Elliptic fibrations on cubic surfaces

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ABSTRACT

We classify elliptic fibrations birational to a nonsingular, minimal cubic surface over a field of characteristic zero. Our proof is adapted to provide computational techniques for the analysis of such fibrations, and we describe an implementation of this analysis in computer algebra.

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1. Introduction

Let $X \subset \mathbb{P}^3$ be a nonsingular projective cubic surface over a field k of characteristic zero. An *elliptic fibration on X*, sometimes called an *elliptic fibration birational to X*, is a dominant rational map $\varphi: X \dashrightarrow B$ to a normal variety B, where φ is defined over k, it has connected fibres, and its general geometric fibre is birational to a curve of genus 1.

We describe in Section 2.1 a class of elliptic fibrations called *Halphen fibrations*. Conversely, given an elliptic fibration on a *minimal X* (see below) we relate it to an Halphen fibration as follows:

Theorem 1.1. *Let $X \subset \mathbb{P}^3$ be a nonsingular, minimal cubic surface over a field k of characteristic zero. If $\varphi: X \dashrightarrow B$ is an elliptic fibration on X then $B \cong \mathbb{P}^1$ and there exists a composite*

$$X - \overset{i_s}{-} \succ X - \overset{i_{s-1}}{-} \succ \quad \cdots \quad - \overset{i_2}{-} \succ X - \overset{i_1}{-} \succ X$$

of birational selfmaps of X, each of which is a Geiser or Bertini involution, such that $\varphi \circ i_1 \circ \cdots \circ i_s: X \dashrightarrow B \cong \mathbb{P}^1$ is an Halphen fibration.

Geiser and Bertini involutions are birational selfmaps of X described in Section 2.2. This result is proved in Cheltsov [1] and independently in the unpublished [2]. Our aims and methods are different from those of [1], however: we seek to be as explicit as possible, and we have implemented algorithms in the computational algebra system MAGMA [3] for Halphen fibrations and Geiser and Bertini involutions. Our code is available at [4].

All varieties, subschemes, maps and linear systems are defined over the fixed field k of characteristic zero, except where a different field is mentioned explicitly.

Contents of the paper. In the remainder of the introduction we discuss motivation and background for the problem. We build Halphen fibrations on X in Section 2. In Section 3 we discuss the Noether–Fano–Iskovskikh inequalities and then prove

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Theorem 1.1. Section 4 is devoted to algorithmic considerations and an outline of our implementation, while Section 5 contains worked computer examples.

Cubic surfaces and minimality. Throughout this paper, by *cubic surface* we mean a nonsingular surface $X \subset \mathbb{P}^3$ defined by a homogeneous polynomial of degree 3 with coefficients in k. We denote $-K_X$ by A.

When k is algebraically closed, it is well known that X contains 27 straight lines and that these span the Picard group $\mathrm{Pic}(X) \cong \mathbb{Z}^7$. One quickly deduces that there is a birational map $X \dashrightarrow \mathbb{P}^2$; in other words, X is *rational*. On the other hand, if k is not algebraically closed, some of these lines may fail to be defined over k and the Picard group may have smaller rank. A cubic surface X is *minimal* if the Picard number of X, $\rho(X) = \mathrm{rank}\,\mathrm{Pic}(X)$, is 1. It is easy to see that if X is minimal then $\mathrm{Pic}(X) = \mathbb{Z}(-K_X)$; this is the fundamental property we use.

Elliptic fibrations were defined above with apparently arbitrary base B, but in fact it follows from Iitaka's bound on Kodaira dimension that $g(B) = 0$ for any surface X of Kodaira dimension $-\infty$; see [5] Theorem 18.4. In particular this applies to cubic surfaces and so we have:

Proposition. *If* $X \dashrightarrow B$ *is an elliptic fibration on a cubic surface X then $g(B) = 0$.*

There remains the question of whether B has a k-rational point, that is, whether $B \cong \mathbb{P}^1$ over k; we return to this in Section 3.1.

Geometric motivation. Our main motivation for studying elliptic fibrations on cubic surfaces is geometric. This is best explained from a broader perspective.

A *Fano n-fold* is a normal projective variety X of dimension n, with at worst $\mathbb{Q}$-factorial terminal singularities and Picard number 1, such that $-K_X$ is ample. A fundamental question in Mori theory is whether a given Fano n-fold X admits birational maps to other Mori fibre spaces — see [6] for a discussion, noting that a key example of such *birational nonrigidity* is a rational map $\varphi\colon X \dashrightarrow S$ whose generic fibre is a curve of genus 0 rather than 1. We regard the search for elliptic fibrations as a limiting case in Mori theory — a point of view we learned from papers of Iskovskikh [7] and Cheltsov [1], and one that becomes clearer when we discuss the Noether–Fano–Iskovskikh inequalities in Section 3.2. For more on how our problem fits into modern birational geometry, see [7] and the introduction to [8].

Arithmetic motivation. Cases of the more general problem of classifying elliptic fibrations on Fano varieties also have arithmetic applications. From this point of view a cubic surface is a baby case; but scaled-up versions of our methods attack, for example, the same problem for some Fano 3-folds; see [1,9].

In arithmetic a basic question concerning Fano varieties is the existence, or at least potential density, of rational points. Elliptic fibrations offer one approach; see Bogomolov and Tschinkel [10], for instance.

History. In contrast to the modern motivation, some of the methods are ancient. In his paper [11] of 1882 Halphen considered the problem of finding a plane curve G of degree 6 with nine prescribed double points $P_1, \ldots, P_9$. The question is: for which collections of points $\{P_i\}$ is there a solution apart from $G = 2C$, where C is the (in general unique) cubic containing all the P_i? Halphen's answer is that C must indeed be unique and – in modern language and supposing for simplicity that C is nonsingular, and so elliptic – $P_1 \oplus \cdots \oplus P_9$ must be a nonzero 2-torsion point of C, where any inflection point is chosen as the zero for the group law. He proceeds to consider higher torsion as well. Translated to a cubic surface, this is essentially Theorem 2.4. A natural next step is the result analogous to Theorem 1.1 for $X = \mathbb{P}^2$, and this was proved by Dolgachev [12] in 1966.

The approach of [1] to Theorem 1.1 is considerably more highbrow than ours: he uses general properties of mobile log pairs and does not spell out the construction of elliptic fibrations in detail. The paper [2], on the other hand, was originally conceived as a test case for [13] and [9], which concern similar problems for Fano 3-folds.

2. Constructing elliptic fibrations

We fix a nonsingular, minimal cubic surface X defined over k, with $A = -K_X$. Linear equivalence of divisors is denoted by $\sim$ and $\mathbb{Q}$-linear equivalence by $\sim_{\mathbb{Q}}$.

2.1. Halphen fibrations

The simplest elliptic fibrations arise as the pencil of planes through a given line. That is, if $L = (f = g = 0)$ is a line in $\mathbb{P}^3$ defined by two independent linear forms f, g and not lying wholly in X, then the map $\varphi = (f, g)$ is an elliptic fibration. In this section we make a larger class of fibrations which includes these linear fibrations as a simple case.

Definition 2.1. A pair (G, D) is called *Halphen data on X* when $G \in |A|$ is (reduced and) irreducible over k and $D \in \mathrm{Div}(G)$ is an effective k-rational divisor of degree 3, supported in the nonsingular locus of G, satisfying $\mathcal{O}_G(\mu D) \cong \mathcal{O}_G(\mu A)$ for some integer $\mu \geqslant 1$. The smallest such $\mu \geqslant 1$ is called the *index* of (G, D).

Since X is minimal, G may be any irreducible plane cubic or the union of three conjugate lines (it is required to be irreducible over k, not over $\bar{k}$). Since $\mathrm{Supp}(D) \subset \mathrm{Nonsing}(G)$, the sheaf isomorphism condition says that $A_{|G} - D$ is a torsion class of order μ in $\mathrm{Pic}(G)$.

