On the second variation of the spectral zeta function of the Laplacian on homogeneous Riemaniann manifolds

by

Omenyi, Louis Okechukwu

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Dedication

To

my darling wife

Nnenna

and my lovely children

Kosisochi, Ihechiruru, Chisimdi and Onyinyechi.

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CONTENTS

Abstract

The spectral zeta function, introduced by Minakshisundaram and Pleijel in [36] and denoted by $\zeta_g(s)$, encodes important spectral information for the Laplacian on Riemannian manifolds. For instance, the important notions of the determinant of the Laplacian and Casimir energy are defined via the spectral zeta function. On homogeneous manifolds, it is known that the spectral zeta function is critical with respect to conformal metric perturbations, (see e.g Richardson ([47]) and Okikiolu ([41])). In this thesis, we compute a second variation formula of $\zeta_g(s)$ on closed homogeneous Riemannian manifolds under conformal metric perturbations. It is well known that the quadratic form corresponding to this second variation is given by a certain pseudodifferential operator that depends meromorphically on s. The symbol of this operator was analysed by Okikiolu in ([42]). We analyse it in more detail on homogeneous spaces, in particular on the spheres S^n. The case $n = 3$ is treated in great detail. In order to describe the second variation we introduce a certain distributional integral kernel, analyse its meromorphic properties and the pole structure. The Casimir energy defined as the finite part of $\zeta_g(-\frac{1}{2})$ on the n-sphere and other points of $\zeta_g(s)$ are used to illustrate our results. The techniques employed are heat kernel asymptotics on Riemannian manifolds, the associated meromorphic continuation of the zeta function, harmonic analysis on spheres, and asymptotic analysis.

CHAPTER 1

Introduction

The systematic use of the spectral zeta function to give sense to infinite series (regularization) is believed to be begun by Godfrey H. Hardy and John E. Littlewood from the second decade of the last century. Elizalde [22] among other literature posits that they actually established the convergence and equivalence of series regularized with the heat and zeta functions methods. As Hardy realized his surprise, Srinivasa Ramanujan had also found for himself the functional equation of the zeta function. In the 1930s, Torsten Carleman [14] went one step further by constructing the zeta function encoding the eigenvalues of the Laplacian of compact Riemannian manifold, for the case of a compact region of the plane. The most significant improvement, published in specialized literature in 1949, was due to Subharamiah Minakshisundaram and Åke Pleijel [36] who extended Carlemam's result, showing that for the Laplacian of a compact Riemannian manifold, the corresponding zeta function has a meromorphic continuation to the whole complex plane. This is what has come to be known and called Minakshisundaram-Pleijel spectral zeta function, (see e.g [23]). In the mid 1960s, Robert T. Seeley [52] extended these important results of Minakshisundaram and Pleijel to elliptic pseudodifferential operators on compact Riemannian manifolds, showing that for such operators, one can define the determinant using the zeta function regularization. Daniel B. Ray and

Isadore M. Singer [45] in 1971 used Seeley's results to define the determinant of a positive self-adjoint operator of the Laplace-type on a Riemannian manifold.

This thesis addresses the question of how a point $s = s_0$ behaves under volume-preserving conformal second-order variation of the spectral zeta function $\zeta_g(s)$ on closed n-dimensional homogeneous Riemannian manifold (M, g). The Casimir energy defined by $\zeta_g(-\frac{1}{2})$ on the n-sphere is used to illustrate our results, as this is of particular interest in physics.

The study of Casimir energy (or Vacuum energy as it is also referred to) is believed to originate from the work of Hendrik B. G. Casimir (1909 - 2000), who in the year 1948 pointed out the existence of a force between a pair of neutral perfectly conducting parallel plates. The Casimir energy may be thought-of as the energy difference due to the distortion of the vacuum; (see e.g [15] and [23]). The energy difference gives rise to what is known as the Casimir force. Although Casimir energy is a concept arising in quantum field theory with observable consequences in physics, research is on-going in the modern aspects of spectral geometry, formulating the notion in a purely mathematical framework; see e.g. [23] and the numerous literature cited there-in.

In most physics literature, the Casimir (Vacuum) energy, popularly denoted by E_{cas} is written as sum over the eigenvalues $\omega_k = \sqrt{\lambda_k}$ of the Laplacian on smooth functions on M, i.e $E_{\text{cas}} = \sum_k \omega_k$ which is the spectral zeta function at $s = -\frac{1}{2}$. Because the spectral zeta function depends on the choice of metric g, the Casimir energy $\zeta_g(-\frac{1}{2})$ is defined via the spectral zeta function as a function on the set of metrics on the manifold M, see [22]. However, this sum is usually divergent and has to be regularized. A very simple and elegant way of performing the regularization is via the analytic continuation of the zeta function. Sometimes, it happens that after the analytic continuation, the zeta function at the desired value has a pole. For instance, the meromorphic continuation of the spectral zeta function $\zeta_g(s)$ of the Laplacian on M computed in section 2 of chapter three of this thesis shows that the points $s = -\frac{1}{2}, \frac{1}{2}$ are simple poles for odd dimensions but are not for even dimensions. Thereby, further regularization techniques are required.

This work on variation of any point $s = s_0$ of the spectral zeta function e.g the Casimir energy is motivated by analogous work done for the determinant of the Laplacian. Ray and Singer [45] introduced and characterised the regularised determinant of the Laplacian as $\det \Delta_g = e^{-\zeta'(0)}$. Ray and Singer then studied its connections with topology through studying it as a function over the space of metrics on a fixed smooth manifold, M, an idea a physicist A. Polyakov generalised and applied to string theory [44]. The result of these works is the following variational formula, called then Polyakov-Ray-Singer variation formula:

$$\log\det\Delta_{g_\epsilon} - \log\det\Delta_g = -\frac{1}{6\pi}\Big[\frac{1}{2}\int_M |\nabla_g\dot{\phi}_0(x)|^2 dA_g + \int_M \dot{\phi}_0(x) k_g(x) dV_g\Big] + \log A_{g_\epsilon} - \log A_g.$$

Here M is a fixed smooth Riemannian manifold, g and $g_\epsilon = e^{\phi_\epsilon} g$ are two equal volume conformally equivalent smooth metrics on M and $\dot{\phi}_0(x) = \frac{\partial}{\partial\epsilon}\phi_\epsilon(x)|_{\epsilon=0}$. Then ∇_g, A, dA_g and k_g are the gradient, area, area element and Gaussian curvature associated with g, while A_{g_ϵ} is the area associated with (M, g_ϵ), etc.

More recently, Osgood, Philips and Sarnak in [43] found that among all fixed volume conformal class of metrics $\{g_\epsilon = e^{\phi_\epsilon} g\}$ on a Riemannian surface M, the constant curvature metric has maximal determinant. That is,

$$\frac{d}{d\epsilon}(-\log\det\Delta_{g_\epsilon})|_{\epsilon=0} = \frac{1}{12\pi}\int_M \dot{\phi}_g(x) k_g(x) dV_g = 0$$

if and only if k_g is constant since the area is constant. A similar result was obtained by Richardson [47] on 3-dimensional Riemannian manifolds. He found that the 3-sphere with the standard round metric is a local maximum for a fixed-volume conformal deformation $\{g_\epsilon\}$. In 2001, Kate Okikiolu [41] generalised the result of Richardson to all closed odd n-dimensional Riemannian manifolds. Note that in all the results cited above, the extremal properties of the manifolds under conformal variations depend on the dimensions of the manifolds.

The work in this thesis raises and addresses a related question about the behaviour of the second variation of the spectral zeta function at any point $s = s_0$ of the spectral zeta function. For example, "**how does the Casimir energy behave under such volume-preserving conformal variation of the metric of a smooth, compact and connected n-dimensional Riemannian manifold** (M, g)**?**" Our results are illustrated with the n-sphere. The techniques of the heat and spectral zeta kernels are utilized to investigate the spectral properties of the Laplacian and other Laplace-type operators on compact Riemannian manifolds with particular reference to the unit n-sphere. In particular, we computed the variation of the zeta function and the Casimir energy under conformal variation of the Riemannian metric. We find conditions that prove that the round metric g on the n-sphere, S^n, is a critical point for the Casimir energy under such variations of the metric.

Consider how a conformal change of metric affects the Laplacian. Let (M, g) be a smooth homogeneous Riemannian manifold and $0 < \rho \in C^\infty(M)$. Then the Laplacian with respect to the conformally equivalent metric $h = \rho g$ is given by

$$\Delta_h \psi = \rho^{-1} \Delta_g \psi + (1 - \frac{n}{2}) \rho^{-2} \mathrm{div}(\rho \nabla_g) \psi$$

with

$$\mathrm{div}(\rho \nabla_g) := g^{ij} (\partial_i \rho) \partial_j$$

where the so-called Einstein summation convention of summing over repeated indices is used. Then we can see that the Casimir energy is not invariant under change of metric. Consider for instance a scaling of the metric with a constant $c > 0$, one quickly sees that $\Delta_g \mapsto \frac{1}{c} \Delta_g$ and

$$\zeta_{cg}(s) = \sum_{k=1}^{\infty} \frac{c^s}{\lambda_k^s} = c^s \zeta_g(s).$$

So, the Casimir energy changes as

$$\zeta_{cg}(-\frac{1}{2}) = c^{-\frac{1}{2}} \zeta_g(-\frac{1}{2}).$$

Consequently, it becomes of much interest to study how the Casimir energy and other points of the spectral zeta function vary under more general deformation of the metric such as conformal perturbation of the Riemannian manifold, and in fact, it is sensible to fix the volume so as to factor out this trivial scaling.

Now, choose

$$\phi : M \times (-c, c) \to \mathbb{R}$$

a family of functions smooth in the first variable x and real analytic in the second ϵ. Write

$$\phi_\epsilon(x) = \phi(x, \epsilon) \text{ with } \phi_0 = 0.$$

Define the corresponding family of conformal metrics g_ϵ such that $g_\epsilon = e^{\phi_\epsilon} g$, with the condition that

$$g_\epsilon^{(1)} = \frac{\partial}{\partial \epsilon}(g_\epsilon)|_{\epsilon=0} = \dot{\phi}_0 g, \quad \dot{\phi}_0 \in C^\infty(M); \quad \text{where } \; \dot{\phi}_\epsilon = \frac{\partial}{\partial \epsilon}(\phi_\epsilon).$$

It is well known among other properties of such perturbation that there exists a sequence of eigenvalues $\{\Lambda_k(\epsilon)\} \subset \mathbb{R}$ (counted with multiplicities) and $\psi_k(\epsilon)$ on $C^\infty(M)$, such that $\Delta_{g(\epsilon)}\psi_k(\epsilon) = \Lambda_k(\epsilon)\psi_k(\epsilon)$ and $\Lambda_k(0) = \lambda_k$ where λ_k is the eigenvalue associated with the unperturbed metric g; see e.g Zelditch [67], Bando and Urakawa ([5]). One can now write the associated spectral zeta kernel of Δ_ϵ on M as

$$\zeta_{g_\epsilon}(s, x, y) = \sum_{k=1}^{\infty} \frac{\psi_{k,j}(\epsilon, x)\bar{\psi}_{k,j}(\epsilon, y)}{\big(\Lambda_k(\epsilon)\big)^s}; \quad \Re(s) > \frac{n}{2}.$$

Our first theorem is:

Theorem 1.0.1. *(Theorem 5.3.2 in the text): Let (M, g) be a smooth, compact and connected Riemannian manifold and Δ_g the Laplacian on it with eigenvalues $\{\lambda_k\}$ listed according to their multiplicities. If $\{g_\epsilon = e^{\phi_\epsilon} g\}$ is a family of volume-preserving conformally equivalent metrics, then the first order variation $\zeta_g^{(1)}(s) := \frac{\partial}{\partial \epsilon}\big(\zeta_{g_\epsilon}(s)\big)|_{\epsilon=0}$ of the spectral zeta function $\zeta_g(s)$ of Δ_ϵ, is given by*

$$\zeta_g^{(1)}(s) = s \int_M \dot{\phi}_0(x)\zeta_g(s, x, x)dV_g + \frac{1}{2}(\frac{n}{2} - 1)s \int_M (\Delta_g \dot{\phi}_0(x))\zeta_g(s + 1, x, x)dV_g.$$

The Casimir energy has the first order variation $\mathrm{FP}[\zeta_g^{(1)}(s)]\big|_{s=-\frac{1}{2}}$ *given by*

$$\begin{aligned}\mathrm{FP}[\zeta_g^{(1)}(s)]\big|_{s=-\frac{1}{2}} &= -\frac{1}{2}\int_M \dot{\phi}_0(x)\mathrm{FP}[\zeta_g(s,x,x)]\big|_{s=-\frac{1}{2}} dV_g \\ &- \frac{1}{4}(\frac{n}{2}-1)\int_M (\Delta_g\dot{\phi}_0(x))\mathrm{FP}[\zeta_g(s+1,x,x)]\big|_{s=-\frac{1}{2}} dV_g,\end{aligned}$$

where

$$\mathrm{FP}[f](s) := \begin{cases} f(s) \text{ if } s \text{ is not a pole} \\ \lim_{\epsilon\to 0}\left(f(s+\epsilon) - \frac{\text{Residue}}{\epsilon}\right), \text{ if } s \text{ is a pole of order 1.}\end{cases}$$

Some interesting results about the criticality of the metric g under the perturbations follow from this. First we have the following theorem.

Theorem 1.0.2. *(Theorem 5.3.6 in the text): If Δ_ϵ is the Laplacian on (M, g_ϵ) with zeta kernel $\zeta_g(s,x,y)$, then g is a critical point of the Casimir energy $\zeta_g(-\frac{1}{2})$ for all constant-volume conformal variations of the metric if* $\mathrm{FP}[\zeta_g(s,x,x)]\big|_{s=-\frac{1}{2}}$ *is a constant.*

A corollary now follows when we consider the class of homogeneous manifolds that comes from quotients of Lie groups with bi-invariant metrics. For example, the natural action of $SO(n+1)$ on the n-sphere S^n is transitive, hence $S^n \approx SO(n+1)/SO(n)$ is a homogeneous manifold.

Corollary 1.0.3. *(Corollary 5.3.8 in the text): The metrics on homogeneous smooth Riemannian manifolds are critical points of the variation of the Casimir energy $\zeta_g(-\frac{1}{2})$ under fix-volume conformal variation of the homogeneous metric.*

To decide the extremal nature of the canonical metric g on M for the volume- preserving conformal deformation $\{g_\epsilon\}$, we compute the second order variation of the spectral zeta function. The main result in this regard is the following, again relating to volume preserving conformal perturbations of the usual metric g on a closed homogeneous Riemannian manifold M.

Theorem 1.0.4. *(Theorem 6.1.1 in the text): Let $\{g_\epsilon = e^{\phi_\epsilon} g\}$ be a family of volume-preserving conformal metrics on a closed homogeneous Riemannian manifold M with its canonical metric*

g. Choose $\phi_\epsilon \in C^\infty(M)$ such that $\int_M \dot\phi_0 dV_g = 0$ and $\int_M (\dot\phi_0)^2 dV_g > 0$. Then for $\Re(s) > \frac{n}{2}$, the second order variation $\zeta_g^{(2)}(s) := \frac{\partial^2}{\partial\epsilon^2}\big(\zeta_{g_\epsilon}(s)\big)|_{\epsilon=0}$ of the spectral zeta function $\zeta_g(s)$ is given by

$$\begin{aligned}
\zeta_g^{(2)}(s) &= s\int_M\int_M \dot\phi_0(x)\Psi_{s-1}(x,y)\dot\phi_0(y)dV_g(x)dV_g(y) \\
&- (1-\frac{n}{2})s\int_M\int_M \dot\phi_0(x)\Psi_s(x,y)(\Delta_g\dot\phi_0(y))dV_g(x)dV_g(y) \\
&+ \frac{(n-2)^2}{16}s\int_M\int_M (\Delta_g\dot\phi_0(x))\Psi_{s+1}(x,y)(\Delta_g\dot\phi_0(y))dV_g(x)dV_g(y) \\
&- \frac{1}{8}(n+2)^2 s\frac{1}{V}\int_M\int_M \dot\phi_0(x)\dot\phi_0(y)\zeta(s,x,y)dV_g(x)dV_g(y) \\
&- \frac{1}{8}(n-2)^2 s\zeta_n(s+1)\frac{1}{V}\int_M \dot\phi_0(x)(\Delta_g\dot\phi_0(x))dV_g(x) \\
&+ (1-\frac{n}{2})s\zeta_n(s)\frac{1}{V}\int_M (\dot\phi_0(x))^2 dV_g(x),
\end{aligned}$$

where

$$\Psi_s(x,y) = \frac{1}{\Gamma(s)}\int_0^\infty\int_0^\infty (K(u,x,y)-\frac{1}{V})(K(v,x,y)-\frac{1}{V})(u+v)^{s-1}\mathrm{d}u\mathrm{d}v.$$

In order to answer the question of how a select point $s = s_0$ behaves under volume-preserving conformal second-order variation of the spectral zeta function $\zeta_g(s)$ on a fixed manifold (M, g); we constructed a meromorphic continuation of Ψ_s as a bi-distribution to the whole of the complex s-plane. The result is the next theorem.

Theorem 1.0.5. *(Theorem 6.2.3 in the text): As a distribution, $\Psi_s(f_1 \otimes f_2)$ has a meromorphic continuation to the whole complex s-plane given by*

$$\begin{aligned}
\langle\Psi_s(f_1\otimes f_2), h\rangle &= 2\int_M\int_M \zeta_g(s+1,x,y)f_1(x)f_2(y)\mathrm{d}V_g(x)\mathrm{d}V_g(y) \\
&+ \frac{1}{2\pi^{3/2}\Gamma(s)}\sum_{k=0}^{\infty}\frac{1}{k!}\Big\{\frac{2^{s+k+1/2}-4}{(2s+2k-3)(2s+2k-1)}\cdot h(0) \\
&+ \frac{3\times 2^{s+k+1/2}(2s+2k-7)+48}{(2s+2k-5)(2s+2k-3)}\cdot h''(0) \\
&+ \frac{30\big(2^{s+k+1/2}[4k^2+8k(s-4)+4s(s-8)+71]\big)}{(2s+2k-7)(2s+2k-5)(8s+8k+12)}\cdot h^{(\mathrm{iv})}(0)\Big\}
\end{aligned}$$

where h is a test function.

It has simple poles at $s = \frac{1}{2} - k, \quad k = 0, 1, 2, \cdots$ *with residues at each pole* $s = s_0$ *with* $s_0 = \frac{1}{2} - k$ *given by*

$$\mathrm{Res}_{s=s_0}\langle \Psi_s(f_1 \otimes f_2), h\rangle = \lim_{s \to s_0} \frac{1}{\Gamma(s_0)}\big[(s - s_0) f(s,k)\big]$$

where

$$\begin{aligned} f(s,k) &= \frac{1}{2\pi^{3/2}} \frac{1}{k!}\Big\{ \frac{2^{s+k+1/2} - 4}{(2s + 2k - 3)(2s + 2k - 1)} \cdot h(0) \\ &+ \frac{3 \times 2^{s+k+1/2}(2s + 2k - 7) + 48}{(2s + 2k - 5)(2s + 2k - 3)} \cdot h''(0) \\ &+ \frac{30\big(2^{s+k+1/2}[4k^2 + 8k(s - 4) + 4s(s - 8) + 71]\big)}{(2s + 2k - 7)(2s + 2k - 5)(8s + 8k + 12)} \cdot h^{(\mathrm{iv})}(0)\Big\}. \end{aligned}$$

The values of the distribution $\Psi_s(f_1 \otimes f_2)$ *at* $s = 0, -1, -2, \cdots, -k$; *with* $k \in \mathbb{Z}^+$ *are given by*

$$\Psi_s(f_1 \otimes f_2) = \begin{cases} 0 & \text{at} \quad s = 0 \\ \int_M f_1(x) f_2(x) \mathrm{d}V_g(x) & \text{at} \quad s = -1 \\ \int_M f_1(x) \Delta_g f_2(x) \mathrm{d}V_g(x) & \text{at} \quad s = -2 \\ \int_M f_1(x) \Delta_g^2 f_2(x) \mathrm{d}V_g(x) & \text{at} \quad s = -3 \\ \int_M f_1(x) \Delta_g^3 f_2(x) \mathrm{d}V_g(x) & \text{at} \quad s = -4 \\ \vdots & \qquad\qquad . \end{cases}$$

This thesis is organized as follows. Chapter one is the introduction while in chapter two, notations are fixed and some basic preliminary concepts explained with the intention of making the thesis more self-contained. In chapter three, spectral functions, namely the zeta functions, heat and the zeta kernels which are the basic tools used throughout the work are presented. The explicit computation of the Casimir energy of the Laplacian Δ_g and the Laplace-type operator $\Delta_g + \frac{n-1}{2}$ on the n-sphere is done in chapter four. Also, the heat and zeta kernels are expressed explicitly here in terms of the so-called Gegenbauer polynomials. Chapter five discusses the conformal variations of the Riemanninan metrics and criticality conditions of the Casimir energy vis-à-vis the variations. Finally, the thesis concludes in chapter six with the computation of the second variation of the spectral zeta function. Some concluding remarks relating the results of this work to other known works are also included.

