JACOBI–TRUDY FORMULA FOR GENERALISED SCHUR POLYNOMIALS

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ABSTRACT. Jacobi–Trudy formula for a generalisation of Schur polynomials related to any sequence of orthogonal polynomials in one variable is given. As a corollary we have Giambelli formula for generalised Schur polynomials.

To Grisha Olshanski on his 65-th birthday with admiration

1. INTRODUCTION

The classical Jacobi–Trudy formula expresses the Schur polynomials

$$S_\lambda(x_1,\dots,x_n) = \frac{\begin{vmatrix} x_1^{\lambda_1+n-1} & x_2^{\lambda_1+n-1} & \dots & x_n^{\lambda_1+n-1} \\ x_1^{\lambda_2+n-2} & x_2^{\lambda_2+n-2} & \dots & x_n^{\lambda_2+n-2} \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{\lambda_n} & x_2^{\lambda_n} & \dots & x_n^{\lambda_n} \end{vmatrix}}{\Delta_n(x)} \tag{1}$$

where $\lambda = (\lambda_1,\dots,\lambda_n)$ is a partition and $\Delta_n(x) = \prod_{i<j}^n (x_i - x_j)$, as the determinant

$$S_\lambda(x_1,\dots,x_n) = \begin{vmatrix} h_{\lambda_1} & h_{\lambda_1+1} & \dots & h_{\lambda_1+l-1} \\ h_{\lambda_2-1} & h_{\lambda_2} & \dots & h_{\lambda_2+l-2} \\ \vdots & \vdots & \ddots & \vdots \\ h_{\lambda_l-l+1} & h_{\lambda_l-l+2} & \dots & h_{\lambda_l} \end{vmatrix} \tag{2}$$

where $l = l(\lambda)$ and $h_i = h_i(x_1,\dots,x_n)$ are the complete symmetric polynomials (see [1]). Note that these polynomials h_i are particular case of Schur polynomials S_λ, corresponding to the partition $\lambda = (i)$ consisting of one part.

In this note we give a version of this formula, which is valid for the following generalisation of Schur polynomials related to any sequence of orthogonal polynomials in one variable.

More precisely, let $\{\varphi_i(z)\},\ i = 0,1,2,\dots$ be a sequence of polynomials in one variable, which satisfy a three-term recurrence relation

$$z\varphi_i(z) = \varphi_{i+1}(z) + a(i)\varphi_i(z) + b(i)\varphi_{i-1}(z) \tag{3}$$

with $\varphi_0 \equiv 1$, $\varphi_{-1} \equiv 0$ (for example, a sequence of the orthogonal polynomials [2]). The corresponding *generalised Schur polynomials* $S(x_1,\dots,x_n|a,b)$ are defined for any partition λ and two infinite sequences $a = \{a_i\}$, $b = \{b_i\}$ by the Weyl-type formula

$$S_\lambda(x_1, \dots, x_n|a, b) = \frac{\begin{vmatrix} \varphi_{\lambda_1+n-1}(x_1) & \varphi_{\lambda_1+n-1}(x_2) & \dots & \varphi_{\lambda_1+n-1}(x_n) \\ \varphi_{\lambda_2+n-2}(x_1) & \varphi_{\lambda_2+n-2}(x_2) & \dots & \varphi_{\lambda_2+n-2}(x_n) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_{\lambda_n}(x_1) & \varphi_{\lambda_n}(x_2) & \dots & \varphi_{\lambda_n}(x_n) \end{vmatrix}}{\Delta_n(u)}. \tag{4}$$

For $n = 1$ we assume that $\Delta_1 \equiv 1$, so for $\lambda = (i)$ we have $S_\lambda(x_1|a, b) = \varphi_i(x_1)$. If the initial sequence $\varphi_i(z)$ was orthogonal with measure $d\mu(z)$ the polynomials $S_\lambda(x_1, \dots, x_n|a, b)$ are orthogonal with respect to the measure

$$\Omega(z) = \Delta_n^2(z) \prod_{i=1}^{n} d\mu(z_i). \tag{5}$$

Alternatively, the generalised Schur polynomials can be defined in this case as the polynomials of the triangular form

$$S_\lambda(x_1, \dots, x_n|a, b) = \sum_{\mu \preceq \lambda} K_{\lambda,\mu}(a, b) m_\mu(x_1, \dots, x_n), \tag{6}$$

which are orthogonal with respect to the measure (5). Here $m_\mu(x_1, \dots, x_n)$ are monomial symmetric polynomials [1] and the notation $\mu \preceq \lambda$ means that $\mu_1 + \dots + \mu_k \leq \lambda_1 + \dots + \lambda_k$ for all $1 \leq k \leq n$.