Definition 2.2. Let (G, D) be Halphen data on X. The *resolution of* (G, D) is the blowup $\pi: Y \to X$ of a set of up to three points P_i that lie on varieties dominating X and are determined as follows:

A1. If D is a sum of distinct k-rational points of G then let $\{P_1, P_2, P_3\} = \mathrm{Supp}(D)$ (as points of X) and let π be the blowup of these points.
A2. If $D = p + 2q$, where $p \neq q$ are k-rational points of G, then let $P_1 = p$ and $P_2 = q$ (as points of X); also let $\xi: Y' \to X$ be the blowup of these points and let E_2' be the exceptional curve lying over P_2. Now define P_3 to be the point $G' \cap E_2'$ on Y', where G' is the strict transform of G; let $o: Y \to Y'$ be the blowup of P_3 and set $\pi = \xi \circ o$.
A3. If $D = 3p$ with p a k-rational point of G then let $P_1 = p$ and let $\nu: Y' \to X$ be the blowup of P_1. Next define $P_2 = E_1' \cap G'$ where $E_1', G' \subset Y'$ are respectively the exceptional curve of ν and the strict transform of G, and let $\xi: Y'' \to Y'$ be the blowup of P_2. Now, similarly, define $P_3 = E_2'' \cap G''$ where $E_2'', G'' \subset Y''$ are respectively the exceptional curve of ξ and the strict transform of G'. Finally let $o: Y \to Y''$ be the blowup of P_3 and let $\pi = \nu \circ \xi \circ o: Y \to X$.
 B. If $D = p_1 + p_2$ with p_1 a k-rational point of G and $\deg(p_2) = 2$ then let $P_i = p_i$ for $i = 1$, 2 and let $\pi: Y \to X$ be the blowup of P_1 and P_2.
 C. If $D = p$, a single k-closed point of G of degree 3, then let $P_1 = p$ and let $\pi: Y \to X$ be the blowup of P_1.

In each case we fix the following notation: let $E_i \subset Y$ be the *total* transform on Y of the exceptional curve over P_i. So in case A2, for example, $E_2 = o^*(E_2') = E_2'' + E_3$ has two irreducible components, $E_2'' = o_*^{-1}(E_2')$ and $E_3 = \mathrm{Exc}(o)$. Furthermore let $E = \sum_i E_i$, the relative canonical class of π.

It can easily be checked in the above definition that E_i is the reduced preimage of P_i on Y. Note, though, that this is a consequence of our positioning of each subsequently defined P_j on the strict transform of G; the corresponding statement no longer holds, for example, in the closely related notation of Section 3.2 below.

Definition 2.3. Let (G, D) be Halphen data on X of index μ, and let $\pi: Y \to X$ be the resolution introduced above with relative canonical class E. We define $\mathcal{H}_Y$ to be the linear system $|\mu\pi^*(A) - \mu E|$ on Y. The *Halphen system $\mathcal{H}$ associated with* (G, D) is the birational transform of $\mathcal{H}_Y$ on X.

Notice that $\mathcal{H}$ is the set of divisors in $|\mu A|$ that have multiplicity μ at every point P_i. It would be natural to write $\mathcal{H} = |\mu A - \mu D|$, but we don't.

Theorem 2.4. *Let* (G, D) *be Halphen data on X of index μ, and let $\mathcal{H}$ be the linear system described in* Definition 2.3. *Then $\mathcal{H}$ is a mobile pencil, and the rational map $\varphi = \varphi_{\mathcal{H}}$ is an elliptic fibration $\varphi: X \dashrightarrow \mathbb{P}^1$ that has μG as a fibre. The set-theoretic base locus of φ is* $\mathrm{Supp}(D)$ *and the resolution of* (G, D) *is its minimal resolution of indeterminacies.*

Following Cheltsov [1], fibrations $\varphi_{\mathcal{H}}$ arising in this way are called *Halphen fibrations*. We give the proof of this theorem in Section 2.3.

2.2. Twisting by Geiser and Bertini involutions

Not all elliptic fibrations are Halphen: we can precompose, or *twist*, Halphen fibrations by elements of $\mathrm{Bir}(X)$, and usually the result will have more than three basepoints (counted with degree).

We describe two particular classes of birational selfmap of X: Geiser and Bertini involutions, also described at greater length in [8, Section 2]. In fact, the group $\mathrm{Bir}(X)$ (in the case of minimal X) is generated by Geiser and Bertini involutions together with all regular automorphisms, although we do not use this fact explicitly; see [14] Chapter 5.

Geiser involutions. Let $P \in X$ be a point of degree 1. We define a birational map $i_P: X \dashrightarrow X$ as follows. Let Q be a general point of X, and let $L \subset \mathbb{P}^3$ be the line joining P to Q. Then $L \cap X$ consists of three distinct points, P, Q and a new point R. Define $i_P(Q) = R$. In fact, i_P is the map defined by the linear system $|2A - 3P|$.

Bertini involutions. Let $P \in X$ be a point of degree 2. Let $L \subset \mathbb{P}^3$ be the unique line that contains P. Since X is minimal, L intersects X in P and exactly one other point R of degree 1. We define a birational map $i_P: X \dashrightarrow X$ as follows. Let Q be a general point of X. If $\Pi \cong \mathbb{P}^2$ is the plane spanned by P and Q, then $C = \Pi \cap X$ is a nonsingular plane cubic curve containing R. Then $i_P(Q) = -Q$, the inverse of Q in the group law on C with origin R. In fact, i_P is the map defined by the linear system $|5A - 6P|$.

2.3. Proof of Theorem 2.4

Comments about G. We are given Halphen data (G, D) on X. The curve G is a Gorenstein scheme with $\omega_G \cong \mathcal{O}_G$ and $\chi(\mathcal{O}_G) = 0$.

When $\mu > 1$, G cannot be a cuspidal cubic since in that case the Picard group $\mathrm{Pic}(G) \cong \mathbb{G}_a$ is torsion free; here we use $\mathrm{char}(k) = 0$. This restriction on G also follows from Theorem 2.4, given Kodaira's classification of multiple fibres of elliptic fibrations: multiple cusps do not occur. Our G may be a nodal cubic (with Picard group $\mathbb{G}_m$) or a triangle of conjugate lines (with Picard group an extension of $\mathbb{Z}^3$ by $\mathbb{G}_m$). If $\mu = 1$ then G can be cuspidal; but in this case we are free to re-choose G as we please from the pencil $\mathcal{H}$ of Definition 2.3, so without loss of generality G is nonsingular.

Proof of Theorem 2.4. The case $\mu = 1$ is trivial, so let $\mu \geqslant 2$.

Let $\pi \colon Y \to X$ together with the points P_i be the resolution of (G, D) of Definition 2.2. We have the Halphen system $\mathcal{H}$ on X of Definition 2.3 and, by construction, $\mu G \in \mathcal{H}$.

Suppose at first that we are in case A1, B or C. Define $\mathcal{F}$ on X as the tensor product of all $\mathcal{I}_{P_i}^{\mu}$. There is a map between exact sequences of sheaves of $\mathcal{O}_X$-modules:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{F}(\mu A) & \longrightarrow & \mathcal{O}_X(\mu A) & \longrightarrow & \mathcal{G} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_G(\mu A - \mu D) & \longrightarrow & \mathcal{O}_G(\mu A) & \longrightarrow & \mathcal{O}_{\mu D}(\mu A) & \longrightarrow & 0
\end{array} \qquad (1)$$

where $\mathcal{G} = (\mathcal{O}_X/\mathcal{F}) \otimes \mathcal{O}_X(\mu A)$. (The left-most vertical arrow is from the definition of $\mathcal{F}$, the central one is clear, and the final one follows from the others.) By assumption, $\mathcal{O}_G(\mu A - \mu D) \cong \mathcal{O}_G$.