CHAPTER 2

Preliminaries and basic concepts

This chapter comprises of basic notations, definitions and concepts used throughout this thesis.

2.1. Notations

The symbol " := " is adopted for equality by definition.

- The following sets of numbers are denoted thus: $\mathbb{N}$ - the set of positive integers, $\mathbb{N}_0$ - the set of non-negative integers, $\mathbb{Z}$ - the set of integers, $\mathbb{R}$ - the set of real numbers, $\mathbb{R}_+$ - the set of non-negative real numbers and $\mathbb{C}$ - the set of complex numbers.

- Let $x \in \mathbb{R}$, $[x]$ denotes the integer part of x.

- Let $m, n \in \mathbb{N}_0$; $m \geq n$, the binomial coefficient is given by

$$\binom{m}{n} = \frac{m!}{(m-n)!n!}$$

  and the notation for the double factorial is given by

$$m!! = \begin{cases} m(m-2)(m-4)\cdots 2 \text{ for } & m \text{ even} \\ m(m-2)(m-4)\cdots 1 \text{ for } & m \text{ odd} \\ 1 \quad \text{for} & m = 0. \end{cases}$$

For Riemannian manifolds, we have the following notations:

- The pair (M, g) shall denote a Riemannian manifold with the metric g throughout the work. $n \in \mathbb{N}$ is used for the dimension of the manifold.

- When the Riemannian manifold is the unit sphere of dimension n with the usual round metric also called "n-sphere" in this work, we write S^n. For $0 \neq x \in \mathbb{R}^n$, we have $x = ||x||y$ with $y \in S^{n-1}$. Note that the lower case alphabets x and y are used to indicate points both on $\mathbb{R}^n$ and on the unit sphere S^n. In any of the cases, we indicate which space we mean.

- The Euclidean distance between two points $x, y \in S^n$ is given by

$$||x - y|| = \sqrt{2(1 - x \cdot y)}$$

- The geodesic distance between two points $x, y \in S^n$ is the angle between x and y on S^n, i.e

$$\theta(x, y) := \arccos(x \cdot y) \in [0, \pi].$$

  It is also the arc-length of the shortest path connecting x and y on S^n.

- The volume of the n-sphere is denoted by either V_n or $|S^n|$ and the volume form by dV_n. We use V and dV_g for the volume and volume form on (M, g).

- The Laplacian on (M, g) is written as Δ_g while that on the n-sphere is Δ_n.

- The space of complex-valued or real-valued continuous functions on S^n is denoted by $C(S^n)$. This is a Banach space with canonical norm

$$||f||_\infty := \sup\{|f(x)| : x \in S^n\}.$$

  Similarly, the space of complex-valued or real-valued k-times continuously differentiable functions on S^n is denoted by $C^k(S^n)$ and $C^\infty(S^n)$ when they are infinitely differentiable.

- The space of complex-valued or real-valued square integrable functions on S^n is denoted by $L^2(S^n)$. This is a Hilbert space with canonical inner-product

$$\langle f_1, f_2\rangle_{L^2} = \int_{S^n} f_1\bar{f}_2 dV_n$$

and induced norm $||f||_{L^2} = \langle f, f\rangle_{L^2}^{1/2}$. Note, $L^2(S^n)$ is the completion of $C(S^n)$ with respect to this norm.

It is also convenient to use the multi-index notation. A multi-index with n components is

$$\alpha = (\alpha_1, \cdots, \alpha_n),\ \ \alpha_1, \cdots, \alpha_n \in \mathbb{N}_0.$$

When we indicate explicitly the dependence on dimension, we write $\alpha_{(n)}$ instead of α. The length of α is

$$|\alpha| = \sum_{j=1}^{n} \alpha_j.$$

We write $\alpha!$ to mean $\alpha_1! \cdots \alpha_n!$ and with $x = (x_1, \cdots, x_n)^T$, x^α is defined to be

$$x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}.$$

Similarly, with the gradient operator

$$\nabla_g = (\partial_{x_1}, \cdots, \partial_{x_n})^T \ \text{ where } \ \partial_{x_j} = \frac{1}{i}\frac{\partial}{\partial x_j},$$

we define

$$\nabla_g^\alpha := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.$$

Basic spectral functions used in this work are denoted thus:

- The heat kernel is denoted by $K(t, x, y)$ for $t > 0$, x and y in M.

- The Riemann zeta function is written as $\zeta_R(s)$ while the Hurwitz zeta function is written as $\zeta_H(s, a)$ with $s \in \mathbb{C}$ and a a positive integer.

- The spectral zeta function of the Laplacian on functions on (M, g) is denoted by $\zeta_g(s)$ and its kernel denoted by $\zeta_g(s, x, y)$ where $s \in \mathbb{C}$, x and y in M.

When the zeta function and zeta kernel are defined on S^n, we denoted them by $\zeta_{S^n}(s)$ and $\zeta_{S^n}(s,x,y)$ respectively.

Other notations will be explained at the section where they may be introduced.

2.2. Transforms

Here we give the definitions of basic transforms that will be used throughout this thesis.

Definition 2.2.1. *For any Lebesgue integrable function $f \in L^1(\mathbb{R}^n)$, the Fourier transform $\hat{f}$ is defined by*

$$\hat{f}(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} f(x) e^{-i\langle x,\xi\rangle} \mathrm{d}x \tag{2.2.1}$$

with inversion formula

$$f(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \hat{f}(\xi) e^{i\langle x,\xi\rangle} \mathrm{d}\xi. \tag{2.2.2}$$

Definition 2.2.2. *The Mellin transform $M(f)$ of a function f is defined by*

$$M(f)(s) = \int_0^\infty f(t) t^{s-1} \mathrm{d}t, \qquad s \in \mathbb{C}. \tag{2.2.3}$$

Definition 2.2.3. *The gamma function is the integral (Mellin) transform of $f(t) = e^{-t}$ over the right-half complex plane through the Euler integral*

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} \mathrm{d}t; \quad \Re(s) > 0 \tag{2.2.4}$$

with poles at $s = 0, -1, -2, -3, \cdots$.

The gamma function has a number of useful properties, for example the Euler reflection formula allows to obtain values of $\Gamma(s)$ in the left-half of the complex plane, namely

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}; \quad 0 < \Re(s) < 1. \tag{2.2.5}$$

Also it satisfies the recurrence relation $\Gamma(s+1) = s\Gamma(s)$. It is the generalization of the factorial function in the sense that $\Gamma(n+1) = n!$ for $n \in \mathbb{N}_0$. c.f: [36, 62, 18, 61] and [6].

2.3. Manifolds

Let M be a topological space. Firstly, we recall some basic topological notions. M is called **Hausdorff** if for any two distinct points $x, y \in M$, there exists disjoint open subsets $U, V \subset M$ containing x and y respectively. A covering $(U_\alpha)_{\alpha \in I}$ (I an arbitrary index set) is called **locally finite** if each $x \in M$ has a neighbourhood that intersects only finitely many U_α. M is said to be **paracompact** if any open covering possesses a locally finite refinement. This means that for any open covering $(U_\alpha)_{\alpha \in I}$ there exists a locally finite open covering $(U'_\beta)_{\beta \in I'}$ (I' an arbitrary index set) with

$$\forall \beta \in I' \; \exists \alpha \in I : U'_\beta \subset U_\alpha.$$

M is said to be **connected** if there are no two or more disjoint open subsets whose union is M. It is **second countable** if it admits a countable bases for its topology. M is **compact** if its every open covering has a subcovering. A map between topological spaces is called **continuous** if the preimage of any open set is again open. A bijective map which is continuous in both directions is called a **homeomorphism**. If the map is bijective and of class C^∞ with a differentiable inverse of class C^∞ then we say it is a **diffeomorphism**. For more details, one may see Jost [32].

Definition 2.3.1. *An n-dimensional chart on M is any pair (U, ψ), where U is an open subset of M and ψ is a homeomorphism of U onto an open subset of $\mathbb{R}^n$ called the image of the chart.*

Definition 2.3.2. *A Hausdorff, second countable, connected topological space M is called an n-dimensional topological manifold (with a countable basis) if any point of M belongs to an n-dimensional chart.*

Let M be an n-dimensional manifold. For any chart (U, φ) on M, the local coordinate system $(x^1, x^2, \cdots, x^n)$ is defined in U by taking the φ-pullback of the Cartesian coordinate system in $\mathbb{R}^n$. Consequently, one can say that a chart is an open set $U \subset M$ with a local coordinate system.

For any two charts (U, φ) and (V, ψ) on M for which the intersection $U \cap V$ is not empty, the map $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ is called the chart transition from one chart to the other.

A chart transition map on an n-dimensional manifold is called C^k if its k^{th} derivatives exist and are continuous for a positive integer $k \leq n$. When this condition holds for all positive integers, we say the map is C^∞ or simply "smooth". A family of charts on the manifold M is called a C^k-atlas when the associated chart transition is C^k and if it covers all of M. Two charts (U, φ) and (V, ψ) are said to be compatible if $U \cap V \neq \emptyset$ and the transition map $\psi \circ \varphi^{-1}$ is a diffeomorphism. Similarly, two C^k-atlases are called compatible if their union is again a C^k-atlas. The union of all compatible C^k-atlases determines a C^k-structure on M. We collect these notions together as the following definition.

Definition 2.3.3. *A differentiable n-dimensional manifold M is a connected paracompact Hausdorff topological space for which every point has a neighbourhood $\mathcal{U}$ that is homeomorphic to an open subset $\Omega \subset \mathbb{R}^n$. Such a homeomorphism $\psi : \mathcal{U} \to \Omega$ is called a chart. Again, a family $\{\mathcal{U}_\alpha, \psi_\alpha\}$ of charts for which the $\mathcal{U}_\alpha$ constitute an open covering of M is called an atlas. The atlas $\{\mathcal{U}_\alpha, \psi_\alpha\}$ of M is called differentiable if all charts transitions $\psi_\beta \circ \psi_\alpha^{-1}$: $\psi_\alpha(\mathcal{U}_\alpha \cap \mathcal{U}_\beta) \to \psi_\beta(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)$ are differentiable of class $C^\infty(M)$. A maximal differentiable atlas is called a differentiable structure and a manifold with differentiable structure is called a differentiable manifold; see [32] and c.f: [16, 49, 19] and [32].*

Definition 2.3.4. *An n-dimensional topological manifold with boundary is a Hausdorff second countable topological manifold M in which every point has a neighbourhood homeomorphic to an open subset of the upper half space $\mathbb{H}^n := \{(x^1, \cdots, x^n) \in \mathbb{R}^n : x^n \geq 0\}$.*

We denote the boundary by ∂M. If $\partial M = \emptyset$, we call M ***manifold without boundary****. If M is without boundary and in addition compact then we call it a* ***closed manifold****.*

2.3.1. Tangent Space

Let M be a smooth manifold. Following Bär [7] and Chavel [16], we let a linear map

$$\xi : C^\infty(M) \to \mathbb{R}$$

be such that at a point $x \in M$,

$$\xi(fg)(x) = \xi(f)(g(x)) + \xi(g)(f(x)) \text{ for all } f, g \in C^\infty(M). \tag{2.3.1}$$

We denote the set of all such maps by T_xM. For all $\xi, \eta \in T_xM$ and $\lambda \in \mathbb{R}$, we define the sum and scalar multiplication $\xi + \eta$ and $\lambda\xi$ so that T_xM is a linear space over $\mathbb{R}$.

Definition 2.3.5. *The linear space T_xM is called the tangent space of M at x and we denote by $TM := \sqcup_x T_xM$ the disjoint union of all the tangent spaces at the point x. TM is called the tangent bundle of M.*

Theorem 2.3.6. *If M is an n-smooth manifold, then the tangent space T_xM is an n-dimensional vector space.*

We will denote the components of ξ in local coordinate chart $(x^1, \cdots, x^n)$ by ξ^i and write

$$\xi(f) = \xi^i \frac{\partial f}{\partial x^i} \quad \forall f \in C^\infty(M). \tag{2.3.2}$$

An alternative notation for (2.3.2) which we will also adopt in this thesis is $\xi(f) = \frac{\partial f}{\partial \xi}$ and then the identity 2.3.2 takes the form

$$\frac{\partial f}{\partial \xi} = \xi^i \frac{\partial f}{\partial x^i} \tag{2.3.3}$$

which allows to think of ξ as a directional derivative at x and to interpret $\frac{\partial f}{\partial \xi}$ as a directional derivative; see e.g Grigor'yam [26].

Definition 2.3.7. *[26](vector field): A vector field on a smooth manifold M is a family $\{v(x)\}_{x \in M}$ of tangent vectors such that $v(x) \in T_xM$ for any $x \in M$. In local coordinates, it can be represented in the form $v(x) = v^i \frac{\partial}{\partial x^i}$.*

The vector field $v(x)$ is called smooth if all the functions v^i are smooth in any chart.

2.3.2. Riemannian Metrics

A Riemannian metric (also called Riemannian metric tensor) on M is a family of symmetric positive definite bilinear forms $g = \{g(x)\}$ on T_xM which depends on $x \in M$ smoothly. The metric enables to define an inner product $\langle .,. \rangle_{g(x)}$ on TM by $\langle \xi, \eta \rangle_{g(x)} \ \ \forall \ \xi, \eta \in T_xM$. Hence, T_xM becomes an Euclidean space. For any $\xi \in T_xM$, its length $||\xi|| = \sqrt{\langle \xi, \xi \rangle}$.

In local coordinates, the inner product has the form $\langle \xi, \eta \rangle_{g(x)} = g_{ij}(x)\xi^i\eta^j$ where $(g_{ij})_{ij=1}^n$ is a square symmetric positive-definite matrix expressing the metric in the local coordinates.

These can be summarised in form of definition thus:

Definition 2.3.8. *(Riemannian metric): A Riemannian metric on a smooth manifold M is an assignment of an inner product*

$$\langle .,. \rangle_{g(x)} : T_xM \times T_xM \to \mathbb{R}$$

for all $x \in M$ such that

(1.) $\langle \xi, \xi \rangle_{g(x)} > 0$ *for* $\xi \neq 0$ *(positive definite);*

(2.) $\langle \xi, \eta \rangle_{g(x)} = \langle \eta, \xi \rangle_{g(x)}$ $\forall \eta, \xi \in T_xM$ *(symmetric);*

(3.) $g_x(a_i\xi_i + a_j\xi_j, b_i\eta_i + b_j\eta_j) = \sum_{i,j=1}^n a_ib_jg_x(\xi_i, \eta_j)$
$\forall \xi_i, \eta_j \in T_xM$ *and* $\forall a_i, a_j, b_i, b_j \in \mathbb{R}$ *(bilinear);*

(4.) $\langle \xi, \eta \rangle_{g(x)} = 0$ *if and only if either* $\xi = 0$ *or* $\eta = 0$ *or* $\xi = 0 = \eta$ *(non-degenerate); and*

(5.) for all $x \in M$, there exist local coordinates $\{x^i\}$ such that $g_{ij}(x) = \langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \rangle_{g(x)}$ are smooth functions.

Definition 2.3.9. *(Riemannian Manifold): A Riemannian manifold is a pair (M, g), where g is a Riemannian metric on the smooth manifold M.*

We recall that a curve γ on M is a continuous map $\gamma : [a, b] \to M$. Let $\gamma : [a, b] \to M$ be a parametrized differentiable curve on M with velocity γ'. The length of γ is given by

$$L(\gamma) = \int_a^b \sqrt{g\langle \gamma'(t), \gamma'(t)\rangle} dt = \int_a^b ||\gamma'(t)|| dt; \quad t \in [a, b]. \tag{2.3.4}$$

Definition 2.3.10. *[34] The geodesic distance between $x, y \in M$ is defined by*

$$d(x, y) := \inf\{L(\gamma) : \gamma : [0, 1] \to M \text{ piecewise smooth, } \gamma(0) = x, \text{ and } \gamma(1) = y\} \tag{2.3.5}$$

that is, the shortest curve in M connecting x and y. The diameter of M is $\sup_{x,y \in M} d(x, y)$.

A topological manifold is complete if it is complete as a topological space. The geodesic in a complete Riemannian manifold go on indefinitely; that is, it is isometric to the real line.

Definition 2.3.11. *[34] Let $\xi \in T_xM$ for $x \in M$. Then there is a unique geodesic γ_ξ satisfying $\gamma_\xi(0) = x$ and $\gamma'_\xi(0) = \xi$. The exponential map at x is the map*

$$\exp_x : T_xM \to M, \quad \xi \mapsto \gamma_\xi(1)$$

where 1 is an identity point in M.

A Riemannian manifold M is geodesically complete if for all $x \in M$, the exponential map $\exp_x$ is defined for all $\xi \in T_xM$. This means that any geodesic $\gamma(t)$ starting from x is defined for all values of the parameter $t \in \mathbb{R}$.

The Hopf-Rinow theorem given below gives conditions for the metric to be complete.

Theorem 2.3.12. *(Hopf-Rinow): Let M be a connected Riemannian manifold. Then, the following are equivalent:*

(1.) The closed and bounded subsets of M are compact.

(2.) M is a complete metric space with the metric $d(x, y)$ defined by (2.3.5).

(3.) M is geodesically complete. That is, for every $x \in M$, the exponential map $\exp_x$ is defined on the entire tangent space T_xM.

A connected Riemannian manifold M can be thought-of as a complete metric space whose distance function is the arclength of the geodesic between any two points $x, y \in M$. The manifold is compact if and only if it is complete and has finite diameter; (see e.g. [7]).

2.4. The Laplacian

Let M be a smooth, compact and connected n-dimensional Riemannian manifold without boundary and let g be a smooth Riemannian metric on M. For a coordinate chart on M,

$$(x^1, x^2, \cdots, x^n) : \mathcal{U} \to \mathbb{R}^n, \quad (\mathcal{U} \subset M \ \text{ open}),$$

we represent g by the matrix-valued function (g_{ij}) where

$$g_{ij} = \langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \rangle_g$$

and $\langle \cdot, \cdot \rangle_g$ is the inner product on the tangent space $T_x M$.

The volume form dV_g of (M, g) is defined as $dV_g = \sqrt{|g|}dx$; with $|g| = \det(g_{ij})$ and $dx = dx^1 \wedge \cdots \wedge dx^n$. To give definition of the Laplacian on smooth functions over the Riemannian manifold M, we need the following definitions.

Definition 2.4.1. *(Differential)[26]: For a fixed $x \in M$, let f be a smooth function in a neighbourhood of x. The differential df of f at x is a linear functional on $T_x M$ given by the pairing*

$$\langle df, \xi \rangle = \xi(f) \quad \text{for any } \xi \in T_x M. \tag{2.4.1}$$

Hence, df is an element of the dual space $T_m^* M$, (called the cotangent space). Elements of $T_x^* M$ are called **covectors**. A basis $\{e_1, \cdots, e_n\}$ in $T_x M$ has dual basis $\{e^1, \cdots, e^n\}$ in $T_x^* M$ which is defined by

$$\langle e^i, e_j \rangle = \delta_j^i = \begin{cases} 1; & j = i \\ 0; & j \neq i. \end{cases}$$

For example, the basis $\{\frac{\partial}{\partial x^i}\}$ has dual basis $\{dx^i\}$ because $\langle dx^i, \frac{\partial}{\partial x_j} \rangle = \delta_j^i$.

The covector df can be represented in the basis $\{dx^i\}$ as follows:

$$df = \frac{\partial f}{\partial x^i} dx^i \tag{2.4.2}$$

that is, the partial derivatives $\frac{\partial f}{\partial x_i}$ are the components of the differential df. Indeed,

$$\langle \frac{\partial f}{\partial x^i} dx^i, \frac{\partial}{\partial x_j} \rangle = \frac{\partial f}{\partial x^i} \langle dx^i, \frac{\partial}{\partial x_j} \rangle = \frac{\partial f}{\partial x^i} \delta_j^i = \langle df, \frac{\partial}{\partial x_j} \rangle.$$

Definition 2.4.2. *(Gradient): For any smooth function f on M, its gradient $\nabla_g f$ at a point $x \in M$ is defined by*

$$(\nabla_g f)(x) = g^{-1}(x)df(x) \tag{2.4.3}$$

where $g^{-1}(x) := g^{ij}(x)$ is the inverse of g. If we let $\xi = \nabla_g f(x)$ then for any $\eta \in T_xM$ one writes

$$\langle \nabla_g f, \eta \rangle_g = \langle df, \eta \rangle = \frac{\partial f}{\partial \eta} \tag{2.4.4}$$

which is another way of defining the gradient.

