ON KRAMERS-KRONIG RELATIONS FOR GUIDED AND SURFACE WAVES

VV Krylov   Loughborough University, Loughborough, Leicestershire LE11 3TU, UK.

1   INTRODUCTION

It is well known that in unbounded media the acoustic attenuation as function of frequency is linked to the frequency-dependent sound velocity (dispersion) via Kramers-Kronig dispersion relations, or simply Kramers-Kronig relations. These relations have been named after Kramers[1] and Kronig[2] who established them independently for attenuation and dispersion in the field of Optics. In Acoustics, these relations have been first described by Ginzburg[3] in respect of sound attenuation and dispersion. This was followed by a number of papers published by different authors who examined their applications to various problems linked to investigations of acoustic wave attenuation and dispersion in unbounded media (see e.g. Refs. 4-13). Note that Kramers-Kronig relations are the sequence of causality and linearity of the medium[14]. This is why they are very general and fundamentally important for better understanding the nature of wave attenuation and dispersion and as a tool in physical acoustics measurements, where they can be used for verification purposes.

Physical acoustics measurements are often carried out not in unbounded media, but in acoustic waveguides, e.g. inside liquid-filled pipes. Surface acoustic waves are also used frequently for physical acoustics measurements. As a rule, guided or surface waves are dispersive even in the idealised situation of absence of material losses in the medium, which seems to be in contradiction with Kramers-Kronig relations. The problem of applicability of Kramers-Kronig relations to waveguides has been first examined by the present author[15] using the inversion theorem of the theory of functions of complex variable. It has been shown that Kramers-Kronig relations have limited applicability to guided and surface waves. In particular, they are not applicable to waves propagating in waveguides characterised by the possibility of wave energy leakage from the waveguides into the surrounding medium. For waveguides without leakages, e.g. liquid-filled pipes with rigid walls, Kramers-Kronig relations may remain valid for both ideal and viscous liquids. In the former case, Kramers-Kronig relations express the exponential decay of non-propagating (evanescent) higher-order acoustic modes below the cut-off frequencies via the dispersion of the same modes above the cut-off frequencies. Note that the latter result has been rediscovered in the relatively recent paper by Haakestad et al.[16]. Apparently, the authors of Ref. 16 were unaware of the earlier work[15], and they used a different approach in their analysis.

In the present paper, the applicability of Kramers-Kronig relations to guided and surface waves is further investigated using the approach developed in Ref. 15. Examples of numerical calculations of wave dispersion and attenuation using Kramers-Kronig relations are presented for two cases where they are applicable: for unbounded media with relaxation and for guided waves in ideal waveguides with rigid walls. Comparison of these numerical results with the direct analytical calculations of sound attenuation and dispersion, which are possible in these cases, illustrate the importance of Kramers-Kronig relations for both general wave theory and practical applications.

2   KRAMERS-KRONIG RELATIONS FOR UNBOUNDED MEDIA

It is instructive to start a discussion of Kramers-Kronig relations with a common case of sound propagation in unbounded media. It is convenient to apply Kramers-Kronig relations to a complex coefficient of refraction $n(\omega) = c(\infty)/c(\omega)$, where $c(\infty)$ is the sound velocity in the medium at an infinitely large frequency, and $c(\omega)$ is the sound velocity in the same medium at arbitrary circular

frequency ω. The velocity $c(\omega)$ is generally assumed to be complex in order to take account of dissipation. Kramers-Kronig dispersion relations can be written in the form (see e.g. Ref. 14)

$$\operatorname{Re} n(\omega) - 1 = \frac{1}{\pi} V.P. \int\limits_{-\infty}^{\infty} \frac{\operatorname{Im} n(\omega')}{\omega' - \omega} d\omega' , \tag{1}$$

$$\operatorname{Im} n(\omega) = -\frac{1}{\pi} V.P. \int\limits_{-\infty}^{\infty} \frac{\operatorname{Re} n(\omega') - 1}{\omega' - \omega} d\omega' . \tag{2}$$

