}{\hat{\sigma}_i}$	95% confidence interval		
						Lower	Upper	differenc
1	1.015	-0.0102	1.0252	0.0075	1.3641	1.0082	1.0300	0.0218
2	1.1505	-0.0078	1.1583	0.0252	0.3093	1.1345	1.2288	0.0943
3	1.0308	-0.0039	1.0347	0.0138	0.2831	1.0166	1.0616	0.045
4	1.1359	-0.0168	1.1527	0.0203	0.8291	1.1324	1.2054	0.073
5	1	0	1	0	-	1	1	0
6	1.8621	-0.0335	1.8956	0.0540	0.6208	1.8329	2.0255	0.1926
7	1.5914	-0.0242	1.6156	0.0462	0.5235	1.5641	1.7303	0.1662
8	1.7987	-0.0381	1.8368	0.0574	0.6640	1.778	1.9707	0.1927
9	1.714	-0.0345	1.7485	0.0543	0.6351	1.6947	1.8756	0.1809
10	1	0	1	0	-	1	1	0
11	1.1116	-0.0306	1.1422	0.0562	0.5449	1.0455	1.2232	0.1777
12	1.9331	-0.0405	1.9736	0.0908	0.4460	1.8541	2.1884	0.3343
13	1	0	1	0	-	1	1	0
14	1.036	-0.0207	1.0567	0.0208	0.9929	1.0101	1.0720	0.0619
15	1.143	-0.0439	1.1869	0.0489	0.8971	1.1200	1.2860	0.166
16	1.2532	-0.0494	1.3026	0.0616	0.8017	1.2297	1.4617	0.2320
17	1.0331	-0.0128	1.0459	0.0133	0.9652	1.0277	1.0664	0.0387
18	1.3196	-0.00797	1.3276	0.0178	0.4489	1.2216	1.3560	0.1344
19	1.3974	-0.0156	1.4130	0.0184	0.8466	1.3948	1.4571	0.0623
20	1.0031	-0.0010	1.0041	0.0037	0.2675	1.0021	1.0062	0.0041
21	1.3265	-0.0137	1.3402	0.0193	0.7114	1.3193	1.3852	0.0659
22	1.2344	-0.0129	1.2473	0.0178	0.7249	1.2277	1.2889	0.0612
23	1	0	1	0	-	1	1	0
24	1	0	1	0	-	1	1	0
25	1.0006	-0.0005	1.0011	0.0002	2.0229	1.0006	1.0012	0.0006
26	1.4387	-0.0290	1.4677	0.0296	0.9809	1.4379	1.5458	0.1079
27	1.5747	-0.0239	1.5986	0.0297	0.8053	1.5701	1.6800	0.1099
28	1	0	1	0	-	1	1	0
29	1.0593	-0.0266	1.0859	0.0261	1.0204	1.0593	1.1186	0.0593

Table 4.5.1 shows the results for the bootstrap exercise for B = 2000 and h = 0.2186. Column 1 shows the fund number, while columns 2 to 5 give the original DEA efficiency estimates, the bias and the bias-corrected estimates, the standard deviations of the bootstrapped values and the value of $\frac{|bias_i|}{\hat{\sigma}_i}$; and the last column gives 95% confidence intervals for the efficiency estimates, showing the upper and lower bounds and the difference between them. From Table 4.5.1, the initial DEA model gave an average uncorrected efficiency score of 1.2470, while the bootstrap model generated an average bias-corrected score of 1.2642. The minimum uncorrected score was 1 and the maximum was 1.9331, while the minimum bias corrected score was 1 and the maximum was 1.9736. For the most efficient funds- fund 5,10,13,23,24 and fund 28, the 2000 bootstrap estimators are all equal to one, therefore the bias equal to zero and the bias corrected estimators are also equal to one. The 95% confidence intervals for these funds become a single point. The results also reveal that all the estimated biases are negative, which is as expected, because according to Simar and Wilson (1998), the DEA estimate is upwardly biased using an input oriented model and downwardly biased for an output oriented model. The original scores had a mean bias of -0.0168. And the standard deviations for all the estimators are quite small with the maximum standard deviation equal to 0.0908. All the funds satisfy the condition of $\frac{|bias_i|}{\hat{\sigma}_i} > \frac{1}{4}$ (4.2.18), except for the most efficient funds which have both the bias and the standard deviation equal to zero. Lower bounds for the estimated 95% confidence intervals range from 1 for the six most efficient funds to 1.8541 for the 12th fund. The estimated upper 95% confidence bounds range from 1 for the most efficient funds to 2.1884 for the 12th fund. In addition, the differences between the upper and lower bounds range from 0 for the frontier funds to 0.3343 for Fund 12. From the results, all of the original DEA scores are within the lower and upper bounds of 95% confidence interval, with the maximum range is as small as 0.3343. Therefore the statistical test indicates that the DEA scores are very reliable.

This conclusion can be further proved by the comparison between rankings of the funds from original DEA scores and rankings based on bias-corrected DEA scores. The results are showed as follows:

Table 4.5.2 Rankings before and after bias correction

Fund No. And Name	Ranking from original DEA scores	Ranking from bias-corrected DEA scores
(1) Aberdeen UK Mid Cap A Acc	9	9
(2) AEGON Ethical Equity A	17	16
(3) AEGON Ethical Equity B	10	10
(4) Allianz RCM UK Mid Cap A	15	15
(5) Artemis UK Special Situations	1	1
(6) Aviva Investors SF UK Growth SC1	28	28
(7) Aviva Investors SF UK Growth SC2	25	25
(8) Aviva Investors UK Ethical SC1	27	27
(9) Aviva Investors UK Ethical SC2	26	26
(10) BlackRock UK Special Situations A Acc	1	1
(11) Ecclesiastical Amity UK C	14	14
(12) F&C Stewardship Growth 1 Acc	29	29
(13) F&C Stewardship Income 1 Acc	1	1
(14) GAM Exempt Trust UK Opportunities	12	12
(15) GAM UK Diversified Acc	16	17
(16) Henderson UK Equity Income I Acc	19	19
(17) HSBC FTSE 250 Index Retail Acc	11	11
(18) Marlborough UK Primary Opps A Acc	20	20
(19) Marlborough UK Primary Opps B Acc	22	22
(20) MFM Bowland	8	8
(21) Saracen Growth Alpha	21	21
(22) Saracen Growth Beta	18	18
(23) Schroder UK Mid 250 Acc	1	1
(24) Standard Life UK Eq High Alpha Inst Acc	1	1
(25) Standard Life UK Eq High Alpha R Acc	7	7
(26) Standard Life UK Ethical Inst	23	23
(27) Standard Life UK Ethical R	24	24
(28) SVM UK Opportunities Instl	1	1
(29) SVM UK Opportunities Retail	13	13

From Table 4.5.2 it can be seen that except fund 2 and fund 15, all the other funds have exactly the same rankings before and after bias correction. And the 2^{nd} fund has the ranking of 17 based on original estimators while the bias corrected estimator for the 2^{nd} fund gives the ranking of 16. For the fund 15, it has the rankings of 16 and 17 before and after the bias correction respectively. Therefore, for funds 2 and 15, the different rankings using original DEA scores and bias corrected DEA scores are very close.

4.6 **Conclusion**

Nonparametric efficiency measures are criticized for lacking a statistical basis. This is based on the fact that the efficiency scores obtained from DEA models are likely to be overestimated/ underestimated. However, it is demonstrated that boostrap methods can be used to provide the statistical inference for the DEA scores by focusing on the underlying DGP.

This application provides a statistical test of the DEA scores obtained from Morey and Morey (1999) quadratic model by using bootstrap techniques introduced by Simar and Wilson (1998, 2000). Algorithms of smoothed bootstrap for this quadratic DEA model are designed. The results show that the DEA scores from Morey and Morey (1999) quadratic model are downwardly biased with a mean bias of -0.0168, which is quite small. Also, the confidence intervals are fairly narrow for all the funds with the original estimators lie between the lower bounds and upper bounds. In addition, after the bias correction, 27 funds have the rankings unchanged compared with the rankings based on the original estimators with 2 funds have slight change in their rankings. Therefore, the conclusion could be drawn from this statistical test that the DEA scores obtained from Morey and Morey (1999) mean augmentation approach are very reliable.

Based on Morey and Morey (1999), which laid a foundation for the development of quadratic and cubic DEA models, Briec et al. (2004) applied a directional distance function which allowed simultaneous changes in the direction of reducing inputs and expanding outputs. Briec et al. (2007) extended the work of Briec et al. (2004) into mean-variance-skewness space using cubic programming. However, relevant empirical papers applying these methods are very few, and none of these papers discuss the statistical properties of DEA estimators. Therefore, statistical inferences need to be provided for these quadratic and cubic DEA models to test the reliability of the estimators and correct the biases. Bootstrap Algorithms for these quadratic and cubic DEA models have not been developed.

Chapter 5 Evaluating contextual variables affecting investment trust performance in second stage DEA efficiency analyses

5.1 Introduction and motivation

One type of investment fund is investment trust, which is close ended fund. It is actually a listed company, and differs from unit trusts in the sense that it issues equity itself and the number of shares is fixed as with any other company that issues shares. An investment trust normally only invests in specific types of assets for example UK technology shares and is banned from switching to other segments. There are over 350 investment trusts quoted in London, with total assets of over £60 billion.

Therefore it is meaningful to examine the efficiency of investment trusts, and to analyze the factors contributing to investment trusts performance and detect the determinants of inefficiency. Therefore second stage DEA efficiency analyses are used to evaluate contextual variables affecting the fund performance. This framework involves two-stages. In the first stage, efficiency scores are calculated using Morey and Morey (1999) quadratic DEA model. And then in the second stage, these scores are correlated with other explanatory variables which also have an impact on the funds' performances. In this stage the DEA efficiency scores are regressed on potential factors to test the statistical significance of those factors.

Sharpe ratio and Jensen's alpha are two traditional measurements that are often used to rank the performance of investment portfolios. Sharpe ratio is calculated by dividing a fund's annualized excess returns by the standard deviation of a fund's annualized excess returns and mutual funds with lager Sharpe Ratios are assumed to have better historical risk-adjusted performance than those with small ratios. Jensen's alpha, which is derived from the market model and the CAPM, is calculated by taking the excess funds return over the risk free rate and subtracting beta times the excess return of the benchmark over the risk free rate. Jensen's alpha represents the average incremental rate of return on the portfolio which is due solely to the manager's stock-selection abilities. A positive Alpha figure indicates the portfolio has performed better than the market beta would predict, and a negative Alpha indicates the portfolio has underperformed compared with the expectations established by beta.

The Sharpe ratio and Jensen's alpha are based on the risk adjusted return analysis with Sharpe ratio having the standard deviation of the excess return of the fund over risk free rate as the risk measure and Jensen's alpha using beta representing the systematic risk in the market. Therefore they are likely to be significantly related to the efficiency scores of the investment trusts obtained by applying the quadratic DEA model with risk as input and return as output.

Also, one may be interested to see whether the fund expenses have an impact on the fund performance, for example, whether more efficient funds have higher expenses. The fund expense ratio is used, which is reflected in the fund's NAV. It is the percentage of fund assets that pays for operating expenses and management fees, including administrative fees, and all other asset-based costs incurred by the fund, except brokerage costs. Sales charges are not included in the expense ratio.

Stattman (1980) and Rosenberg, Reid, and Lanstein (1985) and Chan, Hamao and Lakonishok (1991) find that book-to-market ratio makes a positive contribution in explaining the average returns on stocks. Basu(1983) shows that earnings-price ratios(E/P) are positively related to the average returns of U.S. stocks. Inspired by the above research, book-to-market ratio and price earnings ratio are also included in this application as potential factors to examine their influence on the investment trusts performance.

Another potential variable is market capitalization (a stock's price times its shares outstanding) which may have an impact on the funds efficiency in terms of the size effect. Among the second stage DEA literature, OLS-robust, Tobit models and Papke-Wooldridge (PW) model are most commonly used second stage models. In this application, they are conducted and compared to evaluate contextual variables affecting the performance of investment funds.

Chapter 5 is organized as follows: Section 2 provides a literature review and methodology of second stage DEA models; Section 3 shows the data collection; Section 4 presents the results; and Section 5 gives the conclusion.

5.2 Literature review and methodology

Analysis of factors contributing to productivity efficiency has been an important area of research in DEA. Many studies have used the two-stage analysis of first calculating efficiency scores and then relating these scores to contextual variables. Examples are Ray (1991) and Forsund (1999). Among the literature, the commonly used two-stage methods include ordinary least squares (OLS), Tobit regression and the Papke-Wooldridge approach based on quasi-maximum likelihood estimation (QMLE).

Banker and Natarajan (2008) is a very important paper which provides a statistical foundation for two-stage analyses. Banker (2008) develops a basic model in which the contextual variables are linked to inefficiency. In this model, a single output, y, is specified as a general function of multiple inputs and an error term. It is illustrated as follows:

$$y = \phi(X) * e^{\varepsilon^*} \tag{5.2.1}$$

where $\phi(X)$ is the true production function and ε is an error term. The production function $\phi(\cdot)$ is monotone increasing and concave in input X. In this framework, consider j decision-making units (DMUs); $j = 1,...,N$ $X_j \equiv (x_{1j},...,x_{Ij})$ is a vector inputs and $Z_j \equiv (z_{1j},...,z_{Sj})$ is a vector of contextual variables that may influence the overall productivity in transforming the inputs into the output. X_j and Z_j are nonnegative vectors.

The random variable ε^* from (5.2.1) is generated by the process

$$\varepsilon^* = v - u - \sum_{s=1}^{S} \beta_s z_s \tag{5.2.2}$$

Where v represents random noise and has a two sided distribution, and u represents technical inefficiency and is asymmetrically distributed. β_s, $s = 1,...,S,$ are nonnegative weights of the contextual variables z_s . Also it has $\varepsilon = v - u.$ It assumes that the input variable vector X , the contextual variable vector z , the inefficiency u, and the noise v are independently distributed. Therefore, the random variable consists of three components: a linear function of contextual variables; a technical inefficiency term and a random noise.

Banker and Natarajan (2008) claims that if it is assume $\phi(X)$ can be specified as $\phi(X;\gamma)$, where γ is a parameter vector; the relationship in (5.2.2) can be transformed to

$$\ln y = \ln \phi(X;\gamma) - \sum_{s=1}^{S} \beta_s Z_s + \varepsilon \tag{5.2.3}$$

In the spirit of the 'DEA+method' suggested by Gstach (1998), Banker and Natarajan (2008) defines

$$\ln \tilde{\phi}(X) = \ln \phi(\cdot) + V^M \tag{5.2.4a}$$

and

$$\begin{aligned} \ln \tilde{\theta} &= (\varepsilon - V^M) - \sum_{i=1}^{S} \beta_i z_i \\ &= (v - V^M) - u - \sum_{i=1}^{S} \beta_i z_i \le 0 \end{aligned} \tag{5.2.4b}$$

where $\tilde{\phi}(X)$ is also monotone increasing and concave. Inserting (5.2.4a) and (5.2.4b) into (5.2.1) yields

$$\ln y = \ln \tilde{\phi}(X) + \ln \tilde{\theta} \tag{5.2.5}$$

Let $\tilde{\varepsilon} = V^M - \varepsilon$, then (3.2.4) could be expressed as

$$\ln \tilde{\theta} = \sum_{i=1}^{S} \beta_i z_i - \tilde{\varepsilon} \tag{5.2.6}$$

Thus the DEA scores obtained from the first stage are related to the contextual variables.