In local coordinates,

$$(\nabla_g f)^i = g^{ij} \frac{\partial f}{\partial x^j}.$$

If f_1 on M is another smooth function, set $\eta = \nabla f_1$ then,

$$\langle \nabla_g f, \nabla_g f_1 \rangle = \langle df, \nabla_g f_1 \rangle = g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial f_1}{\partial x^j}. \tag{2.4.5}$$

At this point, we may recall again that the Riemannian measure (volume form) $dV_g(x)$ on (M, g) can be represented in local coordinates as $\sqrt{|g|}dx^1 \wedge \cdots \wedge dx^n$ where $|g| = \det(g_{ij})$. We have the next theorem.

Theorem 2.4.3. *([26])(Divergence theorem): Let v be any smooth vector field on M and dV a Riemannian measure. Then, there exists a unique smooth function on M called divergence and denoted by* $\mathrm{div} v$ *such that the following identity holds:*

$$\int_M (\mathrm{div} v) f dV = -\int_M \langle v, \nabla_g f \rangle dV \qquad \forall f \in C_0^\infty(M). \tag{2.4.6}$$

In local coordinates,

$$\mathrm{div} v = \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^j} \left(\sqrt{|g|}\ v^j \right). \tag{2.4.7}$$

Note also that for any continuous function f on M if $\int_M f\psi dV = 0$ for all $\psi \in C_0^\infty(M)$ then $f \equiv 0$.

Proposition 2.4.4. *Let (M, g) be a Riemannian manifold without boundary. Then for every smooth vector field v on M, $\int_M \operatorname{div} v \, dV_g = 0$; where dV_g is the volume form on M induced by the metric g.*

The Laplacian on smooth functions on (M, g) is the operator

$$\Delta_g : C^\infty(M) \to C^\infty(M) \tag{2.4.8}$$

defined in local coordinates by

$$\Delta_g = -\operatorname{div}(\text{grad}) = -\frac{1}{\sqrt{|g|}} \sum_{i,j} \frac{\partial}{\partial x^i} (\sqrt{|g|} g^{ij} \frac{\partial}{\partial x^j}) \tag{2.4.9}$$

where g^{ij} are the components of the dual metric on the cotangent bundle $T_p^* M$.

Lemma 2.4.5. *The Laplacian on the space of compactly supported smooth functions on M with inner product*

$$\langle f, f_1 \rangle_{L^2(M)} = \int_M f(x)\overline{f_1(x)} dV_g$$

is symmetric.

The operator Δ_g extends to a self-adjoint operator on $L^2(M) \supset H^2(M) \to L^2(M)$ with compact resolvent. This implies that there exists an orthonormal basis $\{f_k\} \in L^2(M)$ consisting of eigenfunctions such that

$$\Delta_g f_k = \lambda_k f_k \tag{2.4.10}$$

where the eigenvalues are listed with multiplicities

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_k \leq \cdots \nearrow \infty; \tag{2.4.11}$$

see for example, ([26], [16] and [35]). The Laplacian Δ_g thus, has one-dimensional null space consisting precisely of constant functions.

Proposition 2.4.6. *Let $f_1, f_2 \in M$ be smooth. Then,*

(i.) $\Delta_g(f_1 + f_2) = \Delta_g(f_1) + \Delta_g(f_2)$,

(ii.) $\Delta_g(f_1 f_2) = f_2 \Delta_g(f_1) + f_1 \Delta_g(f_2) - 2\langle \nabla_g f_1, \nabla_g f_2 \rangle$.

2.4.1. The Laplacian on the unit n-sphere

The n-dimensional sphere of radius r, $r \in \mathbb{R}^+$ is the set of points in $\mathbb{R}^{n+1}$ at a distance r from a given central point; i.e $S^n(r) = \{x \in \mathbb{R}^{n+1} : \|x\| = r\}$. We call S^n a unit n-sphere or simply an n-sphere when the radius $r = 1$ and write the unit n-sphere as the set

$$S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}. \tag{2.4.12}$$

The n-sphere is an n-dimensional compact manifold in $(n+1)$-space of constant positive sectional curvature, namely $+1$, $\quad n \geq 2$. So, in particular, the 0-sphere, 1-sphere and the 2-sphere are respectively a pair of points on a line segment, a circle on a plane and the ordinary sphere in 3-dimension.

Proposition 2.4.7. *(Volume of S^n): The volume of the unit n-sphere is given by*

$$V_n = |S^n| = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})}, \tag{2.4.13}$$

where Γ is the Euler gamma function (2.2.4).

Observe therefore that from this proposition, we have $|S^0| = 2$; $|S^1| = 2\pi$; $|S^2| = 4\pi$; $|S^3| = 2\pi^2$; $|S^4| = \frac{8\pi^2}{3}$; $|S^5| = \pi^3$ etc, as is well known.

Let $f \in S^n$ be any function on the n-sphere and $\tilde{f}$ be its extension to an open neighbourhood of S^n that is constant along rays from the centre of S^n. We say that $f \in C^2(S^n)$ if $\tilde{f}$ is a C^2 function of that neighbourhood. For such functions (not containing $\{0\}$) on S^n the Laplacian Δ_n equals

$$\Delta_n f = \Delta_g \tilde{f} \tag{2.4.14}$$

where Δ_g on the right-hand side of (2.4.14) is the usual Laplacian in $\mathbb{R}^{n+1}$.

In $\mathbb{R}^n$, $n \geq 2$, every point $x \neq 0$ can be represented in polar coordinates as a couple (r, θ) where $r := |x| > 0$ is the polar radius and $\theta := \frac{x}{|x|} \in S^{n-1}$ is the polar angle.

Claim 2.4.8. *[26]: The canonical metric $g_{\mathbb{R}^n}$ on $\mathbb{R}^n$ has the following representation in polar coordinates: $g_{\mathbb{R}^n} = dr^2 + r^2 g_{S^{n-1}}$ where $g_{S^{n-1}}$ is the canonical spherical metric on S^{n-1}.*

Note that the metric $g_{S^{n-1}}$ is obtained by restricting the metric $g_{\mathbb{R}^n}$ to S^{n-1}. On S^{n-1}, the polar coordinate is $(\theta^1, \cdots, \theta^{n-1})$ whilst $r = 1$ and $dr = 0$. Indeed, for any $\xi \in T_x S^{n-1}$, $\langle dr, \xi \rangle = \xi(r) = \xi(r|_{S^{n-1}}) = \xi(1) = 0$.

Consider now the polar coordinates on S^n : $(\theta^1, \cdots, \theta^n)$. Let p be the north pole and q be the south pole of S^n, i.e ($p = (0, 0, \cdots, 0, 1)$) and $q = -p$. For any $x \in S^n \setminus \{p, q\}$, define $r \in (0, \pi)$ and $\theta \in S^{n-1}$ by $\cos r = x^{n+1}$ and $\theta = \frac{x'}{|x'|}$ where x' is the projection of x onto $\mathbb{R}^n = \{x \in \mathbb{R}^{n+1} : x^{n+1} = 0\}$. Clearly, the polar radius is the angle between the position vectors x and p, and r can be regarded as the latitude of the point x measured from the pole. The polar angle θ can be regarded as the longitude of the point x; see figure (2.1).

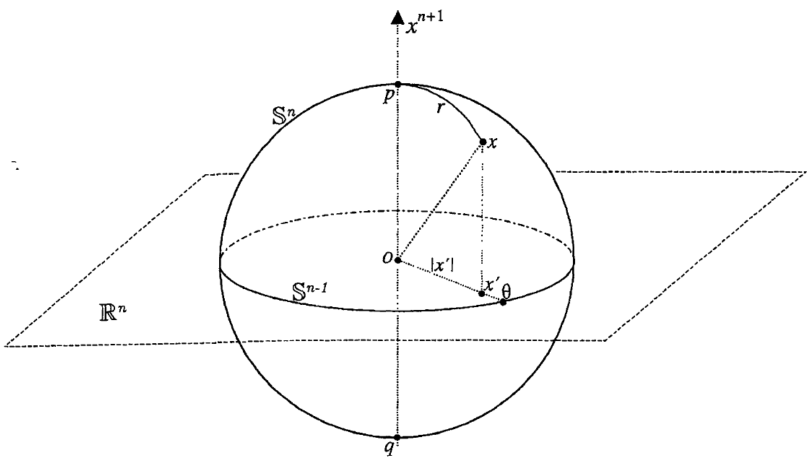

Figure 2.1: Polar coordinates on S^n [26].

Claim 2.4.9. *[26]: The canonical metric g_{S^n} on S^n has the following representation in polar coordinates: $g_{S^n} = dr^2 + \sin^2 r g_{S^{n-1}}$.*

In the polar coordinates, the Riemannian measure on S^n is given by $dV = \sin^{n-1} r dr d\theta$.

For the unit n-sphere, the Laplacian (2.4.14) in polar coordinates reduces to

$$\Delta_n = \frac{1}{\sin^{n-1}\theta} \frac{\partial}{\partial \theta} \{\sin^{n-1}\theta \frac{\partial}{\partial \theta}\} + \frac{1}{\sin^2 \theta} \Delta_{n-1} \tag{2.4.15}$$

where Δ_{n-1} is the Laplacian on S^{n-1}.

Example 2.4.10. *The Laplacian in polar coordinates* (θ, ϕ) *on* S^2 *endowed with the round metric*

$$g_{S^2} = d\theta^2 + \sin^2\theta d\phi^2 = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}$$

using equation (2.4.9) is $\Delta_2 = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\{\sin\theta\frac{\partial}{\partial\theta}\} + \frac{1}{\sin^2\theta}\Delta_1$ *where* $\Delta_1 = \frac{\partial^2}{\partial\theta^2}$ *is the Laplacian on* S^1.

On S^3*, where the round metric is*

$$g_{S^3} = d\theta^2 + \sin^2\theta d\phi^2 + \sin^2\theta\sin^2\phi d\psi^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sin^2\theta & 0 \\ 0 & 0 & \sin^2\theta\sin^2\phi \end{pmatrix}$$

using equation (2.4.9) is $\Delta_3 = \frac{1}{\sin^2\theta}\frac{\partial}{\partial\theta}\{\sin^2\theta\frac{\partial}{\partial\theta}\} + \frac{1}{\sin^2\theta}\Delta_2$ *where* Δ_2 *is the Laplacian on* S^2.

Continuing this way, one arrives at (2.4.15).

2.4.2. Eigenfunctions of the Laplacian on the n-spheres.

Following ([29], [53], [54], [27] and [3]), we give a brief discussion of the harmonic homogeneous polynomials restricted to the n-sphere which are the eigenfunctions of the Laplacian on S^n.

Definition 2.4.11. *A function* $f : \mathbb{R}^n \to \mathbb{C}$ *is called homogeneous of degree* k *if it satisfies* $f(tx) = t^k f(x)$ *for all* $x \in \mathbb{R}^n$ *and* $t > 0$ *fixed.*

Definition 2.4.12. *Let* $\mathcal{P}_k(n)$ *denote the space of homogeneous polynomials of degree* k *in* $(n+1)$ *variables. The space* $\mathcal{H}_k(n) := \{p_k \in \mathcal{P}_k(n) : \Delta_g p_k = 0, p_k \text{ homogeneous}\}$ *is called the space of harmonic homogeneous polynomials.*

Note, if $p_k \in \mathcal{H}_k(n)$ then $p_k(x) = |x|^k \cdot p_k(\frac{x}{|x|})$; where $\frac{x}{|x|} \in S^n$; $x \neq 0$.

Remark 2.4.13. *The following are basic properties of the harmonic homogeneous polynomials.*

- *If* $p_k\big|_{S^n} = q_k\big|_{S^n}$; i.e $p_k(x) = q_k(x) \quad \forall\, x \in S^n$*, then*

$$p_k(x) = |x|^k p_k(\frac{x}{|x|}) = |x|^k q_k(\frac{x}{|x|}) = q_k(x) \quad \forall x \neq 0; \Rightarrow p_k = q_k$$

since they are both polynomials.

- *The space $\mathcal{H}_k(S^n) := \{p_k \in \mathcal{H}_k(n) : \Delta_n p_k = 0\}$ is called the space of spherical harmonic homogeneous polynomials.*

- *It is known that there is a linear isomorphism between $\mathcal{H}_k(n)$ and $\mathcal{H}_k(S^n)$.*

- *Every polynomial p_k of degree k in n variables can be uniquely written as*

$$p_k(x) = \sum_{|\alpha|=k} c_\alpha x^\alpha \text{ for } \alpha \text{ a multiindex and } c_\alpha \in \mathbb{C}.$$

- *The space of polynomials p_k is a vector space of dimension $\begin{pmatrix} n+k-1 \\ n \end{pmatrix}$.*

Definition 2.4.14. *The Hermitian form on $p_k \in \mathcal{P}_k(n)$ is defined to be the differential operator*

$$p_k(\partial) = \sum_{|\alpha|=k} c_\alpha \frac{\partial^k}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}. \tag{2.4.16}$$

So for any two polynomials $p_k, q_k \in \mathcal{P}_k(n)$, let $\langle p_k, q_k \rangle := p_k(\partial)\bar{q}_k$.

Thus,

$$\langle p_k, q_k \rangle = \langle \sum_{|\alpha|=k} a_\alpha x^\alpha, \sum_{|\alpha|=k} b_\alpha x^\alpha \rangle = \sum_{|\alpha|=k} \langle a_\alpha, b_\alpha \frac{\partial^k}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}} x^\alpha \rangle = \sum_{|\alpha|=k} \alpha! a_\alpha \bar{b}_\alpha.$$

Proposition 2.4.15. *Let $p \in \mathcal{P}_k$. Write $p = |x|^k f(x)$. Then,*

$$p \in \mathcal{H}_k(S^n) \Leftrightarrow \Delta_n f(x) + k(k+n-1) f(x) = 0. \tag{2.4.17}$$

Theorem 2.4.16.

(i.) The map $\Delta_g : \mathcal{P}_k(n) \to \mathcal{P}_{k-2}(n)$ is surjective for all $n, k \geq 2$.

(ii.) We have the (so-called Calderon-decomposition) orthogonal decomposition

$$\begin{aligned} \mathcal{P}_k(n) &= \mathcal{H}_k(n) \oplus |x|^2 \mathcal{H}_{k-2}(n) \oplus \cdots \oplus |x|^{2j} \mathcal{H}_{k-2j}(n) \oplus \\ \cdots &\oplus |x|^{2[k/2]} \mathcal{H}_{[k/2]}(n) = \oplus_{0 \leq j \leq [k/2]} |x|^{2j} \mathcal{H}_{k-2j}(n). \end{aligned} \tag{2.4.18}$$

For proof, one may see ([37]).

The theorem (2.4.16) above has a number of important corollaries. Since every polynomial in $(n+1)$ variables is the sum of homogeneous polynomials, we have the following:

Corollary 2.4.17. *The restriction to S^n of every polynomial in $n \geq 1$ variables is a sum of restrictions to S^n of harmonic polynomials.*

Theorem 2.4.18. *The n-dimensional spherical harmonic polynomials $\mathcal{H}_k$ constitute a subset of a complete orthonormal system of functions in $\mathcal{P}_k$ with respect to the $L^2(S^n)$ inner product.*

For proof, one can see [50], pages 350-351.

The restriction of elements of $\mathcal{H}_k$ to S^n are called **spherical harmonic polynomials of degree k**, and are therefore eigenfunctions of Δ_n with eigenvalues $k(k+n-1)$.

Corollary 2.4.19. *[53] The dimension $d_k(n)$ of the space of harmonic polynomial $\mathcal{H}_k$ is given by the formula*

$$d_k(n) = \binom{k+n}{n} - \binom{k+n-2}{n}. \tag{2.4.19}$$

Lemma 2.4.20. *The multiplicities $d_k(n)$ of the eigenspace of the spectrum $\{\lambda_k\}$ can be expressed as*

$$d_k(n) = \frac{(2k+n-1)(k+n-2)!}{k!(n-1)!} \tag{2.4.20}$$

where $k \in \mathbb{N}_0$ and $n \geq 1$ is the dimension of the manifold S^n.

Proof. It is clear that

$$\begin{aligned} d_k(n) &= \binom{k+n}{n} - \binom{k+n-2}{n} = \frac{(k+n)!}{k!n!} - \frac{(k+n-2)!}{(k-2)!n!} \\ &= \frac{(k+n-2)!}{n!}\left[\frac{(k+n)(k+n-1)}{k!} - \frac{1}{(k-2)!}\right] \end{aligned}$$

which simplifies as

$$\begin{aligned} &\frac{(k+n-2)!}{n!}\left[\frac{(k+n)(k+n-1)}{k!} - \frac{1}{(k-2)!}\right] = \frac{(k+n-2)!}{n!(k-2)!}\left[\frac{(k+n)(k+n-1)}{k(k-1)} - 1\right] \\ &= \frac{(k+n-2)!}{k!}\frac{n}{n!}(2k+n-1) = \frac{(2k+n-1)(k+n-2)!}{k!(n-1)!} \end{aligned}$$

□

2.5. Conformal perturbation of metrics on M

Definition 2.5.1. *(Pull-back Riemannian metric)([7]): Let (M, g) and $(N, \tilde{g})$ be two Riemannian manifolds and let $\Phi : M \to N$ be a smooth diffeomorphism. The pull-back Riemannian metric $\Phi^* \tilde{g}$ on M is defined by $(\Phi^* \tilde{g})_x \langle \xi, \eta \rangle := \tilde{g}_{\Phi(x)} \langle d_x \Phi(\xi), d_x \Phi(\eta) \rangle$ for all $x \in M$ and $\xi, \eta \in T_x M$; where $d_x \Phi(\cdot)$ is the differential of $\Phi(\cdot)$ at the point $x \in M$.*

Note, $\Phi^* \tilde{g}$ is a Riemannian metric on M since $\tilde{g}$ is. Indeed, $\Phi^* \tilde{g}$ is the unique Riemannian metric on M for which $\Phi : M \to N$ is an isometry.

Definition 2.5.2. *(Conformal mapping)([7]): Let (M, g) and $(N, \tilde{g})$ be two Riemannian manifolds with $g = \sum_{i,j} g_{ij} dx^i dx^j; \quad \tilde{g} = \sum_{i,j} \tilde{g}_{ij} dx^i dx^j$ respectively. A smooth diffeomorphism $\Phi : M \to N$ is called conformal (angle - preserving) if there exists a positive function $\psi : M \to \mathbb{R}$ such that $\Phi^* \tilde{g} = \psi \cdot g$.*

If (M, g) is an n-dimensional smooth and connected Riemannian manifold with metric tensor g, the Riemannian metric g defines, in the tangent space at each point of the manifold, the inner product $\langle \xi, \eta \rangle_{g(x)}; \quad \xi, \ \eta \in T_x M$ at the point $x \in M$. The angle θ between any two tangent vectors ξ, η is given by

$$\cos \theta = \frac{\langle \xi, \eta \rangle_{g(x)}}{\sqrt{\langle \xi, \xi \rangle_{g(x)}} \sqrt{\langle \eta, \eta \rangle_{g(x)}}}.$$

Let g and $\tilde{g}$ be two metrics on M. If the angles between two tangent vectors with respect to g and $\tilde{g}$ are always equal to each other at each point of the manifold, we say g and $\tilde{g}$ are conformal.

The preceding definition can be thought of as a special case when $M = N$: the two metrics g and $\tilde{g}$ on M are called conformal if there exists a positive function $\Phi : M \to M$ such that $\tilde{g} = \Phi g$. In this case, we say (M, g) and $(M, \tilde{g})$ are conformal.

The definition also implies that the angle between any two tangent vectors $\xi, \ \eta \in T_x M$ at the point $x \in M$ is the same as the angle between $d_x \Phi(\xi), d_x \Phi(\eta) \in T_{\Phi(x)}(M)$; where the conformal factor Φ in general depends on x.

2.6. Distributions on M

In addition to conformal structures on M, the notion of distribution can as well be ascribed to such manifolds. We therefore briefly clarify the general notion of distribution on any smooth manifold M. To do this, define the space of test functions, which we denote by $\mathcal{D}(M)$, as $C^\infty(M)$ with the following notion of convergence: $\psi_k \to \psi$ if the following two conditions are satisfied:

(1). In any chart $\mathcal{U}$ and for any multi-index α, $\partial^\alpha \psi_k \to \partial^\alpha \psi$ in $\mathcal{U}$.

(2). All supports Suppψ_k of ψ_k are contained in a compact subset of M.