When $\varphi_i(z)$ is the sequence of classical Jacobi polynomials [2], the corresponding generalised Schur polynomials coincide with the multidimensional Jacobi polynomials with parameter $\theta = 1$ (see Lassalle [4] and Okounkov-Olshanski [5]).

Denote the polynomials $S_\lambda(x_1, \dots, x_n|a, b)$ with $\lambda = (i, 0, \dots, 0)$ as $h_i(x)$ and extend this sequence for negative i by assuming that $h_i(x) \equiv 0$ for $i < 0$. Extend also the sequence of coefficients $a(i)$ and $b(i)$ to the negative i arbitrarily and define recursively the polynomials $h_i^{(r)}(x_1, \dots, x_n)$ by the relation

$$h_i^{(r+1)} = h_{i+1}^{(r)} + a(i + n - 1)h_i^{(r)} + b(i + n - 1)h_{i-1}^{(r)} \tag{7}$$

with initial data $h_i^{(0)}(x) = h_i(x)$. One can check that $h_i^{(r)}(x) \equiv 0$ whenever $i + r < 0$ and that the definition of the polynomials $h_i^{(r)}(x_1, \dots, x_n)$ does not depend on the extension of the coefficients to the negative i provided

$$r \leq i + 2n - 2. \tag{8}$$

In particular, all the entries of the formula (9) below are well defined.

Our main result is the following

Theorem 1.1. *The generalised Schur polynomials satisfy the following Jacobi-Trudy formula:*

$$S_\lambda(x_1,\dots,x_n|a,b) = \begin{vmatrix} h_{\lambda_1} & h^{(1)}_{\lambda_1} & \dots & h^{(l-1)}_{\lambda_1} \\ h_{\lambda_2-1} & h^{(1)}_{\lambda_2-1} & \dots & h^{(l-1)}_{\lambda_2-1} \\ \vdots & \vdots & \ddots & \vdots \\ h_{\lambda_l-l+1} & h^{(1)}_{\lambda_l-l+1} & \dots & h^{(l-1)}_{\lambda_l-l+1} \end{vmatrix} \tag{9}$$

where $l = l(\lambda)$.

This gives a universal proof of the Jacobi–Trudy and Giambelli formulas for usual Schur polynomals as well as for the characters of symplectic and orthogonal Lie algebras (see [3]) and for the factorial Schur polynomials [1, 6]. Another interesting case, which seems to be new, is the Jacobi–Trudy formula for the multidimensional Jacobi polynomials with parameter $\theta = 1$.

2. Proof

We start with the following lemma.

Lemma 2.1. *The following equality holds*

$$h^{(r)}_i(x_1,\dots,x_n) - x_1 h^{(r-1)}_i(x_1,\dots,x_n) = h^{(r-1)}_{i+1}(x_2,\dots,x_n) \tag{10}$$

for all r, i satisfying the relation (8).

Proof. The proof is by induction in r. When $r = 1$ we have from definition

$$h^{(1)}_i(x_1,\dots,x_n) - x_1 h^{(0)}_i(x_1,\dots,x_n) = h_{i+1}(x_1,\dots,x_n) + a(i+n-1)h_i(x_1,\dots,x_n)$$
$$+ b(i+n-1)h_{i-1}(x_1,\dots,x_n) - x_1 h_i(x_1,\dots,x_n) =$$
$$= \Delta_n(x)^{-1} \begin{vmatrix} 0 & (x_2-x_1)\varphi_{i+n-1}(x_2) & \dots & (x_n-x_1)\varphi_{i+n-1}(x_n) \\ \varphi_{n-2}(x_1) & \varphi_{n-2}(x_2) & \dots & \varphi_{n-2}(x_n) \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{vmatrix}.$$