In practice, the corresponding DEA estimator $\ln \hat{\tilde{\theta}}$ will replace $\ln \tilde{\theta}$ as the dependent variable in (5.2.6). And then OLS can be used. Banker (1993) proves that the generated estimators of β_i are consistent; the corresponding t-statistic provides significance of a particular contextual variable. Alternatively, if a specific parametric form is assumed for the p.d.f. of ε, e.g. v is normally distributed u is either half normal or exponential, then maximum likelihood estimation (MLE) can be used. And the generated estimators β_i are consistent, efficient and asymptotically normally distributed; the corresponding t-statistics can provide the significant test of contextual variables.

(1) OLS estimation

Define $\beta_0 = E(\varepsilon) - V^M$ and $\delta = \varepsilon - E(\varepsilon)$, (5.2.6) can be rewritten as

$$\ln \theta = \beta_0 - \sum_{i=1}^{S} \beta_i z_i + \delta. \tag{5.2.7}$$

where the error term δ in (5.2.7) has a zero mean and a finite variance.

Banker(2009) proves that if $Q = p\lim(Z^{'}Z/n)$ is a positive definite matrix, then the OLS estimator of $\tilde{\beta}$ in

$$\ln\hat{\tilde{\theta}} = \tilde{\beta}_0 - Z\tilde{\beta} + \tilde{\delta} \tag{5.2.8}$$

yields a consistent estimator of the parameter vector β.

(2) MLE estimation

(5.2.6) can be rewritten as

$$\varepsilon = \sum_{i=1}^{S}\beta_i z_i + V^M + \ln\theta \tag{5.2.9}$$

The log-likelihood function can be formed as

$$\sum_{j=1}^{N} Ln\, f(\sum_{i=1}^{S}\beta_i Z_{ij} + V^M + \ln\theta) \tag{5.2.10}$$

Banker (2009) proves that maximizing (5.2.10) yields consistent estimators of β.

Another very commonly used method in the second stage analysis is the Tobit model. Tobit model is a censored regression model, and is employed when the dependent variables are limited within a particular range. For DEA models, the output oriented DEA have the scores bounded above 1, and the input oriented DEA have the scores limited between zero and one. Therefore, the Tobit model is often used instead of OLS to evaluate the effects of contextual variables. Researchers who have applied tobit at second stage DEA to explain the efficiency distributions include Bravo-Ureta et al. (2007), Latruffe et al. (2004), Oum and Yu (2004), Fethi et al. (2002), Vestergaard et al. (2002), Ruggiero and Vitaliano (1999), Viitala and Hanninen (1998), Kirjavainen and Loikkanen (1998), Gillen and Lall (1997), Luoma et al. (1996), Chilingerian (1995) and Bjurek et al. (1992).

Followed by an input oriented DEA model, which gives the DEA scores ranging from zero to one, a two-limit tobit method is used.

The two-limit Tobit model is defined as

$$
\begin{aligned}
& y_i^{*} = x_i \beta_i + \varepsilon_i \\
& y_i = y_i^{*} \quad if \quad 0 \le y_i^{*} \le 1 \\
& y_i = 0 \quad if \quad y_i^{*} < 0 \\
& y_i = 1 \quad if \quad y_i^{*} > 1
\end{aligned}
\tag{5.2.11}
$$

Where y_i^{*} is the latent dependent variable, y_i is the observed dependent variable, x_i is the vector of the independent variables, and the ε_i are assumed to be independently normally distributed: $\varepsilon_i \sim \quad N(0,\sigma)$.

Given (5.2.11) is the data generating process (DGP), the combined likelihood for a sample containing some $y_i -$observations=0, some=1 and some between 0 and 1 is given by:

$$L = \prod_{y_i=0} P(y_i = 0) \prod_{y_i=1} P(y_i = 1) \prod_{0<0\, y_i<1} P(0 < y_i < 1) \tag{5.2.12}$$

According to Hoff(2007),

$$P(y_i = 0) = F(-\sum \beta_i x_i | 0,\sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{-\sum \beta_i x_i} e^{-t^{2/(2\sigma^2)}} dt \tag{5.2.13}$$

Where $F(x|\mu,\sigma)$ is the cumulative distribution function.

Likewise:

$$P(y_i = 1) = F(-(1\sum - \beta_i x_i)|0,\sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{-(1-\sum \beta_i x_i)} e^{-t^{2/(2\sigma^2)}} dt. \tag{5.2.14}$$

$$P(y_i | 0 < y < 1) = f(y_i - \sum \beta_i x_i | 0, \sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(y_i - \sum \beta_i x_i)^2 / (2\sigma^2)} \qquad (5.2.15)$$

Where $f(x|\mu,\sigma)$ is the probability density function.

If there are no $y_i -$observations=0, which is the case in DEA models, then the first term in (5.2.12) will disappear, thus the likelihood functions for two-limit tobit (2LT) and one-limit tobit (1LT), with a limit at one, will be identical. Furthermore, if there are no $y_i -$ observations=0 and 1, both of the first two terms will disappear, leaving the third term alone to be maximized in the likelihood function. Thus in this case the 2LT and 1LT MLE and OLS estimates are identical.

There are some drawbacks about Tobit models. Many researchers argued that the Tobit model is misspecified when applied to DEA scores, because no efficiency score would be equal to zero. Therefore the first multiplication in (5.2.12) will be left out. Although it has been argued as misspecified, the two-limit Tobit model has been one of the most often used method in second stage DEA.

Besides the two-limit Tobit model, Greene (1993) suggests the use of censoring at zero, which gives the computational convenience. Fethi et al (2002) describes a model which makes a transformation to the original DEA score to allow the censoring point concentrating at zero.

The dependent variable is obtained by taking the reciprocal of DEA score minus one, and this model is illustrated as follows:

$$y_i = (1/\theta) - 1 \qquad (5.2.16)$$

$$\begin{aligned} & y_i^* = \beta_i x_i + \varepsilon_i \\ & y_i = y_i^* \quad if \quad y_i^* \phi\ 0, and \\ & y_i = 0, otherwise, \end{aligned} \qquad (5.2.17)$$

Where $\varepsilon \sim N(0,\sigma^2)$

The corresponding likelihood function (L) becomes

$$L = \prod_{y_i=0}(1 - F_i)\prod_{y_i \phi 0}\frac{1}{(2\pi\sigma^2)^{1/2}} \times e^{-\left[1/(2\sigma^2)\right](y_i - \beta_i x_i)^2} \qquad (5.2.18)$$

Where

$$F_i = \int_{-\infty}^{\beta_i x_i/\sigma}\frac{1}{(2\pi\sigma^2)^{1/2}}e^{-t^2/2}dt \qquad (5.2.19)$$

Take the logarithm form of the likelihood function:

$$LLF = \ln L = \sum \ln(1 - F_i) - \frac{1}{2}\ln\sigma^2 - \sum(Y_i - \beta_i x_i)/(2\sigma^2)$$

(5.2.20)

To obtain the estimators of β_i, the logarithm likelihood function is maximised as follows:

$$LLF(\hat{\beta}, \hat{\sigma}^2) = \max_{\beta,\sigma^2} LLF(\beta, \sigma^2) \qquad (5.2.21)$$

To analyze the effect of the contextual variables to the productivity, the marginal effects of the independent variables need to be examined. Hoff (2007) derives the marginal effect of the individual explanatory variable x_m on the expectation of y in a two-limit tobit case. It is given by

$$\frac{\partial E(y/x)}{\partial x_m} = \beta_m\left[\Phi(\frac{1-\sum\beta_i x_i}{\sigma}) - \Phi(\frac{-\sum\beta_i x_i}{\sigma})\right] \qquad (5.2.22)$$

Where $\Phi(\cdot)$ is the standard normal cumulative distribution function.

McDonald and Moffitt (1980) provides another useful interpretation of the marginal effects in a one-limit tobit case which has the censoring point concentrating at zero. It is expressed as follows:

$$\frac{\partial E(y/x)}{\partial x_m} = P(y>0)\frac{\partial E[y|y>0]}{\partial x_m} + (\partial E[y|y>0])\frac{\partial P(y>0)}{\partial x_m} \qquad (5.2.23)$$

(5.2.23) shows the decomposition of Tobit marginal effects: a change in x_m affects two parts: (1) it affects the conditional mean of y_i in the non-limit part of the distribution; and (2) it affects the probability that the observation being above the limit.

It is argued that the DEA scores are not generated by a censoring process in which case Tobit regression is inappropriate. Instead the DEA scores are fractional data. Simar and Wilson (2007) and McDonald (2009) claim that the DEA programming, as an efficiency score generating process, gives a normalization process rather than censoring. The result that all the efficiency scores lies within the range (0,1] is a product of the way the programming defines, which is not out of a censoring mechanism. The regression dependent variable, then, is not censored data, but fractional data. Papke and Wooldridge (1996) propose a fractional response model that extends the generalized linear model (GLM) literature from statistic. They construct a model based on quasi-maximum likelihood estimation (QMLE) to deal with fractional dependent variable, which confined to the (0,1] interval, with many observations at the right boundary, 1.

The fractional response model is based on the Bernoulli distribution function, which is a subset of the binomial distribution function. Assume there are sequences of n independent success/failure experiments (also called n 'trials') for N DEA units, Define

$$Z = \begin{cases} 1 & \textit{if the outcome is a success} \\ 0 & \textit{if the outcome is a failure} \end{cases} \qquad (5.2.24)$$

The probability of success in unit i is $\Pr(Z=1)=\pi_i \; i=1,...,N$ and the probability of failure is therefore $\Pr(Z=0)=1-\pi_i$. An important characteristic of π is that it's restricted to the interval [0,1]. The number of successes for unit i in n trials is denoted by Y_i . There is $E(Y_i) = n_i \pi_i$ and the corresponding share in each trial is $y_i = \dfrac{Y_i}{n_i}$. Thus $E(y_i) = \pi_i$ with $0 \leq y_i \leq 1$.

The conditional binomial probability density function for group i is given by

$$f(y_i | X_i, n_i) = \begin{pmatrix} n_i \\ n_i y_i \end{pmatrix} \pi_i^{n_i y_i} (1-\pi_i)^{n_i - n_i y_i} \qquad i = 1,..., N \tag{5.2.25}$$

Where X_i refers to a set of explanatory variables with the corresponding parameter vector β . When n = 1, the binomial distribution is a Bernoulli distribution and probability density function becomes,

$$f(y_i | X_i) = \pi_i^{y_j} (1-\pi_i)^{1-y_i} \qquad i = 1,..., N \tag{5.2.26}$$

The joint density function is as follows,

$$\prod_{i=1}^{n} \pi_i^{y_j} (1-\pi_i)^{1-y_i} = \exp\left[\sum_{i=1}^{n} y_i \ln\left(\frac{\pi_i}{1-\pi_i} \right) + \sum_{i=1}^{n} \ln(1-\pi_i) \right] \qquad i = 1,..., N \tag{5.2.27}$$

Or

$$\prod_{i=1}^{n} \pi_i^{y_i} (1-\pi_i)^{1-y_i} = \exp\left[\sum_{i=1}^{n} y_i \ln(\pi_i) + \sum_{i=1}^{n} (1-y_i) \ln(1-\pi_i) \right] \qquad i = 1,...., N \tag{5.2.28}$$

which is a member of the exponential family. The conditional expectation of the fractional response variable y_i is specified as

$$E(y_i | X_i) = G(X_i \beta), \qquad i = 1,..., N \tag{5.2.29}$$

Logistic or logit regression and probit regression are chosen as link functions to ensure that π is restricted to the interval [0,1]. Define

$$g(\pi_i) = G(X_i \beta) \qquad i = 1,..., N \tag{5.2.30}$$

and $g(\pi_i)$ is called the link function.

The Probit model is constructed as follows,

$$\begin{aligned}\pi &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left[-\frac{1}{2} (\frac{x-\mu}{\sigma})^2 \right] dx \\ &= \Phi(\frac{x-\mu}{\sigma})\end{aligned} \tag{5.2.31}$$

Where Φ denotes the cumulative probability function for the standard normal distribution $N(0,1)$. Thus

$$g(\pi_j) = \Phi^{-1}(\pi) \tag{5.2.32}$$

and the link function $g(\pi_j)$ is the inverse cumulative normal probability function Φ^{-1}.

Another model that gives similar distribution from the probit model is the logistic model. It is constructed as:

$$\pi_i = \frac{1}{1+e^{-G(x_i \beta)}} = \frac{e^{G(x_i \beta)}}{1+e^{G(x_i \beta)}} \qquad i = 1,..., N \tag{5.2.33}$$

Therefore,

$$\log it\pi_i = \log(\frac{\pi_i}{1-\pi_i}) = G(x_i, \beta) \qquad i = 1,...,N \tag{5.2.34}$$

And $\log[\pi_i /(1-\pi_i)]$ is called the logit function.

In the QMLE the mean is substituted for π_i, therefore, substituting $G(X_i\beta)$ for π_i in (3.2.28), the the Bernoulli likelihood is given by the following formula:

$$\ln(f_i(\beta)) = y_i \ln(G(X_i\beta)) + (1-y_i)\ln(1-G(X_i\beta)) \tag{5.2.35}$$

The likelihood is a re-parameterization of the probability distribution function to estimate parameter vector β. By taking the natural log of (35), the Bernoulli log likelihood is as follows,

$$\sum_{i=1}^{N}\ln(f_i(\beta)) = \sum_{i=1}^{N}(y_i \ln(G(X_i\beta)) + (1-y_i)\ln(1-G(X_i\beta))) \tag{5.2.36}$$

And the quasi-maximum likelihood estimator (QMLE) of β could be obtained by maximizing (36). Practically, it means to take the first derivative of the log-likelihood function and to solve the following equation.

$$\frac{\partial \sum_{i=1}^{N}\ln(f_i(\beta))}{\partial\beta} = 0 \tag{5.2.37}$$

Hoff (2007) derives the marginal effect of the explanatory variable x_m on the expectation of y when logit function is used is given by

$$\frac{\partial E(y/x)}{\partial x_m} = \beta_m \left[\frac{\exp(-\sum \beta_i x_i)}{1+\exp(-\sum \beta_i x_i))^2} \right] \tag{5.2.38}$$

There is only one paper being found using the second stage models to analyze the performance about exchange traded funds. Tsolas (2011) employs a Tobit model to measure the performance of a sample of natural resources exchange traded funds using time-series data. The DEA approaches of Haslem and Scheraga (2003, 2006) and of Kerstens and Van de Woestyne (2011) are applied. The former use Sharpe ratio as output and variables with positive user costs are chosen as inputs; the latter use Jensen's alpha as a single output or both the Sharpe ratio and Jensen's alpha as outputs and some ETFs characteristics are identified as inputs. In Tsolas (2011) the chosen explanatory variables include PE ratio, beta coefficient, persistence and size. In this thesis, the second stage models including the Papke-Wooldridge approach are estimated using the STATA software application.