Here *Re n(ω)* and *Im n(ω)* are real and imaginary parts of $n(\omega)$ respectively, and symbol *'V.P.'* indicates that the integrals in (1) and (2) are understood in the sense of the Cauchy principal value ('valeur principale'). Note that mathematical derivation of the relations (1) and (2) is based on the Cauchy theorem and on the analyticity of the function $n(\omega)$ in the upper or lower half spaces of the complex variable ω, which in turn usually follows from the causality principle[14], which means that the reaction of the medium cannot take place before the action. Also, a 'good' behaviour of $n(\omega)$ for large values of ω is assumed. In many cases functions $n(\omega)$ have some symmetry in respect of frequency, namely $n(-\omega) = n^*(\omega)$. In such cases Kramers-Kronig relations (1) and (2) can be rewritten in the form of the integrals over positive frequencies only:

$$\operatorname{Re} n(\omega) - 1 = \frac{2}{\pi} V.P. \int\limits_{0}^{\infty} \frac{\omega' \operatorname{Im} n(\omega')}{(\omega')^2 - \omega^2} d\omega' , \tag{3}$$

$$\operatorname{Im} n(\omega) = -\frac{2}{\pi} V.P. \int\limits_{0}^{\infty} \frac{\omega [\operatorname{Re} n(\omega') - 1]}{(\omega')^2 - \omega^2} d\omega' . \tag{4}$$

As an example, let us consider application of Kramers-Kronig relations to the case of a medium with relaxation, for which a full complex function for the wavenumber $k(\omega)$ is well known (see e.g. Ref. 17):

$$k(\omega) = \frac{\omega}{c(0)} \left[1 - \frac{m}{2} \frac{\omega^2 \tau^2}{1 + \omega^2 \tau^2} + i \frac{m}{2} \frac{\omega \tau}{1 + \omega^2 \tau^2} \right]. \tag{5}$$

Here τ is the time of relaxation for a given medium, and $m = [(c(\infty))^2 - (c(0))^2]/(c(0))^2$. Expressing the coefficient of refraction $n(\omega) = c(\infty)/c(\omega) = k(\omega)/k(\infty)$ from (5), one can obtain

$$n(\omega) = \frac{1}{1 - m/2} \left[1 - \frac{m}{2} \frac{\omega^2 \tau^2}{1 + \omega^2 \tau^2} + i \frac{m}{2} \frac{\omega \tau}{1 + \omega^2 \tau^2} \right]. \tag{6}$$

Plots of the real and imaginary parts of $n(\omega)$ defined analytically by (6) are shown in Figures 1 and 2 respectively. Calculations have been carried out for $m = 0.089$ and $\tau = 0.0003$ s.

Let us now calculate the imaginary part of $n(\omega)$ numerically using Kramers-Kronig relations. Substituting the real part of $n(\omega)$ from (6) into (4) and performing numerical integration, one can obtain the imaginary part, *Im n(ω)*. The results of the calculations are shown in Figure 3 by a dashed curve. For comparison, on the same figure the imaginary part of $n(\omega)$ defined by (6) is also displayed by a solid curve. As expected, both these curves are close to each other, which illustrates that Kramers-Kronig relations work well in this case. A small discrepancy at peak values of *Im n(ω)* can be explained by inaccuracy in the numerical calculation of the principal value of the integral.
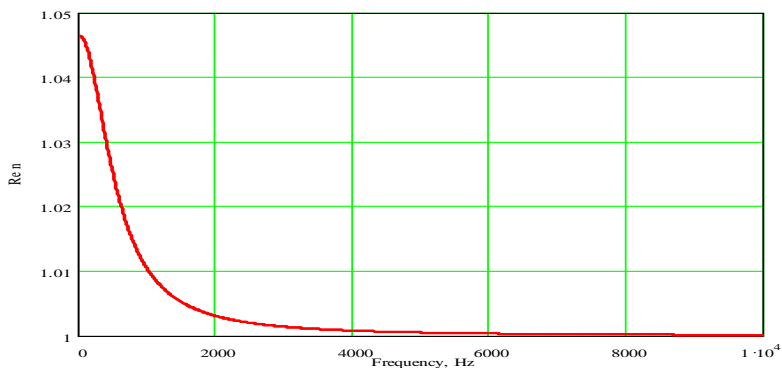
Figure 1. Real part of the refraction coefficient $n(\omega)$ for a medium with relaxation defined by Equation (6).
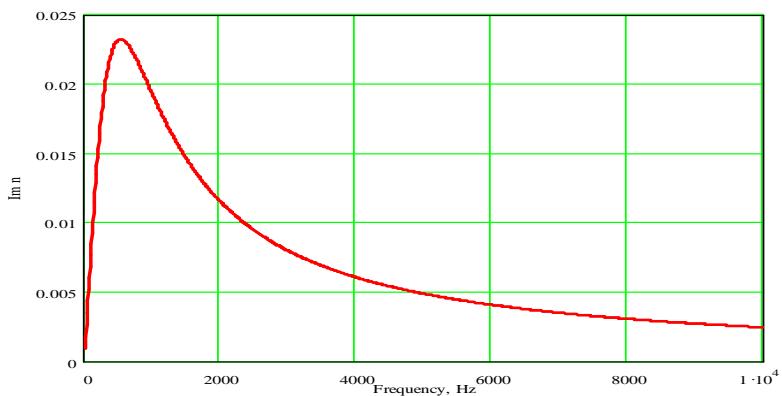