A distribution is a continuous linear functional on $\mathcal{D}(M)$. If u is a distribution then its value at a function $\psi \in \mathcal{D}(M)$ is denoted by the pairing $\langle u, \psi \rangle$. The space $\mathcal{D}'(M)$ of all distributions is clearly linear. The convergence in $\mathcal{D}'(M)$ is defined as $u_k \to u$ if $\langle u_k, \psi \rangle \to \langle u, \psi \rangle$ for all $\psi \in \mathcal{D}(M)$.

2.7. Duhamel's formula

To compute the second-order variation of the spectral zeta function, we shall need the Duhamel's formula:

Theorem 2.7.1. *(Duhamel's formula). Let $A(\epsilon)$ be a matrix-valued smooth function in $\epsilon \in (0, t)$. Then*

$$\frac{d}{d\epsilon} e^{tA(\epsilon)} = \int_{u=0}^{t} e^{uA(\epsilon)} A'(\epsilon) e^{(t-u)A(\epsilon)} du. \tag{2.7.1}$$

Proof. Using Taylor's formula with remainder, i.e for a smooth function f with

$$f(\epsilon + h) = f(\epsilon) + f'(\epsilon)h + \int_{\epsilon}^{\epsilon+h} (\epsilon + h - \tau) f''(\tau) d\tau$$

we write

$$tA(\epsilon + h) = tA(\epsilon) + tA'(\epsilon)h + th^2 \int_0^1 (1 - x) A''(\epsilon + hx) dx$$

where we changed variable τ to $\epsilon + hx$ and set $f(\epsilon) = tA(\epsilon)$ with $t \equiv 1$ fixed in the Taylor's formula.

Now define

$$E(u) = e^{uf(\epsilon+h)}e^{(1-u)f(\epsilon)}$$

then, we see that

$$\int_0^1 E'(u)du = E(1) - E(0) = e^{tA(\epsilon+h)} - e^{tA(\epsilon)}.$$

So,

$$\begin{aligned}
\int_0^1 E'(u)du &= \int_0^1 \left[tA(\epsilon+h)e^{utA(\epsilon+h)}e^{(1-u)tA(\epsilon)} - e^{utA(\epsilon+h)}tA(\epsilon)e^{(1-u)tA(\epsilon)}\right]du \\
&= \int_0^1 e^{utA(\epsilon+h)}\left[tA(\epsilon+h) - tA(\epsilon)\right]e^{(1-u)tA(\epsilon)}du \\
&= t\int_0^1 e^{utA(\epsilon+h)}\left[A(\epsilon+h) - A(\epsilon)\right]e^{(1-u)tA(\epsilon)}du
\end{aligned}$$

where $E'(u)$ has been computed using the Leibnitz rule and the fact that for any constant square matrix C, it holds that $\frac{d}{du}e^{uC} = Ce^{uC} = e^{uC}C$.

Thus,

$$\begin{aligned}
\frac{1}{h}\left[e^{utA(\epsilon+h)} - e^{utA(\epsilon)}\right] &= \int_0^1 e^{utA(\epsilon+h)}\frac{1}{h}\left[tA(\epsilon+h) - tA(\epsilon)\right]e^{(1-u)tA(\epsilon)}du \\
&= \int_0^1 e^{utA(\epsilon+h)}\left[t\big(A'(\epsilon) + B(\epsilon,h)h\big)\right]e^{(1-u)tA(\epsilon)}du
\end{aligned}$$

where $B(\epsilon,h) = \int_0^1 (1-x)A''(\epsilon+hx)dx$ which is smooth in ϵ and h. Taking limit as $h \to 0$ gives the formula $\square$

To make sense of the Duhamel's formula (2.7.1) for differential operators, we collect, summarily, the following facts; following Taylor [57].

Theorem 2.7.2. *[57]. If A is a bounded, self-adjoint operator on a separable Hilbert space $\mathcal{H}$, then there is a $\sigma-$Compact space Ω (i.e a union of countably many compact subspaces), a Borel measure μ, a unitary map*

$$T : L^2(\Omega, \mathrm{d}\mu) \to \mathcal{H}, \tag{2.7.2}$$

and a real-valued function $a \in L^\infty(\Omega, \mathrm{d}\mu)$ such that

$$(T^{-1}ATf)(x) = a(x)f(x); \quad \forall f \in L^\infty(\Omega, \mathrm{d}\mu). \tag{2.7.3}$$

The proof of theorem (2.7.2) requires the operator

$$U(\epsilon) = \mathrm{e}^{i\epsilon A} \tag{2.7.4}$$

defined by the power-series expansion

$$\mathrm{e}^{i\epsilon A} = \sum_{k=0}^{\infty} \frac{(i\epsilon)^k}{k!} A^k. \tag{2.7.5}$$

$U(\epsilon)$ is uniquely characterized as the solution to the differential equation

$$\left.\begin{array}{rcl} \frac{\mathrm{d}}{\mathrm{d}\epsilon}U(\epsilon) & = & iAU(\epsilon) \\ U(0) & = & \mathrm{id.} \end{array}\right\} \tag{2.7.6}$$

The semi-group property

$$U(\epsilon + \tilde{\epsilon}) = U(\epsilon)U(\tilde{\epsilon}) \tag{2.7.7}$$

follows, since both sides satisfy the ordinary differential equation

$$\left.\begin{array}{rcl} \frac{\mathrm{d}}{\mathrm{d}\tilde{\epsilon}}z(\tilde{\epsilon}) & = & iAz(\epsilon) \\ z(0) & = & U(\epsilon). \end{array}\right\} \tag{2.7.8}$$

If $A = A^*$, then applying the adjoint to (2.7.4) gives $U(\epsilon) = U(-\epsilon)$ which is the inverse of $U(\epsilon)$ in view of (2.7.7). Thus, $\{U(\epsilon) : \epsilon \in \mathbb{R}\}$ is a group of unitary operators.

Definition 2.7.3. *(Cyclic vector). Let A be an endormorphism of a finite-dimensional vector space X. A cyclic vector for A is a vector v such that $v, Av, \cdots, A^{k-1}v$ form a basis for X.*

A vector v in an infinite-dimensional Banach or Hilbert space with the operator A on it is called cyclic if the linear combination of vectors $A^k v; \quad k = 0, 1, \cdots$ form a dense subspace.

Proposition 2.7.4. *If $U(\epsilon)$ is a strongly continuous unitary group on $\mathcal{H}$, having a cyclic vector v, take $\Omega = \mathbb{R}$. Then there exists a positive Borel measure μ on $\mathbb{R}$ and a unitary map*

$$T : L^2(\Omega, \mathrm{d}\mu) \to \mathcal{H}$$

such that

$$\left.\begin{array}{rcl}(T^{-1}U(\epsilon)Tf)(x) & = & \mathrm{e}^{i\epsilon x}f(x)\\ \forall f\in L^{\infty}(\Omega,\mathrm{d}\mu). & & \end{array}\right\}\qquad(2.7.9)$$

The measure μ on $\mathbb{R}$ will be the Fourier transform

$$\mu=\hat{\xi}\qquad(2.7.10)$$

where

$$\xi(\epsilon)=(2\pi)^{-n/2}\langle\mathrm{e}^{i\epsilon A}v,v\rangle;\quad \xi\in L^{\infty}(\mathbb{R})\qquad(2.7.11)$$

and μ a tempered distribution.

The map T is first defined $T:S(\mathbb{R})\to\mathcal{H}$ where $S(\mathbb{R})$ is the Schwartz space of rapidly decreasing functions, by $T(f)=f(A)v$ with the operator $f(A)$ defined by

$$f(A)=(2\pi)^{-1/2}\int_{-\infty}^{\infty}\hat{f}(\epsilon)\mathrm{e}^{i\epsilon A}\mathrm{d}\epsilon.\qquad(2.7.12)$$

The tempered distribution μ defined by (2.7.10) - (2.7.11) is a positive measure on $\mathbb{R}$.

It follows from (2.7.12) that if $f\in S(\mathbb{R})$ then

$$\left.\begin{array}{rcl}\mathrm{e}^{i\epsilon A}f(A) & = & f_{\epsilon}(A)\\ \text{with }\ f_{\epsilon}(\tau) & = & \mathrm{e}^{i\epsilon\tau}f(\tau)\end{array}\right\}\qquad(2.7.13)$$

hence,

$$T^{-1}\mathrm{e}^{i\epsilon A}Tf=T^{-1}f_{\epsilon}(A)v=\mathrm{e}^{i\epsilon\tau}f(\tau),\qquad(2.7.14)$$

noting that $S(\mathbb{R})$ is dense in $L^2(\mathbb{R},\mathrm{d}\mu)$.

Now let A be an unbounded self-adjoint operator on $\mathcal{H}$ whose domain $D(A)$ is a dense linear subspace of $\mathcal{H}$. This extension is due to von Neumann and thus uses the so-called von Neumann's unitary trick. Its statement is the following:

Proposition 2.7.5. *[56]. Let $A:\mathcal{H}_0\to\mathcal{H}_1$ be closed and densely defined. If A^*A is self-adjoint and* $\mathrm{id}+A^*A$ *has a bounded inverse, then*

- $A\pm i:D(A)\to\mathcal{H}$ *bijectively;*

- *$U = (A - i)(A + i)^{-1}$ is unitary on $\mathcal{H}$ and*

- *$A = i(\mathrm{id} + U)(\mathrm{id} - U)^{-1}$ where the range of $(\mathrm{id} - U) = 2i(A + i)^{-1}$ is $D(A)$.*

Theorem 2.7.6. *If A is an unbounded self-adjoint operator on a separable Hilbert space $\mathcal{H}$, there is a measure space (Ω, μ), a unitary map $T : L^2(\Omega, \mathrm{d}\mu) \to \mathcal{H}$, and a real-valued measurable function $a \in \Omega$ such that*

$$(T^{-1}ATf)(x) = a(x)f(x); \quad \forall Tf \in D(A). \tag{2.7.15}$$

In this situation, given $f \in L^2(\Omega, \mathrm{d}\mu)$, $Tf \in D(A)$ if and only if $a(x)f(x) \in L^2(\Omega, \mathrm{d}\mu)$.

The formula (2.7.15) is called the "spectral representation" of the self-adjoint operator A. Using it, one can extend the functional calculus defined by (2.7.12) as follows. For a Borel function $f : \mathbb{R} \to \mathbb{C}$, define $f(A)$ by

$$T^{-1}f(A)Tg(x) = f(a(x))g(x). \tag{2.7.16}$$

If f is a bounded Borel function, this is defined for all $g \in L^2(\Omega, \mathrm{d}\mu)$ and provides a bounded operator $f(A)$ on $\mathcal{H}$. More generally,

$$D(f(A)) = \{Tg \in \mathcal{H} : g \in L^2(\Omega, \mathrm{d}\mu) \;\text{ and }\; f(a(x))g \in L^2(\Omega, \mathrm{d}\mu)\}. \tag{2.7.17}$$

In particular, we define $\mathrm{e}^{i\epsilon A}$ for unbounded self-adjoint A by

$$T^{-1}\mathrm{e}^{i\epsilon A}Tg(x) = \mathrm{e}^{i\epsilon a(x)}g(x). \tag{2.7.18}$$

Then, $\mathrm{e}^{i\epsilon A}$ is a strongly continuously unitary group and we have the so- called Stone's theorem:

Theorem 2.7.7. *(Stone's theorem)[57]: If A is self-adjoint, then iA generates a strongly continuous unitary group $U(\epsilon) = \mathrm{e}^{i\epsilon A}$.*

Proposition 2.7.8. *If (M, g) is a complete closed Riemannian manifold, then the Laplacian Δ_g is essentially self-adjoint on $C^\infty(M)$. Its self-adjoint extension has the Sobolev space of order 2 (which we may denote as $\mathcal{H}^2(M)$) as its domain.*

2.8. Interchanging limits

The following theorems will be used to switch limits and integrals; and sum and integrals.

Theorem 2.8.1. *(Dominated Convergence Theorem). Let $\Omega \subseteq \mathbb{R}^n$ be open and let $\{\psi_k\}$ be a sequence of integrable functions on Ω. Suppose that $\lim_{k\to\infty} \psi_k(x) = \psi(x)$ μ-almost everywhere. Further suppose that there exists $\omega \geq 0$ with $\int_\Omega \omega(x)\mathrm{d}\mu(x) < \infty$ such that $\psi_k(x) \leq \omega(x) \;\; \forall k$. Then $\psi(x) \leq \omega(x)$ μ-almost everywhere and*

$$\lim_{k\to\infty} \int_\Omega \psi_k(x)\mathrm{d}\mu(x) = \int_\Omega \psi(x)\mathrm{d}\mu(x);$$

where $\mathrm{d}\mu(x)$ is the measure form on Ω.

Theorem 2.8.2. *(Fubini - Tonelli theorem). Let $\{\psi_k\}$ be a sequence of measurable functions. Sum and integral such as $\sum_k \int \psi_k(x)dx$ can be interchanged in either of the following cases:*

$$\psi_k \geq 0, \forall k \in \mathbb{N} \;\; \text{or} \;\; \sum_k \int |\psi_k(x)|dx < \infty.$$

2.9. Trace Class Operators

Following Roe ([48]), we give a brief explanation of the notion of Hilbert-Schmidt and trace-class operators. We begin with the general theory of traces. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be (separable infinite dimensional) Hilbert spaces and choose orthonormal bases $\{\psi_i\}$ and $\{\psi_j\}$ in $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. A bounded linear operator $A : \mathcal{H}_1 \to \mathcal{H}_2$ can be represented by an "infinite matrix" with coefficients

$$c_{ij}(A) = \langle A\psi_i, \psi_j\rangle. \tag{2.9.1}$$

Proposition 2.9.1. *The value of*

$$||A||^2_{HS} = \sum_{ij} |c_{ij}(A)|^2 \tag{2.9.2}$$

(which could be finite or infinite), is independent of choice of the orthonormal basis in $\mathcal{H}_1$ and $\mathcal{H}_2$.

To prove the proposition (2.9.1), we will need the Parseval's theorem:

Theorem 2.9.2. *(Parseval theorem)[46]: Let $\mathcal{H}$ be a Hilbert space and $\{\psi_j\}$ be an orthonormal basis. Then for each $y \in \mathcal{H}$,*

$$\left.\begin{array}{rcl} y & = & \sum_j \langle\psi_j, y\rangle\psi_j; \quad \text{and} \\ ||y||^2 & = & \sum_j |\langle\psi_j, y\rangle|^2 \end{array}\right\} \tag{2.9.3}$$

Proof. of proposition (2.9.1): By the Parseval's theorem

$$||A||_{HS}^2 = \sum_{ij} |c_{ij}(A)|^2 = \sum_i ||A\psi_i||^2 \tag{2.9.4}$$

which is certainly independent of the choice of basis in $\mathcal{H}_2$. But since $c_{ij}(A) = \bar{c}_{ji}(A^*)$, we also have that $||A||_{HS}^2 = ||A^*||_{HS}^2$ which is independent of the choice of basis in $\mathcal{H}_1$ by the same argument ☐

Definition 2.9.3. *An operator A such that $||A||_{HS} < \infty$ is called a Hilbert-Schmidt operator and $||A||_{HS}$ is called its Hilbert-Schmidt norm.*

Proposition 2.9.4. *(1.) The Hilbert-Schmidt norm is induced by an inner product*

$$\langle A, B\rangle = \sum_{ij} \bar{c}_{ij}(A)c_{ij}(B). \tag{2.9.5}$$

(2.) Relative to the inner product (2.9.5), the space of Hilbert-Schmidt operators is a Hilbert space.

(3.) The Hilbert-Schmidt norm dominates the operator norm, i.e $||A|| \leq ||A||_{HS}$.

(4.) Hilbert-Schmidt operators are compact.

(5.) The sum of two Hilbert-Schmidt operators, and the product of (in either order) of a Hilbert-Schmidt operator and a bounded operator are Hilbert-Schmidt.

For proof, one can see Shubin ([53]) or Reed and Simon ([46]).

Definition 2.9.5. *A bounded operator T on a Hilbert space $\mathcal{H}$ is said to be of trace-class if there are Hilbert-Schmidt operators A and B on $\mathcal{H}$ with $T = AB$. Its trace* $\mathrm{Tr}(T)$ *is defined to be the Hilbert-Schmidt inner product $\langle A^*, B\rangle$.*

A priori, the trace depends on the choice of basis of A and B; however,

$$\mathrm{Tr}(T) = \sum_{ij} \bar{c}_{ij}(A^*)c_{ij}(B) = \sum_{ij} c_{ji}(A)c_{ij}(B) = \sum_{jj} c_{jj}(T) \tag{2.9.6}$$

in fact depends only on T.

Remark 2.9.6. *We have now defined several classes of operators:*

$$(\text{trace}-\text{class}) \subset (\text{Hilbert}-\text{Schmidt}) \subset (\text{compact}) \subset (\text{bounded}).$$

This sequence of inclusion should be seen as the "non-commutative analogue" of the sequence of inclusion $l^1 \subset l^2 \subset C^0 \subset l^\infty$. For details, see Roe [48] for example.

Proposition 2.9.7. *Let T be self-adjoint and trace-class. Then,* $\mathrm{Tr}(T)$ *is the sum of the eigenvalues of T i.e* $\mathrm{Tr}(T) = \sum_j \lambda_j$.

The most important fact about the trace is its commutator property:

Proposition 2.9.8. *Let T and B be bounded operators on $\mathcal{H}$, and suppose either T is of trace-class or both T and B are Hilbert-Schmidt. Then,* $\mathrm{Tr}(TB) = \mathrm{Tr}(BT)$.

Examples of Hilbert-Schmidt and trace-class operators come from integral operators on manifolds.

Proposition 2.9.9. *Let (M, g) be a compact manifold equipped with a smooth volume form dV_g and let A be a bounded operator on $L^2(M)$ defined by*

$$(A\psi)(x) = \int_M K(x, y)\psi(y)dV_g(y) \tag{2.9.7}$$

where K is continuous on $M \times M$. Then, A is a Hilbert-Schmidt operator; and

$$||A||_{HS}^2 = \int_M \int_M |K(x, y)|^2 dV_g(x)dV_g(y). \tag{2.9.8}$$

Theorem 2.9.10. *(Mercer's theorem : [9]) Let $K(x, y)$ be a smooth kernel on $M \times M$ defining an integral operator T by $(T\psi)(x) = \int_M K(x, y)\psi(y)dV(y)$ on $L^2(M)$. Then T is trace-class with trace given by* $\mathrm{Tr}(T) = \int_M K(x, x)dV(x)$.

CHAPTER 3

Spectral functions: Zeta and Heat kernels

This chapter gives a brief but clear discussion of the spectral functions namely the spectral zeta function of the Laplacian Δ_g and its kernel. It also presents the heat kernel and discusses how it is of use to this study.

3.1. Zeta functions

The Riemann zeta function ζ_R is the function defined as

$$\zeta_R : \{s \in \mathbb{C} : \Re(s) > 1\} \to \mathbb{C}$$

with

$$\zeta_R(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}; \quad \Re(s) > 1. \tag{3.1.1}$$

From the Riemann zeta function $\zeta_R : \{s \in \mathbb{C} : \Re(s) > 1\} \to \mathbb{C}$, notice that since

$$\sum_{k=1}^{\infty} \Big|\frac{1}{k^s}\Big| = \sum_{k=1}^{\infty} \frac{1}{k^{\Re(s)}}, \tag{3.1.2}$$

the series on the right-hand-side of (3.1.2) converges absolutely if and only if $\Re(s) > 1$. The Riemann zeta function defined by (3.1.1) above is holomorphic in the region indicated. It, however, admits a meromorphic continuation to the whole s-complex plane with only simple pole at $s = 1$ and has residue 1. It satisfies the functional equation:

$$\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta_R(s) = \pi^{-\frac{(1-s)}{2}}\zeta_R(1-s) \tag{3.1.3}$$

(see e.g. Titchmarsh [58]). Other variant definitions of zeta functions abound. Any other such zeta function used in this report is defined at the point of usage.

Hurwitz zeta function $\zeta_H(s,a)$ is a generalization of the Riemann zeta function (3.1.1).