Kodaira vanishing shows that $H^1(X, \mathcal{O}_X(\mu A)) = 0$. By Serre duality (since G is Gorenstein) we have

$$H^1(G, \mathcal{O}_G(\mu A)) \cong \mathrm{Hom}(\mathcal{O}_G(\mu A), \mathcal{O}_G)^*,$$

and this Hom is zero because A is ample on every component of G. So, taking cohomology, we have a map between exact sequences of k-vector spaces:

$$\begin{array}{ccccccccc}
0 \to & H^0(X, \mathcal{F}(\mu A)) & \to & H^0(X, \mathcal{O}_X(\mu A)) & \longrightarrow & H^0(X, \mathcal{G}) & \longrightarrow & H^1(X, \mathcal{F}(\mu A)) & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow{\scriptstyle \beta_1} & & \downarrow{\scriptstyle \beta_2} & \\
0 \longrightarrow & H^0(G, \mathcal{O}_G) & \longrightarrow & H^0(G, \mathcal{O}_G(\mu A)) & \to & H^0(G, \mathcal{O}_{\mu D}(\mu A)) & \xrightarrow{\alpha} & H^1(G, \mathcal{O}_G) & \longrightarrow 0
\end{array}$$

Since both α and β_1 are surjective, we have that β_2 is surjective. Now $\chi(\mathcal{O}_G) = 0$ so $h^1(G, \mathcal{O}_G) = 1$ and we conclude that $H^1(X, \mathcal{F}(\mu A)) \neq 0$.

From a local calculation at the geometric points of D we have

$$h^0(X, \mathcal{G}) \leqslant 3 \binom{\mu + 1}{2}$$

and by Riemann–Roch

$$h^0(X, \mathcal{O}_X(\mu A)) = 3\mu(\mu + 1)/2 + 1 \geqslant h^0(X, \mathcal{G}) + 1.$$

Thus $h^0(X, \mathcal{F}(\mu A)) \geqslant 2$.

The linear system $\mathcal{H}$ is the system associated with $H^0(X, \mathcal{F}(\mu A))$, and so it has positive dimension; $\mathcal{H}_Y$ has the same dimension. Since $\mu G \in \mathcal{H}$, the only possible fixed curve of $\mathcal{H}$ is some multiple $\mu' G$, but then $(\mu - \mu')G$ contradicts the minimality of μ; therefore $\mathcal{H}$ is mobile. Let $H_Y \in \mathcal{H}_Y$ be a general element. Since $H_Y \sim \mu \pi^*(A) - \mu E$, and $E^2 = 3$, we have $H_Y^2 = 0$. So the map $\varphi_Y = \varphi_{\mathcal{H}_Y}$ is a morphism to a curve. Furthermore, $H_Y \sim -K_Y$ so the general fibre is a nonsingular curve (over k) with trivial canonical class. Since $\mu \pi_*^{-1}(G)$ is a fibre of φ_Y, the image curve B has a rational point $Q \in B$. The minimality of μ implies that $\mathcal{H}_Y$ is the pencil $\varphi_Y^*|\mathcal{O}_B(Q)|$.

In cases A2 and A3, we make similar calculations on a blowup of X. For example, in case A2 let $\tau \colon X' \to X$ be the blowup of P_2 with exceptional curve L. Define G' and $\mathcal{H}'$ to be the birational transforms on X' of G and $\mathcal{H}$ respectively. The point P_3 lies on X', and we identify P_1 with its preimage under τ. Let $A' = \tau^*A - L$, and let $D' = P_1 + 2P_3$ as a divisor on G'.

Define $\mathcal{F}'$ as the sheaf $\mathcal{I}_{P_1}^{\mu} \otimes \mathcal{I}_{P_3}^{\mu}$ on X'. There is a map between exact sequences of sheaves of $\mathcal{O}_{X'}$-modules analogous to (1) above (involving A', G', etc.) with $\mathcal{G}' = \left(\mathcal{O}_{X'}/\mathcal{F}'\right) \otimes \mathcal{O}_{X'}(\mu A')$. Since $A' \sim -K_{X'}$, the argument works as before in cohomology, with the conclusion that $H^1(X', \mathcal{F}'(\mu A')) \neq 0$. The dimension calculation differs slightly, giving instead that

$$h^0(X', \mathcal{G}') \leqslant 2 \binom{\mu + 1}{2}$$

and $h^0(X', \mathcal{O}_{X'}(\mu A')) = 2\mu(\mu + 1)/2 + 1 \geqslant h^0(X', \mathcal{G}') + 1$. The conclusion is again that $h^0(X', \mathcal{F}'(\mu A')) \geqslant 2$, and the rest of the proof follows verbatim. In case A3, the only change is again the dimension calculation. $\qquad \square$

3. Proof of the main theorem

Let $\varphi \colon X \dashrightarrow B$ be as in the statement of Theorem 1.1.

3.1. Rationality of the base

Let H_B be a very ample divisor on B. We may choose it to have minimal possible degree; since B has genus 0, this is either 1 or 2. We first show that in fact the minimal degree is always 1, and so $B \cong \mathbb{P}^1$.

Suppose $\deg H_B = 2$; in particular, this means that B has no rational points. We let $\mathcal{H} = \varphi^*|H_B|$. A general element $H \in \mathcal{H}$ splits over $\bar{k}$ as a sum $D_1 + D_2$ of two conjugate curves each of genus 1. Over $\bar{k}$, $D_1 \sim D_2$, so the class of D_1 in $\mathrm{Pic}(\overline{X})$ is Galois invariant. In particular, D_1 defines a divisor class in $\mathrm{Pic}(X)$ over k. So H is divisible by 2 in $\mathrm{Pic}(X)$: say $H \sim 2F$ where F is an effective divisor defined over k. So $F \sim D_1$ over $\bar{k}$ and therefore, over k, $|F|$ determines a map $X \dashrightarrow \mathbb{P}^1$ which factorises φ. So B has a rational point, contrary to our assumption.

3.2. More preliminaries

We know now that B has a rational point, so we may assume $B = \mathbb{P}^1$. We denote by $\mathcal{H}$ the mobile linear system $\varphi^*|\mathcal{O}_{\mathbb{P}^1}(1)|$, a linear system that defines φ. Since X is minimal, $\mathcal{H} \subset |\mu A|$ for some fixed $\mu \in \mathbb{N}$. The anticanonical degree μ is also denoted $\deg \mathcal{H}$.

Let $P_1, \ldots, P_r$ be the distinct basepoints of $\mathcal{H}$ and $m_1, \ldots, m_r \in \mathbb{N}$ their multiplicities: so a general $C \in \mathcal{H}$ has $\mathrm{mult}_{P_i}(C) = m_i$ for all i. The list $P_1, \ldots, P_r$ may include infinitely near basepoints that lie on surfaces dominating X; compare with Definition 2.2. Note that any P_i may have degree greater than 1.

Let $f \colon W \to X$ be the blowup (in any appropriate order) of all the P_i; f is a minimal resolution of indeterminacy for φ. As in Section 2.1, we denote by E_i the *total transform* on W of the exceptional curve over P_i: that is, if L is the exceptional curve of the blowup of P_i then E_i is the total transform of L on W. (Note that E_i may be reducible or, in contrast to Section 2.1, even nonreduced.) Then denoting $\deg P_i$ by d_i, we have

$$E_i^2 = -d_i \quad \text{and} \quad E_i E_j = 0 \text{ for } i \neq j. \tag{2}$$

With this notation, the adjunction formula for f reads

$$K_W \sim f^* K_X + E_1 + \cdots + E_r \tag{3}$$

and the birational transform $\mathcal{H}_W$ of $\mathcal{H}$ on W satisfies

$$\mathcal{H}_W \sim f^* \mathcal{H} - m_1 E_1 - \cdots - m_r E_r. \tag{4}$$

Theorem 3.1 (*Noether–Fano–Iskovskikh inequalities*). *Under the hypotheses of Theorem 1.1, $\mathcal{H}$ has a basepoint of multiplicity at least $\mu = \deg \mathcal{H}$.*

Remark 3.2. In the notation above, this theorem says that $m_i \geqslant \mu$ for some i; in particular, there is at least one basepoint P_i. Moreover, when applying the theorem later, we may assume the basepoint P_i with $m_i \geqslant \mu$ is a point of X, not an infinitely near point, because multiplicities of linear systems on nonsingular surfaces are nonincreasing under blowup.