Figure 2. Imaginary part of the refraction coefficient $n(\omega)$ for a medium with relaxation defined by Equation (6).
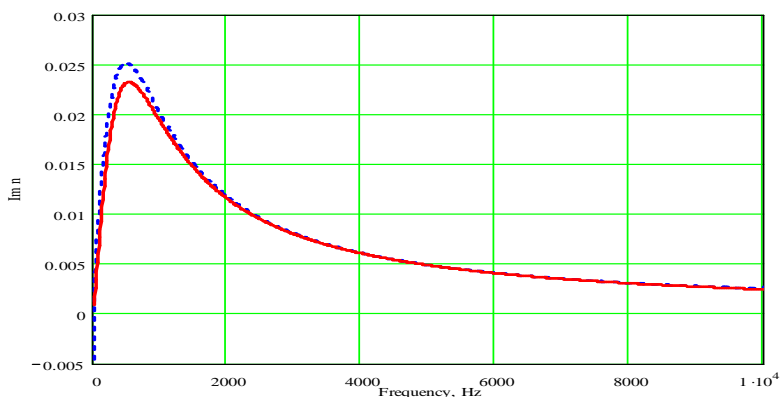

Figure 3. Imaginary part of the refraction coefficient $n(\omega)$ for a medium with relaxation defined by Equation (6) (solid curve) and calculated using Kramers-Kronig relation (4) (dashed curve).

3   KRAMERS-KRONIG RELATIONS FOR GUIDED WAVES

In the case of guided waves, the behaviour of the complex refraction index $n(\omega)$ is determined not only by the material properties of the medium, like in the case of unbounded media considered above, but also by the dispersion equations defined for specific waveguiding systems, e.g. elastic layers, liquid-filled pipes, etc. This makes the behaviour of the functions $n(\omega)$ for guided modes much more complicated. As a result of this, their analyticity can be violated, causing the breakdown of Kramers-Kronig relations.

Some general conclusions for such waveguiding systems have been obtained in Ref. 15 using the inversion theorem of the theory of functions of complex variable. In particular, it has been shown that one of the common reasons for violation of analyticity can be the appearance of branch points on the complex plain ω that are associated with some physical features of the waveguides. According to the inversion theorem (see e.g. Ref. 18), functions $n(\omega)$ may have a branch point of order j at the point $\alpha \neq \infty$ if the derivative of the inverse function $d\omega/dn$ has a zero of order j or a pole of order $j+2$ at the point $n(\alpha)$. Since zeros of the function $d\omega/dn$ coincide with zeros of the function $d\omega/dk$ describing a complex group velocity (here $k = \omega/c(\omega)$ is the wavenumber), important conclusions about the positions of the branch points of the function $n(\omega)$ can be made from the positions of zero group velocity. Generally speaking, branch points can be located at arbitrary positions on the complex plain ω. In particular, they can be located away from the real axis ω. In such cases, contour integration used for derivation of Kramers-Kronig relations would require adding integrals over the branch cuts, which means that Kramers-Kronig relations become invalid.

An important exception are the cases where branch points of the functions $n(\omega)$ for normal waves are located on the real axis ω. This means that the group velocities of such normal modes are equal to zero at critical (cut off) frequencies, which can take place for waveguides without possible leakage of energy from waveguides[19]. In such cases Kramers-Kronig relations may remain valid.