Definition 3.1.1. *Let $s \in \mathbb{C}$ and $0 < a \leq 1$. Then for $\Re(s) > 1$, the Hurwitz zeta function is defined by*

$$\zeta_H(s,a) = \sum_{k=0}^{\infty} \frac{1}{(k+a)^s}; \quad \Re(s) > 1. \tag{3.1.4}$$

Clearly, $\zeta_H(s,1) = \zeta_R(s)$. Expressions for $a = b+1$ follows by observing that

$$\zeta_H(s,1+b) = \sum_{k=0}^{\infty} \frac{1}{(k+1+b)^s} = \zeta_H(s,b) - \frac{1}{b^s}. \tag{3.1.5}$$

Theorem 3.1.2. *For $0 < a \leq 1$, we have*

$$\zeta_H(s,a) = \frac{1}{a^s} + \sum_{m=0}^{\infty} (-1)^m \frac{\Gamma(s+m)}{m!\Gamma(s)} a^m \zeta_R(s+m). \tag{3.1.6}$$

Proof. Note that for $|z| < 1$ the following binomial expansion holds

$$(1-z)^{-s} = \sum_{m=0}^{\infty} \frac{\Gamma(s+m)}{m!\Gamma(s)} z^m.$$

So for $\Re(s) > 1$, we have

$$\begin{aligned} \zeta_H(s,a) &= \frac{1}{a^s} + \sum_{k=1}^{\infty} \frac{1}{k^s} \frac{1}{(1+\frac{a}{k})^s} \\ &= \frac{1}{a^s} + \sum_{k=1}^{\infty} \frac{1}{k^s} \sum_{m=0}^{\infty} (-1)^m \frac{\Gamma(s+m)}{m!\Gamma(s)} (\frac{a}{k})^m \\ &= \frac{1}{a^s} + \sum_{m=0}^{\infty} (-1)^m \frac{\Gamma(s+m)}{m!\Gamma(s)} (a)^m \sum_{k=1}^{\infty} \frac{1}{k^{s+m}} \end{aligned}$$

which gives the expansion $\square$

Definition 3.1.3. *For any $x \in \mathbb{C}$, the Bernoulli polynomial $B_k(x)$ is defined by the generating function*

$$\frac{ze^{xz}}{e^z - 1} = \sum_{k=0}^{\infty} \frac{B_k(x)}{k!} z^k; \quad |z| < 2\pi. \tag{3.1.7}$$

Examples of these polynomials are $B_0(x) = 1,\ B_1(x) = x - \frac{1}{2}$, etc. $B_k(0)$ are the Bernoulli numbers denoted by B_k. Thus,

$$\frac{z}{e^z - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} z^k; \quad |z| < 2\pi.$$

Theorem 3.1.4. *For $\Re(s) > 1,$ we have*

$$\zeta_H(s,a) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{e^{-at}}{1 - e^{-t}} t^{s-1} dt. \tag{3.1.8}$$

Also, for all $k \in \mathbb{N}$,

$$\zeta_H(s,a) = -\frac{B_{k+1}(a)}{k+1}. \tag{3.1.9}$$

Another generalisation of the Riemann zeta function is the spectral zeta function, which is the function of interest in this dissertation. The Laplacian defined in (2.4.9) acting on functions on the closed and connected Riemannian n-manifold is a non-negative operator and has the discrete spectrum $\{\lambda_k\}_{k=1}^{\infty}$ listed with multiplicities. We define

$$\zeta_g(s) = \sum_{k=1}^{\infty} \frac{1}{\lambda_k^s}\ ; \qquad \Re(s) > \frac{n}{2}. \tag{3.1.10}$$

Hermann Weyl in 1911 has shown, (see, for example [35]), that the dimension and volume of a bounded domain X in $\mathbb{R}^n$ with smooth boundary is determined by its Dirichlet or Neumann spectrum. He specifically proved that if $N(\lambda) = \#\{\lambda_k \leq \lambda\}$ is the eigenvalue counting function, then one has the following asymptotic formula

$$N(\lambda) \sim \frac{\omega_n \mathrm{Vol}(X)}{(2\pi)^n} \lambda^{n/2}, \quad \text{as } \lambda \to \infty; \tag{3.1.11}$$

that is, on taking $\lambda = \lambda_k$:

$$\lambda_k \sim c_n \mathrm{Vol}(X)^{-2/n} k^{2/n}\ ; \quad \text{as} \quad k \to \infty$$

where $\omega_n = \frac{\pi^{n/2}}{\Gamma(n/2+1)}$ is the volume of the unit ball in $\mathbb{R}^n$, $c_n = 4\pi^2 \omega_n^{-2/n}$, n is the dimension of X and $\sim$ means the ratio of the right-hand-side to the left-hand-side approaches 1 as $k \to \infty$. This result is what is now known as the Weyl's law and it generalises to closed Riemannian manifolds such as the ones we consider in this work. It is usually paraphrased in this case as: **"can one hear the dimension and volume of a Riemannian manifold?"** From the Weyl's law, one sees that the sum in equation (3.1.10) converges absolutely for $\Re(s) > \frac{n}{2}$. If the manifold is, for example, the unit circle S^1, with the usual metric; then the eigenvalues of the Laplacian are $k^2,\ k \in \mathbb{Z}$. So, $\zeta_g(s) = 2\zeta_R(2s)$, where ζ_R is the Riemann zeta function. The zeta function $\zeta_g(s)$, as will be shown later, admits a meromorphic continuation to the entire complex s-plane and is holomorphic at $s = 0$.

Note that since the Laplacian, and hence its eigenvalues, are constructed using only the metric g, the zeta function is a geometric invariant. That is, given a diffeomorphism

$$I : M \to M; \quad \text{then} \quad \zeta_{I^*g}(s) = \zeta_g(s).$$

To show that the spectral zeta function has a meromorphic continuation, we use the operator Δ_g^{-s} and its integral kernel $\zeta_g(s, x, y)$, also called the zeta kernel. The operator Δ_g^{-s} is uniquely defined by the following properties (see e.g [43] and [21]):

(1.) it is linear on $L^2(M)$ with 1-dimensional null space consisting of constant functions. This ensures that the smallest eigenvalue of Δ_g^{-s} is 0 of multiplicity 1 with corresponding eigenfunction $\frac{1}{\sqrt{V}}$ where V is the volume of M;

(2.) the image of Δ_g^{-s} is contained in the orthogonal complement of constant functions in $L^2(M)$ i.e.

$$\int_M \Delta_g^{-s}\psi dV_g = 0 \ \ \forall\ \psi \in L^2(M) \ \text{ constant; } \ \text{ and}$$

(3.) $\Delta_g^{-s}\psi_k(x) = \lambda_k^{-s}\psi_k(x)$ for all $\psi_k;\ k > 0$ an orthonormal basis of eigenfunction of Δ_g.

Then for $\Re(s) > \frac{n}{2}$, we see by property (3.) that Δ_g^{-s} is trace class, with trace given by the spectral zeta function, namely

$$\zeta_g(s) = \sum_{k=1}^{\infty} \frac{1}{\lambda_k^s} = \mathrm{Tr}(\Delta_g^{-s}) = \int_M \zeta_g(s, x, x) dV \ ; \quad \Re(s) > \frac{n}{2}. \tag{3.1.12}$$

Theorem 3.1.5. *[35]. Let $\{\psi_k\}_{k=1}^{\infty}$ be an orthonormal eigenbasis for Δ_g corresponding to the eigenvalues $\{\lambda_k\}_{k=1}^{\infty}$ listed with multiplicities. Then the zeta kernel, $\zeta_g(s,x,y)$, also called the* ***point-wise zeta function****, is equal to*

$$\zeta_g(s,x,y) = \sum_{k=1}^{\infty} \frac{\psi_k(x)\bar{\psi}_k(y)}{\lambda_k^s}; \quad \Re(s) > \frac{n}{2}. \tag{3.1.13}$$

From here on, we suppress the subscript g in $\zeta_g(s)$ and Δ_g. We simply write $\zeta(s)$ and Δ for $\zeta_g(s)$ and Δ_g respectively, unless for purpose of emphasis.

3.2. The heat kernel

The meromorphic extension of $\zeta_g(s)$ is proved by showing a relationship between the zeta kernel and the heat kernel. Thus in this section, we will review properties of the heat kernel. The heat kernel, $K(t,x,y) : (0,\infty) \times M \times M \to \mathbb{R}$, is a continuous function on $(0,\infty) \times M \times M$. It is the so-called fundamental solution to the heat equation, i.e, it is the unique solution to the following system of equations:

$$\left.\begin{array}{rcl} (\frac{\partial}{\partial t} + \Delta_x)K(t,x,y) & = & 0 \\ \lim_{t\to 0}\int_M K(t,x,y)\psi(y)dV_y & = & \psi(x) \end{array}\right\} \tag{3.2.1}$$

for $t > 0;\ x, y \in M$ and Δ_x is the Laplacian acting on any $\psi \in L^2(M)$, where the limit in the second equation of (3.2.1) is uniform for every $\psi \in C^\infty(M)$.

The heat operator $e^{-t\Delta} : L^2(M) \to L^2(M)$ is the operator defined by the integral kernel $K(t,x,y)$ as

$$(e^{-t\Delta}\psi)(y) := \int_M K(t,x,y)\psi(x)dV_x$$

for $\psi \in L^2(M)$. The heat kernel is symmetric in the space variables, that is $K(t,x,y) = K(t,y,x)\ \ \forall\ x,y \in M$. Thus the heat operator is self-adjoint, that is, for $\psi_1, \psi_2 \in L^2(M)$ we have

$$\begin{aligned} \langle e^{-t\Delta}\psi_1, \psi_2\rangle_{L^2(M)} &= \int_M \{\int_M K(t,x,y)\psi_1(y)dV_y\}\bar{\psi}_2(x)dV_x \\ &= \int_M \{\int_M K(t,y,x)\psi_2(x)dV_x\}\bar{\psi}_1(y)dV_y = \langle \psi_1, e^{-t\Delta}\psi_2\rangle_{L^2(M)}. \end{aligned}$$

Now returning to the heat kernel, let $\{\psi_k\}_{k=0}^{\infty}$ with $\int_M \psi_k(x)\bar{\psi}_l(x)dV_g(x) = \delta_{kl}$ be orthonormal basis of eigenfunctions of Δ with corresponding eigenvalues $\{\lambda_k\}$ listed with multiplicities. Then $\{\psi_k\}_{k=0}^{\infty}$ are also eigenfunctions of the heat operator with corresponding eigenvalues $\{e^{-\lambda_k t}\}$. In terms of these eigenfunctions, the Mercer's theorem implies that $e^{-t\Delta}$ is trace-class for all $t > 0$ and one can write the heat kernel as

$$K(t,x,y) = \sum_{k=0}^{\infty} e^{-\lambda_k t}\psi_k(x)\bar{\psi}_k(y).$$

The convergence for all $t > 0$ is uniform on $M \times M$. In particular, the trace of the heat operator

$$\mathrm{Tr}(e^{-\Delta_g t}) = \sum_{k=0}^{\infty} e^{-\lambda_k t}|\psi_k(x)|^2 = \sum_{k=0}^{\infty} e^{-\lambda_k t} = \int_M K(t,x,x)dV_g(x) < \infty. \tag{3.2.2}$$

There are a number of salient properties of the heat kernel that will play crucial roles in this work. First, the heat kernel admits an expansion along the diagonal. This is the so-called Minakshisundaram-Pleijel expansion of the trace of the heat kernel:

$$\mathrm{Tr}(e^{-\Delta_g t}) = \frac{1}{(4\pi t)^{n/2}}\Big\{a_0 + a_1 t + a_2 t^2 + \cdots + a_N t^N + O\big(t^{N+1}\big)\Big\}; \tag{3.2.3}$$

as $t \to 0^+$; where $a_j = \int_M \mathcal{O}_j(x)dV_g(x)$ with $\mathcal{O}_j(x)$ being some smooth functions on M which depend only on the geometric data at the point $x \in M$. For example, $a_0 = \text{volume}(M)$ and $a_1(x) = \frac{(4\pi)^{-n/2}c(x)}{6}$ with $c(x)$ being the scalar curvature of M at the point x; see e.g ([36]). This asymptotic expansion is also written as

$$K(t,x,x) = \frac{1}{(4\pi t)^{n/2}}\Big\{1 + a_1(x)t + a_2(x)t^2 + \cdots + a_N(x)t^N + O\big(t^{N+1}\big)\Big\}; \tag{3.2.4}$$

as $t \to 0^+$ in some literature; (see e.g [66]).

Lemma 3.2.1. *[16] Let K be the heat kernel on a Riemannian manifold M. Then the following properties hold:*

(1.) K is a strictly positive smooth (C^∞) function on $(0,\infty) \times M \times M$.

(2.) K is symmetric in the space variables $x, y \in M$.

(3.) $\int_M K(t,x,y)dV(y) = 1$.

(4.) $\int_M K(s,x,y)K(t,y,z)dV(y) = K(s+t,x,z)$ *(semigroup property).*

Here we will prove only the semi-group property, which will also be of help in computing the variations of the spectral zeta function.

Proposition 3.2.2. *(c.f: [49]) The heat operator satisfies*

$$e^{-t\Delta}e^{-u\Delta} = e^{-(t+u)\Delta} \tag{3.2.5}$$

with associated kernel

$$K(t,x,y) = \int_M K(t-u,x,z)K(u,z,y)dV(z) \text{ for } \ 0 < u < t. \tag{3.2.6}$$

Proof. For $0 < u < t$ and $\psi \in L^2(M)$ we have

$$\begin{aligned}
\big(e^{-(t-u)\Delta}e^{-u\Delta}\psi\big)(x) &= e^{-(t-u)\Delta|_z}\big(\int_M K(u,z,y)\psi(y)dV(y)\big)(x) \\
&= \int_M K(t-u,x,z)\big(\int_M K(u,z,y)\psi(y)dV(y)\big)dV(z) \\
&= \int_M \big(\int_M K(t-u,x,z)K(u,z,y)dV(z)\big)\psi(y)dV(y).
\end{aligned}$$

Thus, $e^{-t\Delta}e^{-u\Delta}$ has the integral kernel

$$\tilde{K}(t,x,y) = \int_M K(t-u,x,z)K(u,z,y)dV(z).$$

Hence,

$$\big(\partial_t + \Delta_x\big)\big(\int_M K(t-u,x,z)K(u,z,y)dV(z)\big) = 0$$

and

$$\begin{aligned}
&\lim_{t\searrow 0}\int_M \big(\int_M K(t-u,x,z)K(u,z,y)dV(z)\big)\psi(y)dV(y) \\
&= \lim_{t\searrow 0}\int_M K(t-u,x,z)\big(\lim_{u\searrow 0}\int_M K(u,z,y)\psi(y)dV(y)\big)dV(z) \\
&= \lim_{t\searrow 0}\int_M K(t,x,z)\psi(z)dV(z) = \psi(z)
\end{aligned}$$

where we have used that $u \to 0$ as $t \to 0$ and

$$\left.\begin{array}{rcl} (\partial_t + \Delta_x)K(t,x,z) & = & 0 \\ \lim_{t\to 0}\int_M K(t,x,z)\psi(z)dV(z) & = & \psi(x). \end{array}\right\}$$

Thus, $\tilde{K}(t,x,y) = \int_M K(t-u,x,z)K(u,z,y)dV(z)$ has the two defining properties (3.2.1) of the heat kernel. Therefore, $\tilde{K}(t,x,y) = K(t,x,y)$ and $e^{-t\Delta}e^{-u\Delta} = e^{-(t+u)\Delta}$ which proves the proposition □

Lemma 3.2.3. *([16]) The semigroup (the heat operator) $e^{-\Delta t}$ on $L^2(M)$ is a symmetric Markov semigroup; that is*

(1.) Δ is positive and self-adjoint,

(2.) If $0 < \psi \in L^2(M)$, then $e^{-\Delta t}\psi \geq 0$; and

(3.) $e^{-\Delta t}$ is a contraction on $L^\infty(M)$.

The proofs of these properties are contained in Chapter VIII of ([16]).

3.3. Meromorphic continuation of the zeta kernel

Recall the Mellin transform of a measurable function ψ is defined by

$$(M\psi)(s) := \int_0^\infty \psi(t)t^{s-1}dt.$$

Furthermore, the Mellin transform as a function of s is meromorphic on a region of $\mathbb{C}$ that depends upon the decay properties of ψ. Specifically, if

$$\psi(t) = \begin{cases} \mathcal{O}(t^\alpha) & t \to 0 \\ \mathcal{O}(t^\beta) & t \to \infty, \end{cases} \tag{3.3.1}$$

then $(M\psi)(s)$ is meromorphic for $\Re(s) \in (-\alpha, -\beta)$, c.f: [24]. We use this to prove the meromorphic continuation of the zeta kernel using the following relationship between the two kernels.

Lemma 3.3.1. *[23] The zeta kernel and the heat kernel are related by*

$$\zeta_g(s,x,y) = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}(K(t,x,y) - \frac{1}{V})dt,$$

$\Re(s) > \frac{n}{2}$.

Proof. Observe that for any $x > 0$ and $\Re(s) > 0$,

$$x^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty e^{-xt} t^{s-1} dt$$

since a change of variable from, say, xt to τ gives x^{-s} and since $\Gamma(s)$ is holomorphic for $\Re(s) > 0$.

Consequently,

$$\lambda_k^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty e^{-\lambda_k t} t^{s-1} dt.$$

Thus,

$$\zeta_g(s, x, y) = \sum_{k=1}^\infty \left[\psi_k(x) \overline{\psi}_k(y) \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-\lambda_k t} dt \right]; \quad \Re(s) > \frac{n}{2}.$$

Therefore, using Theorem (2.8.2) to switch the order of the sum and the integral, we have

$$\begin{aligned} \zeta_g(s, x, y) &= \frac{1}{\Gamma(s)} \int_0^\infty \left(\sum_{k=1}^\infty e^{-\lambda_k t} \psi_k(x) \overline{\psi}_k(y) \right) t^{s-1}\, dt \\ &= \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \left(\int_M K(t, x, y) dV_g - \frac{1}{V} \right) dt. \end{aligned}$$

Thus,

$$\zeta_g(s, x, y) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \left(K(t, x, y) - \frac{1}{V} \right) dt. \tag{3.3.2}$$

□

Now let's consider how to meromorphically continue the zeta kernel. Lemma 3.3.1 says that up to the Gamma function, $\zeta_g(s)$ is the Mellin transform of the heat kernel minus $\frac{1}{V}$ We want to use this relationship for $\Re(s) > \frac{n}{2}$ to extend the definition of the zeta kernel meromorphically to the rest of $\mathbb{C}$. We do this by breaking up the integrand into an infinite sum that has the appropriate decay and a finite sum that we can examine directly.

We will use the following theorem about the decay of the heat kernel as $t \to \infty$.

Theorem 3.3.2. *There is a constant $\beta = \beta(M) > 0$, depending only on M, such that for all $x, y \in M$ and $t \geq 1$, the following inequality holds:*

$$\left| K(t, x, y) - \frac{1}{V} \right| \leq \beta e^{-\lambda_1 t}.$$

Proof. Since M is compact, the claim is a consequence of the fact that

$$\begin{aligned}\left|K(t,x,y)-\frac{1}{V}\right| &= \left|\sum_{k=1}^{\infty} e^{-\lambda_k t}\psi_k(x)\bar{\psi}_k(y)\right| \\ &= e^{-\lambda_1 t}\left|\sum_{k=1}^{\infty} e^{-(\lambda_k-\lambda_1)t}\psi_k(x)\bar{\psi}_k(y)\right| \\ &\leq e^{-\lambda_1 t}\Big(\sum_{k=1}^{N} e^{-(\lambda_k-\lambda_1)t}\left|\psi_k(x)\bar{\psi}_k(y)\right| \\ &+ \left|\sum_{k=N+1}^{\infty} e^{-(\lambda_k-\lambda_1)t}\psi_k(x)\bar{\psi}_k(y)\right|\Big).\end{aligned}$$

As $K(t,x,y)$ is uniformly convergent on $[1,\infty)\times M\times M$, one can find a positive integer N such that

$$e^{-\lambda_1 t}\Big(\left|\sum_{k=N+1}^{\infty} e^{-(\lambda_k-\lambda_1)t}\psi_k(x)\bar{\psi}_k(y)\right|\Big)=1.$$

Again, using that M is compact, we have

$$e^{-\lambda_1 t}\Big(\sum_{k=1}^{N} e^{-(\lambda_k-\lambda_1)t}\left|\psi_k(x)\bar{\psi}_k(y)\right|\Big)\leq \beta e^{-\lambda_1 t};$$

thus, the claim follows □

On the diagonal, we can write a modified version of this result:

Corollary 3.3.3. *Let (M,g) be smooth, compact and connected Riemannian manifold with volume V and $K(t,x,y)$ the heat kernel. Then for any fixed $t_0>0$ given,*

$$\Big(K(t,x,x)-\frac{1}{V}\Big)\leq\Big(K(t_0,x,x)-\frac{1}{V}\Big)\cdot e^{-\lambda_1(t-t_0)} \tag{3.3.3}$$

provided $t\geq t_0$ and λ_1 is the first positive eigenvalue of the Laplacian on M.