5.3 Data collection

The databases used in this application are Morningstar Direct database and Datastream. The funds chosen were one specific type of the UK investment trusts: those classified by Morningstar as 'UK equity Mid/Small cap'. Another important criterion is that the funds selected must have at least 3 years of monthly returns, Sharp ratio, Jensen's Alpha, net expense ratio, price to earning ratio, market capitalization, price, and net asset value data available. Price and net asset value data are needed to calculate the book to market ratio. The sample period is from January1, 2008 to December 31, 2010, therefore, each fund selected need to have an inception date at or before January 1, 2008.

There are 33 funds categorized as UK equity Mid/Small cap in the Morningstar Direct database, with 4 funds being deleted from the sample because monthly returns are missing. 3 funds were further deleted because of negative mean returns, with 26 funds remaining in the sample. Data of Sharp ratio, Jensen's Alpha, net expense ratio are obtained from Morningstar direct database, while data of price to earnings ratio, market capitalization, price, and net asset value ratio are drawn from Datastream. However, only 13 funds with complete data of Sharp ratio, Jensen's Alpha, net expense ratio, price to earning ratio, market capitalization, price, and net asset value data can be found in the Morningstar Direct database and Datastream. For each of the 13 funds, the following figures were calculated for the 3-year time periods: (i) Mean monthly returns; (ii) Covariances; (iii) Variances. These values were calculated using monthly return data from Morningstar Direct database. Expressed in percentage terms, Morningstar's calculation of monthly return is determined by taking the change in monthly net asset value, reinvesting all income and capital-gains distributions during that month, and dividing by the starting net asset value. The total returns account for management, administrative, fees and other costs taken out of fund assets. The mean return is the time-series average of the 36 monthly returns. The 3 year mean returns for selected funds are presented in table 1.

Table 5.3.1 three year mean returns for selected funds

Fund Number	**Fund name**	**3 year mean returns**
1	Small Companies Dividend Trust Ord.	0.08
2	JPMorgan Mid-Cap IT ORD	0.08
3	Dunedin Smaller Companies Ord	0.90
4	Lowland Inv Tr	0.44
5	Schroder UK Mid Cap	0.76
6	Standard Life UK Smaller Companies	1.51
7	Invesco Perpetual UK Smaller	0.63
8	Aurora Investment Trust PLC	1.46
9	JPMorgan Smaller Companies IT ORD	0.74
10	The Throgmorton Trust PLC	1.00
11	Henderson Opportunities Trust	0.37
12	BlackRock Smaller Companies Trust	1.46
13	Artemis Alpha Trust PLC	1.31

The Sharpe ratio, Jensen’s alpha, net expense ratio, price/earnings ratio, market capitalization (market value) and book to market ratio are showed in table 5.3.2. They are all time-series average of yearly data from 2008 to 2010. Book to market ratio are calculated using price and net asset value (book value) data from Datastream.

Table 5.3.2 time-series average of yearly data from 2008 to 2010

Fund Number	**Sharpe**	**Alpha**	**NER**	**PE**	**MV**	**BTM**
1	-0.20	0.50	2.02	7.60	14.71	1.14
2	-0.24	-5.21	0.62	23.87	105.38	1.15
3	0.05	9.47	1.49	22.83	44.43	1.30
4	-0.11	8.38	0.80	24.23	172.50	1.07
5	-0.01	3.11	0.91	35.67	65.06	1.32
6	0.30	10.90	1.21	377.93	46.89	1.31
7	-0.06	0.59	1.32	36.80	87.82	1.22
8	0.20	21.65	2.08	118.37	19.52	1.17
9	-0.01	9.17	0.76	47.77	69.08	1.28
10	0.07	8.05	1.23	54.03	120.47	1.30
11	-0.12	8.08	1.02	47.20	28.24	1.25
12	0.22	13.67	1.07	40.27	126.03	1.31
13	0.22	15.13	1.03	95.47	62.82	1.32

5.4 Results analysis

In the first stage, the DEA scores are obtained using 3-year period mean monthly returns. Morey and Morey (1999) combined 3-year, 5-year and 10 year data in the programming, but only 3-year period data is used in this application, firstly because for 10 year-period, even fewer funds with complete data could be found since it is a very long period; secondly, if DEA score as dependent variable is derived from a combination of 3-year, 5-year and 10 year data, then the period for the independent variables would be difficult to choose.

Risk contraction approach is chosen in this application because in this approach, the DEA scores as dependent variable have values between zero and one, therefore it is more representatable. The results are showed in Table 5.4.1:

Table 5.4.1 DEA scores from Risk contraction approach.

Risk contraction approach		
	Fund name	**DEA scores**
1	Small Companies Dividend Trust Ord.	0.3139
2	JPMorgan Mid-Cap IT ORD	0.5008
3	Dunedin Smaller Companies Ord	0.7269
4	Lowland Inv Tr	0.4482
5	Schroder UK Mid Cap	0.7744
6	Standard Life UK Smaller Companies	1
7	Invesco Perpetual UK Smaller	0.8076
8	Aurora Investment Trust PLC	0.3718
9	JPMorgan Smaller Companies IT ORD	0.4794
10	The Throgmorton Trust PLC	0.5267
11	Henderson Opportunities Trust	0.3779
12	BlackRock Smaller Companies Trust	0.5024
13	Artemis Alpha Trust PLC	0.8916

Table 5.4.2 Summary statistics

Factor	**Mean**	**Standard deviation**	**Minimum**	**Maximum**
Number of observations=13				
Sharp Ratio	0.02	0.17	-0.24	0.30
Alpha	7.96	7.02	-5.21	21.65
NER	1.20	0.45	0.62	2.08
PE	71.70	96.85	7.60	377.93
MV	74.07	46.69	14.71	172.50
BTM	1.24	0.08	1.07	1.32

Robust-OLS regression, Tobit models and Papke-Wooldridge (PW) models are conducted and compared to evaluate contextual variables affecting the performance of investment trusts. PW quasi-maximum-likelihood model is designed to address the problem of non-i.i.d. errors (specifically including heteroscedasticity) in the regression in an optimal estimation procedure. The DEA efficiency scores are regressed on potential variables including Sharpe ratio, Jensen's alpha, expense ratio, Price/Earnings ratio, market capitalization, book to market ratio of the investment funds to test the statistical significance of those factors.

Positive relations between the DEA scores and Sharpe ratio, DEA scores and Jensen's alpha are expected. Sharpe ratio is calculated by dividing a fund's annualized excess returns by the standard deviation of a fund's annualized excess returns and mutual funds with lager Sharpe Ratios are assumed to have better historical risk-adjusted performance than those with small ratios. Jensen's alpha is calculated by taking the excess funds return over the risk free rate and subtracting beta times the excess return of the benchmark over the risk free rate. Jensen's alpha represents the average incremental rate of return on the portfolio which is due solely to the manager's stock-selection abilities. Sharpe ratio and Jensen's alpha are two measurements that are also in mean and variance space; therefore, positive relationships between both measures and the DEA scores are expected. The DEA scores and net expense ratio however, should be negatively related because the higher the expense, the less profitable the investment funds. Book to market ratio is a ratio between book value or net tangible assets per share and the price. If the ratio is above 1 then the stock is undervalued; if it is less than 1, the stock is overvalued. In the long run, the undervalued investment trusts should be more efficient.

Therefore, a positive relationship between the book to market value and the efficiency measure would be expected. The price/Earnings ratio is obtained by dividing the company's market capitalization by its total annual earnings. In general, a high P/E suggests that investors are expecting higher earnings growth in the future compared to companies with a lower P/E, therefore, the higher PE ratio is, the more efficient the investment trust would be, which indicates that there would be a positive relationship between the DEA score and the PE ratio. For the market capitalisation, it's not clear what relationship it would be because it's hard to predict whether the smaller fund or the larger fund is more efficient than the other type. Results from different models are presented in table 5.4.3.

Table 5.4.3 Results from different models

Factor	**OLS**	**Two Limit Tobit/ One Limit Tobit**	**Tobit censoring at zero**	**PW(logit)**
Sharp Ratio	2.0697 (1.2631)	1.9306 (0.9072)	-5.6383 (2.4381)	7.7289 (4.3902)
Alpha	-0.0347 (0.0168)	-0.0356 (0.0121)	0.1016 (0.0327)	-0.1562 (0.0584)
NER	-0.1912 (0.1877)	-0.1679 (0.1381)	0.6014 (0.3405)	-0.6423 (0.6835)
PE	-0.0002 (0.0010)	0.0006 (0.0010)	-0.0014 (0.0043)	0.0063 (0.0060)
MV	-0.0016 (1.1771)	-0.0013 (0.0014)	0.0014 (0.0032)	-0.0047 (0.0072)
BTM	-0.5086 (1.8924)	-0.3505 (0.90371)	0.1408 (2.2171)	-0.9597 (4.5314)

Notes: the quantities in () are the standard errors robust to variance misspecification.

The third column of table 5.4.3 contains the results of estimating equation (5.2.11), followed by the fourth column which shows the results of estimating (5.2.16)-(5.2.17). And the fifth column gives the results of (5.2.29) given (5.2.33) as the link function.

The one limit tobit with a limit at one has the same result as that of two limit tobit model because there is no DEA score equal to be zero, therefore the first term in (5.2.12) will disappear, thus the likelihood functions for two-limit tobit (2LT) and one-limit tobit (1LT), with a limit at one, will be identical.

Heteroscedasticity is expected in all models because cross sectional data is used. Therefore the heteroscedasticity-robust standard errors are reported in brackets below the coefficients. Also it is very often in the second stage models that the independent variables are to some extent correlated with each other. The correlation matrix among six independent variables is showed in table 5.4.4.

Table 5.4.4 Correlation matrix

	Sharpe	Alpha	NER	PE	MV	BTM
Sharpe	1	0.8016	0.1859	0.6556	-0.1038	0.6139
Alpha	0.8016	1	0.3155	0.3609	-0.1650	0.2822
NER	0.1859	0.3155	1	0.0863	-0.6061	-0.1587
PE	0.6556	0.3609	0.0863	1	-0.2473	0.3015
MV	-0.1038	-0.1650	-0.6061	-0.2473	1	-0.2052
BTM	0.6139	0.2822	-0.1587	0.3015	-0.2053	1

Results from table 5.4.4 show that Sharpe ratio and Jensen's alpha are most highly correlated. This is because despite all other relations, Sharpe ratio and Jensen's alpha both have expected returns in their models. The correlation coefficient is as high as 0.6556 between PE ratio and Sharpe ratio, with any other two variables more or less correlated with each other.

Marginal effects of Robust-OLS are regression coefficients corresponding to each variables; and marginal effects of two limit tobit, one limit model censoring at zero and PW model are given by (5.2.22), (5.2.23) and (5.2.38). The results are showed in the following table.

Table 5.4.5 Average marginal effects

Factor	Robust-OLS	Two Limit Tobit/ One Limit Tobit	Tobit censoring at zero	PW
Sharp Ratio	2.0697 (0.152)	1.9306* (0.054)	-5.6383* (0.066)	1.6133 (0.660)
Alpha	-0.0347* (0.084)	-0.0356*** (0.008)	0.1016** (0.012)	-0.0326 (0.487)
NER	-0.1912 (0.348)	-0.1679 (0.283)	0.6014 (0.209)	-0.1341 (0.820)
PE	-0.0002 (0.833)	0.0006 (0.638)	-0.0014 (0.783)	0.0013 (0.813)
MV	-0.0016 (0.420)	-0.0013 (0.430)	0.0014 (0.775)	-0.0010 (0.875)
BTM	-0.5086 (0.681)	-0.3504 (0.747)	0.1408 (0.966)	-0.2003 (0.961)

Notes: The quantities in () below correlation coefficients are the corresponding p-values;
* p<0.1, ** p<0.05, *** p<0.01.

The results show that firstly, Shape ratio and price/earnings ratio have positive impact on the fund performance under Robust-OLS and three tobit models, but not statistically significant, while Jensen's alpha, net expense ratio, market value, and book to market ratio of the fund have negative impact on the fund performance, but only Jensen's alpha is statistically significant in three tobit models. Secondly, for the average marginal effects, and magnitude of all the factors are fairly close for OLS and Two limit and One limit tobit models, and PW model with the same sign, while the results from tobit model censoring at zero has the opposite sign and different magnitude. This is because in tobit model censoring at zero, the dependent variable is obtained by taking the reciprocal of DEA score minus one, therefore the positive relation between DEA scores and the dependent variable in other regressions turn to negative in this model. Note that the p values of the marginal effects from PW model are much larger than other models except of the last factor book to market ratio, where Tobit model censoring at zero has slightly larger p value.

From table 5.4.5, except the Jensen's alpha in Tobit models, all the other factors have p values larger than the critical value 0.05 for 95% confidence level, which means that these factors contribute very little to the explanation of investment fund mean return. This could be due to model misspecification, for example, inclusion of irrelevant variables. In addition, because Sharpe ratio and Jensen's alpha are two measurements that are based on the mean-variance framework, and Morey and Morey (1999) quadratic DEA model is also constructed in mean and variance space. One may be interested in the correlation of the rankings of funds using DEA scores, Shape ratio and Jensen's alpha. Table 5.4.6 shows rankings of the 13 sample funds from different models while Table 5.4.7 gives the correlation among rankings from different models.

Table 5.4.6 Rankings from different models

Fund number	**DEA**	**Sharpe**	**Alpha**
1	13	12	12
2	8	13	13
3	5	6	5
4	10	10	7
5	4	7	10
6	1	1	4
7	3	9	11
8	12	4	1
9	9	7	6
10	6	5	9
11	11	11	8
12	7	2	3
13	2	2	2

Table 5.4.7 Correlation among rankings from different models

	DEA	Sharpe	Alpha
DEA	1	0.5543** (0.0493)	0.1538 (0.6158)
Sharp	0.5543** (0.0493)	1	0.82*** (0.0006)
Alpha	0.1538 (0.6158)	0.8181*** (0.0006)	1

Notes: The quantities in () below correlation coefficients are the corresponding p-values;
* p<0.1, ** p<0.05, *** p<0.01.

The results show that there is quite high correlation between the ranking from the quadratic DEA model and Sharpe ratio, which equals 0.5543, with a p-value of 0.0493 and the correlation between Jensen's alpha and DEA is 0.1538, but not statistically significant as indicated by a p-value as high as 0.6158. The correlation between rankings from Sharpe ratio and Jensen's alpha is very high, which equals 0.8181, and it is highly significant at 1% significance level with the p-value equals 0.0006.

In addition, a recursive model is applied. Regressing DEA scores, Sharpe Ratio and Jensen's Alpha on net expense ratio, price/earnings ratio, market value and book to market ratio respectively using robust-OLS gives the following results. They are indicated as model (1), model (2) and model (3) respectively in Table 5.4.8.

Table 5.4.8 Statistics from the recursive model

Dependent variables	DEA scores	Sharpe Ratio	Jensen's Alpha
Factor	**Model (1)**	**Model (2)**	**Model (3)**
NER	-0.0511 (0.743)	0.2049** (0.042)	8.1751 (0.243)
PE	0.0010*** (0.005)	0.0010*** (0.005)	0.0207 (0.288)
MV	0.0006 (0.739)	0.0018** (0.039)	0.0437 (0.380)
BTM	1.2107* (0.067)	1.3077*** (0.006)	28.4235 (0.300)
R-square	0.5438	0.7914	0.3150
F-value	0.0006***	0.0020***	0.4232

Notes: The quantities in () below estimation coefficients are the corresponding p-values;
* $p<0.1$, ** $p<0.05$, *** $p<0.01$.