As an example of waveguide without energy leakage, let us consider a waveguide made up by two rigid walls separated by the distance h and filled with air (see e.g. Ref. [19]). The dispersion equation for such a waveguide is well known, and in terms of $n(\omega)$ it can be written in the form

$$n(\omega) = \left(1 - \frac{c^2(\infty)\pi^2 m^2}{\omega^2 h^2}\right)^{1/2} , \qquad (7)$$

where $m = 0, 1, 2, \ldots$. As can be seen from (7), the function $n(\omega)$ has branch points at the cut off frequencies $\omega_m = \pm c(\infty)\pi m/h$ and a simple pole at $\omega = 0$.

Kramers-Kronig relations for this case can be written in the form[15]

$$\operatorname{Re} n(\omega) - 1 = \frac{2}{\pi} V.P. \int_0^\infty \frac{\omega' \operatorname{Im} n(\omega')}{(\omega')^2 - \omega^2} d\omega', \qquad (8)$$

$$\operatorname{Im} n(\omega) = -\frac{2}{\pi} V.P. \int_0^\infty \frac{\omega [\operatorname{Re} n(\omega') - 1]}{(\omega')^2 - \omega^2} d\omega' + \frac{c(\infty)\pi m}{h\omega} . \qquad (9)$$

The only difference of these relations from the relations (3) and (4) is the presence of the last term in (9) caused by the pole at $\omega = 0$. As was first mentioned in Ref. 15, if the material attenuation in (7) is neglected, the dispersion relations (8) and (9) describe the relationship between the phase velocities of the propagating normal modes above the cut off frequencies (for $m \neq 0$) with the

attenuation decrements of the non-propagating (evanescent) waves below these cut off frequencies (the case of $m = 0$ corresponding to $n(\omega) = 1$ is trivial).
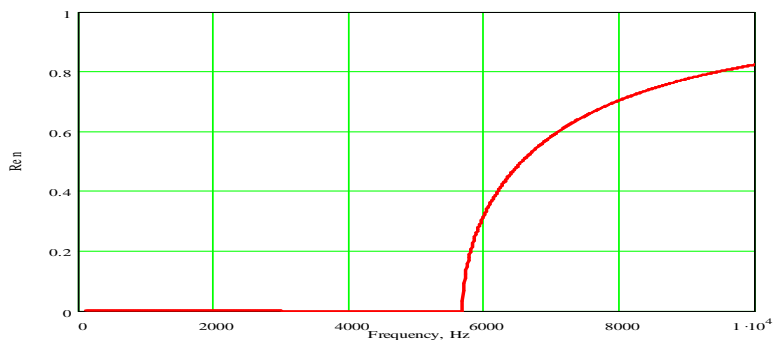

Figure 4. Real part of the refraction coefficient $n(\omega)$ for the first mode ($m = 1$) of a waveguide with rigid walls defined by Equation (7).
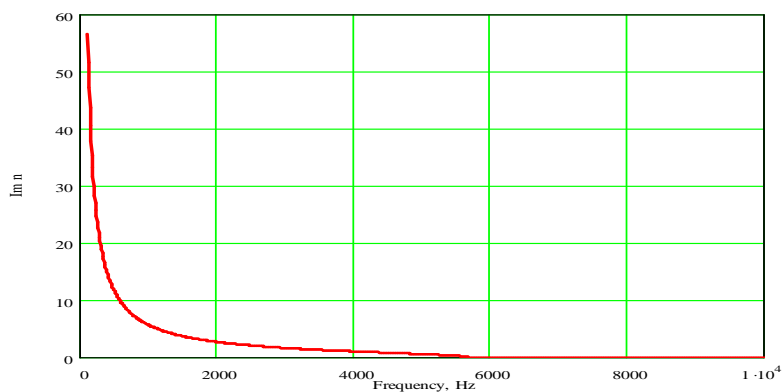