Since $\lambda_1>0$, these results imply that as $t\to\infty$, we have exponential decay, which means $-\beta=\infty$ in Equation (3.3.1) above. Thus we only need to split up the heat kernel to control the decay as $t\to 0$. Before doing this, we note that we also get the following corollary:

Corollary 3.3.4. *c.f [9]: Let (M,g) be smooth, compact and connected n-dimensional Riemannian manifold with volume V and $K(t,x,y)$ the heat kernel. Then, the zeta kernel is given* *by the sum:*

$$\zeta_g(s,x,y) = \sum_{k=1}^{\infty} \frac{\psi_k(x)\bar{\psi}_{k},(y)}{\lambda_k^s}$$

where this sum converges absolutely and uniformly on $M \times M$ for $\alpha = \Re(s) > \frac{n}{2}$.

Proof. Because $K(t,x,y) - \frac{1}{V}$ decays exponentially fast as $t \to 0$, and for $\Re(s) > n/2$ the integrand in equation 3.3.2 also is bounded at $t = 0$, the integral $\zeta_g(s,x,y)$ converges uniformly in x and y to a smooth function on $M \times M$. Then by Mercer's Theorem, the corollary follows. □

Now we will complete the proof of meromorphic continuation of the zeta kernel.

Proposition 3.3.5. *The zeta kernel $\zeta_g(s,x,y)$ admits a meromorphic continuation to the whole of the complex s-plane.*

Proof. For $x \neq y$, we have that $K(t,x,y) - \frac{1}{V}$ decays exponentially .

For $x = y$, using the asymptotic expansion of the heat kernel (3.2.3), we have

$$\begin{aligned}
\Gamma(s)\zeta_g(s,x,x) &= \int_0^{\infty} (K(t,x,x) - \frac{1}{V})t^{s-1}dt \\
&= \int_0^{\infty} \left(t^{-n/2}\left\{ a_1(x)t + \cdots + a_N(x)t^N \right\} + R_N(x,t) - \frac{1}{V} \right) t^{s-1}\, dt \\
&= \int_0^{\infty} \big(t^{-n/2}\left\{ a_1(x)t + \cdots + a_N(x)t^N \right\} - \frac{1}{V} \big) t^{s-1}\, dt + M(R_N)(s),
\end{aligned}$$

where $R_N(x,t) = O\big(t^{N+1-(n/2)}\big)$. Thus $M(R_N)(s)$ is meromorphic for $\Re(s) \in ((n/2) - N - 1, \infty)$. Thus we need to show the first integral can be meromorphically continued.

To do that, we have to split up the whole heat kernel integral since its asymptotic expansion is only good near $t = 0$. To do that, we fix the following continuous cut-off function.

$$\chi(t) = \begin{cases} 1 & \text{if } t < \frac{1}{2} \\ 2 - 2t & \text{if } \frac{1}{2} \leq t \leq 1 \\ 0 & \text{if } t > 0. \end{cases}$$

Then we split up the integral into

$$\int_0^{\infty} \chi(t)\big(K(t,x,x) - \frac{1}{V}\big)t^{s-1}\mathrm{d}t + \int_0^{\infty} (1-\chi(t))(K(t,x,x) - \frac{1}{V})t^{s-1}\mathrm{d}t.$$

Now the second of these integrals is the Mellin transform of a function that decays exponentially both as $t \to 0$ and as $t \to \infty$. So, the critical strip in the Mellin transform is the whole complex plane. Thus this second integral is holomorphic in s.

In the first integral, we can use the asymptotic expansion to get that it is equal to

$$\int_0^1 \chi(t) R_N(x,t) t^{s-1} \mathrm{d}t + \sum_{j=1}^{N} a_j(x) \int_0^1 \chi(t) t^{j-(\frac{n}{2}+s-1)} \mathrm{d}t - \frac{1}{sV}$$

where $R_N(x,t) = O\Big(t^{N+1-\frac{n}{2}}\Big)$ as $t \to 0$. We can therefore rewrite the first integral as

$$\int_0^\infty \chi(t) R_N(x,t) t^{s-1} \mathrm{d}t,$$

which is now the Mellin transform of a continuous function that vanishes to infinite order as $t \to \infty$ because of the cutoff function $\chi(t)$ and vanishes like $t^{N+1-\frac{n}{2}}$ as $t \to 0$. Thus the critical strip is $(\frac{n}{2} - N - 1, \infty)$. So, taking N sufficiently large, we can also make this piece extend holomorphically as far as we like out the right side of the complex plane.

We are left with

$$\sum_{j=1}^{N} a_j(x) \int_0^1 \chi(t) t^{j-(\frac{n}{2}+s-1)} \mathrm{d}t - \frac{1}{sV}$$

which we can directly integrate for $s > j - \frac{n}{2}$, although we have to do it in two pieces. We can just look at the integral, but when we calculate the residues, we will have to keep track of the coefficients $a_j(x)$. The integral

$$\begin{aligned} \int_0^1 \chi(t) t^{j-(\frac{n}{2}+s-1)} \mathrm{d}t &= \int_0^{\frac{1}{2}} t^{j-(\frac{n}{2}+s-1)} \mathrm{d}t + \int_{\frac{1}{2}}^1 (2-2t) t^{j-(\frac{n}{2}+s-1)} \mathrm{d}t \\ &= \int_0^1 t^{j-(\frac{n}{2}+s-1)} \mathrm{d}t + 2\int_{\frac{1}{2}}^1 t^{j-(\frac{n}{2}+s-1)} \mathrm{d}t - 2\int_{\frac{1}{2}}^1 t^{j-(\frac{n}{2}+s)} \mathrm{d}t. \end{aligned}$$

The last two pieces of the integral are holomorphic and the first for $\Re(s) > \frac{n}{2} - j$ yields

$$\sum_{j=0}^{N} \frac{a_j(x)}{s - n/2 + j}$$

Thus, we have

$$\zeta_g(s,x,x) = \frac{1}{\Gamma(s)} \left\{ \sum_{j=0}^{N} \frac{a_j(x)}{s-n/2+j} - \frac{1}{sV} \right\} + G_k(s); \tag{3.3.4}$$

where $G_k(s)$ is a holomorphic function for $\Re(s) > n/2 - N - 1$. $\square$

3.4. Consequences for the zeta function

We want to know something about the poles of $\zeta_g(s)$. Since,

$$\lim_{s\to 0}\frac{1}{s\Gamma(s)} = \lim_{s\to 0}\frac{1}{\Gamma(s+1)} = 1,$$

$\zeta_g(s)$ is holomorphic at $s=0$. Furthermore, since the poles of $\Gamma(s)$ are at $s=-j, \quad j\in\mathbb{N}$, the poles of $\zeta_g(s)$ which are all simple are located at

$$s = \begin{cases} \frac{n}{2}, \frac{n}{2}-1, \frac{n}{2}-2, \cdots, 2, 1 & \text{for } n \text{ even; and} \\ \frac{n}{2}, \frac{n}{2}-1, \cdots, \frac{1}{2}, -\frac{1}{2}, \cdots, [\frac{n-1}{2}]-\frac{n}{2} & \text{for } n \text{ odd.} \end{cases} \tag{3.4.1}$$

The residue of $\zeta_g(s)$ at $s=k$ is given by

$$\mathrm{Res}_{s=k}\zeta_g(s) = \frac{a_{\frac{n}{2}-k}(x)}{\Gamma(k)}. \tag{3.4.2}$$

Observe also that from (3.3.4), that at $s=0$, the pole in $\Gamma(s)$ forces $G_k(s)$ to vanish and so,

$$\zeta_g(0,x,x) = a_{\frac{n}{2}}(x) - \frac{1}{V}.$$

Thus, a knowledge of $a_{\frac{n}{2}}$ is sufficient to determine $\zeta_g(0,x,x)$ for example,

$$\zeta_g(0,x,x) = \begin{cases} -\frac{1}{V} & \text{for } n \text{ odd;} \\ a_{\frac{n}{2}}(x) - \frac{1}{V} & \text{for } n \text{ even} \end{cases}$$

where V is the volume of M.

However, in order to determine the Casimir energy, one evaluates $\zeta_g(s,x,x)$ at $s=-\frac{1}{2}$ and automatically a pole with a non-zero residue is encountered since if the dimension n of the manifold M is odd we have

$$a_{\frac{n}{2}+\frac{1}{2}} \neq 0 \quad \text{and} \quad a_{\frac{n}{2}-\frac{1}{2}} \neq 0 \tag{3.4.3}$$

since $s=\frac{1}{2}$ is a pole too. If n is even then, automatically

$$a_{\frac{n}{2}+\frac{1}{2}} = 0 = a_{\frac{n}{2}-\frac{1}{2}} \tag{3.4.4}$$

and the zeta regularization can be used to determine the Casimir energy on M; see e.g. ([21]) and ([23]).

CHAPTER 4

Spectral decompositions and spectral zeta function on spheres

4.1. Gegenbauer Polynomials

The zeta and heat kernels discussed in the previous chapter can be expressed explicitly for the n-sphere in terms of Gegenbauer polynomials. Explicit forms of those kernels will enable us carry-out the computation of the variation of the zeta function and the Casimir energy for spheres successfully in the next chapter. Thus, following Atkinson ([3]) and Morimoto ([37]), we give a brief review of them here and show their relation with the zeta and heat kernels.

The Gegenbauer polynomials are the generalizations of the Legendre polynomials to higher dimensions. These polynomials can be characterized by a formula generalizing the Rodrigues representation of the Legendre polynomials, namely

$$P_m(x) = \frac{1}{2^m m!}\frac{d^m}{dx^m}(x^2-1)^m. \tag{4.1.1}$$

The polynomials (4.1.1) solve the Legendre equation

$$(1-x^2)P_m(x)'' - 2xP_m(x)' + (k(k+1))P_m(x) = 0. \tag{4.1.2}$$

These generalise as follows.

Definition 4.1.1. *The Gegenbauer polynomial $P_k^n(t)$ is given by*

$$P_k^n(t) = k!\Gamma(\frac{n-1}{2})\sum_{j=0}^{[\frac{k}{2}]}(-1)^j \frac{(1-t^2)^j t^{k-2j}}{4^j j!(k-2j)!\Gamma(j+\frac{n-1}{2})}, \tag{4.1.3}$$

or given through the extended Rodrigues formula

$$P_k^n(t) = (-1)^k R_{k,n}(1-t^2)^{\frac{3-n}{2}} \frac{d^k}{dt^k}(1-t^2)^{k+\frac{n-3}{2}} \quad \text{with} \quad n \geq 2, \tag{4.1.4}$$

where the Rodrigues constant $R_{k,n}$ is given by

$$R_{k,n} = \frac{\Gamma(\frac{n-1}{2})}{2^k\Gamma(k+\frac{n-1}{2})}.$$

Gegenbauer polynomials are relevant to the study of the heat and zeta kernels because of the following result.

Lemma 4.1.2. *(Addition formula, c.f: Morimoto [37])*

Let $\{\psi_{k,j} : 1 \leq j \leq d_k(n)\}$ be an orthonormal basis of the space of n-dimensional spherical harmonics $\mathcal{H}_k(S^n)$, i.e:

$$\int_{S^n} \psi_{k,j}(x)\bar{\psi}_{k,l}(x)dV_g(x) = \delta_{jl}; \quad 1 \leq j, l \leq d_k(n). \tag{4.1.5}$$

Then

$$\sum_{j=1}^{d_k(n)} \psi_{k,j}(x)\bar{\psi}_{k,l}(y) = \frac{d_k(n)}{|S^n|} P_k^{\frac{(n-1)}{2}}(x \cdot y) \tag{4.1.6}$$

where as before, $P_k^n(t)$ are the Gegenbauer polynomials of degree k in n dimensions.

For proof, one may see Morimoto [37]. Note in particular, this means that $P_k^{(n-1)/2}(x \cdot y)$ is a harmonic function on S^n with eigenvalue $\lambda_k = k(k+n-1)$.

The Gegenbauer polynomials enable one to write the heat kernel on S^n explicitly namely, for all $t > 0$, and $x, y \in S^n$:

$$\begin{aligned} K(t,x,y) &:= \frac{1}{V}\sum_{k=0}^{\infty}\sum_{j=1}^{d_k(n)} e^{-k(k+n-1)t}\psi_{k,j}(x)\bar{\psi}_{k,j}(y) && (4.1.7) \\ &= \frac{1}{V}\sum_{k=0}^{\infty} e^{-k(k+n-1)t} \frac{d_k(n)}{P_k^{\frac{(n-1)}{2}}(1)} P_k^{\frac{(n-1)}{2}}(x \cdot y). && (4.1.8) \end{aligned}$$

where V is the volume of S^n, and $d_k(n)$ is the dimension of the λ_k eigenspace. It is also known that the zeta kernel $\zeta_{S^n}(s,x,y)$ on S^n is explicitly given by

$$\zeta_{S^n}(s,x,y) = \frac{1}{V}\sum_{k=1}^{\infty} \frac{d_k(n)}{(k(k+n-1))^s} \cdot \frac{1}{P_k^{\frac{(n-1)}{2}}(1)} P_k^{\frac{(n-1)}{2}}(x\cdot y) \tag{4.1.9}$$

(see e.g Wogu [64], Camporesi [10] and Morimoto [37]).

The Gegenbauer polynomials have many useful properties, proved in Morimoto [37] and Atkinson [3], which we summarise here.

Proposition 4.1.3. *The Gegenbauer polynomials $P_k^{(n-1)/2}(t)$ are solutions of the differential equation*

$$(1-t^2)(P_k^{(n-1)/2})''(t) - nt(P_k^{(n-1)/2})'(t) + k(k+n-1)P_k^{(n-1)/2}(t) = 0. \tag{4.1.10}$$

The differential equation (4.1.10) is known as the Gegenbauer differential equation. Observe that for $n = 2$, one gets the Legendre equation (4.1.2).

Proof. For a fixed $x \in S^n$, the function ψ_k given by $P_k^n(x\cdot y) = P_k^{(n-1)/2}(\cos\theta)$ belongs to $\mathcal{H}_k(S^n)$ where θ is the geodesic angle between x and $y \in S^n$. So we have

$$\Delta_n\psi_k = k(k+n-1)\psi_k$$

where for $\psi \in C^\infty(S^n)$ we have that $\Delta_n\psi$ is given by

$$\Delta_n\psi = \frac{1}{\sin^{n-1}\theta}\frac{\partial}{\partial\theta}\big(\sin^{n-1}\frac{\partial\psi}{\partial\theta}\big) + \frac{1}{\sin^2\theta}\Delta_{n-1}\psi$$

in local coordinates where $y = \sin\theta\tilde{y} + \cos\theta\ x_{n+1}$

with $\tilde{y} \in S^{n-1}$; $0 \le \theta \le \pi$ and $x_{n+1} = (0,\cdots,0,1) \in \mathbb{R}^n$. But ψ_k only depends on t, so, $\Delta_{n-1}\psi = 0$ and thus

$$k(k+n-1)P_k^n(t) = \Delta_n P_k^n(t) = (1-t^2)\frac{\partial^2 P_k^n(t)}{\partial t^2} - nt\frac{\partial P_k^n(t)}{\partial t} \tag{4.1.11}$$

which yields the proposition $\square$

From (4.1.3), we get that $P_k^n(1) = 1$. Also note that

$$\frac{d^l}{dt^l} P_k^n(t) = 2^l n^l P_{k-1}^{n+1}(t). \tag{4.1.12}$$

Proposition 4.1.4. *For all r and t such that $-1 < r < 1$ and $-1 \leq t \leq 1$, for all $n \geq 1$, we have the following generating formulas of the Gegenbauer polynomial*

$$\sum_{k=0}^{\infty} d_k(n) r^k P_k^n(t) = \frac{1 - r^2}{(1 - 2rt + r^2)^{\frac{(n+1)}{2}}}. \tag{4.1.13}$$

For $n \geq 2$, this may be rewritten as:

$$\sum_{k=0}^{\infty} \frac{n-1}{2k+n-1} d_k(n) r^k P_k^n(t) = \frac{1}{(1 - 2rt + r^2)^{\frac{(n-1)}{2}}}. \tag{4.1.14}$$

In fact for all $\lambda > 0$, the generating formula is also represented as

$$\sum_{k=0}^{\infty} r^k P_k^{\lambda}(t) = \frac{1}{(1 - 2rt + r^2)^{\lambda}}. \tag{4.1.15}$$

Combining equations (4.1.14) and (4.1.15) when $\lambda = (n-1)/2$ gives

$$P_k^{\frac{(n-1)}{2}}(t) = \frac{n-1}{2k+n-1} d_k(n) P_k^n(t). \tag{4.1.16}$$

The addition formula gives the integral formula:

$$\int_{S^n} |P_k^n(x \cdot y)|^2 dV_g(y) = \frac{|S^n|}{d_k(n)}. \tag{4.1.17}$$

To see this integral formula, observe from the addition formula (4.1.6) that

$$\begin{aligned}
\int_{S^n} |P_k^n(x \cdot y)|^2 dV_g(y) &= \left(\frac{|S^n|}{d_k(n)}\right)^2 \int_{S^n} \Big| \sum_{j=1}^{d_k(n)} \psi_{k,j}(x) \bar{\psi}_{k,l}(y) \Big|^2 dV_g(y) \\
&= \left(\frac{|S^n|}{d_k(n)}\right)^2 \sum_{j=1}^{d_k(n)} \big|\psi_{k,j}(x)\big|^2 \\
&= \left(\frac{|S^n|}{d_k(n)}\right)^2 \frac{d_k(n)}{|S^n|}.
\end{aligned}$$

The addition formula also implies that for any $\psi_k \in \mathcal{H}_k(S^n)$ and $x, y \in S^n$

$$\int_{S^n} P_k^n(x \cdot y) \psi_k(x) dV_g(x) = \frac{|S^n|}{d_k(n)} \psi_k(y). \tag{4.1.18}$$

The Gegenbauer polynomials satisfy the following orthogonality condition for any $x, y \in S^n$:

$$\int_{S^n} P_k^n(x\cdot y)P_l^n(x\cdot y)dV_g(x) \quad = \quad \begin{cases} \frac{|S^n|}{d_k(n)} & \text{if } \ k=l. \\ 0 & \text{if } \ k\neq l \end{cases} \tag{4.1.19}$$

Explicit examples of the Gegenbauer polynomials are often given in terms of other special polynomials, for instance, the Gegenbauer polynomial $P_k^n(t)$ is proportional to the Jacobi polynomial $P_k^{(\alpha,\alpha)}(t)$ defined by

$$P_k^{(\alpha,\beta)}(t) = \frac{(-1)^k}{2^k k!}(1-t)^{-\alpha}(1+t)^{-\beta}\frac{d^k}{dt^k}[(1-t)^{-\alpha+k}(1+t)^{-\beta+k}] \tag{4.1.20}$$

when $\alpha = \frac{n-2}{2}$. The Jacobi polynomial $P_k^{(\alpha,\alpha)}(t)$ is also written in terms of the hypergeometric function as

$$P_k^{(\alpha,\beta)}(t) = \frac{(\alpha+1)_k}{k!}\,{}_2F_1\big(-k, k+\alpha+\beta+1; \alpha+1; \frac{(1-t)}{2}\big) \tag{4.1.21}$$

where

$${}_2F_1\big(a,b;c;t\big) = \sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k}\frac{t^k}{k!} \quad \text{and} \quad (\alpha)_k = \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)} \tag{4.1.22}$$

is the Pochhammer symbol.