The theorem contrasts with the familiar case, explained in [8] and [15] Section 5.1, for instance, when $\mathcal{H} \subset |\mu A|$ induces a birational map from X to a nonsingular surface Y that is minimal over k: in this case the NFI inequalities tell us there is a basepoint of multiplicity strictly larger than μ. In Mori theory the latter statement is that $(X, \frac{1}{\mu}\mathcal{H})$ has a noncanonical singularity; the case we need, Theorem 3.1, says that $(X, \frac{1}{\mu}\mathcal{H})$ has a nonterminal singularity. For the modern viewpoint on NFI for elliptic and K3 fibrations birational to Fano varieties, see [9], whose approach follows Cheltsov [1] and is based on ideas of Shokurov [16].

Proof of Theorem 3.1. By Eqs. (3) and (4)

$$0 \sim_{\mathbb{Q}} f^*\left(K_X + \frac{1}{\mu}\mathcal{H}\right) \sim_{\mathbb{Q}} K_W + \frac{1}{\mu}\mathcal{H}_W - \sum_{i=1}^{r}\left(1 - \frac{m_i}{\mu}\right)E_i$$

where $\sim_{\mathbb{Q}}$ denotes $\mathbb{Q}$-linear equivalence of $\mathbb{Q}$-divisors. Now the intersection number $\mathcal{H}_W^2$ is zero since the morphism $\varphi \circ f$ is a fibration, which implies that

$$\sum_{i=1}^{r} d_i m_i^2 = 3\mu^2. \tag{5}$$

In particular, $r \geqslant 1$, and so $\mathcal{H}$ has at least one basepoint. Also $K_W \mathcal{H}_W = 0$ by the adjunction formula, and expanding $\mathcal{H}_W(K_W + (1/\mu)\mathcal{H}_W) = 0$ gives

$$\sum_{i=1}^{r} d_i m_i\left(1 - m_i/\mu\right) = 0. \tag{6}$$

Now (6) implies the result, since if any of the coefficients $(1 - m_i/\mu)$ is nonzero then at least one must be negative. $\quad\square$

3.3. Proof of Theorem 1.1

First we describe the logical structure of the argument. It falls into two parts according to equation (6): either $m_i > \mu$ for some i, in which case we sketch a standard induction step; or $m_i = \mu$ for every i, and we work this base case out in detail.

Induction step. This is essentially the proof of the birational rigidity of X, as given in [8], for example. We are given a point $P_i \in X$ (by Remark 3.2) with multiplicity $m_i > \mu$ — by definition, P_i is a *maximal centre* of $\mathcal{H}$. So

$$3\mu^2 = (\mu A)^2 = \mathcal{H}^2 \geqslant m_i^2 d_i > \mu^2 d_i,$$

where $d_i = \deg P_i$, and the inequality $\mathcal{H}^2 \geqslant m_i^2 d_i$ is the global-to-local comparison of intersection numbers $\mathcal{H}^2 \geqslant (\mathcal{H})_{P_i}^2$. It follows that $d_i = 1$ or 2.

We precompose φ with the Geiser or Bertini involution i_{P_i}. It can be shown – Lemma 2.9.3 of [8] – that this *untwists* $\mathcal{H}$, in other words that $\deg(i_{P_i})_*^{-1}\mathcal{H} < \deg \mathcal{H} = \mu$, and we conclude by induction on the degree μ. (Note that if $\mu = 1$ then all $m_i = 1$ by (6).)

Base case. Eq. (5) implies that $\sum d_i = 3$, i.e., if we count over an algebraic closure $\bar{k}$ of k then there are three basepoints; we must show they arise from Halphen data (G, D).

So let $\psi = f \circ \varphi \colon W \to \mathbb{P}^1$ be the morphism obtained by blowing up the base locus $P_1, \ldots, P_r$ of φ. We work over $\bar{k}$ for the remainder of this paragraph. Take a general fibre F of ψ; by Bertini's Theorem, F is a nonsingular curve of genus 1. Now

$$F \sim \mu f^*(A) - \mu \sum_{i=1}^r E_i \sim -\mu K_W.$$

By Kodaira's canonical bundle formula applied to ψ,

$$K_W \sim \psi^*(K_{\mathbb{P}^1} + M) + \sum_j (n_j - 1)G_j$$

where M is a divisor of degree $\chi(\mathcal{O}_W)$ on $\mathbb{P}^1$ and the $n_j G_j \sim F$, with $n_j \geqslant 2$, are the multiple fibres of ψ. Now $\chi(\mathcal{O}_W) = \chi(\mathcal{O}_X) = 1$ so M is a point and we have

$$-\frac{1}{\mu}F \sim_{\mathbb{Q}} -F + \sum_j \left(1 - \frac{1}{n_j}\right)F.$$

Therefore $1 - \frac{1}{\mu} = \sum_j \left(1 - \frac{1}{n_j}\right)$. So either $\mu = 1$ and there are no multiple fibres, or there is a single multiple fibre $n_1 G_1 = \mu G_1 \sim F$ of multiplicity μ. Since the subscheme of multiple fibres is Galois invariant, G_1 is in fact defined over k. From here on, we work exclusively over k.

In the case $\mu = 1$, $\mathcal{H}$ is a pencil contained in $|A|$ so it gives a linear fibration and we are done. The main case is $\mu > 1$. Let $G_W = G_1$ and $G = f_*(G_W)$: then

$$G \sim_{\mathbb{Q}} f_* \left(\frac{1}{\mu}F\right) \sim_{\mathbb{Q}} f_*(-K_W) \sim -K_X = A$$

so G is a plane section of X. By minimality of X, G is irreducible over k; also $\mu G = f_*(\mu G_W) \in f_*(\mathcal{H}_W) = \mathcal{H}$, so $\text{mult}_{P_i}(G) \geqslant 1$ for each basepoint P_i. (We are abusing notation here: if P_i is an infinitely near point, let Z denote any surface between X and W on which P_i lies and define $\text{mult}_{P_i}(G)$ to be $\text{mult}_{P_i}(G_Z)$, where G_Z is the pushforward of G_W to Z.) We claim that in fact $\text{mult}_{P_i}(G) = 1$ for each P_i. Indeed, first note that G_W is the strict transform of G on W, since otherwise G_W would contain some E_i with multiplicity at least 1; but then E_i would be contained in a fibre of ψ, contradicting

$$FE_i = -\mu K_W E_i = \mu d_i > 0.$$

Therefore the claim $\text{mult}_{P_i}(G) = 1$ for each P_i is equivalent to

$$G_W = f^*(G) - \sum_i E_i;$$

but the latter follows from the facts $\mu G \in \mathcal{H}$, $\mu G_W \in \mathcal{H}_W$ and $\mathcal{H}_W = f^*(\mathcal{H}) - \sum \mu E_i$.

We now construct an effective k-rational divisor D of degree 3 on G by the inverse of the procedure in Definition 2.2. We define D to be $\sum \ell_i P_i$ as a divisor on G, where the sum extends over basepoints P_i that lie on X (rather than on a surface dominating X) and ℓ_i is some factor 1, 2 or 3 that we specify. If the P_i are all points of X then we set all $\ell_i = 1$, so $D = P_1 + P_2 + P_3$ (this is one of cases A1, B and C). If $P_1, P_2 \in X$ and P_3 lies above P_2, possibly after renumbering, then we set $\ell_1 = 1$ and $\ell_2 = 2$, so $D = P_1 + 2P_2$ (case A2). Notice that in this case P_3 must be the unique intersection point of the exceptional curve above P_2 and the birational transform of G, so this procedure is indeed the inverse of the construction in Definition 2.2. If $P_1 \in X$, P_2 lies over P_1 and P_3 lies over P_2, then we set $\ell_1 = 3$, so $D = 3P_1$ (case A3); again the points P_i lie on the strict transform of G at every stage.