Subtracting the first column from the others we get

$$= \Delta_n(x)^{-1} \begin{vmatrix} 0 & (x_2-x_1)\varphi_{i+n-1}(x_2) & \dots & (x_n-x_1)\varphi_{l+n-1}(x_n) \\ \varphi_{n-2}(x_1) & \varphi_{n-2}(x_2)-\varphi_{n-2}(x_1) & \dots & \varphi_{n-2}(x_n)-\varphi_{n-2}(x_1) \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \dots & 0 \end{vmatrix} =$$

$$= \Delta_{n-1}(x)^{-1} \begin{vmatrix} \varphi_{i+n-1}(x_2) & \dots & \varphi_{i+n-1}(x_n) \\ \frac{\varphi_{n-2}(x_2)-\varphi_{n-2}(x_1)}{x_2-x_1} & \dots & \frac{\varphi_{n-2}(x_n)-\varphi_{n-2}(x_1)}{x_n-x_1} \\ \vdots & \ddots & \vdots \\ \frac{\varphi_1(x_2)-\varphi_1(x_1)}{x_2-x_1} & \dots & \frac{\varphi_1(x_n)-\varphi_1(x_1)}{x_n-x_1} \end{vmatrix} = h_{i+1}(x_2,\dots,x_n).$$

The induction step is straightforward check using the relation (7). $\square$

Now we are ready to prove the Jacobi–Trudy formula. The proof is by induction in $l = l(\lambda)$. If $l = 1$ then the formula follows from the definition of h_i. Suppose that $l > 1$. We will use the bracket $\{g(x_1, \ldots, x_n)\}$ to denote the result of the alternation:

$$\{g(x_1, \ldots, x_n)\} = \sum_{w \in S_n} \varepsilon(w) g(x_{w(1)}, \ldots, x_{w(n)}).$$

We claim that

$$\{h_i^{(r)}(x_1, \ldots, x_n) x_1^{n-1} x_2^{n-2} \ldots x_n^0\} = \{x_1^r \varphi_{i+n-1}(x_1) x_2^{n-2} \ldots x_n^0\}$$

for any $r \leq i + 2n - 2$. Indeed, for $r = 0$ this true by definition and the induction step follows easily from relations (3) and (7). From this we have

$$\left\{ \begin{vmatrix} h_{\lambda_1} & h_{\lambda_1}^{(1)} & \ldots & h_{\lambda_1}^{(l-1)} \\ h_{\lambda_2-1} & h_{\lambda_2-1}^{(1)} & \ldots & h_{\lambda_2-1}^{(l-1)} \\ \vdots & \vdots & \ddots & \vdots \\ h_{\lambda_l-l+1} & h_{\lambda_l-l+1}^{(1)} & \ldots & h_{\lambda_l-l+1}^{(l-1)} \end{vmatrix} x_1^{n-1} x_2^{n-2} \ldots x_n^0 \right\}$$

$$= \left\{ \begin{vmatrix} \varphi_{\lambda_1+n-1}(x_1) & x_1 \varphi_{\lambda_1+n-1}(x_1) & \ldots & x_1^{l-1} \varphi_{\lambda_1+n-1}(x_1) \\ h_{\lambda_2-1} & h_{\lambda_2-1}^{(1)} & \ldots & h_{\lambda_2-1}^{(l-1)} \\ \vdots & \vdots & \ddots & \vdots \\ h_{\lambda_l-l+1} & h_{\lambda_l-l+1}^{(1)} & \ldots & h_{\lambda_l-l+1}^{(l-1)} \end{vmatrix} x_2^{n-2} \ldots x_n^0 \right\}.$$

Multiply every column except the last one by x_1 and subtract it from the next column. Then by lemma this expression is equal to

$$\left\{ \begin{vmatrix} \varphi_{\lambda_1+n-1}(x_1) & 0 & \ldots & 0 \\ h_{\lambda_2-1} & \hat{h}_{\lambda_2-1} & \ldots & \hat{h}_{\lambda_2-1}^{(l-2)} \\ \vdots & \vdots & \ddots & \vdots \\ h_{\lambda_l-l+1} & \hat{h}_{\lambda_l-l+1} & \ldots & \hat{h}_{\lambda_l-l+1}^{(l-2)} \end{vmatrix} x_2^{n-2} \ldots x_n^0 \right\}$$

$$= \left\{ \varphi_{\lambda_1+n-1}(x_1) \begin{vmatrix} \hat{h}_{\lambda_2-1} & \ldots & \hat{h}_{\lambda_2-1}^{(l-2)} \\ \vdots & \ddots & \vdots \\ \hat{h}_{\lambda_l-l+1} & \ldots & \hat{h}_{\lambda_l-l+1}^{(l-2)} \end{vmatrix} x_2^{n-2} \ldots x_n^0 \right\},$$

where $\hat{h}_i^{(r)} = h_i^{(r)}(x_2, \ldots, x_n)$. By induction this is equal to

$$\{\varphi_{\lambda_1+n-1}(x_1) \varphi_{\lambda_2+n-2}(x_2) \ldots \varphi_{\lambda_n}(x_n)\},$$

which by definition coincides with f_λ. This completes the proof of the main theorem.