The results of model (1) indicates that the net expense ratio has a negative impact on the efficiency score indicated by the DEA score as expected, but not statistically significant, with PE ratio, Market value and book to market ratio impact the efficiency score positively, but only the coefficient of the PE ratio is statistically significant. Model (2) gives very good results, in a way that all the coefficients are statistically significant, but the net expense ratio is positively related to the efficiency measure indicated by Sharpe ratio. This maybe because usually better performing fund companies locate their offices in better area which incur higher rents or giving more budget in the advertisement etc. In model (2), the PE ratio, market value, book to market ratio make significant positive contribution to explaining the efficiency indicated by Sharpe ratio, this is consistent with the findings in Stattman (1980) and Rosenberg, Reid, and Lanstein (1985) and Hamao, and Lakonishok (1991) which find that book-to-market ratio makes a positive contribution in explaining the average returns on stocks; and also Basu(1983) which shows that PE ratio are positively related to the average returns of U.S. stocks.

From model (3), all the factors have a positive impact on the efficiency measure indicated by Jensen's Alpha, but none of the coefficients are statistically significant. Also, model (2) shows the highest R-square among those three models, which equals 0.7914. This means that the model fits well. The prob>F gives the overall significance level of the regression model. Specifically, it indicates the probability of the null hypothesis that all of the regression coefficients are equal to zero is rejected. Model (1) has the smallest prob>F value, which equals 0.0006; while that in model (2) is slightly higher, but still highly significant at 1% significance level.

To illustrate the results better, Figure 5.4.1, Figure 5.4.2 and Figure 5.4.3 show three models' forecasting abilities. Line graphs are chosen to give a clear picture about the prediction abilities of the above models to track the actual DEA scores, irrespective of the fact that DEA scores are discrete data. The horizontal axis is the fund number while the vertical axis gives the dependent variable in each model. It can be seen that the prediction from model (2) which is the regression of the Sharpe ratio on net expense ratio, price/earnings ratio, market value and book to market ratio tracks the actual data most closely.

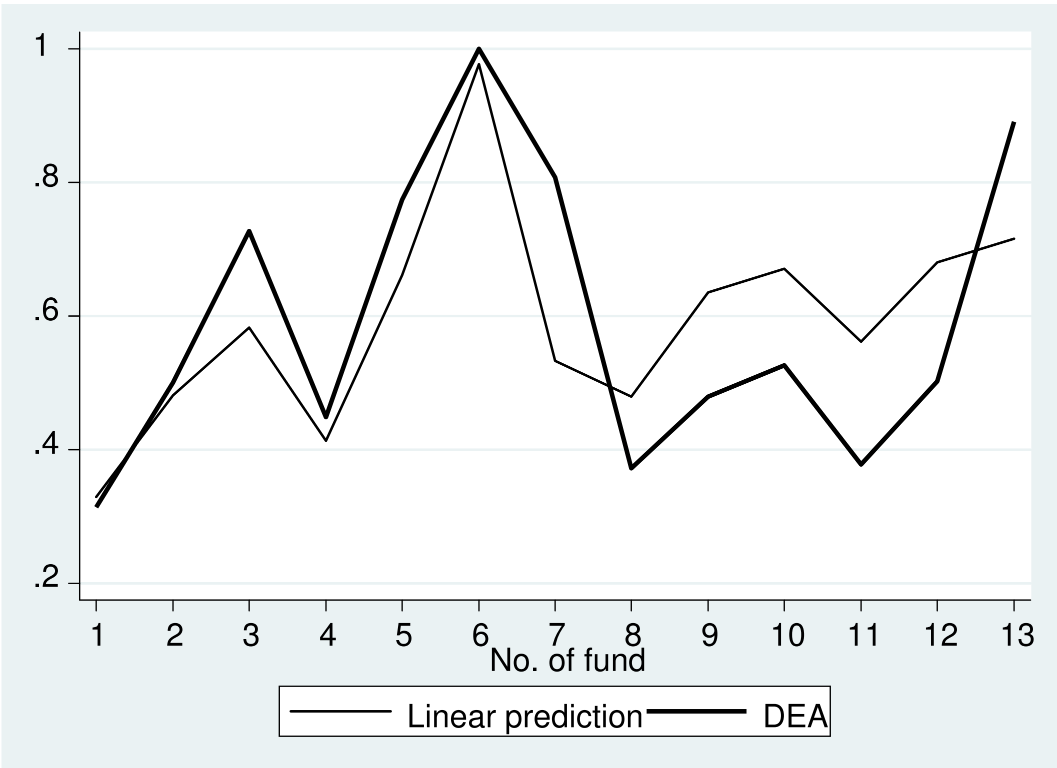

Figure 5.4 .1 forecasting abilities from recursive model (1)

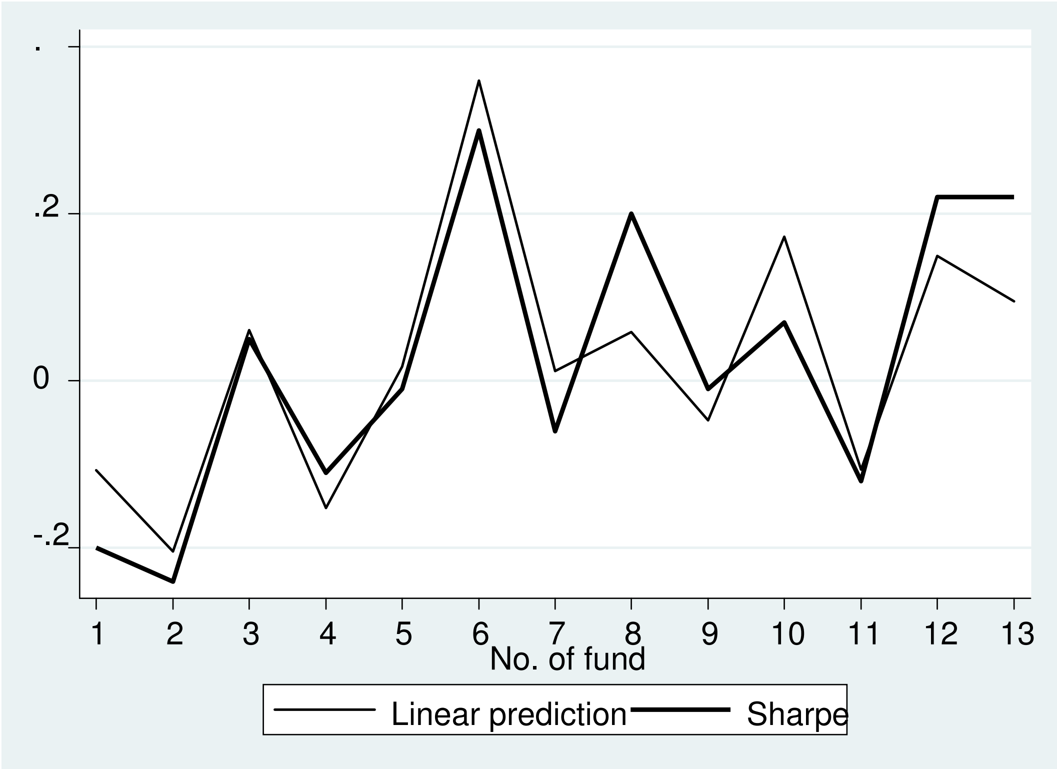

Figure 5.4.2 forecasting abilities from recursive model (2)

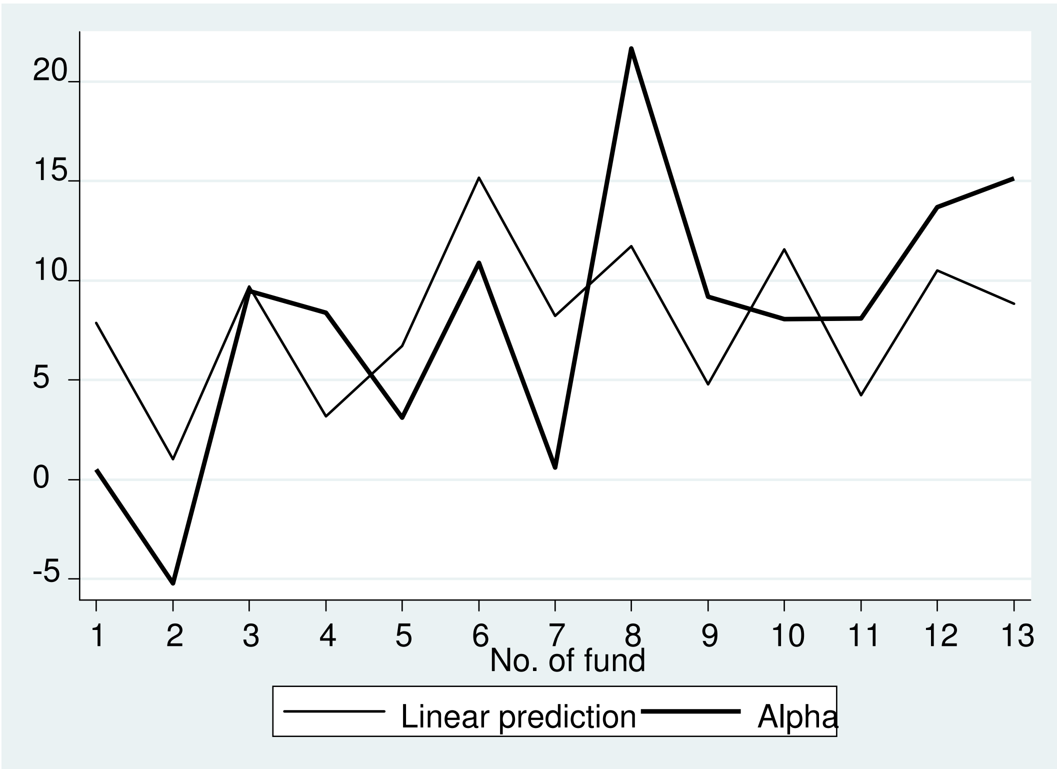

Table 5.4.3 forecasting abilities from recursive model (3)

5.5 Conclusion

This application examines two issues, one is to detect the factors influencing the investment trust efficiency utilise second stage DEA models; the other is to compare and rank three investment trust efficiency indicators- DEA score, Sharpe ratio and Jensen's alpha based on a recursive model.

Firstly of all, six potential factors including Sharpe ratio, Jensen's alpha, net expense ratio, price/earnings ratio, market value and book to market ratio that may have an influence on the investment trusts performance are examined. The efficiency scores of the investment trusts are obtained from a quadratic DEA model with risk as input and mean return as output. Five second stage DEA models- Robust-OLS, three tobit models and Papke-Wooldridge model are applied, and the results show that Sharpe ratio is positively related to the efficiency scores of the investment trusts, Jensen's alpha has a negative impact on the DEA scores, while all other factors contribute very little in explaining the efficiency of the investment trusts. The marginal effects from robust-OLS, Tobit two limit and one limit model are very close, with PW model has similar coefficients with much larger p values. The coefficient from Tobit model with censoring at zero has the opposite sign and different magnitude compared with those from other models. This is due to the difference in modeling. Although it seems only Sharpe ratio contributes significantly to the investment trust efficiency from this application, it may be caused by the fact that there are only one input-risk, one output- return being considered in the DEA program.

In the second part of this application, a recursive model is applied when DEA scores, Sharpe ratio and Jensen's alpha are used as dependent variables respectively while net expense ratio, PE ratio, market value and book to market ratio are explaining factors in all three regressions. The results show that the Sharpe ratio as an efficiency measure can be explained very well by net expense ratio, PE ratio, market value and book to market ratio, while the other two regressions have lower R-squares and insignificant coefficients. Therefore from this recursive model Sharpe ratio is found to be a good efficiency measure considering net expense ratio, PE ratio, market value and book to market ratio as explaining factors. And the results show that the DEA score is worse than Sharpe ratio but better than Jensen's Alpha as efficiency indicator. And another way to improve the modelling is to look for a 'better' range of potential factors which may have impacts on the efficiency of the investment trusts.

Limitations of this empirical work include, firstly, turnover ratio and beta are intended to be included as potential factors, but relevant data cannot be found in the database; secondly the DEA scores as dependent variable are obtained from a quadratic DEA model considering only risk and return. Therefore, the results could be very different if applying linear DEA models with multiple inputs and multiple outputs. All these problems are left for further research.

Chapter 6 Conclusions

6.1 Introduction

Interest in the efficient performance of investment funds has been ongoing for many years, but it is only in the last decade and a half that the topic has been seriously address in the non-parametric performance literature.

The core purpose of this thesis is to apply a quadratic data envelopment analysis model with bootstrap and second stage regression to estimate the efficiency of a sample of investment trusts, obtain the statistical inference of the efficiency scores and detect the determinants of inefficiency. The motivation of this thesis comes from the drawback of the traditional portfolio analysis, which is its sensitivity to chosen benchmarks. For example, the market portfolio in Capital Asset Pricing Model is an ideal portfolio that only exists in theory. In practice certain indexes are used as approximations, but this causes problems since different indexes are likely to give different results in empirical work. For multi-index models, the difficulties lie in justifying how many and which indexes should be included in the model and defining which category a particular equity belongs to, especially for some equities with properties that suit more than one category. In comparison, the DEA models are practically feasible. In the DEA models, there is no theoretical benchmark like the market portfolio of the CAPM. Instead, the benchmarking fund consists of certain funds in the group, each with a particular weight; rather than being compared with an idealised fund that requires information about all the equities in the market, DEA models benchmark the funds under evaluation against themselves. This makes DEA models practically feasible and easier to test.

The contribution of the first chapter of this thesis is that it applies the procedures in Morey and Morey (1999) to a new modern data set comprising a multi-year sample of investment funds and identifies six efficient funds among 29 funds. The relative ranking of all 29 funds are obtained, and the marginal contributions of the mean return and variance in each period to the fund efficiency are examined.

Morey and Morey (1999) quadratic DEA models are particularly chosen because of the unique characteristics of investment trusts. They utilise the insights from Markowitz portfolio theory that there is correlation between different assets which should not be ignored, and these co-movements between different securities affect the relationship between expected return and risk of the combined portfolio. On one hand, the quadratic DEA models Morey and Morey (1999) developed do not completely abandon the traditional portfolio theory but instead relate the non-parametric methodologies to the foundations of traditional performance measurement in mean-variance space. On the other hand the model is derived from the standard data envelopment analysis but differs from it in having non-linear constraints in the envelopment version of the model's structure.

The contribution of the second chapter is that it tested the statistical significance of DEA scores obtained from Morey and Morey (1999) by utilising the Simar-Wilson (2008) bootstrapping algorithms to develop statistical inference and confidence intervals for the indexes of efficient investment fund performance. Algorithms of smoothed bootstrap for this quadratic DEA model are designed. Biases are corrected, and confidence intervals are obtained. And the results indicate that even with slight bias, the DEA scores obtained from Morey and Morey (1999) mean augmentation approach are very reliable.