Next we check that (G, D) is Halphen data: the outstanding point is that $\mathcal{O}_G(H) \cong \mathcal{O}_G(\mu D)$ for a general curve $H \in \mathcal{H}$, that is, that H cuts out exactly μD on G. At a point P, the divisor of H on G is $i_P(H, G)P$, where $i_P(H, G)$ denotes the local intersection number of H and G. So we must show that for basepoints P_i that lie on X, we have $i_{P_i}(H, G) = \ell_i \text{mult}_{P_i}(H)$ for the ℓ_i defined above. In cases A1, B and C, H can be chosen so that at any basepoint P_i none of its branches is tangent to G at

P_i – otherwise there would be an additional infinitely near basepoint above P_i – so $i_{P_i}(H, G) = \mathrm{mult}_{P_i}(H)$ and all $\ell_i = 1$ as required. In case A2, using the notation above with P_3 the infinitely near point, again $i_{P_1}(H, G) = \mathrm{mult}_{P_1}(H)$. So

$$i_{P_2}(G, H) = GH - i_{P_1}(G, H) = 3\mu - \mu = 2\mu = 2\,\mathrm{mult}_{P_2}(H)$$

and $\ell_2 = 2$ as required. Case A3 is similar.

Finally, let μ' be the index of (G, D); μ' is a divisor of μ. The construction of Theorem 2.4 now applies to (G, D) to give a pencil $\mathcal{P}$ on X containing $\mu'G$. On W, the multiple $(\mu/\mu')\pi_*^{-1}\mathcal{P}$ is contained in $\mathcal{H}_W$; since $\mathcal{H}_W$ is a pencil, we have $\mu' = \mu$ and $\mathcal{H} = \mathcal{P}$.

4. Algorithms

We describe algorithms to carry out our analysis of elliptic fibrations; we assume without comment standard routines of computer algebra such as Taylor series expansions, ideal quotients and primary decomposition. We also need the field k to be computable; that is, we must be able to make standard computations in linear algebra over k and work with polynomials, rational functions and power series over k and in small finite extensions of k. The routines are expressed here in a modular way; we have implemented them in the computer algebra system MAGMA [3] closely following this recipe. Our descriptions below are self-contained and we include them to support the code.

The initial setup of the cubic surface is this: $R = k[x, y, z, t]$ is the homogeneous coordinate ring of $\mathbb{P}^3$ and $R(X) = R/F = \bigoplus_{n\in\mathbb{N}} H^0(X, \mathcal{O}(n))$ is the homogeneous coordinate ring of X; here $F = F(x, y, z, t)$ is the defining equation of X, a homogeneous polynomial of degree 3.

Overview of the computer code. The code can be used to build examples of Halphen fibrations, as in Section 2.1, and Geiser and Bertini involutions in order to twist Halphen fibrations, as in Section 2.2; using these in conjunction, one can realise Theorem 1.1 for particular examples. The central point in all of these is to impose conditions on linear systems on X. We describe an algorithm to do this in Section 4.1; this follows our code very closely. Then we explain the applications in Section 4.2.

Finally we give an implementation of Theorem 1.1 in Section 4.3. This requires two additional elements: we need to compute the multiplicity of a linear system (not just a single curve) at a point $P \in X$ and to analyse the base locus of a linear system on X.

4.1. Imposing conditions on linear systems

This is the central algorithm: given a (nonsingular, rational) point $P \in X$ and positive integers d and m, return the space of forms of degree d on $\mathbb{P}^3$ that vanish to order m at P when regarded as functions on X in a neighbourhood of P.

Step 1: *A good patch on the blowup of X at P.* Change coordinates so that $P = (0 : 0 : 0 : 1) \in X \subset \mathbb{P}^3$ and so that the projective tangent space T_pX to X at P is the hyperplane $y = 0$. Then consider the blowup patch (xz, yz, z) in local coordinates on X at P. Altogether, this determines a map $f: \mathbb{A}^3 \to \mathbb{P}^3$ with exceptional divisor $E_{\mathrm{amb}} = (z = 0)$. The birational transform $\widetilde{X}$ satisfies $f^*(X) = \widetilde{X} + E_{\mathrm{amb}}$ and the exceptional curve of $f_{|\widetilde{X}}: \widetilde{X} \to X$ is $E = E_{\mathrm{amb}} \cap \widetilde{X}$, which is the x-axis in E_{amb}.

Step 2: *Parametrise $\widetilde{X}$ near the generic point of E.* The local equation of $\widetilde{X}$ is $g = f^*(F)/z$. The exceptional curve E is the x-axis. Working over $K = k(x)$, $\widetilde{X}$ is the curve $g(y, z) = 0$ in $\mathbb{A}_K^2$, and this is nonsingular at the origin (the generic point of E). Cast g into the ring $k(x)[\![z]\!][y]$ and compute a root Y of g as a polynomial in y – this is the implicit function $Y = y(z) \in K[\![z]\!]$ implied by $g(y, z) = 0$ (with coefficients in K).

Step 3: *Pull a general form of degree d back along the blowup.* Let N be the binomial coefficient $d + 3$ choose 3 and let $p = a_1x^d + a_2x^{d-1}y + \cdots + a_Nt^d$ be a form of degree d with indeterminate coefficients $a_1, \ldots, a_N$. Compute $q(x, y, z) = f^*(p)$.

Step 4: *Impose vanishing conditions on q.* Evaluate q at $y = Y$. The result is a power series in z with coefficients in $k(x)$ and the indeterminates $a_1, \ldots, a_N$. The condition that p vanishes to order at least m at $P \in X$ is just that the coefficient of z^i vanishes identically for $i = 0, \ldots, m - 1$. Each such coefficient is of the form $p_i(x, a_1, \ldots, a_N)/q_i(x)$, where $q_i(x)$ is a polynomial in x and p_i is polynomial in x but linear in $a_1, \ldots, a_N$. Writing $p_i = \sum_j \ell_{i,j}(a_1, \ldots, a_N)x^j$, the coefficient of z^i is zero if and only if $\ell_{i,j}(a_1, \ldots, a_N) = 0$ for each j. This is finitely many k-linear conditions on the a_i.

Step 5: *Interpret the linear algebra on X.* Choose a basis of the solution space U_0 of the linear conditions on $a_1, \ldots, a_N$. This is almost the solution; if $d \geqslant 3$, however, we must work modulo the equation F of the surface X. This is trivial linear algebra: compute the span $W_d = F \cdot \mathcal{O}(d - 3)$ of F in degree d, intersect with the given solutions $W = W_d \cap U_0$, and then compute a complement U inside U_0 so that $U_0 = W \oplus U$. A basis of U gives the coefficients (in the ordered basis of monomials of degree d) of a basis of the required linear subsystem of $|\mathcal{O}_{\mathbb{P}^3}(d)|$.

Variation 1: *working inside a given linear system.* Rather than working with all monomials of degree d, we can start with a subspace $V \subset H^0(X, \mathcal{O}(d))$ and impose conditions on that. We simply work with a basis of V throughout the calculation in place of the basis of monomials used above.

Variation 2: *nonrational basepoints.* In our applications, the only nonrational basepoints P that we need to consider have degree 2 or 3. In the former case we can make a degree 2 extension $k \subset k_2$ so that P is rational after base change to k_2.

Computing as before at one of the two geometric points of P gives k_2-linear conditions on the coefficients a_i. Picking a basis for k_2 over k, we can split these conditions into 'real and imaginary' parts, and impose them all as linear conditions over k. A similar trick works for points of degree 3.

4.2. Applications of the central algorithm

Building Halphen fibrations from Halphen data. We are given Halphen data (G, D) of index μ on X, as in Definition 2.1, and we need to construct the associated Halphen system $\mathcal{H} \subset |\mu A|$ of Definition 2.3 by imposing conditions on $|\mu A|$.

Recall the points P_i that are blown up in Definition 2.2 to make the resolution of (G, D). In cases A1, B and C, we simply impose the basepoints of X as multiplicity μ basepoints of $\mathcal{H}$, using Variation 2 of the algorithm to handle nonrational basepoints. In case A2, we need to impose the conditions at P_1 and P_3 only — for the latter we must blow up X at P_2 and compute on that new surface. Similarly in case A3 we make two blowups and impose conditions only at P_3.