3. Giambelli formula

As a corollary we have the following Giambelli formula for generalised Schur functions. [1] Let us denote the generalised Schur polynomials corresponding to the hook Young diagrams as

$$S_{(u|v)}(x) = S_{(u+1,1^{(v)})}(x).$$

Theorem 3.1. *The generalised Schur polynomials satisfy the following Giambelli formula*

$$S_\lambda(x_1,\dots,x_n|a,b) = \begin{vmatrix} S_{(\lambda_1-1|\lambda'_1-1)} & S_{(\lambda_1-1|\lambda'_2-2)} & \dots & S_{(\lambda_1-1|\lambda'_r-r)} \\ S_{(\lambda_2-2|\lambda'_1-1)} & S_{(\lambda_2-2|\lambda'_2-2)} & \dots & S_{(\lambda_2-2|\lambda'_r-r)} \\ \vdots & \vdots & \ddots & \vdots \\ S_{(\lambda_r-r|\lambda'_1-1)} & S_{(\lambda_r-r|\lambda'_2-2)} & \dots & S_{(\lambda_r-r|\lambda'_r-r)} \end{vmatrix} \tag{11}$$

where r is the number of the diagonal boxes of λ.

Proof. The proof follows the same line as Macdonald's proof of the usual Giambelli formula (see [1], Ch.1, Section 3, Example 21), but we give the proof here for the reader's convenience.

From the Theorem 1.1 we see that

$$S_{(u|v)}(x) = \begin{vmatrix} h_{u+1} & h^{(1)}_{u+1} & \dots & h^{(v)}_{u+1} \\ h_0 & h^{(1)}_0 & \dots & h^{(v)}_0 \\ \vdots & \vdots & \ddots & \vdots \\ h_{1-v} & h^{(1)}_{1-v} & \dots & h^{(v)}_{1-v} \end{vmatrix}$$

In this formula $u \geq 0$, $v \geq 0$, but we can define the functions $S_{(u|v)}(x)$ by the same formula for all integers u and nonnegative integers v. It is easy to check that this defines them correctly and that for u negative $S_{(u|v)}(x) = 0$ except when $u + v = -1$, in which case $S_{(u|v)}(x) = (-1)^v$.

Now consider the following matrix of the size $j \times (j+1)$

$$H^{(j)} = \begin{pmatrix} h_0 & h^{(1)}_0 & \dots & h^{(j)}_0 \\ h_{-1} & h^{(1)}_{-1} & \dots & h^{(j)}_{-1} \\ \vdots & \vdots & \ddots & \vdots \\ h_{1-j} & h^{(1)}_{1-j} & \dots & h^{(j)}_{1-j} \end{pmatrix}$$

and denote by $\Delta^{(j)}_i$, $1 \leq i \leq j+1$ the determinant of its sub-matrix without the i-th column multiplied by $(-1)^{i-1}$. If $i > j+1$ we set by definition $\Delta^{(j)}_i = 0$. One can check also that $\Delta^{(k-1)}_k = (-1)^{k-1}$.

[1] We are very grateful to G. Olshanski, who pointed out this to us.

For any partition λ consider the matrices

$$A = \begin{pmatrix} h_{\lambda_1} & h^{(1)}_{\lambda_1} & \dots & h^{(l-1)}_{\lambda_1} \\ h_{\lambda_2-1} & h^{(1)}_{\lambda_2-1} & \dots & h^{(l-1)}_{\lambda_2-1} \\ \vdots & \vdots & \ddots & \vdots \\ h_{\lambda_l-l+1} & h^{(1)}_{\lambda_l-l+1} & \dots & h^{(l-1)}_{\lambda_l-l+1} \end{pmatrix}, \quad B = \begin{pmatrix} \Delta^{(l-1)}_1 & \Delta^{(l-2)}_1 & \dots & \Delta^{(0)}_1 \\ \Delta^{(l-1)}_2 & \Delta^{(l-2)}_2 & \dots & \Delta^{(0)}_2 \\ \vdots & \vdots & \ddots & \vdots \\ \Delta^{(l-1)}_l & \Delta^{(l-2)}_l & \dots & \Delta^{(0)}_l \end{pmatrix}.$$