Figure 5. Imaginary part of the refraction coefficient $n(\omega)$ for the first mode ($m = 1$) of a waveguide with rigid walls defined by Equation (7).
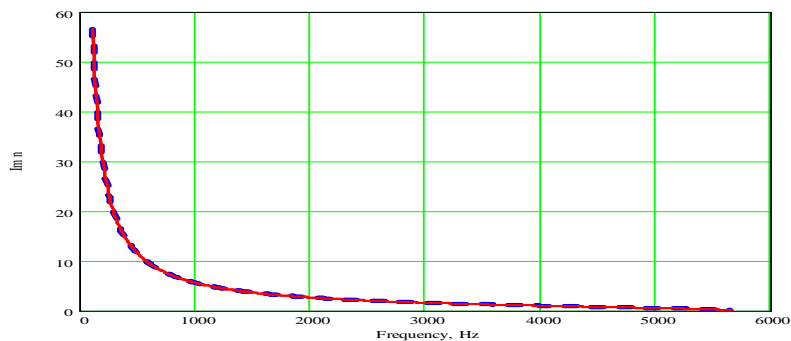

Figure 6. Imaginary part of the refraction coefficient $n(\omega)$ for the first mode ($m = 1$) of a waveguide with rigid walls defined by Equation (7) (solid curve) and calculated using Kramers-Kronig relation (9) (dashed curve).

Numerical calculations carried out for the first mode ($m = 1$) of a waveguide formed by two parallel rigid walls and filled with air illustrate the above points. Calculations have been carried out for $h = 0.03$ m and $c(\infty) = 340$ m/s. Functions $Re\ n(\omega)$ and $Im\ n(\omega)$ that have been calculated directly from the analytical expression (7) are shown in Figures 4 and 5 respectively. The results for $Im\ n(\omega)$ obtained from numerical integration using the dispersion relation (9) (dashed curve) are shown in Figure 6 together with the results directly calculated from the analytical expression (7) (solid curve, see also Figure 5). As one can see, the solid and dashed curves in Figure 6 are almost indistinguishable, which confirms validity of Kramers-Kronig relations in this case.

Note that Kramers-Kronig relations remain valid for waveguides with rigid walls also in the case of taking into account realistic material losses (e.g. if a waveguide is filled with a relaxing medium). This is important for verification of physical acoustic measurements of sound attenuation and velocity in different materials and substances. If modes with $m \neq 0$ are used, Kramers-Kronig relations connect the combined dispersion due to relaxation and waveguide propagation with the combined attenuation of propagating and evanescent waves.

Let us now discuss open waveguides (e. g. acoustic waveguides formed by horizontal layers in the ocean or atmosphere)[20] and consider the case when wave energy leakage from such waveguides is not possible. Typical examples of such waveguides are infinite linear or parabolic layers that can be considered as approximate models for real waveguides. Such layers always have ray turning points, which means that there is no escape for the wave energy. However, the behaviour of complex functions $n(\omega)$ at large ω for some of such waveguides still can be problematic. For example, they can depend on ω exponentially, in which case Kramers-Kronig relations may break down. The behaviour of $n(\omega)$ at large ω should be investigated for each specific open waveguide before applying Kramers-Kronig relations. For the same reason, Kramers-Kronig relations are generally not applicable to Lamb waves propagating in elastic plates (note that such plates do not belong to the class of open waveguides). These include the lowest order symmetric and antisymmetric Lamb modes, also called symmetric plate waves and flexural (bending) waves respectively.

In the case of open waveguides for which the possibilities of energy leakage can exist at some frequency ranges, the analytical properties of the complex refraction indexes of normal modes $n(\omega)$ can be broken down due to the appearance of branch points on the complex plain ω located away from the real axis ω. For this reason, Kramers-Kronig relations for such waveguides generally become invalid. Even though branch cut integrals arising in the process of contour integration could be added in this case to obtain modified Kramers-Kronig relations, this would not make any practical sense as the location of branch points is specific for every particular waveguide and is generally unknown. Examples of open waveguides with the potential possibilities of energy leakage include typical ocean waveguides for underwater acoustic modes[20], seismic waveguides formed by layered ground[21], etc.