Geiser and Bertini involutions. As usual, let $A = \mathcal{O}_X(1)$. The Geiser involution at P is given by the linear system $\mathcal{L} = |2A - 3P|$, and the Bertini involution at P is given by $\mathcal{L} = |5A - 6P|$. Bases of these linear systems are computed by the algorithm of Section 4.3; we start by computing any basis, which determines a map $j_P \colon X \dashrightarrow \mathbb{P}^3$.

However, it is important to choose the right basis. There are two problems that may occur with our initial choice: the image of j_P may not be X; and, even if it is, j_P could be the involution that we want composed with a linear automorphism of X. Our solution is to mimic the geometric definition of i_P in Section 2.2. For both Geiser and Bertini involutions we find five affine-independent points and compute their images under both i_P and j_P, and thus interpolate for the linear automorphism τ of $\mathbb{P}^3$ such that $i_P = \tau \circ j_P$.

In the Geiser case, if L is a general line through P then the two residual points of $X \cap L$ are swapped by the involution. Typically, residual points arising as $X \cap L$ become geometric only after a degree 2 base change, and different lines need different field extensions. This is a bit fiddly in computer code, but is only linear algebra. (There may be a better solution using the projection of X away from P to $\mathbb{P}^2$ and working directly with the equation of X expressed as a quadratic over the generic point of $\mathbb{P}^2$.)

For the Bertini involution, in order to compute a single point and its image under i_P we first find the unique line L though P and the point $R \in X$ such that $L \cap X = \{P, R\}$. Let $\Pi \supset L$ be a general plane containing L; $E = X \cap \Pi$ is a nonsingular cubic curve. We make the Weierstrass model of (E, R) — that is, we embed E in a new plane $\mathbb{P}^2$ with R as a point of inflexion. In that model, we take a general line through R and compute the two other (possibly equal) intersection points (Q_1, Q_2) of that line with E. Then $Q_2 = -Q_1$ in the group law on E with R as zero, and the Bertini involution maps Q_1 to Q_2. Of course it may happen that the points Q_i are not k-rational; but in that case, as for the Geiser involution, we simply make a degree 2 field extension to realise them and separate 'real and imaginary' parts later.

Calculating multiplicities of linear systems. Suppose $\mathcal{H}$ is a linear system on X and $P \in X$ a point of degree 1. To compute the multiplicity of $\mathcal{H}$ at P we run the first three steps of the algorithm of Section 4.1 and the first evaluation of Step 4. The result is a power series in the variable z, and the multiplicity of $\mathcal{H}$ at P is the order of that power series.

Whether this works in practice depends on what implementation of power series is being used. If power series are expanded lazily with precision extended as required then it works as stated; if they are computed to a fixed precision then the algorithm is best applied to compute lower bounds on multiplicities. Fortunately we use it only to identify maximal centres, for which a lower bound is exactly the requirement.

4.3. The main theorem: untwisting elliptic fibrations

We are given a cubic surface $X \subset \mathbb{P}^3$ together with a rational map $\varphi \colon X \dashrightarrow \mathbb{P}^1$ defined by two homogeneous polynomials f, g of common degree d. Equivalently, we may regard φ as a linear system $\mathcal{H} = \langle f, g \rangle \subset H^0(X, \mathcal{O}(\mu))$. In outline, the algorithm is simple; it terminates by the proof of Theorem 1.1, the main point being that Step 3 below cannot be repeated infinitely often.

Step 0: Trivial termination. If the degree μ is equal to 1 then stop: the pencil must be a linear elliptic fibration. Return the pencil and its base locus (which is trivial to compute).

Step 1: Basepoints. Ideally we would compute precisely the base locus of $\mathcal{H}$ as a subscheme of X and work directly with that. But to avoid computing in local rings, our algorithm in Section 4.4 below computes a finite set of reduced zero-dimensional subschemes of X that supports the base locus. (In short, it solves $f = g = 0$ on X and then strips off one-dimensional primary components.) We call these *potential basepoints* of $\mathcal{H}$.

As in Section 3, the degree of a maximal centre is at most 2, so we discard any potential basepoints of higher degree. We refer to any of the remainder as a *potential centre* of φ.

Step 1a: Check termination. If there are no potential centres then stop: the linear system must be an Halphen system, and moreover we must be in case C of Definition 2.2 — that is, there is a single basepoint of degree 3. Return the system and its base locus.

Step 2: *Multiplicities.* Compute the multiplicity of the linear system $\mathcal{H}$ at each potential centre P in turn. (At points of degree 2 we make a quadratic field extension and calculate at one of the two resulting geometric points.) If P has multiplicity $m > \mu$ then go to Step 3. It may happen that no such P exists, in which case:

Step 2a: *Termination.* This is the base case of the proof of Theorem 1.1. The linear system gives an Halphen fibration and its base locus consists of all the potential centres of multiplicity $m = \mu$. Return the linear system and its base locus.

Step 3: *Untwist.* If the maximal centre P has degree 1 then compute the Geiser involution $i_P \colon X \dashrightarrow X$ at that point. If it has degree 2, compute the Bertini involution $i_P \colon X \dashrightarrow X$. In either case, replace φ by $\varphi \circ i_P$ and repeat from Step 0.

4.4. Analysing base loci on surfaces

It remains to provide an algorithm for Step 1 above. We work in slightly more generality with an arbitrary linear system $\mathcal{L}$ on X corresponding to a subspace $V \subset H^0(X, \mathcal{O}(d))$. The base locus $B = \mathrm{Bs}\,\mathcal{L}$ of $\mathcal{L}$ is contained in the subscheme $B' \subset X$ defined by the ideal $I = \langle V \rangle \subset R(X)$; the algorithm below returns the reduced set of associated primes of height $\geqslant 2$ of B'.

Step 0: *Setup.* $\mathcal{L}$ is defined by a basis of V, a finite set of homogeneous polynomials $p_1, \ldots, p_k$ of degree d. Let $I = \langle p_1, \ldots, p_k, F \rangle \subset R$; this is the ideal of B' considered as a subscheme of $\mathbb{P}^3$.

Step 1: *Identify and remove codimension 1 components.* Let I_{red} be the radical of I and let $P_1, \ldots, P_N$ be the height 1 associated primes of I_{red}. Let $J_0 = I$ and, for $i = 1, \ldots, N$, let $J_i = (J_{i-1} : P_i^{n_i})$ where $n_i \in \mathbb{N}$ is minimal such that J_i is not contained in P_i. This removes the codimension 1 base locus without removing any embedded primes there (at least set-theoretically); the radical of J_N is the ideal of the set of all isolated or embedded basepoints.

Step 2: *End.* Let $K = \mathrm{Rad}(J_N)$, the ideal of a reduced zero-dimensional scheme. Let $R_1, \ldots, R_M$ be the associated primes of K. Return this set of primes.

5. Examples

We have implemented computer code in the MAGMA computational algebra system; together with instructions, it can be downloaded at [4]. We present some examples below to illustrate our code. Here we work in $\mathbb{P}^3$ defined over $k = \mathbb{Q}$, which we input as

```
> k := Rationals();
> P3<x,y,z,t> := ProjectiveSpace(k,3);
```

The symbol > is the MAGMA prompt. In some cases below the output has been edited mildly.