The contribution of the the third chapter in this thesis is that it applies second stage DEA models to analyse the factors contributing to investment trusts performance and detect the determinants of inefficiency. Robust-OLS regression, Tobit models and Papke-Wooldridge (PW) models are conducted and compared to evaluate contextual variables affecting the performance of investment funds. In the first stage, efficiency scores are calculated using Morey and Morey (1999) quadratic DEA model, and in the second stage, these scores are regressed on potential explanatory variables including Sharpe ratio, Jensen's alpha, expense ratio, P/E ratio, book to market ratio and market value of the investment funds to test the statistical significance of those factors. Only the Sharpe ratio is found to have a significant positive impact on the efficiency score. This may be however, because of the limitations of the dependent variable, which is obtained from a quadratic DEA model only has risk and return in the consideration. Then a recursive model is applied when DEA scores, Sharpe ratio and Jensen's alpha are used as dependent variables respectively while net expense ratio, PE ratio, market value and book to market ratio are explaining factors in all three regressions. The results show that Sharpe ratio is a good efficiency measure considering net expense ratio, PE ratio, market value and book to market ratio as explaining factors. And the DEA score is worse than Sharpe ratio but better than Jensen's Alpha as an efficiency indicator.

6.2 Contributions to knowledge

This thesis has five main contributions. Firstly of all, it compares in detail traditional portfolio analysis and DEA models, especially Morey and Morey (1999) quadratic DEA model, in the investment fund evaluation.

Secondly, it applies the procedures in Morey and Morey (1999) to a new modern data set comprising a multi-year sample of investment funds and identifies six efficient funds among 29 funds.

Third, it extends the Morey and Morey (1999) quadratic model by utilising the Simar-Wilson (2008) bootstrapping algorithms to develop statistical inference and confidence intervals for the indexes of efficient investment fund performance. Algorithms of smoothed bootstrap for this quadratic DEA model are designed.

Fourth, second stage DEA models are applied to analyse the factors contributing to investment trusts performance and detects the determinants of inefficiency. Only one paper has been found in the literature about practices of second stage DEA on investment trusts so far. Therefore, it is very meaningful to examine different potential factors affecting the performance of investment trusts.

Fifth, for the benefit of other researchers, the new Matlab codes designed by the author of the thesis for Morey and Morey (1999) models are presented. With the Matlab codes, not only the results are obtained, but also how this quadratic model is programming could be very clearly seen, with all the details revealed.

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Appendix A: Covariance matrix in Chapter 3

3-year covariance

Columns 1 through 16

43.4799	33.7459	33.7972	47.0919	31.0737	34.7373	34.6697	35.9146	35.9078	38.2330	34.1033	35.5538	31.4586	30.0476	33.6165	47.5488
33.7459	29.3560	29.3902	37.4539	26.0108	29.0294	28.9621	29.9995	29.9897	31.8721	27.5858	29.1372	26.1404	24.1238	26.8391	37.9503
33.7972	29.3902	29.4252	37.5104	26.0413	29.0683	29.0009	30.0366	30.0268	31.9042	27.6316	29.1763	26.1780	24.1632	26.8833	38.0137
47.0919	37.4539	37.5104	53.5373	34.6125	38.2399	38.1443	39.2622	39.2517	42.0104	37.3252	39.2720	34.5390	32.8421	36.8050	51.9480
31.0737	26.0108	26.0413	34.6125	25.4702	26.4914	26.4224	27.3226	27.3153	29.3508	25.1598	26.8605	23.6398	22.6085	25.2592	34.8759
34.7373	29.0294	29.0683	38.2399	26.4914	30.1008	30.0317	30.9300	30.9191	32.1904	28.5133	29.8853	26.6538	24.6984	27.6049	39.3626
34.6697	28.9621	29.0009	38.1443	26.4224	30.0317	29.9638	30.8605	30.8497	32.1328	28.4409	29.8116	26.5883	24.6409	27.5405	39.2723
35.9146	29.9995	30.0366	39.2622	27.3226	30.9300	30.8605	32.1390	32.1278	33.2567	29.3045	30.9262	27.6929	25.6003	28.5895	40.6889
35.9078	29.9897	30.0268	39.2517	27.3153	30.9191	30.8497	32.1278	32.1167	33.2448	29.3013	30.9170	27.6832	25.5947	28.5837	40.6803
38.2330	31.8721	31.9042	42.0104	29.3508	32.1904	32.1328	33.2567	33.2448	36.6318	30.2582	32.1287	28.5644	27.1290	30.1756	41.9106
34.1033	27.5858	27.6316	37.3252	25.1598	28.5133	28.4409	29.3045	29.3013	30.2582	28.2562	29.0376	25.8427	24.0897	27.0702	38.7869
35.5538	29.1372	29.1763	39.2720	26.8605	29.8853	29.8116	30.9262	30.9170	32.1287	29.0376	30.8515	27.3467	25.5539	28.5264	40.4697
31.4586	26.1404	26.1780	34.5390	23.6398	26.6538	26.5883	27.6929	27.6832	28.5644	25.8427	27.3467	24.8915	22.6551	25.3525	36.1754
30.0476	24.1238	24.1632	32.8421	22.6085	24.6984	24.6409	25.6003	25.5947	27.1290	24.0897	25.5539	22.6551	23.4065	25.7194	33.8119
33.6165	26.8391	26.8833	36.8050	25.2592	27.6049	27.5405	28.5895	28.5837	30.1756	27.0702	28.5264	25.3525	25.7194	28.5594	37.9705
47.5488	37.9503	38.0137	51.9480	34.8759	39.3626	39.2723	40.6889	40.6803	41.9106	38.7869	40.4697	36.1754	33.8119	37.9705	55.6349
44.6263	35.3130	35.3625	49.1692	32.6533	36.1755	36.0986	37.3361	37.3267	39.6733	35.3604	37.0258	32.9297	30.7982	34.7294	49.2747
44.0397	35.7253	35.7681	49.3042	33.3450	36.5037	36.4198	37.6316	37.6244	40.0408	35.5939	37.3192	33.6497	31.0380	35.2535	49.7274

44.3834	36.0196	36.0683	49.7073	33.7082	36.9043	36.8212	37.9570	37.9498	40.4882	35.9948	37.6815	33.9311	31.4737	35.7547	50.4044
33.2435	29.2350	29.2603	38.3103	25.8612	29.3901	29.3291	30.2594	30.2431	31.6129	28.1104	29.0262	27.1057	23.1348	26.7155	39.3068
41.8842	35.0440	35.0786	45.8070	31.8882	35.7281	35.6567	36.8699	36.8637	39.4329	34.2135	35.4242	31.8721	29.5890	33.2154	47.8002
41.8862	35.0475	35.0822	45.8072	31.8896	35.7281	35.6568	36.8716	36.8654	39.4378	34.2127	35.4220	31.8739	29.5905	33.2188	47.7987
49.3605	39.3941	39.4507	55.0316	36.3264	40.0833	39.9812	41.2229	41.2113	43.5973	39.3410	41.0410	36.2099	34.3355	38.6100	54.4788
56.4330	45.9055	45.9560	61.8529	43.2560	46.9038	46.7981	48.6942	48.6851	51.7910	45.3346	47.7284	42.8083	40.2938	45.1478	63.3570
56.4002	45.8882	45.9386	61.8131	43.2353	46.8800	46.7745	48.6714	48.6625	51.7605	45.3183	47.7068	42.7919	40.2704	45.1212	63.3323
43.0116	35.4712	35.5085	46.9729	32.5095	36.1703	36.0900	37.6348	37.6235	39.7982	34.5670	36.4151	32.9677	30.0979	33.6678	47.6097
42.9707	35.4446	35.4820	46.9187	32.4764	36.1434	36.0630	37.6057	37.5944	39.7575	34.5494	36.3868	32.9449	30.0728	33.6402	47.5785
60.4018	49.6081	49.6798	68.9900	46.7514	50.1040	49.9873	51.5409	51.5229	55.4410	48.2733	51.1226	45.9120	42.6306	48.5075	69.1249
60.4242	49.6167	49.6883	69.0405	46.7538	50.1240	50.0068	51.5539	51.5358	55.4445	48.2950	51.1485	45.9145	42.6488	48.5222	69.1421

Columns 17 through 29

44.6263	44.0397	44.3834	33.2435	41.8842	41.8862	49.3605	56.4330	56.4002	43.0116	42.9707	60.4018	60.4242
35.3130	35.7253	36.0196	29.2350	35.0440	35.0475	39.3941	45.9055	45.8882	35.4712	35.4446	49.6081	49.6167
35.3625	35.7681	36.0683	29.2603	35.0786	35.0822	39.4507	45.9560	45.9386	35.5085	35.4820	49.6798	49.6883
49.1692	49.3042	49.7073	38.3103	45.8070	45.8072	55.0316	61.8529	61.8131	46.9729	46.9187	68.9900	69.0405
32.6533	33.3450	33.7082	25.8612	31.8882	31.8896	36.3264	43.2560	43.2353	32.5095	32.4764	46.7514	46.7538
36.1755	36.5037	36.9043	29.3901	35.7281	35.7281	40.0833	46.9038	46.8800	36.1703	36.1434	50.1040	50.1240
36.0986	36.4198	36.8212	29.3291	35.6567	35.6568	39.9812	46.7981	46.7745	36.0900	36.0630	49.9873	50.0068
37.3361	37.6316	37.9570	30.2594	36.8699	36.8716	41.2229	48.6942	48.6714	37.6348	37.6057	51.5409	51.5539
37.3267	37.6244	37.9498	30.2431	36.8637	36.8654	41.2113	48.6851	48.6625	37.6235	37.5944	51.5229	51.5358
39.6733	40.0408	40.4882	31.6129	39.4329	39.4378	43.5973	51.7910	51.7605	39.7982	39.7575	55.4410	55.4445
35.3604	35.5939	35.9948	28.1104	34.2135	34.2127	39.3410	45.3346	45.3183	34.5670	34.5494	48.2733	48.2950
37.0258	37.3192	37.6815	29.0262	35.4242	35.4220	41.0410	47.7284	47.7068	36.4151	36.3868	51.1226	51.1485

32.9297	33.6497	33.9311	27.1057	31.8721	31.8739	36.2099	42.8083	42.7919	32.9677	32.9449	45.9120	45.9145
30.7982	31.0380	31.4737	23.1348	29.5890	29.5905	34.3355	40.2938	40.2704	30.0979	30.0728	42.6306	42.6488
34.7294	35.2535	35.7547	26.7155	33.2154	33.2188	38.6100	45.1478	45.1212	33.6678	33.6402	48.5075	48.5222
49.2747	49.7274	50.4044	39.3068	47.8002	47.7987	54.4788	63.3570	63.3323	47.6097	47.5785	69.1249	69.1421
46.7077	46.6602	46.9879	36.1467	43.6201	43.6261	51.5953	59.1209	59.0898	44.9201	44.8751	64.3494	64.3617
46.6602	50.7876	51.2132	38.5284	44.9278	44.9401	51.2092	61.2257	61.1799	45.9585	45.9241	69.1509	69.1453
46.9879	51.2132	51.8278	38.6181	45.4357	45.4485	51.5363	61.5342	61.4880	46.1404	46.1061	69.7926	69.7865
36.1467	38.5284	38.6181	41.2744	36.6989	36.7116	40.1834	47.0213	46.9964	35.9281	35.9055	56.2517	56.2380
43.6201	44.9278	45.4357	36.6989	46.4054	46.4108	48.1801	57.1465	57.1238	43.6202	43.5887	63.3895	63.3837
43.6261	44.9401	45.4485	36.7116	46.4108	46.4165	48.1870	57.1512	57.1285	43.6246	43.5931	63.4033	63.3972
51.5953	51.2092	51.5363	40.1834	48.1801	48.1870	58.5356	64.4491	64.4114	49.0247	48.9739	72.1247	72.1588
59.1209	61.2257	61.5342	47.0213	57.1465	57.1512	64.4491	79.6227	79.5903	59.3750	59.3326	83.6659	83.6508
59.0898	61.1799	61.4880	46.9964	57.1238	57.1285	64.4114	79.5903	79.5592	59.3546	59.3123	83.6020	83.5866
44.9201	45.9585	46.1404	35.9281	43.6202	43.6246	49.0247	59.3750	59.3546	46.0424	46.0024	61.8121	61.8114
44.8751	45.9241	46.1061	35.9055	43.5887	43.5931	48.9739	59.3326	59.3123	46.0024	45.9629	61.7519	61.7510
64.3494	69.1509	69.7926	56.2517	63.3895	63.4033	72.1247	83.6659	83.6020	61.8121	61.7519	103.6475	103.6304
64.3617	69.1453	69.7865	56.2380	63.3837	63.3972	72.1588	83.6508	83.5866	61.8114	61.7510	103.6304	103.6156

5-year covariance

Columns 1 through 16

34.9157	25.7203	25.7541	34.7696	24.1673	26.4039	26.3469	27.5446	27.5305	28.4665	26.8381	28.3109	24.8660	23.8727	26.3676	35.8785