Note that B is upper-triangular with respect to the anti-diagonal with the anti-diagonal elements $(-1)^{k-1}$, so the determinant of B is identically equal to 1, while the determinant of A by Theorem 1.1 coincides with $S_\lambda(x_1, \dots, x_n|a, b)$. From linear algebra and definition of $S_{(u|v)}$ we have

$$AB = \begin{pmatrix} S_{(\lambda_1-1|l-1)} & S_{(\lambda_1-1|l-2)} & \dots & S_{(\lambda_1-1|0)} \\ S_{(\lambda_2-2|l-1)} & S_{(\lambda_2-2|l-2)} & \dots & S_{(\lambda_2-2|0)} \\ \vdots & \vdots & \ddots & \vdots \\ S_{(\lambda_l-l|l-1)} & S_{(\lambda_l-l|l-2)} & \dots & S_{(\lambda_l-l|0)} \end{pmatrix}.$$

Taking the determinants of both sides we see that $S_\lambda(x_1, \dots, x_n|a, b)$ equals to the determinant of the last matrix. In this matrix there are many zeros since for $k > r$ we have $\lambda_k - k < 0$ and therefore $S_{(\lambda_k-k|l-j)} = (-1)^{l-j}$ if $\lambda_k - k + l - j = -1$ and 0 otherwise. This means that in the k-th row with $k > r$ there is only one non-zero element $(-1)^{l-j}$ with $l - j = k - \lambda_k - 1$. This reduces the calculation of the determinant to the $r \times r$ matrix with the remaining columns having the numbers $\lambda'_j - j,\ j = 1, \dots, r$. Indeed, for any λ of length l with r boxes on the diagonal the union of two sets $\{k - \lambda_k - 1\}, k = r+1, \dots, l$ and $\{\lambda'_j - j\}, j = 1, \dots, r$ is the set $\{0, 1, 2, \dots, l-1\}$ as it follows, for example, from the identity

$$\sum_{i=1}^{l} t^i(1 - t^{-\lambda_i}) = \sum_{j=1}^{r} (t^{\lambda'_j - j+1} - t^{j-\lambda_j})$$

(see [1], Ch.1, Section 1, Example 4). The check of the sign completes the proof. $\square$

4. Particular cases

As a corollary we have the following well-known cases of the Jacobi–Trudy formula.

1. When $a(i) = b(i) = 0$ for all $i \geq 0$ we have $\varphi_i(z) = z^i$ and (9) clearly coincides with the usual Jacobi–Trudy formula for Schur polynomials.

2. The characters of the orthogonal Lie algebra $so(2n+1)$ correspond to the case when $a(i) = 0$, $b(i) = 1$ for $i > 0$ and $a(0) = -1$, $b(0) = 0$ and the polynomials $\varphi_i(z) = x^i + x^{i-1} + \dots + x^{-i}$, $z = x + x^{-1}$. Using the recurrence relation (7), having in this case the form

$$h^{(r+1)}_i = h^{(r)}_{i+1} + h^{(r)}_{i-1}$$

we can rewrite the general Jacobi-Trudy formula (9) in the form known in representation theory (see Prop. 24.33 in Fulton-Harris [3]): the character χ_λ is the determinant of the $l \times l$ matrix whose i-th row is

$$(h_{\lambda_i - i+1} \quad h_{\lambda_i - i+2} + h_{\lambda_i - i} \quad h_{\lambda_i - i+3} + h_{\lambda_i - i-1} \ \dots \ h_{\lambda_i - i+l} + h_{\lambda_i + 2 - l}).$$

The same is true for the characters of the even orthogonal Lie algebra $so(2n)$, where $a(i) = 0$ for all $i \geq 0$ and $b(i) = 1$ for $i > 1$ with $b(1) = 2$ and $\varphi_i(z) = x^i + x^{-i}$, $z = x + x^{-1}$ (see Prop. 24.44 in [3]) and for the symplectic Lie algebra $sp(2n)$, when $a(i) = 0$, $b(i) = 1$ for all $i \geq 0$ and $\varphi_i(z) = x^i + x^{i-2} + \dots + x^{-i}$, $z = x + x^{-1}$ (Prop. 24.22 in [3]). Note that the change of $a(0)$ and $b(1)$ does not affect the definition of the relevant $h_i^{(r)}$ for $r > 0$.