5.1. An Halphen fibration with $\mu = 2$

We start with the surface $X \colon (t^3 - x^3 + y^2 z + 2xz^2 - z^3 = 0) \subset \mathbb{P}^3$.

```
> X := Scheme(P3,t^{3} - x^{3} + y^{2}*z + 2*x*z^{2} - z^{3});
> IsNonsingular(X);
true
```

The surface X is not minimal – for example, $z = x - t = 0$ is a line – but we can still construct interesting elliptic fibrations on it. The $t = 0$ section of X is an elliptic curve G with origin $O = (0 : 1 : 0 : 0)$ and an obvious rational 2-torsion point $R = (1 : 0 : 1 : 0)$. (Of course, to construct the example we started with this curve and extended to X.)

```
> O := X ! [0,1,0,0];
> R := X ! [1,0,1,0];
```

To make Halphen data with $\mu = 2$, we need an effective, k-rational divisor D on G of degree 3 for which $D - 3O$ is 2-torsion in $\mathrm{Pic}(G)$. We construct such D as follows. Let $L \subset \mathbb{P}^3$ be the line $y = t = 0$ and define a point of degree 2 on X by $L \cap X = \{R, P\}$: so P is the union of the two points $(\alpha : 0 : 1 : 0)$ with $\alpha^2 + \alpha - 1 = 0$. Define $D = P + O$ as a divisor on G. The pair (G, D) is Halphen data of index $\mu = 2$. In fact the construction of the Halphen system is in terms of linear systems and points on X, rather than on G, so for the calculation it only remains to construct P.

```
> L := Scheme(P3,[y,t]);
> PandR := Intersection(X,L);
> P := [ Z : Z in IrreducibleComponents(PandR) | Degree(Z) eq 2 ][1];  P;
Scheme over Rational Field defined by x^{2} + x*z - z^{2}, y, t
```

We build the Halphen system by imposing D as base locus of multiplicity 2 on the linear system $|2A|$, where A is a hyperplane section of X.

```
> A2 := LinearSystem(P3,2);
> H0 := ImposeBasepoint(X,A2,P,2);
> H := ImposeBasepoint(X,H0,O,2);
> H;
Linear system on Projective Space of dimension 3
with 2 sections: x^{2} + x*z - z^{2}, t^{2}
```

The resulting fibration is $\varphi = (x^2 + xz - z^2 : t^2) : X \dashrightarrow \mathbb{P}^1$, and we see that $\varphi^{-1}(1:0) = 2G$. We check that the fibre $C = \varphi^{-1}(-1:1)$ is irreducible and has genus 1:

```
> C := Curve(Intersection(X, Scheme(P3, t^{2} + x^{2} + x*z - z^{2})));
> assert IsIrreducible(C);
> Genus(C);
1
```

5.2. Geiser and Bertini involutions

We construct a Geiser involution on the minimal surface $X : (x^3 + y^3 + z^3 + 3t^3 = 0) \subset \mathbb{P}^3$.

```
> X := Scheme(P3,x^{3} + y^{3} + z^{3} + 3*t^{3});
> P := X ! [1,1,1,-1];
> iP := GeiserInvolution(X,P);
> DefiningEquations(iP);
```

returns the equations of the involution i_P:

$$(-xy + y^2 - xz + z^2 - 3xt - 3t^2 : x^2 - xy - yz + z^2 - 3yt - 3t^2 :$$
$$x^2 + y^2 - xz - yz - 3zt - 3t^2 : -x^2 - y^2 - z^2 - xt - yt - zt).$$

Since $P \in X$ is not an Eckardt point – we discuss that case below – the Geiser involution contracts the tangent curve $C_P = T_P(X) \cap X$ to P.

```
> TP := TangentSpace(X,P);
> CP := Curve(Intersection(X,TP));
> iP(CP);
Scheme over Rational Field defined by z + t, y + t, x + t
> Support(iP(CP));
{ (-1 : -1 : -1 : 1) }
```

To make a Bertini involution, we find a point of degree 2.

```
> L := Scheme(P3,[x-y,z+t]);
> XL := Intersection(X,L);
> Q := [ Z : Z in IrreducibleComponents(XL) | Degree(Z) eq 2 ][1];
> iQ := BertiniInvolution(X,Q);
> DefiningEquations(iQ);
```

again returns the equations of i_Q, although in this case they are too large to print reasonably: the first equation has 38 terms, beginning with

$$6x^2y^3 - 5xy^4 + 5y^5 - x^2y^2z - xy^3z - 4x^2yz^2 - 4y^3z^2 + 6x^2z^3 - 4xyz^3 + 11y^2z^3 - \cdots.$$

5.3. Eckardt points

A k-rational point $P \in X$ is an *Eckardt point* if $T_PX \cap X$ splits as three lines through P over a closure $\overline{k} \supset k$. For example, the surface

$$X : (x^3 + y^3 + z^3 + 2t^3 = 0) \subset \mathbb{P}^3$$

is minimal and $P = (1 : -1 : 0 : 0) \in X$ is an Eckardt point: $T_PX \cap X = (x + y = z^3 + 2t^3 = 0)$. Geiser involutions in Eckardt points are in fact biregular, and we see this here:

```
> X := Scheme(P3, x^{3} + y^{3} + z^{3} + 2*t^{3});
> P := X ! [1,-1,0,0];
> iP := GeiserInvolution(X,P);
```

When MAGMA computes a map to projective space, it does not automatically search for common factors between the defining equations and cancel them. To see the map more clearly, we do this by hand.

```
> [ f div GCD(E) : f in E ] where E is DefiningEquations(iP);
[ y, x, z, t ]
```

So the Geiser involution i_P switches x and y in this case, and that is clearly a biregular automorphism of X.

5.4. An example of untwisting

Working on the same surface $X: (x^3 + y^3 + z^3 + 2t^3 = 0)$ as above, consider the fibration $f = (f_1 : f_2): X \dashrightarrow \mathbb{P}^1$ defined by the two polynomials

$$
\begin{aligned}
f_1 = {}& 57645x^2y^3 + 47234xy^4 - 9963y^5 + 23490x^2y^2z + 97322xy^3z + 70056y^4z - 26730x^2yz^2 \\
& - 33603xy^2z^2 + 5751y^3z^2 + 47925x^2z^3 + 85664xyz^3 - 5373y^2z^3 + 41480xz^4 + 72990yz^4 + 4095z^5 \\
& + 8100x^2y^2t + 157516xy^3t + 148392y^4t - 200880x^2yzt - 25896xy^2zt + 182664y^3zt + 9720x^2z^2t \\
& - 10800xyz^2t - 42408y^2z^2t + 118912xz^3t + 194220yz^3t + 109800z^4t - 124740x^2yt^2 - 27990xy^2t^2 \\
& + 96462y^3t^2 - 42120x^2zt^2 - 112938xyzt^2 - 70722y^2zt^2 + 24042xz^2t^2 + 28314yz^2t^2 + 63558z^3t^2 \\
& + 118530x^2t^3 + 111736xyt^3 - 48186y^2t^3 + 157684xzt^3 + 176616yzt^3 + 14958z^2t^3 + 247316xt^4 \\
& + 338796yt^4 + 265536zt^4 + 123444t^5
\end{aligned}
$$

and

$$
\begin{aligned}
f_2 = {}& 20232x^2y^3 + 27216xy^4 + 6600y^5 - 66429x^2y^2z - 29187xy^3z + 40250y^4z + 25596x^2yz^2 - 8532xy^2z^2 \\
& - 42800y^3z^2 + 24507x^2z^3 + 23436xyz^3 + 3585y^2z^3 - 4185xz^4 + 35420yz^4 - 38240z^5 \\
& - 48978x^2y^2t + 77706xy^3t + 128092y^4t - 84456x^2yzt - 85428xy^2zt - 11724y^3zt + 65322x^2z^2t \\
& + 26676xyz^2t - 8214y^2z^2t + 100710xz^3t + 125152yz^3t + 25500z^4t - 196596x^2yt^2 - 75438xy^2t^2 \\
& + 122086y^3t^2 - 106596x^2zt^2 - 104598xyzt^2 + 366y^2zt^2 + 4590xz^2t^2 - 6786yz^2t^2 + 144574z^3t^2 \\
& - 62424x^2t^3 - 63612xyt^3 - 16932y^2t^3 - 105030xzt^3 + 1972yzt^3 - 98056z^2t^3 + 117720xt^4 \\
& + 231884yt^4 + 36888zt^4 + 247412t^5.
\end{aligned}
$$

Amazingly enough, this is an elliptic fibration — although that is by no means obvious, and we gave up on computing the genus of a fibre with MAGMA after a few hours. To understand f, we follow the proof of Theorem 1.1 as the algorithm of Section 4.3. First we look for a maximal centre.

```
> P1 := ProjectiveSpace(k,1);
> f := map< P3 -> P1 | [f1,f2] >;
> time existence, Q := HasMaximalCentre(f,X); assert existence;
Time: 64.240
```

This function, which executes Steps 1 and 2 of Section 4.3, returns either one or two values: first, either true or false according to whether f has a maximal centre or not; and, second, a maximal centre if there is one. In this example there is a maximal centre of degree 2:

```
> Q;
Scheme over Rational Field defined by
z^{2} - 31/4*z*t - 5/4*t^{2},  x + 3/2*z + 3/2*t,  y - 3/2*z - 1/2*t
> Degree(Q);
2
```

We don't need to know it, but in fact Q is the following pair of conjugate points:

```
> k2<w> := Degree2SplittingField(Q);
> Support(Q,k2);
{ (w : -w - 1 : 1/3*(-2*w - 3) : 1),
     (1/8*(-8*w - 117) : 1/8*(8*w + 109) : 1/12*(8*w + 105) : 1) }
```

Here k_2 is the number field $\mathbb{Q}[w]/(8w^2 + 117w + 135)$.