25.7203	22.9126	22.9339	27.6696	20.2759	22.1651	22.1055	23.1258	23.1175	24.4038	21.3329	22.6144	20.3527	18.7310	20.6381	28.2464
25.7541	22.9339	22.9558	27.7045	20.2943	22.1884	22.1287	23.1481	23.1398	24.4213	21.3607	22.6414	20.3779	18.7556	20.6658	28.2870
34.7696	27.6696	27.7045	38.0468	25.4389	27.7395	27.6645	28.7355	28.7232	30.3664	27.5782	29.3369	25.7061	24.1640	26.8021	37.0802
24.1673	20.2759	20.2943	25.4389	20.2340	20.4807	20.4215	21.3075	21.3013	22.5037	19.7917	21.0879	18.7101	18.3027	20.2418	26.5687
26.4039	22.1651	22.1884	27.7395	20.4807	22.9963	22.9359	23.8078	23.8007	24.2624	21.7929	22.8958	20.5283	19.3669	21.3635	29.0315
26.3469	22.1055	22.1287	27.6645	20.4215	22.9359	22.8765	23.7467	23.7396	24.2091	21.7327	22.8348	20.4729	19.3192	21.3104	28.9572
27.5446	23.1258	23.1481	28.7355	21.3075	23.8078	23.7467	24.9409	24.9338	25.2103	22.5705	23.9491	21.5376	20.2426	22.3002	30.2146
27.5305	23.1175	23.1398	28.7232	21.3013	23.8007	23.7396	24.9338	24.9273	25.2026	22.5627	23.9394	21.5272	20.2384	22.2949	30.2049
28.4665	24.4038	24.4213	30.3664	22.5037	24.2624	24.2091	25.2103	25.2026	27.8395	23.0717	24.2500	21.5823	20.7245	22.8524	30.8475
26.8381	21.3329	21.3607	27.5782	19.7917	21.7929	21.7327	22.5705	22.5627	23.0717	22.1825	22.8706	20.3016	19.1375	21.2467	29.2348
28.3109	22.6144	22.6414	29.3369	21.0879	22.8958	22.8348	23.9491	23.9394	24.2500	22.8706	24.6593	21.7831	20.3350	22.4336	30.6777
24.8660	20.3527	20.3779	25.7061	18.7101	20.5283	20.4729	21.5376	21.5272	21.5823	20.3016	21.7831	19.8320	18.1157	20.0142	27.3489
23.8727	18.7310	18.7556	24.1640	18.3027	19.3669	19.3192	20.2426	20.2384	20.7245	19.1375	20.3350	18.1157	19.1263	20.8307	25.7911
26.3676	20.6381	20.6658	26.8021	20.2418	21.3635	21.3104	22.3002	22.2949	22.8524	21.2467	22.4336	20.0142	20.8307	22.9026	28.7091
35.8785	28.2464	28.2870	37.0802	26.5687	29.0315	28.9572	30.2146	30.2049	30.8475	29.2348	30.6777	27.3489	25.7911	28.7091	40.8304
34.1131	26.5978	26.6307	35.7248	24.6645	26.8544	26.7911	27.9558	27.9424	29.2166	26.8864	28.4973	25.1476	23.5188	26.1863	36.1647
32.3953	27.2077	27.2296	35.1737	25.4236	27.2725	27.2008	28.2810	28.2745	30.2837	26.6742	27.7083	25.0216	23.5236	26.3950	36.0634
32.5714	27.3846	27.4100	35.4104	25.6249	27.4976	27.4262	28.4674	28.4611	30.5398	26.8956	27.9152	25.1779	23.7594	26.6676	36.4459
26.0501	22.5372	22.5473	27.7398	20.9922	22.5289	22.4720	23.3051	23.2945	24.8541	22.2076	22.3530	20.7924	19.1950	21.7773	29.8010
29.9927	26.1207	26.1382	32.0877	24.0955	26.2117	26.1504	27.2458	27.2477	29.0664	24.9987	25.9453	23.5261	22.3737	24.8640	33.9959
29.9886	26.1139	26.1315	32.0783	24.0873	26.2055	26.1444	27.2398	27.2416	29.0621	24.9928	25.9358	23.5198	22.3685	24.8594	33.9869
37.6140	29.4147	29.4522	39.7665	27.0684	29.5344	29.4562	30.6910	30.6764	31.8850	29.6363	31.4668	27.5525	25.8918	28.7671	39.6927
40.9496	33.3395	33.3702	43.0428	31.5631	33.6649	33.5812	35.0848	35.0753	37.3524	33.1085	34.7146	31.1159	29.5844	32.8685	45.0776
40.9335	33.3419	33.3726	43.0262	31.5601	33.6636	33.5799	35.0853	35.0759	37.3469	33.1062	34.7094	31.1145	29.5806	32.8621	45.0699
32.1393	26.9686	26.9918	33.7547	24.9055	26.9659	26.8983	28.2600	28.2512	29.7093	26.1356	27.6259	25.0414	23.2325	25.7331	34.9601
32.1051	26.9520	26.9753	33.7099	24.8884	26.9504	26.8828	28.2421	28.2335	29.6835	26.1194	27.6025	25.0292	23.2157	25.7153	34.9403

44.7849	36.7480	36.7917	48.7672	34.7268	36.8652	36.7718	38.1793	38.1668	40.7457	35.8750	38.0448	34.1189	32.0643	35.9865	49.6233
44.7890	36.7482	36.7919	48.7915	34.7297	36.8745	36.7808	38.1847	38.1722	40.7477	35.8803	38.0532	34.1152	32.0736	35.9939	49.6291

Columns 17 through 29

34.1131	32.3953	32.5714	26.0501	29.9927	29.9886	37.6140	40.9496	40.9335	32.1393	32.1051	44.7849	44.7890
26.5978	27.2077	27.3846	22.5372	26.1207	26.1139	29.4147	33.3395	33.3419	26.9686	26.9520	36.7480	36.7482
26.6307	27.2296	27.4100	22.5473	26.1382	26.1315	29.4522	33.3702	33.3726	26.9918	26.9753	36.7917	36.7919
35.7248	35.1737	35.4104	27.7398	32.0877	32.0783	39.7665	43.0428	43.0262	33.7547	33.7099	48.7672	48.7915
24.6645	25.4236	25.6249	20.9922	24.0955	24.0873	27.0684	31.5631	31.5601	24.9055	24.8884	34.7268	34.7297
26.8544	27.2725	27.4976	22.5289	26.2117	26.2055	29.5344	33.6649	33.6636	26.9659	26.9504	36.8652	36.8745
26.7911	27.2008	27.4262	22.4720	26.1504	26.1444	29.4562	33.5812	33.5799	26.8983	26.8828	36.7718	36.7808
27.9558	28.2810	28.4674	23.3051	27.2458	27.2398	30.6910	35.0848	35.0853	28.2600	28.2421	38.1793	38.1847
27.9424	28.2745	28.4611	23.2945	27.2477	27.2416	30.6764	35.0753	35.0759	28.2512	28.2335	38.1668	38.1722
29.2166	30.2837	30.5398	24.8541	29.0664	29.0621	31.8850	37.3524	37.3469	29.7093	29.6835	40.7457	40.7477
26.8864	26.6742	26.8956	22.2076	24.9987	24.9928	29.6363	33.1085	33.1062	26.1356	26.1194	35.8750	35.8803
28.4973	27.7083	27.9152	22.3530	25.9453	25.9358	31.4668	34.7146	34.7094	27.6259	27.6025	38.0448	38.0532
25.1476	25.0216	25.1779	20.7924	23.5261	23.5198	27.5525	31.1159	31.1145	25.0414	25.0292	34.1189	34.1152
23.5188	23.5236	23.7594	19.1950	22.3737	22.3685	25.8918	29.5844	29.5806	23.2325	23.2157	32.0643	32.0736
26.1863	26.3950	26.6676	21.7773	24.8640	24.8594	28.7671	32.8685	32.8621	25.7331	25.7153	35.9865	35.9939
36.1647	36.0634	36.4459	29.8010	33.9959	33.9869	39.6927	45.0776	45.0699	34.9601	34.9403	49.6233	49.6291
34.7964	33.8996	34.0850	27.0009	30.9851	30.9806	38.2784	42.0289	42.0175	32.9492	32.9109	46.5067	46.5061
33.8996	37.6913	37.9399	29.6546	33.0059	33.0043	36.7859	43.4589	43.4451	33.9141	33.8939	49.6020	49.5980
34.0850	37.9399	38.3051	29.6735	33.3124	33.3111	36.9721	43.6242	43.6103	34.0128	33.9926	49.9503	49.9460
27.0009	29.6546	29.6735	33.7285	27.8497	27.8564	29.5668	35.4745	35.4688	27.7860	27.7812	41.5838	41.5761
30.9851	33.0059	33.3124	27.8497	34.0184	34.0167	34.0390	40.0466	40.0485	32.0514	32.0479	45.1647	45.1650

30.9806	33.0043	33.3111	27.8564	34.0167	34.0153	34.0377	40.0409	40.0428	32.0429	32.0396	45.1641	45.1641
38.2784	36.7859	36.9721	29.5668	34.0390	34.0377	43.2719	45.5748	45.5594	35.7614	35.7201	51.7981	51.8094
42.0289	43.4589	43.6242	35.4745	40.0466	40.0409	45.5748	55.9455	55.9385	42.6820	42.6631	58.7177	58.7049
42.0175	43.4451	43.6103	35.4688	40.0485	40.0428	45.5594	55.9385	55.9326	42.6826	42.6637	58.6924	58.6794
32.9492	33.9141	34.0128	27.7860	32.0514	32.0429	35.7614	42.6820	42.6826	34.4351	34.4187	45.0474	45.0451
32.9109	33.8939	33.9926	27.7812	32.0479	32.0396	35.7201	42.6631	42.6637	34.4187	34.4034	45.0127	45.0105
46.5067	49.6020	49.9503	41.5838	45.1647	45.1641	51.7981	58.7177	58.6924	45.0474	45.0127	72.8428	72.8319
46.5061	49.5980	49.9460	41.5761	45.1650	45.1641	51.8094	58.7049	58.6794	45.0451	45.0105	72.8319	72.8225

10-year covariance

Columns 1 through 16

31.2309	24.8478	24.8617	30.7850	24.0031	24.4882	24.3843	25.3883	25.3618	25.1428	23.3790	24.8199	20.2254	23.8075	24.6090	27.1749
24.8478	23.2353	23.2457	26.1301	21.3706	21.9906	21.9042	22.8142	22.7832	22.3608	19.9640	21.0184	17.2003	20.5254	21.1212	22.6525
24.8617	23.2457	23.2569	26.1479	21.3771	22.0013	21.9158	22.8248	22.7938	22.3663	19.9772	21.0308	17.2128	20.5350	21.1319	22.6727
30.7850	26.1301	26.1479	33.1930	24.9410	25.5111	25.3643	26.2259	26.1997	26.1403	23.8175	25.2868	20.6819	24.0430	24.8585	27.7742
24.0031	21.3706	21.3771	24.9410	22.9294	20.6227	20.5367	21.3769	21.3595	21.1526	18.7396	20.0021	16.5368	20.3949	21.0116	21.3178
24.4882	21.9906	22.0013	25.5111	20.6227	22.2428	22.1054	22.8457	22.8139	21.8098	19.7462	20.5877	16.7819	20.1222	20.8060	22.6744
24.3843	21.9042	21.9158	25.3643	20.5367	22.1054	22.0352	22.7275	22.6943	21.7137	19.7107	20.4938	16.7223	20.0851	20.7746	22.6107
25.3883	22.8142	22.8248	26.2259	21.3769	22.8457	22.7275	23.7902	23.7594	22.4862	20.3872	21.4404	17.5301	21.0107	21.7366	23.4904
25.3618	22.7832	22.7938	26.1997	21.3595	22.8139	22.6943	23.7594	23.7303	22.4638	20.3629	21.4213	17.5210	20.9761	21.7004	23.4706
25.1428	22.3608	22.3663	26.1403	21.1526	21.8098	21.7137	22.4862	22.4638	23.7489	19.9589	20.7800	17.1755	20.0395	20.7332	23.4432
23.3790	19.9640	19.9772	23.8175	18.7396	19.7462	19.7107	20.3872	20.3629	19.9589	19.0803	19.5372	16.0565	18.7290	19.4943	21.8360
24.8199	21.0184	21.0308	25.2868	20.0021	20.5877	20.4938	21.4404	21.4213	20.7800	19.5372	21.1475	17.2760	19.5903	20.3179	22.9191

20.2254	17.2003	17.2128	20.6819	16.5368	16.7819	16.7223	17.5301	17.5210	17.1755	16.0565	17.2760	14.9057	16.0206	16.7023	19.1710
23.8075	20.5254	20.5350	24.0430	20.3949	20.1222	20.0851	21.0107	20.9761	20.0395	18.7290	19.5903	16.0206	22.0817	22.6642	20.9874
24.6090	21.1212	21.1319	24.8585	21.0116	20.8060	20.7746	21.7366	21.7004	20.7332	19.4943	20.3179	16.7023	22.6642	23.5739	22.1688
27.1749	22.6525	22.6727	27.7742	21.3178	22.6744	22.6107	23.4904	23.4706	23.4432	21.8360	22.9191	19.1710	20.9874	22.1688	27.6401
30.1961	25.3124	25.3292	31.1379	24.2313	24.6671	24.5782	25.6074	25.5842	25.2982	23.3465	24.7373	20.2782	23.7602	24.7396	27.1365
27.7446	24.3787	24.3916	29.2914	23.8848	23.7122	23.6254	24.5438	24.5260	24.7176	21.9664	23.1417	19.4694	22.0982	23.1653	26.0448
27.8244	24.4614	24.4759	29.4017	23.9835	23.8218	23.7351	24.6332	24.6155	24.8370	22.0709	23.2386	19.5441	22.2107	23.2982	26.2300
21.5467	19.1000	19.1094	22.3399	18.6286	18.5108	18.5119	19.2533	19.2381	19.3999	17.7932	18.1328	15.8690	17.5475	18.5335	20.9003
25.8318	23.2603	23.2710	26.8093	22.2217	22.2982	22.2004	23.1977	23.1796	23.5008	20.7215	21.5941	18.3132	21.6542	22.6181	24.3176
25.8559	23.2994	23.3101	26.8331	22.2593	22.3258	22.2290	23.2278	23.2096	23.5255	20.7385	21.6071	18.3277	21.6909	22.6554	24.3306
30.3773	25.0798	25.1009	31.6726	23.7475	24.5000	24.3976	25.4967	25.4890	25.3830	23.5864	25.1732	20.8281	23.3852	24.5245	27.9769
30.5931	26.1100	26.1236	31.7336	24.7081	25.9342	25.8163	26.7742	26.7461	27.5664	24.3822	25.4613	21.3676	23.6183	24.9755	29.9130
30.8211	26.3571	26.3686	32.1117	24.8708	26.1073	25.9886	27.0069	26.9804	27.7454	24.6163	25.7394	21.5820	23.8727	25.2260	30.1693
26.9012	23.6476	23.6602	27.9810	22.0548	23.1689	23.0857	24.0741	24.0482	24.0197	21.5216	22.6417	18.9482	21.3232	22.2582	25.3181
26.8795	23.6347	23.6474	27.9513	22.0465	23.1566	23.0727	24.0609	24.0350	24.0021	21.5085	22.6258	18.9434	21.3171	22.2523	25.3015
33.8625	29.1226	29.1471	36.2767	27.7867	28.2846	28.1546	29.3795	29.3726	29.7234	26.7579	28.3644	24.1722	26.8795	28.4196	32.9216
33.6136	28.8814	28.9054	36.0203	27.6327	28.0878	27.9475	29.1835	29.1779	29.5323	26.5668	28.1792	24.0283	26.6779	28.2459	32.7539

Columns 17 through 29

30.1961	27.7446	27.8244	21.5467	25.8318	25.8559	30.3773	30.5931	30.8211	26.9012	26.8795	33.8625	33.6136
25.3124	24.3787	24.4614	19.1000	23.2603	23.2994	25.0798	26.1100	26.3571	23.6476	23.6347	29.1226	28.8814
25.3292	24.3916	24.4759	19.1094	23.2710	23.3101	25.1009	26.1236	26.3686	23.6602	23.6474	29.1471	28.9054
31.1379	29.2914	29.4017	22.3399	26.8093	26.8331	31.6726	31.7336	32.1117	27.9810	27.9513	36.2767	36.0203
24.2313	23.8848	23.9835	18.6286	22.2217	22.2593	23.7475	24.7081	24.8708	22.0548	22.0465	27.7867	27.6327
24.6671	23.7122	23.8218	18.5108	22.2982	22.3258	24.5000	25.9342	26.1073	23.1689	23.1566	28.2846	28.0878