4   KRAMERS-KRONIG RELATIONS FOR SURFACE WAVES

From the point of view of applicability of Kramers-Kronig relations, the case of surface waves is similar to the above-mentioned case of open waveguides. If the possibility of energy leakage exists for a particular type of surface waves, this is usually reflected in the appearance of branch points of the complex function $n(\omega)$ that are not located on the real axis ω. Like in the case of open waveguides, Kramers-Kronig relations are generally not applicable to such surface waves. Typical examples of such surface waves are Love waves (see e.g. Ref. 22), which are shear-horizontal elastic waves propagating in an elastic half space covered by an elastic layer. In their mathematical description, these waves are very similar to guided modes of open underwater acoustic waveguides discussed in the previous section. Energy leakage for such waves is possible, and therefore Kramers-Kronig relations are generally not applicable.

Another example is Rayleigh-type surface waves (sometimes called 'generalised Lamb waves') propagating in an elastic half space covered by a stiffer elastic layer that can accelerate the velocity of the Rayleigh waves with frequency until it becomes larger than shear wave velocity in the supporting half space, at which point the wave becomes leaky[22]. Note in this connection that Rayleigh waves in a homogeneous elastic half space exist at all frequencies and do not have velocity dispersion. Kramers-Kronig relations are, of course, applicable to such waves. In this case $n(\omega)=1$, and Kramers-Kronig relation (4) gives a correct albeit trivial result $Im\ n(\omega) = 0$, as expected. However, if material losses, e.g. due to relaxation, are taken into account, these relations become of practical importance since in many cases Rayleigh waves are used for physical acoustic investigations of materials. In this case Kramers-Kronig relations for Rayleigh waves are valid and they are not different from such relations for bulk longitudinal or shear elastic waves propagating in unbounded solids.

Finally, we make a few comments on the applicability of Kramers-Kronig relations to wedge acoustic waves, i.e. localised vibrations propagating along sharp edges of wedge-like elastic structures[22]. Although these waves are not surface waves in a strict sense (they propagate along a line of intersection of two surfaces, and therefore they are often called 'line waves'), it is convenient to discuss them here since many of their properties are similar to the properties of the above-mentioned surface waves and waves propagating in open waveguides. Moreover, the theory of wedge acoustic waves propagating in slender wedges of arbitrary profile can be constructed using the geometrical acoustics approach in the same way as it is done for normal waves in open waveguides[22, 23]. Therefore, like in the latter case, Kramers-Kronig relations have limited applicability to wedge acoustic waves. Similarly to the case of Rayleigh waves in a homogeneous half space, they are applicable to wedge acoustic waves propagating in wedges of linear profile (formed by intersection of two plains) as there is no dispersion in this case. However, for any other profiles dispersion of wedge waves does exist, and this may result both in the appearance of branch points associated with energy leakage and in not 'good' behaviour of the functions $n(\omega)$ at very large frequencies. In such cases Kramers-Kronig relations may become not applicable to wedge acoustic waves. Typical examples are a truncated wedge of linear profile[23] and a wedge of quadratic profile[24]. Note that in the latter case the theory is very similar to that for open underwater acoustic waveguides with a linear dependence of sound velocity on depth. The approximate solutions of the resulting dispersion equation show that $n(\omega)$ contains exponential terms at very large frequencies[24], which means that Kramers-Kronig relations become not applicable.

5   CONCLUSIONS

Kramers-Kronig dispersion relations have limited applicability to normal modes propagating in waveguides. As a rule, they may be applicable to waveguides from which energy leakage to the surrounding media is not possible, e.g. acoustic waveguides with rigid walls. If such waveguides contain ideal media (e.g. ideal liquids), Kramers-Kronig relations provide a general relationship between the velocities of propagating normal modes above the cut off frequencies and the attenuation decrements of non-propagating (evanescent) modes below the corresponding cut-off frequencies. If material losses in the media, e.g. due to relaxation, are taken into account, Kramers-Kronig relations for such waveguides remain valid and connect the combined dispersion due to relaxation and waveguide propagation with the combined attenuation of propagating and evanescent waves.

If waveguides are open, e.g. formed by layers with different sound velocities, and there is no energy leakage from such waveguides, applicability of Kramers-Kronig relations to normal modes of such waveguides still can be questionable, depending on 'good' behaviour of the mode refraction coefficient at high frequencies. If energy leakage from open waveguides is possible at a certain frequency range, Kramers-Kronig relations may break down. The same conclusions can be drawn for surface and wedge acoustic waves. If surface or wedge waves become leaky waves at certain frequency ranges or if their velocities do not show 'good' behaviour at high frequencies, Kramers-Kronig relations generally become invalid.