In general, the spherical functions (4.1.6) on S^n are expressed more compactly in terms of the Jacobi polynomials up to a normalization scaling constant c_k. That is,

$$\sum_{j=1}^{d_k(n)} \psi_{k,j}(x)\bar{\psi}_{k,l}(y) = c_k P_k^{\left(\frac{(n-2)}{2},\frac{(n-2)}{2}\right)}(\cos\theta)$$

where θ is the geodesic angle between x and y on S^n; and

$$c_k = \frac{1}{P_k^{\left(\frac{(n-2)}{2},\frac{(n-2)}{2}\right)}(1)}$$

i.e

$$\sum_{j=1}^{d_k(n)} \psi_{k,j}(x)\bar{\psi}_{k,l}(y) = \frac{1}{P_k^{\left(\frac{(n-2)}{2},\frac{(n-2)}{2}\right)}(1)} P_k^{\left(\frac{(n-2)}{2},\frac{(n-2)}{2}\right)}(\cos\theta). \tag{4.1.23}$$

Continuing this way one may write

$$K(t,x,y) = \frac{1}{V}\sum_{k=0}^{\infty} e^{-k(k+n-1)t}\frac{d_k(n)}{P_k^{\left(\frac{(n-2)}{2},\frac{(n-2)}{2}\right)}(1)} P_k^{\left(\frac{(n-2)}{2},\frac{(n-2)}{2}\right)}(\cos\theta). \tag{4.1.24}$$

Thus, we immediately have concrete examples of these spherical functions as follows:

On S^1 we have

$$P_k^0(\cos\theta) = \frac{1}{P_k^{\left(-\frac{1}{2},-\frac{1}{2}\right)}(1)} P_k^{\left(-\frac{1}{2},-\frac{1}{2}\right)}(\cos\theta) = \cos k\theta. \tag{4.1.25}$$

For S^2 one gets

$$P_k^{1/2}(\cos\theta) = \frac{1}{P_k^{(0,0)}(1)} P_k^{(0,0)}(\cos\theta) = P_k(\cos\theta) = \frac{2}{\pi}\int_0^\pi \frac{\sin(k+1/2)\theta}{\sqrt{2\cos\theta - 2\cos\psi}} \mathrm{d}\psi; \tag{4.1.26}$$

and on S^3 it is

$$P_k^1(\cos\theta) = \frac{1}{P_k^{(\frac{1}{2},\frac{1}{2})}(1)} P_k^{\left(\frac{1}{2},\frac{1}{2}\right)}(\cos\theta) = \frac{\sin(k+1)\theta}{(k+1)\sin\theta}. \tag{4.1.27}$$

Lemma 4.1.5. *(Projection operator).*

Define

$$F_k(x,y) := \sum_{j=1}^{\mathrm{d}_k(n)} \psi_{k,j}(x)\bar{\psi}_{k,j}(y); \quad \forall x,y \in S^n. \tag{4.1.28}$$

Then for any $h \in \mathcal{H}_l(S^n)$, we have

$$\int_{S^n} h(y)F_k(x,y)\mathrm{d}V_g(y) = \delta_{k,l}h(x) \quad \forall x,y \in S^n; \quad k,l \geq 0. \tag{4.1.29}$$

Proof. When $l \neq k$, since $\mathcal{H}_k(S^n) \perp \mathcal{H}_l(S^n)$ and since

$$F_k(x,y) = \sum_{j=1}^{\mathrm{d}_k(n)} \psi_{k,j}(x)\bar{\psi}_{k,j}(y) \quad \forall x,y \in S^n,$$

it is obvious that

$$\int_{S^n} h(y)F_k(x,y)\mathrm{d}V_g(y) = 0.$$

Now suppose $l = k$, we have

$$\begin{aligned}
\int_{S^n} h(y)F_k(x,y)\mathrm{d}V_g(y) &= \int_{S^n} h(y) \sum_{j=1}^{\mathrm{d}_k(n)} \psi_{k,j}(x)\bar{\psi}_{k,j}(y)\mathrm{d}V_g(y) \\
&= \sum_{j=1}^{\mathrm{d}_k(n)} \psi_{k,j}(x) \int_{S^n} h(y)\bar{\psi}_{k,j}(y)\mathrm{d}V_g(y) \\
&= \sum_{j=1}^{\mathrm{d}_k(n)} \psi_{k,j}(x)\langle h(y), \psi_{k,j}\rangle_{L^2(S^2)} = h(x)
\end{aligned}$$

since $\{\psi_{k,j} : 1 \leq j \leq \mathrm{d}_k(n)\}$ are orthonormal basis of $\mathcal{H}_k(S^n)$. Hence, the lemma follows $\square$

The linear operator F_k is a projection of any $\psi \in L^2(S^n)$ onto the space $\mathcal{H}_k(S^n)$ of eigenfunctions of the Laplacian on S^n. Morimoto [37] shows that the projector F_k is independent of choice of orthonormal basis, is invariant under the action of $\mathrm{SO}(n+1)$, and is self-adjoint i.e

$$\langle \psi, F_k \psi \rangle_{L^2(S^n)} = \langle \psi, (F_k)^2 \psi \rangle_{L^2(S^n)} = \langle F_k \psi, F_k \psi \rangle_{L^2(S^n)} = ||\psi||_{L^2(S^n)}. \tag{4.1.30}$$

A direct consequence of the lemma (4.1.5) above is that since $F_k(x,y) \in \mathcal{H}_k(S^n)$, it is immediate that

$$\int_{S^n} \int_{S^n} F_k(x,y) F_l(x,y) \mathrm{d}V_g(y) \mathrm{d}V_g(x) = \delta_{k,l}. \tag{4.1.31}$$

Note that the Gegenbauer polynomials can be computed with the aid of **Mathematica** using the code $GegenbauerC[k, \lambda, z]$ for $P_k^\lambda(z)$, say. These spherical functions make it easy to discuss the notion of Sobolev spaces on the unit n-sphere. We do that in the next section.

4.2. Sobolev spaces on the unit n-sphere

To make sense of the operator $\Delta_g + c$ on S^n with c some constant, we introduce the notion of Sobolev spaces on the n-sphere. Let $\{\psi_{k,j} : 1 \leq j \leq d_k(n); k \geq 0\}$ be orthonormal basis of spherical harmonics over S^n. For a function v, introduce a sequence of numbers

$$v_{k,j} := \langle v, \psi_{k,j} \rangle_{L^2(S^n)}; \quad 1 \leq j \leq d_k(n) \tag{4.2.1}$$

wherever the integral is defined. Firstly, we consider Sobolev spaces corresponding to a given sequence of numbers $\{a_k\},\ k \geq 0$. Introduce an inner product space

$$\langle u, v \rangle_{H(\{a_k\};S^n)} := \sum_{k=0}^{\infty} |a_k|^2 \sum_{j=1}^{d_k(n)} u_{k,j} \bar{v}_{k,j} \tag{4.2.2}$$

wherever the right-hand-side of (4.2.2) makes sense. This definition is independent of choice of basis $\{\psi_{k,j} : 1 \leq j \leq d_k(n); k \geq 0\}$ in $\mathcal{H}_k(S^n)$ since by the addition formula

$$\begin{aligned}
\sum_{j=1}^{d_k(n)} u_{k,j}\bar{v}_{k,j} &= \sum_{j=1}^{d_k(n)} \int_{S^n} u(x)\psi_{k,j}(x)dV_n(x) \int_{S^n} \bar{v}(y)\bar{\psi}_{k,j}(y)dV_n(y) \\
&= \int_{S^n}\int_{S^n} u(x)\bar{v}(y) \sum_{j=1}^{d_k(n)} \psi_{k,j}(x)\bar{\psi}_{k,j}(y)dV_n(x)dV_n(y) \\
&= \frac{d_k(n)}{|S^n|}\int_{S^n}\int_{S^n} u(x)\bar{v}(y)P_k^{\frac{n-1}{2}}(x\cdot y)dV_n(x)dV_n(y)
\end{aligned}$$

where $P_k^{\frac{n-1}{2}}(x \cdot y)$ is the Gegenbauer polynomial of the distance between x and y on S^n.

Let $C^\infty(\{a_k\}, S^n)$ be the space of all infinitely differentiable functions with finite $H(\{a_k\}; S^n)$-norm.

Definition 4.2.1. *The Sobolev space $H(\{a_k\}, S^n)$ is the completion of the space of the smooth functions $C^\infty(\{a_k\}; S^n)$ with respect to the following norm*

$$||v||_{H(\{a_k\};S^n)} := \langle v, v\rangle^{1/2}_{H(\{a_k\};S^n)} \tag{4.2.3}$$

and inner product

$$\langle u, v\rangle_{H(\{a_k\};S^n)} := \sum_{k=0}^{\infty} |a_k|^2 \sum_{j=0}^{d_k(n)} u_{k,j}\bar{v}_{k,j}. \tag{4.2.4}$$

Now we consider Sobolev spaces with particular choice of the numbers $\{a_k\},\ k \geq 0$. Since

$$\Delta_g\psi_{k,j} = k(k+n-1)\psi_{k,j}$$

we have that

$$(\Delta_g + c^2)\psi_{k,j} = (k(k+n-1)+c^2)\psi_{k,j} = (k+c)^2\psi_{k,j} \tag{4.2.5}$$

where

$$c = \frac{n-1}{2}. \tag{4.2.6}$$

For any $s \in \mathbb{R}$, we formally write

$$(\Delta_g + c^2)^{s/2}\psi_{k,j} = (k+c)^s\psi_{k,j}. \tag{4.2.7}$$

So for any function $v \in L^2(S^n)$, we use the expansion

$$v(x) = \sum_{k=0}^{\infty} \sum_{j=0}^{d_k(n)} v_{k,j} \psi_{k,j}(x) \tag{4.2.8}$$

to write

$$(\Delta_g + c^2)^{s/2} v(x) = \sum_{k=0}^{\infty} \sum_{j=0}^{d_k(n)} v_{k,j}(k+c)^s \psi_{k,j}(x) \tag{4.2.9}$$

as long as the the right-hand-side is defined.

Thus,

$$||(\Delta_g + c^2)^{s/2} v||^2_{L^2(S^n)} = \sum_{k=0}^{\infty} \sum_{j=0}^{d_k(n)} (k+c)^{2s} |v_{k,j}|^2. \tag{4.2.10}$$

Definition 4.2.2. *The Sobolev space $H^s(S^n)$ on the unit n-sphere is the completion of the space of smooth function $C^\infty(S^n)$ on the unit n-sphere with respect to the norm*

$$||v||_{H^s(S^n)} := ||(\Delta_g + c^2)^{s/2} v||_{L^2(S^n)} = \big[\sum_{k=0}^{\infty} \sum_{j=0}^{d_k(n)} (k+c)^{2s} |v_{k,j}|^2\big]^{\frac{1}{2}} \tag{4.2.11}$$

which is induced by the inner product

$$\langle u, v\rangle_{H^s(S^n)} := \int_{S^n} (\Delta_g+c^2)^{s/2} u(x) (\Delta_g+c^2)^{s/2} \bar{v}(x) dV_n(x) = \sum_{k=0}^{\infty} \sum_{j=0}^{d_k(n)} (k+c)^{2s} u_{k,j} \bar{v}_{k,j}. \tag{4.2.12}$$

The Sobolev space $H^s(S^n)$ has the following properties:

(1.) From the definition of $H^s(S^n)$ above, the space of smooth functions $C^\infty(S^n)$ is dense in $H^s(S^n)$ for any $s \in \mathbb{R}$.

(2.) If $t < s$, then we have the embedding

$$H^s(S^n) \hookrightarrow H^t(S^n)$$

as well as the inequality

$$||v||_{H^s(S^n)} \leq \max\{1, \tau^{s-t} ||v||_{H^s(S^n)}\}.$$

(3.) Since

$$||(\Delta_g + c^2)^{t/2} v||_{H^s(S^n)} = ||v||_{H^{s+t}(S^n)} \quad \forall v \in H^{s+t}(S^n),$$

we see that $(\Delta_g + c^2)^{t/2}$ is bounded from $H^{s+t}(S^n)$ to $H^s(S^n)$.

Finally, we show the following embedding.

Lemma 4.2.3. *The following embedding is valid $H^s(S^n) \hookrightarrow C(S^n)$; $\forall s > \frac{n}{2}$.*

Proof. To prove this, it suffices to show that $||v||_{C(S^n)} \leq c||v||_{H^s(S^n)}$. We start with the expansion (4.2.8)

$$|v(x)| \leq \sum_{k=0}^{\infty} \sum_{j=0}^{d_k(n)} |v_{k,j}||\psi_{k,j}(x)|. \tag{4.2.13}$$

By the addition formula and the fact that $P_k^n(1) = 1$, we have:

$$\sum_{j=0}^{d_k(n)} |\psi_{k,j}(x)|^2 = \frac{d_k(n)}{|S^n|} \quad \forall x \in S^n.$$

Now apply the Cauchy-Schwartz inequality to the right side of (4.2.13) to get

$$|v(x)| \leq \sum_{k=0}^{\infty} \big(\sum_{j=0}^{d_k(n)} |v_{k,j}|^2 \big)^{1/2} \big(\sum_{j=0}^{d_k(n)} |\psi_{k,j}(x)|^2 \big)^{1/2} = \sum_{k=0}^{\infty} \big(\frac{d_k(n)}{|S^n|} \big)^{1/2} \big(\sum_{j=0}^{d_k(n)} |v_{k,j}|^2 \big)^{1/2}.$$

From the asymptotic behaviour of $d_k(n)$ namely, $d_k(n) = O\big(k^{n-1}\big)$ for sufficiently large k, we have

$$|v(x)| \leq c \sum_{k=0}^{\infty} (k+1)^{(n-1)/2} \cdot \big(\sum_{j=0}^{d_k(n)} |v_{k,j}|^2 \big)^{1/2}$$

for some constant c.

For $t > 1$ we have that $\sum_{k=1}^{\infty} k^{-t} < \infty$.

Therefore, for $s = (n-1-t)/2 > n/2$,

$$\begin{aligned} |v(x)| &\leq c \sum_{k=0}^{\infty} (k+1)^{-t/2} \big((k+1)^{2s} \sum_{j=0}^{d_k(n)} |v_{k,j}|^2 \big)^{1/2} \\ &\leq c \big[\sum_{k=0}^{\infty} (k+1)^{-t} \big]^{1/2} \big[\sum_{k=0}^{\infty} (k+1)^{2s} \sum_{j=0}^{d_k(n)} |v_{k,j}|^2 \big]^{1/2} \end{aligned}$$

Since this holds for all x, we obtain the required inequality. □

4.3. Casimir energy of Δ_g on S^n

The Casimir force has been a fundamental issue in Quantum Field Theory since the prediction of Hendrik B. G. Casimir in the year 1948 that there exists a force between a pair of neutral perfectly conducting parallel plates; see figure 4.1. Small dielectric bodies interacting at a

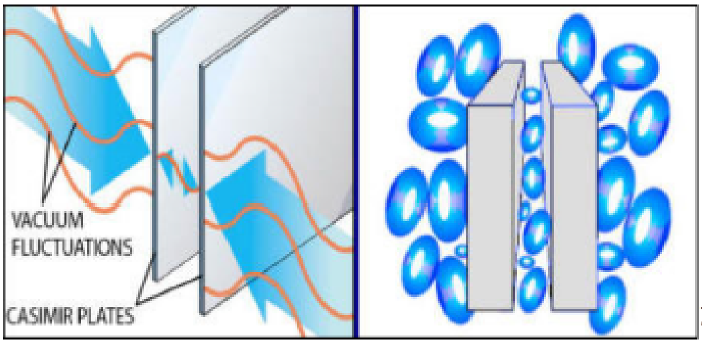

Figure 4.1: The conducting plates [51].

"reasonable" distance attract ([22]) and based on summation of the two-body forces, one may speculate that any two dielectrics would attract. This speculation holds true for example of the Casimir energy of a perfectly conducting hemispheres, see figure (4.2) below.

Let Δ_g be the Laplace operator on smooth functions on a compact Riemannian manifold (M, g). The Casimir energy is defined mathematically via the spectral zeta as a function on the set of metrics on the manifold by $\zeta_g(-\frac{1}{2})$. The Casimir force for the hemispheres in Figure

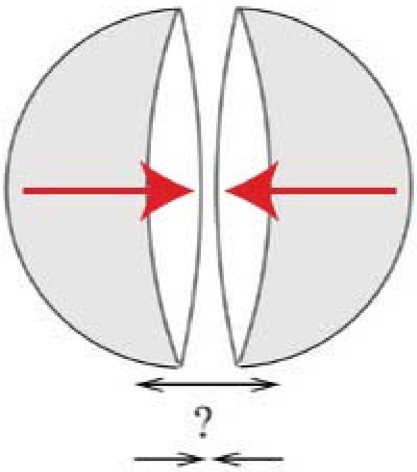

Figure 4.2: Conducting hemispheres.

(4.2) has been computed using this definition, see e.g. ([23]) and the numerous literature cited there-in.

Using the properties of the Riemann and Hurwitz zeta functions and the binomial expansion reviewed in Chapter 3, we can now compute the Casimir energy of the Laplacian Δ_g for the round metric on S^n. As mentioned in the introduction, sometimes zeta regularisation is not sufficient to define the Casimir energy, because ζ_g has a pole at $-1/2$. In this case, we define the Casimir energy as the finite part of ζ_g at $s = -1/2$, where FP is the finite part function defined by

$$\mathrm{FP}[f](s) := \begin{cases} f(s) \;\text{ if }\; s \;\text{ is }\; \text{not }\; \text{a }\; \text{pole} \\ \lim_{\epsilon \to 0} \left(f(s+\epsilon) - \frac{\text{Residue}}{\epsilon} \right), \text{ if } s \text{ is a pole} \end{cases} \tag{4.3.1}$$

see for example ([18]). In addition, in our calculations of the second variation of Casimir energy on spheres, we will need to understand the finite part of ζ at other values of s, thus in this section, we also calculate some of these.

Write the zeta function on the n-sphere as

$$\zeta_{S^n}(s) = \sum_{k=1}^{\infty} \frac{d_k(n)}{(k(k+n-1))^s}; \qquad \Re(s) > \frac{n}{2} \tag{4.3.2}$$

with $d_k(n)$ defined by equation (2.4.19). So for $n = 1$, ie the unit circle, we have

$$\zeta_{S^1}(s) = \sum_{k=1}^{\infty} \frac{2}{(k)^{2s}} = 2\zeta_R(2s). \tag{4.3.3}$$

Thus,

$$\zeta_{S^1}(-\frac{1}{2}) = 2\zeta_R(-1) \approx -0.166667. \tag{4.3.4}$$

On S^2, we have

$$\zeta_{S^2}(s) = \sum_{k=1}^{\infty} \frac{2k+1}{(k(k+1))^s} = \sum_{k=1}^{\infty} \frac{2k}{(k(k+1))^s} + \sum_{k=1}^{\infty} \frac{1}{(k(k+1))^s} = A(s) + B(s). \tag{4.3.5}$$

Using the Mellin transform of an exponential, we have

$$\begin{aligned} A(s) &= \sum_{k=1}^{\infty} \frac{2k}{(k(k+1))^s} = \sum_{k=1}^{\infty} 2k \frac{1}{\Gamma(s)} \int_0^{\infty} e^{-(k^2+k)t} t^{s-1} dt \\ &= \frac{2}{\Gamma(s)} \sum_{k=1}^{\infty} \int_0^{\infty} k e^{-k^2 t} e^{-kt} t^{s-1} dt = \frac{2}{\Gamma(s)} \sum_{k=1}^{\infty} \int_0^{\infty} k e^{-k^2 t} \sum_{m=0}^{\infty} \frac{(-1)^m (kt)^m}{m!} t^{s-1} dt \\ &= \frac{2}{\Gamma(s)} \sum_{m=0}^{\infty} \sum_{k=1}^{\infty} \int_0^{\infty} k e^{-k^2 t} \frac{(-1)^m (kt)^m}{m!} t^{s-1} dt. \end{aligned}$$

The re-arranging of integrals and sums in this computation follows from the Fubini - Tonelli theorem (2.8.2) since the exponential of a negative number is bounded.

Change variable $k^2t \mapsto \tau$ and use the Mellin tranform again to obtain

$$\begin{aligned} A(s) &= \frac{2}{\Gamma(s)} \sum_{m=0}^{\infty} \sum_{k=1}^{\infty} \frac{(-1)^m}{m!} k^{1-m-2s} \int_0^{\infty} e^{-\tau} \tau^{s+m-1} d\tau \\ &= \frac{2}{\Gamma(s)} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \left(\int_0^{\infty} e^{-\tau} \tau^{m+s-1} \, d\tau \right) \sum_{k=1}^{\infty} k^{1-m-2s} \\ &= \frac{2}{\Gamma(s)} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \Gamma(s+m) \zeta_R(2s+m-1). \end{aligned}$$

Recall that the Riemann zeta function has a pole at $s = 1$ with residue 1, and the gamma function has a pole at 0. Thus the only term in this sum with a pole at $s = -1/2$ is the term with $m = 3$. The residue from this term is 1, so $A(s)$ has a simple pole with residue 1 for value of $A(-\frac{1}{2})$.

Similarly, we get

$$B(s) = \frac{1}{\Gamma(s)} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \Gamma(s+m) \zeta_R(2s+m)$$

which again has a simple pole at $s = -1/2$ coming from the $m = 2$ term. Overall, B has residue 1 for the pole at $s = -1/2$. So,

$$\zeta_{S^2}(s) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \frac{\Gamma(s+m)}{\Gamma(s)} \big\{ 2\zeta_R(2s+m-1) + \zeta_R(2s+m) \big\}. \tag{4.3.6}$$

Therefore, the Casimir energy $\zeta_{S^2}(-\frac{1}{2})$ of Δ_g on S^2 becomes

$$\begin{aligned} \zeta_{S^2}(-\frac{1}{2}) &= \sum_{m=0}^{1} \frac{(-1)^m}{m!} \frac{\Gamma(m-\frac{1}{2})}{\Gamma(-\frac{1}{2})} \big\{ 2\zeta_R(m-2) + \zeta_R(m-1) \big\} \\ &+ \mathrm{FP}\Big[\frac{\Gamma(\frac{1}{2})}{2!\Gamma(-\frac{1}{2})} \big\{ 2\zeta_R(0) + \zeta_R(1) \big\} \Big] \\ &- \mathrm{FP}\Big[\frac{\Gamma(\frac{5}{2})}{3!\Gamma(-\frac{1}{2})} \big\{ 2\zeta_R(1) + \zeta_R(2) \big\} \Big] \\ &+ \sum_{m=4}^{\infty} \frac{(-1)^m}{m!} \frac{\Gamma(m-\frac{1}{2})}{\Gamma(-\frac{1}{2})} \big\{ 2\zeta_R(m-2) + \zeta_R(m-1) \big\}. \end{aligned}$$

Hence,

$$\mathrm{FP}[\zeta_{S^2}(-\frac{1}{2})] \approx -0.265096. \tag{4.3.7}$$

On the 3-sphere, we have

$$\zeta_{S^3}(s) = \sum_{k=1}^{\infty} \frac{(k+1)^2}{(k(k+2))^s} = \sum_{k=2}^{\infty} \frac{k^2}{(k^2-1)^s}. \tag{4.3.8}$$

Using the same method as the case of S^2 yields

$$\zeta_{S^3}(s) = \sum_{m=0}^{\infty} \frac{\Gamma(s+m)}{m!\Gamma(s)} \big[\zeta_R(2s+2m-2) - 1\big] \tag{4.3.9}$$

which has simple pole at $m = 2$ with residue $\frac{1}{2}$ for $\zeta_{S^3}(-\frac{1}{2})$.