24.5782	23.6254	23.7351	18.5119	22.2004	22.2290	24.3976	25.8163	25.9886	23.0857	23.0727	28.1546	27.9475
25.6074	24.5438	24.6332	19.2533	23.1977	23.2278	25.4967	26.7742	27.0069	24.0741	24.0609	29.3795	29.1835
25.5842	24.5260	24.6155	19.2381	23.1796	23.2096	25.4890	26.7461	26.9804	24.0482	24.0350	29.3726	29.1779
25.2982	24.7176	24.8370	19.3999	23.5008	23.5255	25.3830	27.5664	27.7454	24.0197	24.0021	29.7234	29.5323
23.3465	21.9664	22.0709	17.7932	20.7215	20.7385	23.5864	24.3822	24.6163	21.5216	21.5085	26.7579	26.5668
24.7373	23.1417	23.2386	18.1328	21.5941	21.6071	25.1732	25.4613	25.7394	22.6417	22.6258	28.3644	28.1792
20.2782	19.4694	19.5441	15.8690	18.3132	18.3277	20.8281	21.3676	21.5820	18.9482	18.9434	24.1722	24.0283
23.7602	22.0982	22.2107	17.5475	21.6542	21.6909	23.3852	23.6183	23.8727	21.3232	21.3171	26.8795	26.6779
24.7396	23.1653	23.2982	18.5335	22.6181	22.6554	24.5245	24.9755	25.2260	22.2582	22.2523	28.4196	28.2459
27.1365	26.0448	26.2300	20.9003	24.3176	24.3306	27.9769	29.9130	30.1693	25.3181	25.3015	32.9216	32.7539
30.5817	28.3840	28.4673	21.9313	26.2782	26.3067	30.9212	30.8755	31.3269	27.3223	27.2982	34.8717	34.6488
28.3840	29.8626	29.9819	22.7222	26.5645	26.5951	28.5098	30.6787	30.7970	26.5095	26.4992	35.1617	34.9554
28.4673	29.9819	30.1611	22.7172	26.7126	26.7435	28.5963	30.7466	30.8688	26.5502	26.5400	35.3234	35.1177
21.9313	22.7222	22.7172	26.3080	21.6527	21.6763	22.4863	24.1893	24.1832	21.0257	21.0181	29.4027	29.1627
26.2782	26.5645	26.7126	21.6527	28.5161	28.5574	26.4093	28.2016	28.4432	24.9203	24.9221	33.0241	32.8022
26.3067	26.5951	26.7435	21.6763	28.5574	28.6015	26.4393	28.2251	28.4661	24.9395	24.9413	33.0675	32.8450
30.9212	28.5098	28.5963	22.4863	26.4093	26.4393	33.0490	31.4301	31.9347	27.5990	27.5750	36.6239	36.4499
30.8755	30.6787	30.7466	24.1893	28.2016	28.2251	31.4301	37.1673	36.9934	30.0983	30.0798	38.0363	37.8495
31.3269	30.7970	30.8688	24.1832	28.4432	28.4661	31.9347	36.9934	37.5331	30.5198	30.5010	38.4335	38.2668
27.3223	26.5095	26.5502	21.0257	24.9203	24.9395	27.5990	30.0983	30.5198	26.8481	26.8344	32.0864	31.8747
27.2982	26.4992	26.5400	21.0181	24.9221	24.9413	27.5750	30.0798	30.5010	26.8344	26.8215	32.0632	31.8515
34.8717	35.1617	35.3234	29.4027	33.0241	33.0675	36.6239	38.0363	38.4335	32.0864	32.0632	48.6614	48.4155
34.6488	34.9554	35.1177	29.1627	32.8022	32.8450	36.4499	37.8495	38.2668	31.8747	31.8515	48.4155	48.2020

>

Appendix B: matlab code for Chapter 3

Name the first matlab file: Main_RunThisFile.m

```
clc
clear
close all

% Lagrange function for maximizing theta and minimizing z, run this file to
display DEA scores, weight for each sample fund and lagrange multiplier;

[ERjt,CovRitRjt]=Condition();

for j0 = 1:29

    disp(['=========solve j0=',num2str(j0),' ========='])

    disp('--------- maximize theta ----------')
    [W1,theta,Lambda1,Alpha1] = Lagrange1(ERjt,CovRitRjt,j0)

    disp('------------ minimize z -----------')
    [W2,z,Lambda2,Alpha2] = Lagrange2(ERjt,CovRitRjt,j0)

end
```

Name the second matlab file: Condition.m

```
% condition function including the information of expected return and
covariance matrix
function [ERjt,CovRitRjt]=Condition()
% expected return for three year periods
ExpCov3=[…]
% expected return for five year periods
ExpCov5=[…]
% expected return for ten year periods
ExpCov10=[…]
R3=[…]
R5=[…]
R10=[…]
% covariance matrix
```

```
CovRitRjt(:,:,1) = ExpCov3;
CovRitRjt(:,:,2) = ExpCov5;
CovRitRjt(:,:,3) = ExpCov10;
ERjt(:,1) = R3;
ERjt(:,2) = R5;
ERjt(:,3) = R10;
end
```

Name the third matlab file Lagrange1.m

```
% the first lagrange function
function [W,theta,Lambda,Alpha] = Lagrange1(ERjt,CovRitRjt,j0)
X0 = 0.5*ones(30,1);
X0(30) = 1;
lb = zeros(30,1);
% lower bound restriction
lb(30) = 1;
ub = ones(30,1);
% upper bound restriction
ub(30) = Inf;
% optimset creates an options structure that you can pass as an input
argument to the following optimization functions
options = optimset('Display','off','Algorithm','interior-
point','LargeScale','on','MaxIter',5000,'MaxFunEvals',1e+5,'TolCon',1e-
16,'TolFun',1e-16,'TolX',1e-16);
%fmincon finds minimum of constrained nonlinear multivariable function
[X,fval,exitflag,output,lambda] = fmincon(@MyFun1,X0,[],[],[],[],lb,ub,@(X)
MyCon1(X,ERjt,CovRitRjt,j0),options);
W = X(1:29);
theta = X(30);
Lambda = [lambda.eqnonlin;lambda.ineqnonlin(1:3)];
Alpha = lambda.ineqnonlin(4:6);
end
```

Name the fourth matlab file Lagrange2.m

```
% the second lagrange function
function [W,z,Lambda,Alpha] = Lagrange2(ERjt,CovRitRjt,j0)
X0 = 0.5*ones(30,1);
X0(30) = 1;
```

```
lb = zeros(30,1);
% lower bound restriction
ub = ones(30,1);
% upper bound restriction
ub(30) = 1;
% optimset creates an options structure that you can pass as an input
argument to the following optimization functions
options = optimset('Display','off','Algorithm','interior-
point','LargeScale','on','MaxIter',5000,'MaxFunEvals',1e+5,'TolCon',1e-
16,'TolFun',1e-16,'TolX',1e-16);
 %fmincon finds minimum of constrained nonlinear multivariable function
[X,fval,exitflag,output,lambda] = fmincon(@MyFun2,X0,[],[],[],[],lb,ub,@(X)
MyCon2(X,ERjt,CovRitRjt,j0),options);
W = X(1:29);
z = X(30);
Lambda = [lambda.eqnonlin;lambda.ineqnonlin(1:3)];
Alpha = lambda.ineqnonlin(4:6);
end
```

Name the fifth matlab file MyCon1.m

```
% the first condition function
function [c,ceq] = MyCon1(X,ERjt,CovRitRjt,j0)
W = X(1:29);
theta = X(30);
% covariance matrix
CovRitRjt1 = CovRitRjt(:,:,1);
CovRitRjt2 = CovRitRjt(:,:,2);
CovRitRjt3 = CovRitRjt(:,:,3);
% variance of each sample fund
Sigma2jt1=diag(CovRitRjt1);
Sigma2jt2=diag(CovRitRjt2);
Sigma2jt3=diag(CovRitRjt3);
CovRitRjt1 = CovRitRjt1-diag(Sigma2jt1);
CovRitRjt2 = CovRitRjt2-diag(Sigma2jt2);
CovRitRjt3 = CovRitRjt3-diag(Sigma2jt3);
tmp = theta*ERjt(j0,:)-W'*ERjt;
c = [tmp'
     (W.^2)'*Sigma2jt1+W'*CovRitRjt1*W-Sigma2jt1(j0)
     (W.^2)'*Sigma2jt2+W'*CovRitRjt2*W-Sigma2jt2(j0)
```

```
      (W.^2)'*Sigma2jt3+W'*CovRitRjt3*W-Sigma2jt3(j0)];
ceq = sum(W)-1;
end
```

Name the sixth matlab file MyCon2.m

```
% the second condition function
function [c,ceq] = MyCon2(X,ERjt,CovRitRjt,j0)
W = X(1:29);
% weights for each sample fund
z = X(30);
% covariance matrix
CovRitRjt1 = CovRitRjt(:,:,1);
CovRitRjt2 = CovRitRjt(:,:,2);
CovRitRjt3 = CovRitRjt(:,:,3);
Sigma2jt1=diag(CovRitRjt1);
Sigma2jt2=diag(CovRitRjt2);
Sigma2jt3=diag(CovRitRjt3);
CovRitRjt1 = CovRitRjt1-diag(Sigma2jt1);
CovRitRjt2 = CovRitRjt2-diag(Sigma2jt2);
CovRitRjt3 = CovRitRjt3-diag(Sigma2jt3);
tmp = ERjt(j0,:)-W'*ERjt;
c = [tmp'
      (W.^2)'*Sigma2jt1+W'*CovRitRjt1*W-z*Sigma2jt1(j0)
      (W.^2)'*Sigma2jt2+W'*CovRitRjt2*W-z*Sigma2jt2(j0)
      (W.^2)'*Sigma2jt3+W'*CovRitRjt3*W-z*Sigma2jt3(j0)];
ceq = sum(W)-1;
end
```

Name the seventh matlab file Myfun1.m

```
% passing output DEA score theta to myfun1
function theta = MyFun1(X)
theta = -X(30);
end
```

Name the eighth matlab file Myfun2.m

```
% passing input DEA score z to myfun2
function z = MyFun2(X)
z = X(30);
end
```

Appendix C: matlab codes for Chapter 4

Step1: form kernel density function using the original DEA scores and obtain the optimal bandwidth h.

File name: kde.m

(kde.m is downloaded from Mathwork website:
http://www.mathworks.com/matlabcentral/fileexchange/14034-kernel-density-estimator)

```
function [bandwidth,density,xmesh,cdf]=kde(data,n,MIN,MAX)
data=[1.015
1.1505
1.0308
1.1359
1
1.8621
1.5914
1.7987
1.714
1
1.1116
1.9331
1
1.036
1.143
1.2532
1.0331
1.3196
1.3974
1.0031
1.3265
1.2344
1
1
1.0006
1.4387
1.5747
1
1.0593];
```

```
% Reliable and extremely fast kernel density estimator for one-dimensional
data;
%         Gaussian kernel is assumed and the bandwidth is chosen
automatically;
%         Unlike many other implementations, this one is immune to problems
%         caused by multimodal densities with widely separated modes (see
example). The
%         estimation does not deteriorate for multimodal densities, because
we never assume
%         a parametric model for the data.
% INPUTS:
%      data    - a vector of data from which the density estimate is
constructed;
%           n  - the number of mesh points used in the uniform
discretization of the
%                interval [MIN, MAX]; n has to be a power of two; if n is
not a power of two, then
%                n is rounded up to the next power of two, i.e., n is set to
n=2^ceil(log2(n));
%                the default value of n is n=2^12;
%   MIN, MAX  - defines the interval [MIN,MAX] on which the density
estimate is constructed;
%                the default values of MIN and MAX are:
%                MIN=min(data)-Range/10 and MAX=max(data)+Range/10, where
Range=max(data)-min(data);
% OUTPUTS:
%   bandwidth - the optimal bandwidth (Gaussian kernel assumed);
%     density - column vector of length 'n' with the values of the density
%                estimate at the grid points;
%     xmesh   - the grid over which the density estimate is computed;
%             - If no output is requested, then the code automatically
plots a graph of
%                the density estimate.
%         cdf  - column vector of length 'n' with the values of the cdf
%  Reference:
% Kernel density estimation via diffusion
% Z. I. Botev, J. F. Grotowski, and D. P. Kroese (2010)
% Annals of Statistics, Volume 38, Number 5, pages 2916-2957.


%
%  Example:
```

```
%           data=[randn(100,1);randn(100,1)*2+35 ;randn(100,1)+55];
%              kde(data,2^14,min(data)-5,max(data)+5);


%  Notes:   If you have a more reliable and accurate one-dimensional kernel
density
%           estimation software, please email me at botev@maths.uq.edu.au


data=data(:); %make data a column vector
if nargin<2 % if n is not supplied switch to the default
    n=2^14;
end
n=2^ceil(log2(n)); % round up n to the next power of 2;
if nargin<4 %define the default  interval [MIN,MAX]
    minimum=min(data); maximum=max(data);
    Range=maximum-minimum;
    MIN=minimum-Range/10; MAX=maximum+Range/10;

end
% set up the grid over which the density estimate is computed;
R=MAX-MIN; dx=R/(n-1); xmesh=MIN+[0:dx:R]; N=length(unique(data));
%bin the data uniformly using the grid defined above;
initial_data=histc(data,xmesh)/N;
initial_data=initial_data/sum(initial_data);
a=dct1d(initial_data); % discrete cosine transform of initial data
% now compute the optimal bandwidth^2 using the referenced method
I=[1:n-1]'.^2; a2=(a(2:end)/2).^2;
% use  fzero to solve the equation t=zeta*gamma^[5](t)
try
    t_star=fzero(@(t)fixed_point(t,N,I,a2),[0,.1]);
catch
    t_star=.28*N^(-2/5);
end
% smooth the discrete cosine transform of initial data using t_star
a_t=a.*exp(-[0:n-1]'.^2*pi^2*t_star/2);
% now apply the inverse discrete cosine transform
if (nargout>1)|(nargout==0)
    density=idct1d(a_t)/R;
end
```

```
% take the rescaling of the data into account
bandwidth=sqrt(t_star)*R;
if nargout==0
    figure(1), plot(xmesh,density)
end
% for cdf estimation
if nargout>3
    f=2*pi^2*sum(I.*a2.*exp(-I*pi^2*t_star));
    t_cdf=(sqrt(pi)*f*N)^(-2/3);
    % now get values of cdf on grid points using IDCT and cumsum function
    a_cdf=a.*exp(-[0:n-1]'.^2*pi^2*t_cdf/2);
    cdf=cumsum(idct1d(a_cdf))*(dx/R);
    % take the rescaling into account if the bandwidth value is required
    bandwidth_cdf=sqrt(t_cdf)*R;
end

end
%################################################################
function  out=fixed_point(t,N,I,a2)
% this implements the function t-zeta*gamma^[l](t)
l=7;
f=2*pi^(2*l)*sum(I.^l.*a2.*exp(-I*pi^2*t));
for s=l-1:-1:2
    K0=prod([1:2:2*s-1])/sqrt(2*pi);  const=(1+(1/2)^(s+1/2))/3;
    time=(2*const*K0/N/f)^(2/(3+2*s));
    f=2*pi^(2*s)*sum(I.^s.*a2.*exp(-I*pi^2*time));
end
out=t-(2*N*sqrt(pi)*f)^(-2/5);
end



%##############################################################
function out = idct1d(data)

% computes the inverse discrete cosine transform
[nrows,ncols]=size(data);
% Compute weights
weights = nrows*exp(i*(0:nrows-1)*pi/(2*nrows)).';
% Compute x tilde using equation (5.93) in Jain
data = real(ifft(weights.*data));
```

```
% Re-order elements of each column according to equations (5.93) and
% (5.94) in Jain
out = zeros(nrows,1);
out(1:2:nrows) = data(1:nrows/2);
out(2:2:nrows) = data(nrows:-1:nrows/2+1);
%   Reference:
%      A. K. Jain, "Fundamentals of Digital Image
%      Processing", pp. 150-153.
end
%###############################################################

function data=dct1d(data)
% computes the discrete cosine transform of the column vector data
[nrows,ncols]= size(data);
% Compute weights to multiply DFT coefficients
weight = [1;2*(exp(-i*(1:nrows-1)*pi/(2*nrows))).'];
% Re-order the elements of the columns of x
data = [ data(1:2:end,:); data(end:-2:2,:) ];
% Multiply FFT by weights:
data= real(weight.* fft(data));
end
```