REFERENCES

1.   R.D.L. Kronig, On the theory of dispersion of X-rays, Journal of the Optical Society of America, 12, 547–557 (1926).
2.   H.A. Kramers, La diffusion de la lumiere par les atomes, Atti del Congresso Internazionale dei Fisici, Como, 2, 545–557 (1927).
3.   V.L. Ginzburg, Concerning the general relationship between absorption and dispersion of sound waves, Soviet Physics - Acoustics, 1, 32–41 (1955).
4.   M. O'Donnel, E.T. Jaynes, J.G. Miller, General relationships between ultrasonic attenuation and dispersion, Journal of the Acoustical Society of America, 63(6), 1935-1937 (1978).
5.   M. O'Donnell, E. T. Jaynes, and J. G. Miller, Kramers-Kronig relationship between ultrasonic attenuation and phase velocity, Journal of the Acoustical Society of America, 69(3), 696–701 (1981).
6.   R.l. Weaver, Y.-H. Pao, Dispersion relations for linear wave propagation in homogeneous and inhomogeneous media, Journal of Mathematical Physics, 22(9), 1909-1918 (1981).
7.   S. Temkin, Attenuation and dispersion of sound in bubbly fluids via the Kramers-Kronig relations, Journal of Fluid Mechanics, 211, 61-72 (1990)
8.   Z. Ye, Acoustic dispersion and attenuation in many spherical scatterer systems and the Kramers–Kronig relations, Journal of the Acoustical Society of America, 101(6), 3299-3305 (1997).
9.   H. J. Wintle, Kramers-Kronig analysis of polymer acoustic data, Journal of Applied Physics, 85, 44–48 (1999).
10.   K.R. Waters, M.S. Hughes, J. Mobley, G.H. Brandenburger, J.G. Miller, On the applicability of Kramers–Kronig relations for ultrasonic attenuation obeying a frequency power law, Journal of the Acoustical Society of America, 108, 556–563 (2000).
11.   K. R. Waters, B. K. Hoffmeister, Kramers-Kronig analysis of attenuation and dispersion in trabecular bone, Journal of the Acoustical Society of America, 118, 3912–3920 (2005).
12.   T. Pritz, Unbounded complex modulus of viscoelastic materials and the Kramers–Kronig relations, Journal of Sound and Vibration, 279, 687–697 (2005)
13.   J. Mobley, Simplified expressions of the subtracted Kramers–Kronig relations using the expanded forms applied to ultrasonic power-law systems, Journal of the Acoustical Society of America, 127(1), 166-173 (2010).
14.   H.M. Nussenzveig, Causality and Dispersion Relations, Academic Press, New York (1972).
15.   V.V. Krylov, Dispersion relations for refractive indices of normal waves, Moscow University Physics Bulletin, 37(2), 48-51(1982).
16.   M.W. Haakestad, J. Skaar, Causality and Kramers-Kronig relations for waveguides, Optics Express, 13(24), 9922-9934 (2005).
17.   V.A. Krasil'nikov, V.V. Krylov, Introduction to Physical Acoustics (in Russian), Nauka, Moscow (1984).
18.   A.G. Gurvitz, R. Kurant, Function Theory (in Russian), Nauka, Moscow (1968).
19.   M.A. Isakovich, General Acoustics (in Russian), Nauka, Moscow (1973).
20.   I. Tolstoy, C.S. Clay, Ocean Acoustics, McGraw-Hill, New York (1966).
21.   K. Aki, P.G. Richards, Quantitative Seismology: Theory and Methods, W.H. Freeman and Company, San Francisco (1980).
22.   S.V. Biryukov, Y.V. Gulyaev, V.V. Krylov, V.P. Plessky, Surface Acoustic Waves in Inhomogeneous Media, Springer-Verlag, Berlin Heidelberg (1995).
23.   V.V. Krylov, Geometrical-acoustics approach to the description of localized vibrational modes of an elastic solid wedge, Soviet Physics – Technical Physics, 35(2), 137-140 (1990).
24.   V.V. Krylov, Localized acoustic modes of a quadratic solid wedge, Moscow University Physics Bulletin, 45(6), 65-69 (1990).