3. The *factorial Schur polynomials* [6] correspond to the special case when $b_i = 0$, so that

$$\varphi_i(z) = (z - a(0))(z - a(1)) \dots (z - a(i-1)), \quad i > 0.$$

The Jacobi–Trudy formula for them can be found in [1], Ch.1, Section 3, Example 20.

4. For $a(i)$, $b(i)$ given by

$$a(x) = -\frac{2p(p + 2q + 1)}{(2x - p - 2q - 1)(2x - p - 2q + 1)}, \tag{12}$$

$$b(x) = \frac{2x(2x - 2q - 1)(2x - 2p - 2q - 1)(2x - 2p - 4q - 2)}{(2x - p - 2q)(2x - p - 2q - 1)^2(2x - p - 2q - 2)} \tag{13}$$

we have the Jacobi-Trudy formula for the multidimensional Jacobi polynomials with $k = -1$, which seems to be new.

5. Infinite-dimensional and super versions

Let us assume now that the coefficients $a(i)$ and $b(i)$ of the recurrence relation are *rational functions* of i. In that case we can define the *generalised Schur functions* (which are the infinite-dimensional version of $S_\lambda(x|a, b)$) in the following way (cf. Okounkov-Olshanski [5]).

First note that the generalised Schur polynomials (4) are the linear combination of the usual Schur polynomials

$$S_\lambda(x_1, \dots, x_n|a, b) = \sum_{\mu \subseteq \lambda} c_{\lambda,\mu}(n|a, b) S_\mu(x_1, \dots, x_n),$$

where $c_{\lambda,\mu}(n|a, b)$ are some rational functions of n. The generalised Schur functions depend on the additional parameter d and defined by

$$S_\lambda(x|a, b; d) = \sum_{\mu \subseteq \lambda} c_{\lambda,\mu}(d|a, b) S_\mu(x), \tag{14}$$

where $S_\mu(x)$ are the usual Schur functions [1]. They satisfy the following infinite dimensional version of the Jacobi–Trudy formula:

$$S_\lambda(x|a,b;d) = \begin{vmatrix} h_{\lambda_1} & h^{(1)}_{\lambda_1} & \dots & h^{(l-1)}_{\lambda_1} \\ h_{\lambda_2-1} & h^{(1)}_{\lambda_2-1} & \dots & h^{(l-1)}_{\lambda_2-1} \\ \vdots & \vdots & \ddots & \vdots \\ h_{\lambda_l-l+1} & h^{(1)}_{\lambda_l-l+1} & \dots & h^{(l-1)}_{\lambda_l-l+1} \end{vmatrix}. \qquad (15)$$

Here $l = l(\lambda)$, $h_i = S_\lambda(x|a,b;d)$ with $\lambda = (i)$ if $i \geq 0$ and $h_i \equiv 0$ if $i < 0$, $h_i^{(r)} = h_i^{(r)}(x|a,b;d)$ are defined for generic d by the recurrence relation

$$h_i^{(r+1)} = h_{i+1}^{(r)} + a(i+d-1)h_i^{(r)} + b(i+d-1)h_{i-1}^{(r)} \qquad (16)$$

with initial data $h_i^{(0)} = h_i$.

The *generalised super Schur polynomials* $S_\mu(x_1, \dots, x_n; y_1, \dots, y_m|a,b)$ can be defined by the same formula (14), where the Schur functions should be replaced by the super Schur polynomials $S_\mu(x_1, \dots, x_n; y_1, \dots, y_m)$ (see e.g. [1]) and d must be specialised as the superdimension $d = n - m$ (provided the coefficients have no poles at $d = n - m$). Alternatively, $S_\mu(x_1, \dots, x_n; y_1, \dots, y_m|a,b)$ is the image of the corresponding generalised Schur function $S_\mu(x|a,b;d)$ under the homomorphism ϕ sending the power sums $p_k \in \Lambda$ to the super power sums $x_1^k + \dots + x_n^k - y_1^k - \dots - y_m^k$ with $d = n - m$. In the case of factorial Schur polynomials their super version had been introduced in a different way by Molev [8].

An important example corresponds to the sequences (12),(13). In this case the generalised Schur functions coincide with the Jacobi symmetric functions with parameter $k = -1$ (see [9]). These functions and their super versions play an important role in representation theory of the orthosymplectic Lie superalgebras [10].

Finally we would like to mention that the Jacobi-Trudy formula (9) can be rewritten in a dual form in terms of the conjugate partition in the spirit of Macdonald [1] (Ch.1, Section 3, Example 21) and Okounkov-Olshanski [7], Section 13.

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