Step 2: creating replicates from original DEA scores and produce $k = \frac{\hat{\theta}_i}{\theta^*_{ib}} i = 1,\ldots 29, b = 1,\ldots 2000$

```
PD=fitdist(x,'kernel','support',[min(x)-Range/10000,max(x)+Range/10000],
'width',0.2186)
% Fit probability distribution object to data
New_X = random(PD, 2000, 29)
for i=1:29
for j=1:2000
k(j,i)=x(i,1)/New_X(j,i)
end
end
```

Step 3: obtain the new DEA scores from the following programming

Name the first file: Main_RunThisFile.m

```
for j= 1:29
   for i=1:2000
X0 = 0.5*ones(30,1);
X0(30) = 1;
% lower bound restrictions
lb = zeros(30,1);
lb(30) = 1;
% upper bound restrictions
ub = ones(30,1);
ub(30) = Inf;
% create or edit optimization options structure
options = optimset('Display','off','Algorithm','interior-
point','LargeScale','on','MaxIter',5000,'MaxFunEvals',1e+5,'TolCon',1e-16,'TolFun',1e-16,'TolX',1e-16);
```

$$ExpCov3 = [...]_{29\times29}$$

$$ExpCov5 = [...]_{29\times29}$$

$$ExpCov10 = [...]_{29\times29}$$

$$R3 = [...]_{29\times1}$$

$$R5 = [...]_{29\times1}$$

$$R10 = [...]_{29\times1}$$

```
% covariance matrix
CovRitRjt(:,:,1) = ExpCov3;
CovRitRjt(:,:,2) = ExpCov5;
CovRitRjt(:,:,3) = ExpCov10;
```

$$k = [...]_{2000\times29}$$

```
% creating bootstrap resamples
z3=[k(i,1)*R3(1,:)
  k(i,2)*R3(2,:)
  k(i,3)*R3(3,:)
```

```
   k(i,4)*R3(4,:)
   k(i,5)*R3(5,:)
   k(i,6)*R3(6,:)
   k(i,7)*R3(7,:)
   k(i,8)*R3(8,:)
   k(i,9)*R3(9,:)
   k(i,10)*R3(10,:)
   k(i,11)*R3(11,:)
   k(i,12)*R3(12,:)
   k(i,13)*R3(13,:)
   k(i,14)*R3(14,:)
   k(i,15)*R3(15,:)
   k(i,16)*R3(16,:)
   k(i,17)*R3(17,:)
   k(i,18)*R3(18,:)
   k(i,19)*R3(19,:)
   k(i,20)*R3(20,:)
   k(i,21)*R3(21,:)
   k(i,22)*R3(22,:)
   k(i,23)*R3(23,:)
   k(i,24)*R3(24,:)
   k(i,25)*R3(25,:)
   k(i,26)*R3(26,:)
   k(i,27)*R3(27,:)
   k(i,28)*R3(28,:)
   k(i,29)*R3(29,:)];
 z5=[k(i,1)*R5(1,:)
```

```
k(i,2)*R5(2,:)
k(i,3)*R5(3,:)
k(i,4)*R5(4,:)
k(i,5)*R5(5,:)
k(i,6)*R5(6,:)
k(i,7)*R5(7,:)
k(i,8)*R5(8,:)
k(i,9)*R5(9,:)
k(i,10)*R5(10,:)
k(i,11)*R5(11,:)
k(i,12)*R5(12,:)
k(i,13)*R5(13,:)
k(i,14)*R5(14,:)
k(i,15)*R5(15,:)
k(i,16)*R5(16,:)
k(i,17)*R5(17,:)
k(i,18)*R5(18,:)
k(i,19)*R5(19,:)
k(i,20)*R5(20,:)
k(i,21)*R5(21,:)
k(i,22)*R5(22,:)
k(i,23)*R5(23,:)
k(i,24)*R5(24,:)
k(i,25)*R5(25,:)
k(i,26)*R5(26,:)
k(i,27)*R5(27,:)
k(i,28)*R5(28,:)
```

```
    k(i,29)*R5(29,:)];
z10=[k(i,1)*R10(1,:)
    k(i,2)*R10(2,:)
    k(i,3)*R10(3,:)
    k(i,4)*R10(4,:)
    k(i,5)*R10(5,:)
    k(i,6)*R10(6,:)
    k(i,7)*R10(7,:)
    k(i,8)*R10(8,:)
    k(i,9)*R10(9,:)
    k(i,10)*R10(10,:)
    k(i,11)*R10(11,:)
    k(i,12)*R10(12,:)
    k(i,13)*R10(13,:)
    k(i,14)*R10(14,:)
    k(i,15)*R10(15,:)
    k(i,16)*R10(16,:)
    k(i,17)*R10(17,:)
    k(i,18)*R10(18,:)
    k(i,19)*R10(19,:)
    k(i,20)*R10(20,:)
    k(i,21)*R10(21,:)
    k(i,22)*R10(22,:)
    k(i,23)*R10(23,:)
    k(i,24)*R10(24,:)
    k(i,25)*R10(25,:)
    k(i,26)*R10(26,:)
```

```
  k(i,27)*R10(27,:)
  k(i,28)*R10(28,:)
  k(i,29)*R10(29,:)];
ERjt(:,1) = z3;
ERjt(:,2) = z5;
ERjt(:,3) = z10;
ERjt0(:,1) = R3;
ERjt0(:,2) = R5;
ERjt0(:,3) = R10;
[X,fval] = fmincon(@MyFun1,X0,[],[],[],[],lb,ub,@(X) MyCon1(X,ERjt,ERjt0,CovRitRjt,j,i),options)
% find minimum of constrained nonlinear multivariable function
eff(i,j)=-fval
end
end
```

Name the second file: Myfun1.m

```
function theta = MyFun1(X)
 % passing theta to myfun1
theta = -X(30);
end
```

Name the third file: Mycon1.m

```
function [c,ceq] = MyCon1(X,ERjt,ERjt0,CovRitRjt,j,i)
```

$$ExpCov3 = [\ldots]_{29\times 29}$$

$$ExpCov5 = [...]_{29\times 29}$$

$$ExpCov10 = [...]_{29\times 29}$$

$$R3 = [...]_{29\times 1}$$

$$R5 = [...]_{29\times 1}$$

$$R10 = [...]_{29\times 1}$$

```
% covariance matrix
CovRitRjt(:,:,1) = ExpCov3;
CovRitRjt(:,:,2) = ExpCov5;
CovRitRjt(:,:,3) = ExpCov10;
```

$$k = [...]_{2000\times 29}$$

```
% creating bootstrap resamples
z3=[k(i,1)*R3(1,:)
  k(i,2)*R3(2,:)
  k(i,3)*R3(3,:)
  k(i,4)*R3(4,:)
  k(i,5)*R3(5,:)
  k(i,6)*R3(6,:)
  k(i,7)*R3(7,:)
  k(i,8)*R3(8,:)
  k(i,9)*R3(9,:)
  k(i,10)*R3(10,:)
  k(i,11)*R3(11,:)
  k(i,12)*R3(12,:)
  k(i,13)*R3(13,:)
  k(i,14)*R3(14,:)
  k(i,15)*R3(15,:)
  k(i,16)*R3(16,:)
```

```
   k(i,17)*R3(17,:)
   k(i,18)*R3(18,:)
   k(i,19)*R3(19,:)
   k(i,20)*R3(20,:)
   k(i,21)*R3(21,:)
   k(i,22)*R3(22,:)
   k(i,23)*R3(23,:)
   k(i,24)*R3(24,:)
   k(i,25)*R3(25,:)
   k(i,26)*R3(26,:)
   k(i,27)*R3(27,:)
   k(i,28)*R3(28,:)
   k(i,29)*R3(29,:)];
z5=[k(i,1)*R5(1,:)
   k(i,2)*R5(2,:)
   k(i,3)*R5(3,:)
   k(i,4)*R5(4,:)
   k(i,5)*R5(5,:)
   k(i,6)*R5(6,:)
   k(i,7)*R5(7,:)
   k(i,8)*R5(8,:)
   k(i,9)*R5(9,:)
   k(i,10)*R5(10,:)
   k(i,11)*R5(11,:)
   k(i,12)*R5(12,:)
   k(i,13)*R5(13,:)
   k(i,14)*R5(14,:)
```

```
  k(i,15)*R5(15,:)
  k(i,16)*R5(16,:)
  k(i,17)*R5(17,:)
  k(i,18)*R5(18,:)
  k(i,19)*R5(19,:)
  k(i,20)*R5(20,:)
  k(i,21)*R5(21,:)
  k(i,22)*R5(22,:)
  k(i,23)*R5(23,:)
  k(i,24)*R5(24,:)
  k(i,25)*R5(25,:)
  k(i,26)*R5(26,:)
  k(i,27)*R5(27,:)
  k(i,28)*R5(28,:)
  k(i,29)*R5(29,:)];
z10=[k(i,1)*R10(1,:)
  k(i,2)*R10(2,:)
  k(i,3)*R10(3,:)
  k(i,4)*R10(4,:)
  k(i,5)*R10(5,:)
  k(i,6)*R10(6,:)
  k(i,7)*R10(7,:)
  k(i,8)*R10(8,:)
  k(i,9)*R10(9,:)
  k(i,10)*R10(10,:)
  k(i,11)*R10(11,:)
  k(i,12)*R10(12,:)
```

```
  k(i,13)*R10(13,:)
  k(i,14)*R10(14,:)
  k(i,15)*R10(15,:)
  k(i,16)*R10(16,:)
  k(i,17)*R10(17,:)
  k(i,18)*R10(18,:)
  k(i,19)*R10(19,:)
  k(i,20)*R10(20,:)
  k(i,21)*R10(21,:)
  k(i,22)*R10(22,:)
  k(i,23)*R10(23,:)
  k(i,24)*R10(24,:)
  k(i,25)*R10(25,:)
  k(i,26)*R10(26,:)
  k(i,27)*R10(27,:)
  k(i,28)*R10(28,:)
  k(i,29)*R10(29,:)];
% expected returns for each fund
ERjt(:,1) = z3;
ERjt(:,2) = z5;
ERjt(:,3) = z10;
Erjt0(:,1) = R3;
Erjt0(:,2) = R5;
Erjt0(:,3) = R10;
W = X(1:29);
% weights for each sample fund
theta = X(30);
```

```
CovRitRjt1 = CovRitRjt(:,:,1);
CovRitRjt2 = CovRitRjt(:,:,2);
CovRitRjt3 = CovRitRjt(:,:,3);
Sigma2jt1=diag(CovRitRjt1);
Sigma2jt2=diag(CovRitRjt2);
Sigma2jt3=diag(CovRitRjt3);
CovRitRjt1 = CovRitRjt1-diag(Sigma2jt1);
CovRitRjt2 = CovRitRjt2-diag(Sigma2jt2);
CovRitRjt3 = CovRitRjt3-diag(Sigma2jt3);
tmp = theta*ERjt0(j,:)-W'*ERjt;
c = [tmp'
    (W.^2)'*Sigma2jt1+W'*CovRitRjt1*W-Sigma2jt1(j)
    (W.^2)'*Sigma2jt2+W'*CovRitRjt2*W-Sigma2jt2(j)
    (W.^2)'*Sigma2jt3+W'*CovRitRjt3*W-Sigma2jt3(j)];
ceq = sum(W)-1;
end
```

Appendix D: Stata code for Chapter 5

regress DEA Sharpe Alpha NER PE MV BTM, vce(robust)

```
% regress DEA score on Shapre Alpha NER PE MV BTM
```

tobit DEA Sharpe Alpha NER PE MV BTM, ll(0) ul(1)

```
% regress DEA score on Shapre Alpha NER PE MV BTM in tobit model with lower bound equal to zero and upper bound equal to one
```

margins, dydx(Sharpe)

```
% margin parameter of Sharpe ratio
```

margins, dydx(Alpha)

```
% margin parameter of alpha
```

margins, dydx(NER)

```
% margin parameter of NER
```

margins, dydx(PE)

```
% margin parameter of PE ratio
```

margins, dydx(MV)

```
% margin parameter of MV
```

margins, dydx(BTM)

```
% margin parameter of BTM
```

tobit DEA Sharpe Alpha NER PE MV BTM, ul(1)

```
% regress DEA score on Shapre Alpha NER PE MV BTM in tobit model with upper bound equal to one
```

margins, dydx(Sharpe)

```
% margin parameter of Sharpe ratio
```

margins, dydx(Alpha)

```
% margin parameter of alpha
```

margins, dydx(NER)

```
% margin parameter of NER
```

margins, dydx(PE)

```
% margin parameter of PE ratio
```

```
margins, dydx(MV)
% margin parameter of MV
margins, dydx(BTM)
% margin parameter of BTM

tobit 1/DEA Sharpe Alpha NER PE MV BTM, ll(0)
% regress DEA score on Shapre Alpha NER PE MV BTM in tobit model with lower bound equal to zero
margins, dydx(Sharpe)
% margin parameter of Sharpe ratio
margins, dydx(Alpha)
% margin parameter of alpha
margins, dydx(NER)
% margin parameter of NER
margins, dydx(PE)
% margin parameter of PE ratio
margins, dydx(MV)
% margin parameter of MV
margins, dydx(BTM)
% margin parameter of BTM

glm DEA Sharpe Alpha NER PE MV BTM, family(binomial 1) link(logit)
% regress DEA score on Shapre Alpha NER PE MV BTM in tobit model with logit function as link function
margins, dydx(Sharpe)
% margin parameter of Sharpe ratio
margins, dydx(Alpha)
% margin parameter of alpha
margins, dydx(NER)
% margin parameter of NER
margins, dydx(PE)
% margin parameter of PE ratio
```

```
margins, dydx(MV)
% margin parameter of MV
margins, dydx(BTM)
% margin parameter of BTM

pwcorr Sharpe Alpha NER PE MV BTM
% Correlations (covariances) of Sharpe Alpha NER PE MV BTM
pwcorr RankingDEA RankingSharpe RankingAlpha, sig
% Correlations of ranking from DEA, Sharpe Alpha
regress DEA NER PE MV BTM, vce(robust)
% regress DEA score on DEA NER PE MV BTM with robust errors
predict NEWDEA, xb
% predict new DEA scores
graph twoway (line DEA NEWDEA numberfund)
% draw graph of DEA scores
regress Sharpe NER PE MV BTM, vce(robust)
% regress Sharpe ratio on NER PE MV BTM with robust errors
predict NEWSharpe, xb
% predict new sharpe ratios
graph twoway (line Sharpe NEWSharpe numberfund)
% draw line graph of new Sharpe ratios
regress  Alpha NER PE MV BTM, vce(robust)
% regress alpha on NER PE MV BTM with robust errors
predict NEWAlpha, xb
% predict new alphas
graph twoway (line Alpha NEWAlpha numberfund)
% draw line graph of new alphas
```