Following Step 3 of Section 4.3, we untwist f using the Bertini involution i_Q centred at Q.

```
> iQ := BertiniInvolution(X,Q);
> g := iQ * f;
```

As before, the defining equations of g have not been simplified by MAGMA, and they are of degree 25 with thousands of terms and no common factor. We can check by cross multiplication that, in fact, g is the rational map $(x : y)$:

```
> Eg := DefiningEquations(g);
> assert IsDivisibleBy(x*Eg[2] - y*Eg[1], DefiningEquation(X));
```

If we didn't suspect this already, we could use linear interpolation to work it out: the untwisted map g is defined by equations of degree strictly less than 5 (that is what untwisting does), and so by computing many points on X together with their images, we can set up a system of linear equations to solve for the unknown coefficients of the lower-degree map. The practical difficulty with this is finding enough points on X. That is easily solved over varying finite extensions of the base field k by intersecting X with random lines. Then it is an elementary, but messy, matter of book-keeping to solve the linear algebra problem directly over k given the various field extensions, rather than passing to a covering field (which would be mathematically trivial but computationally expensive).

5.5. The problem of minimality

Geiser and Bertini involutions exist whether or not the surface X is minimal; the geometric descriptions given in Section 2.2 work regardless. In the nonminimal case, however, the linear systems that determine the involutions need not be $|2A - 3P|$ and $|5A - 6P|$. Here we give an example where $|5A - 6P|$ does not give a Bertini involution.

Let $X = (xt^2 + x^2y + y^3 - z^3 = 0) \subset \mathbb{P}^3$. The point $P = (0 : 0 : 0 : 1)$ is an Eckardt point with tangent curve splitting as a line $x = y - z = 0$ and a conjugate pair of lines $x = y^2 + yz + z^2 = 0$. The point $Q = (1 : 0 : 0 : 0)$ lies on three conics, each defined by $xy = t^2$ together with one of the linear factors of $y^3 - z^3$. Clearly each of the conics meets exactly one of the lines, and that intersection is tangential. The three intersection points are $(0 : 1 : 1 : 0)$, $(0 : \omega : 1 : 0)$ and $(0 : \omega^2 : 1 : 0)$ where ω is some chosen primitive cube root of 1. Let $Z = (x = t = y^2 + yz + z^2 = 0) \subset X$ be the conjugate pair of intersection points. Although X is clearly not minimal, we can compute the linear system $|5A - 6Z|$.

```
> X := Scheme(P3,x*t^{2} + x^{2}*y + y^{3} - z^{3});
> Z := Scheme(P3,[x,t,y^{2}+y*z+z^{2}]);
> L1 := ImposeBasepoint(X, LinearSystem(P3,5), Z, 6);
> L2 := Complement(L1,X);
```

Notice that since the linear system is computed on the ambient $\mathbb{P}^3$, we must work modulo the equation of X by hand, taking a complement of the subspace of degree 6 polynomials that it divides — in previous examples this was hidden inside the function for Bertini involutions.

But this is the wrong linear system; it has (projective) dimension 4:

```
> #Sections(L2);
5
```

Our code cannot compute the Bertini involution in this case. Out of interest, we show instead how to make the map $f: X \dashrightarrow \mathbb{P}^4$ with these five sections and compute its image.

```
> P4<[a]> := ProjectiveSpace(k,4);
> f := map< P3 -> P4 | Sections(L2) >;
> f(X);
```

returns a surface in $\mathbb{P}^4$ defined by three equations, the 2×2 minors of the 2×3 matrix

$$\begin{pmatrix} -a_4 & a_1^2 + a_2^2 + a_2a_3 + a_3^2 & a_1 \\ a_5 & a_4^2 - a_1a_3 & a_2 - a_3 \end{pmatrix}.$$

The third minor is the equation of X; the second is the cone on $\mathbb{P}^1 \times \mathbb{P}^1$ in some coordinates. In fact, this image surface is singular: it has a single Du Val singularity of type A_2. The map f blows up Z and then contracts the two conjugate lines that meet at P, which form a chain of two -2-curves on the blowup.

Acknowledgments

References

[1] I. Cheltsov, Log models of birationally rigid varieties, J. Math. Sci. 102 (2) (2000) 3843–3875.
[2] D.J. Ryder, Elliptic pencils on cubic surfaces (2000) (unpublished manuscript).
[3] W. Bosma, J.J. Cannon, C. Playoust, The Magma algebra system: I. The user language, J. Symbolic Comput. 24 (3–4) (1997) 235–265.
[4] G. Brown, D.J. Ryder, MAGMA code available at www.kent.ac.uk/IMS/personal/gdb/ellfib.mag. At the same URL, the file www.kent.ac.uk/IMS/personal/gdb/examples.ellfib has the examples of Section 5.
[5] W.P. Barth, K. Hulek, C.A.M. Peters, A. Van de Ven, Compact complex surfaces, second edition, in: Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge, vol. 4, Springer-Verlag, 2004.
[6] A. Corti, Singularities of linear systems and 3-fold birational geometry, in: Explicit Birational Geometry of 3-folds, in: LMS Lecture Note Series, vol. 281, CUP, 2000, pp. 259–312.
[7] V.A. Iskovskikh, Birational rigidity of Fano hypersurfaces in the framework of Mori theory, Uspekhi Mat. Nauk 56 (2 (338)) (2001) 3–86.
[8] A. Corti, A. Pukhlikov, M. Reid, Fano 3-fold hypersurfaces, in: Explicit Birational Geometry of 3-folds, in: LMS Lecture Note Series, vol. 281, CUP, 2000, pp. 175–258.
[9] D.J. Ryder, Classification of elliptic and $K3$ fibrations birational to some $\mathbb{Q}$-Fano 3-folds, J. Math. Sci. Univ. Tokyo 13 (1) (2006) 13–42.
[10] F.A. Bogomolov, Yu. Tschinkel, On the density of rational points on elliptic fibrations, J. Reine Angew. Math. 511 (1999) 87–93.
[11] G. Halphen, Sur les courbes planes du sixième degré à neuf points doubles, Bull. Soc. Math. France 10 (1882) 162–172.
[12] I. Dolgachev, Rational surfaces with a pencil of elliptic curves, Izv. Acad. Nauk SSSR Ser. Math. 30 (1966) 1073–1100.
[13] D.J. Ryder, Elliptic and K3 fibrations birational to Fano 3-fold weighted hypersurfaces, Ph.D. Thesis, University of Warwick, 2002, p. viii+122.
[14] Yu.I. Manin, Cubic forms: Algebra, geometry, arithmetic, second edition, in: North-Holland Mathematical Library, vol. 4, North-Holland Publishing Co., 1986, p. x+326.
[15] J. Kollár, K.E. Smith, A. Corti, Rational and nearly rational varieties, in: Cambridge Studies in Advanced Mathematics, vol. 92, CUP, 2004, p. vi+235.
[16] V.V. Shokurov, 3-fold log models, J. Math. Sci. 81 (3) (1996) 2667–2699.
[17] T. Beck, J. Schicho, Adjoint computation for hypersurfaces using formal desingularizations, J. Algebra 320 (11) (2008) 3984–3996.