Therefore,

$$\begin{aligned} \zeta_{S^3}(-\frac{1}{2}) &= \sum_{m=0}^{1} \frac{\Gamma(m-\frac{1}{2})}{m!\Gamma(-\frac{1}{2})} \big[\zeta_R(2m-3) - 1\big] \\ &+ \mathrm{FP}\big[\frac{\Gamma(\frac{3}{2})}{2!\Gamma(-\frac{1}{2})} \zeta_R(1) - 1\big] + \sum_{m=3}^{\infty} \frac{\Gamma(m-\frac{1}{2})}{m!\Gamma(-\frac{1}{2})} \big[\zeta_R(2m-3) - 1\big] \end{aligned}$$

thus,

$$\mathrm{FP}[\zeta_{S^3}(-\frac{1}{2})] \approx -0.411503. \tag{4.3.10}$$

On S^4, we have

$$\zeta_{S^4}(s) = \frac{1}{6} \sum_{k=1}^{\infty} \frac{(k+1)(k+2)(2k+3)}{(k(k+3))^s}. \tag{4.3.11}$$

This can be written as

$$\zeta_{S^4}(s) = \frac{1}{3} \sum_{k=1}^{\infty} \frac{w(w^2-\frac{1}{4})}{(w^2-\frac{9}{4})^s} \quad \text{with} \quad w = k + \frac{3}{2}.$$

Thus, using the properties of the Hurwitz zeta function, we can write

$$\zeta_{S^4}(s) = \frac{1}{3} \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(1-s)}{k!\Gamma(1-s-k)} \big(\frac{9}{4}\big)^k \big[\zeta_H(2s+2k-3, \frac{5}{2}) - \frac{1}{4} \zeta_H(2s+2k-1, \frac{5}{2})\big]. \tag{4.3.12}$$

This gives the Casimir energy of the Laplacian on S^4 as

$$\begin{aligned} \mathrm{FP}[\zeta_{S^4}(-\frac{1}{2})] &= \frac{1}{3} \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(\frac{3}{2})}{k!\Gamma(\frac{3}{2}-k)} \big(\frac{9}{4}\big)^k \big[\zeta_H(2k-4, \frac{5}{2}) - \frac{1}{4} \zeta_H(2k-2, \frac{5}{2})\big] \\ &\approx -0.150622. \end{aligned} \tag{4.3.13}$$

These and other finite part values of $\zeta_{S^n}(s)$ at different points $s = s_0$ of interest computed via the Finite Part scheme (4.3.1) are displayed at table (A.1) of Appendix A. The Casimir energy $\zeta_{\Delta_g}(-\frac{1}{2})$ of the Laplacian Δ_g on higher dimensional unit spheres may be computed similarly.

Carletti and Bragadin ([13]) have expressed the zeta function of the Laplacian on smooth functions on S^n in terms of the Hurwitz zeta function as

$$\zeta_{S^n}(s) = \frac{1}{(n-1)!}\sum_{m=0}^{\infty}(\frac{n-1}{2})^{2m}\binom{-s}{m}\sum_{l=0}^{n-1}B_{n,l}\zeta_H(2s+2m-l,\frac{n+1}{2}); \tag{4.3.14}$$

where $B_{n,l}$ are the Bernoulli polynomials given by

$$B_{n,l} = \sum_{p=0}^{n-l-1}\binom{l+p}{l}(\frac{n-1}{2})^p\big(S_{n,l+p+1}+S_{n-1,l+p}\big); \quad 0\le l\le n-1 \tag{4.3.15}$$

with $S_{k,n}$ being the so-called Stirling numbers of the first kind satisfying

$$S_{n+1,n} = \sum_{l=0}^{n}(-1)^l\frac{k!}{(n-l)!}S_{n-l,n-1}. \tag{4.3.16}$$

Then $\zeta_{S^n}(s)$ has simple poles at $s=\frac{n}{2}-j;\ \ j\in\mathbb{N}$ with residue given by

$$\mathrm{Res}_{s=\frac{n}{2}-j}\zeta_{S^n}(s) = \frac{1}{(n-1)!}\sum_{h=0}^{\frac{n}{2}-1}\sum_{m+h=j;\ m\ge 0}(-1)^m(\frac{n-1}{2})^{2m}\binom{j-\frac{n}{2}}{m}B_{n,n-2h-1}. \tag{4.3.17}$$

We can check that this formula gives the same values as we have calculated above.

4.4. Casimir energy of $\Delta_g+\frac{n-1}{2}$ on S^n

As another example, we can also compute the Casimir energy of Δ_g+c for the n-dimensional spheres, $\zeta_{(\Delta_g+c)}(-1/2)$. One expresses the associated spectral zeta function in terms of the Hurwitz zeta function. Denote by $\{\mu_k\}$ the spectrum of Δ_g+c on S^n :

$$\mu_k = k(k+n-1)+c \tag{4.4.1}$$

with the same eigenfunctions and multiplicities, $d_k(n)$ as for Δ_g. For $c=(n-1)/2$, we define the regularized zeta function as

$$Z_{S^n}(s) = \sum_{k=1}^{\infty}\frac{d_k(n)}{\mu_k^s} = \sum_{k=1}^{\infty}\frac{d_k(n)}{(k+\frac{n-1}{2})^{2s}}; \qquad \Re(s) > \frac{n}{2}. \tag{4.4.2}$$

The regularized zeta function (4.4.2) of the operator $\Delta_g + \frac{n-1}{2}$ on S^n can then be expressed in terms of the Riemann zeta function following Elizalde and others e.g. ([23], [18] and [17]) as follows: For $n = 1$, the regularized zeta function on the unit circle becomes

$$Z_{S^1}(s) = \sum_{k=1}^{\infty} \frac{2}{k^{2s}} = 2\zeta_R(2s)$$

since $d_k(1) = 2$. On S^2 and S^3 the spectra are shifted by $\frac{1}{4}$ and 1 with $d_k(2) = 2k+1$ and $d_k(3) = (k+1)^2$ respectively. Continuing this way gives

$$\begin{aligned}
Z_{S^2}(s) &= \sum_{k=1}^{\infty} \frac{2k+1}{(k+\frac{1}{2})^{2s}} = (2^{2s}-2)\zeta_R(2s-1) - 4^s; \\
Z_{S^3}(s) &= \sum_{k=1}^{\infty} \frac{(k+1)^2}{(k+1)^{2s}} = \zeta_R(2s-1) - 1; \\
Z_{S^4}(s) &= \frac{1}{6}\sum_{k=1}^{\infty} \frac{(k+1)(k+2)(2k+3)}{(k+\frac{3}{2})^{2s}} \\
&= \frac{1}{3}(2^{2s-3}-1)\zeta_R(2s-3) - \frac{1}{3}(2^{2s-3}-\frac{1}{4})\zeta_R(2s-1) - \frac{1}{3}(\frac{2}{3})^{2s-3} + \frac{1}{8}(\frac{2}{3})^{2s}.
\end{aligned}$$

Thus, one can easily read off the Casimir energy $Z_{S^n}(-\frac{1}{2})$ via the regularized zeta function and the Finite Part scheme (4.3.1). These and other finite part values of $Z_{S^n}(s)$ at other points of interest on the n-sphere approximated to the nearest 6-decimal-place are shown in the table (A.2) of Appendix A. Also the poles and residues of Z_{S^n} are also shown in the table. Note, of course that $\mathrm{Res}_{s=\frac{n}{2}-j} Z_{S^n}(s) = \lim_{s\to\frac{n}{2}-j} Z_{S^n}(s)\cdot(s-\frac{n}{2}+j)$; c.f: ([23], [18] and [17]).

CHAPTER 5

The variations and criticality conditions of the spectral zeta function

In this chapter, we compute the variations of the spectral zeta function and the Casimir energy of the Laplacian on a closed Riemannian manifold (M, g) under fixed-volume conformal perturbation of the metric. We begin by considering the changes a fixed-volume conformal perturbation of the metric g makes to the Laplacian and the volume form.

5.1. Change in the Laplacian

Let $\{h = \rho g\}$ be set of Riemannian metrics on M in the conformal class. We immediately have that the volume form dV_h scales as

$$dV_h = \sqrt{\det(h)}dx = \rho^{\frac{n}{2}}\sqrt{\det(g)}dx. \tag{5.1.1}$$

Theorem 5.1.1. *Given* $0 < \rho \in C^{\infty}(M; \rho g)$, *the Laplacian with respect to this conformally changed metric* $h = \rho g$ *is given by*

$$\Delta_h\psi = \rho^{-1}\Delta_g\psi + (1 - \frac{n}{2})\rho^{-2}\mathrm{div}(\rho\nabla_g)\psi \tag{5.1.2}$$

where $\mathrm{div}(\rho\nabla_g)$ *is the operator defined by*

$$\mathrm{div}(\rho\nabla_g) = g^{ij}(\partial_i\rho)\partial_j \tag{5.1.3}$$

where the so-called Einstein summation convention of summing over repeated indices is used. That is,

$$\operatorname{div}(\rho\nabla_g)\psi := \langle\nabla\rho, \nabla\psi\rangle_g.$$

Proof. Using (2.4.9), we have

$$\begin{aligned}
\Delta_h\psi &= -\frac{1}{\sqrt{\det(h)}}\partial_i\big(\sqrt{\det(h)}h^{ij}\partial_j\psi\big)\\
&= -\frac{\rho^{-n/2}}{\sqrt{\det(g)}}\partial_i\big[\rho^{n/2}\sqrt{\det(g)}\rho^{-1}g^{ij}\partial_j\psi\big]\\
&= -\frac{\rho^{-n/2}}{\sqrt{\det(g)}}\partial_i\big[\rho^{n/2-1}\sqrt{\det(g)}g^{ij}\partial_j\psi\big]\\
&= -\frac{\rho^{-n/2}}{\sqrt{\det(g)}}\rho^{n/2-1}\partial_i\big[\sqrt{\det(g)}g^{ij}\partial_j\psi\big]\\
&- (\frac{n}{2}-1)\frac{\rho^{-\frac{n}{2}}}{\sqrt{\det(g)}}\sqrt{\det(g)}\rho^{\frac{n}{2}-2}g^{ij}\partial_j\psi(\partial_i\rho)\\
&= -\frac{\rho^{-1}}{\sqrt{\det(g)}}\partial_i\big[\sqrt{\det(g)}g^{ij}\partial_j\psi\big]\\
&- (\frac{n}{2}-1)\frac{\rho^{-\frac{n}{2}}}{\sqrt{\det(g)}}g^{ij}\sqrt{\det(g)}\rho^{\frac{n}{2}-2}(\partial_j\psi)(\partial_i\rho)
\end{aligned}$$

which simplifies to (5.1.2) □

Corollary 5.1.2. *The operator* $\operatorname{div}(\rho\nabla_g)$ *satisfies the Leibniz product rule. That is, given any* $\psi_1, \psi_2 \in C^\infty(M)$,

$$\operatorname{div}(\rho\nabla_g)(\psi_1\psi_2) = \psi_2\operatorname{div}(\rho\nabla_g)(\psi_1) + \psi_1\operatorname{div}(\rho\nabla_g)(\psi_2) \tag{5.1.4}$$

Proof. From the definition of the operator $\operatorname{div}(\rho\nabla_g)$ (5.1.3), we get

$$\operatorname{div}(\rho\nabla_g)(\psi_1\psi_2) = \sum_{i,j=1}^{n} g^{ij}\frac{\partial\rho}{\partial x_i}\frac{\partial\psi_1}{\partial x_j}(\psi_2) + \sum_{i,j=1}^{n} g^{ij}\frac{\partial\rho}{\partial x_i}\frac{\partial\psi_2}{\partial x_j}(\psi_1)$$

for all $\psi_1, \psi_2 \in C^\infty(M)$ □

Now, let g be a metric on the n-dimensional closed Riemannian manifold M. Let

$$\phi : M \times (-c, c) \to \mathbb{R}$$

be a smooth family of functions $\phi_\epsilon := \phi(\cdot, \epsilon)$ on M with $\phi_0 = 0$, and define the corresponding family of conformal metrics g_ϵ:

$$\{g_\epsilon = e^{\phi_\epsilon} g\},$$

with the condition that $g_\epsilon^{(1)} = \frac{\partial}{\partial \epsilon}(g_\epsilon)|_{\epsilon=0} = \dot{\phi}_0 g, \quad \dot{\phi}_0 \in C^\infty(M)$; where $\dot{\phi}_\epsilon = \frac{\partial}{\partial \epsilon}\big(\phi_\epsilon\big)$.

Then the corresponding family of Laplacians Δ_ϵ are defined as

$$\Delta_\epsilon \psi = e^{-\phi_\epsilon}\Delta_g \psi + (1 - \frac{n}{2}) e^{-2\phi_\epsilon} \mathrm{div}(e^{\phi_\epsilon}\nabla_g)\psi. \tag{5.1.5}$$

It can be seen that Δ_ϵ varies in ϵ as follows:

$$\begin{aligned} \frac{\partial}{\partial \epsilon}\Big(\Delta_\epsilon \psi\Big) &= \frac{\partial}{\partial \epsilon}\Big(e^{-\phi_\epsilon}\Delta_g \psi + (1 - \frac{n}{2}) e^{-2\phi_\epsilon} \mathrm{div}(e^{\phi_\epsilon}\nabla_g)\psi\Big) \\ &= -\dot{\phi}_\epsilon e^{-\phi_\epsilon}\Delta_g \psi - (1 - \frac{n}{2})\dot{\phi}_\epsilon e^{-\phi_\epsilon} \mathrm{div}(\phi_\epsilon \nabla_g)\psi \\ &+ (1 - \frac{n}{2}) e^{-\phi_\epsilon} \mathrm{div}(\dot{\phi}_\epsilon \nabla_g)\psi. \end{aligned}$$

At $\epsilon = 0$, the second term vanishes, so

$$\Delta_0^{(1)} = -\dot{\phi}_0 \Delta_g + (1 - \frac{n}{2})\langle \nabla_g \dot{\phi}_0, \nabla_g \cdot \rangle_g. \tag{5.1.6}$$

Similarly, the second derivative in ϵ is

$$\begin{aligned} \frac{\partial^2}{\partial \epsilon^2}\Big(\Delta_\epsilon \psi\Big) &= \frac{\partial}{\partial \epsilon}\Big(- \dot{\phi}_\epsilon e^{-\phi_\epsilon}\Delta_g \psi - (1 - \frac{n}{2})\dot{\phi}_\epsilon e^{-\phi_\epsilon} \mathrm{div}(\phi_\epsilon \nabla_g)\psi \\ &+ (1 - \frac{n}{2}) e^{-\phi_\epsilon} \mathrm{div}(\dot{\phi}_\epsilon \nabla_g)\psi\Big) \\ &= -\ddot{\phi}_\epsilon e^{-\phi_\epsilon}\Delta_g \psi + (\dot{\phi}_\epsilon)^2 e^{-\phi_\epsilon}\Delta_g \psi - (1 - \frac{n}{2})\ddot{\phi}_\epsilon e^{-\phi_\epsilon} \mathrm{div}(\phi_\epsilon \nabla_g)\psi \\ &+ (1 - \frac{n}{2})(\dot{\phi}_\epsilon)^2 e^{-\phi_\epsilon} \mathrm{div}(\phi_\epsilon \nabla_g)\psi - (1 - \frac{n}{2})\dot{\phi}_\epsilon e^{-\phi_\epsilon} \mathrm{div}(\dot{\phi}_\epsilon \nabla_g)\psi \\ &- (1 - \frac{n}{2})\dot{\phi}_\epsilon e^{-\phi_\epsilon} \mathrm{div}(\dot{\phi}_\epsilon \nabla_g)\psi + (1 - \frac{n}{2}) e^{-\phi_\epsilon} \mathrm{div}(\ddot{\phi}_\epsilon \nabla_g)\psi \end{aligned}$$

so that at $\epsilon = 0$ gives

$$\begin{aligned} \Delta_0^{(2)} &= -\ddot{\phi}_0 \Delta_g + (\dot{\phi}_0)^2 \Delta_g + (n-2)\dot{\phi}_0 \langle \nabla_g \dot{\phi}_0, \nabla_g \cdot \rangle_g \\ &+ (1 - \frac{n}{2})\langle \nabla_g \ddot{\phi}_0, \nabla_g \cdot \rangle_g. \end{aligned} \tag{5.1.7}$$

Because the norms on $L^2(M, dV_g)$ and $L^2(M, dV_\epsilon)$ are different (though equivalent), it is useful to define a unitary transformation between the spaces to relate the operators. From the conformal factor e^{ϕ_ϵ} and the transformed volume form $dV_\epsilon = e^{\frac{n}{2}\phi_\epsilon} dV_g$; define

$$\left.\begin{array}{rcl} T : L^2(M, dV_g) & \longrightarrow & L^2(M, dV_\epsilon) \\ \psi & \longmapsto & e^{-\frac{n}{4}\phi_\epsilon}\psi. \end{array}\right\} \tag{5.1.8}$$

Note, the map T depends on ϵ. With this map T, the family of operators $\tilde{\Delta}_\epsilon := T^{-1}\Delta_\epsilon T$ can be seen by the Rellich-Kato theorem to be self-adjoint for sufficiently small ϵ. Moreover, $\tilde{\Delta}_\epsilon$ and Δ_ϵ have the same spectrum. The Rellich-Kato perturbation theory of unbounded operators consequently applies to the family of operators $\epsilon \mapsto \Delta_\epsilon$, and $\tilde{\Delta}_\epsilon$ see [65].

Proposition 5.1.3. *The transformed Laplacian $\tilde{\Delta}_\epsilon := T^{-1}\Delta_\epsilon T$ is given by*

$$\begin{aligned} \tilde{\Delta}_\epsilon &= e^{-\phi_\epsilon}\Delta_g + \frac{n^2-4n}{16} e^{-\phi_\epsilon}\langle \nabla_g \phi_\epsilon, \nabla_g \phi_\epsilon\rangle_g - \frac{n}{4} e^{-\phi_\epsilon}(\Delta_g \phi_\epsilon) \\ &+ e^{-\phi_\epsilon}\operatorname{div}(\phi_\epsilon \nabla_g) \end{aligned} \tag{5.1.9}$$

Proof. For any $\psi \in C^\infty(M)$, the proposition follows from simplifying

$$T^{-1}\Delta_\epsilon T\psi = e^{\frac{n}{4}\phi_\epsilon}\big\{e^{-\phi_\epsilon}\Delta_g + (1-\frac{n}{2})e^{-2\phi_\epsilon}\operatorname{div}(e^{\phi_\epsilon}\nabla_g)\big\}(e^{-\frac{n}{4}\phi_\epsilon}\psi)$$

□

Lemmata 5.1.4. *The transformed Laplacian $\tilde{\Delta}_\epsilon := T^{-1}\Delta_\epsilon T$ varies as*

$$\tilde{\Delta}_0^{(1)} = -\dot{\phi}_0\Delta_g - \frac{n}{4}(\Delta_g\dot{\phi}_0) + \operatorname{div}(\dot{\phi}_0\nabla_g) \tag{5.1.10}$$

and

$$\begin{aligned} \tilde{\Delta}_0^{(2)} &= -\ddot{\phi}_0\Delta_g + (\dot{\phi}_0)^2\Delta_g + \frac{n^2-4n}{8}\langle \nabla_g\dot{\phi}_0, \nabla_g\dot{\phi}_0\rangle_g \\ &+ \frac{n}{2}\dot{\phi}_0(\Delta_g\dot{\phi}_0) - \frac{n}{4}(\Delta_g\ddot{\phi}_0) - 2\dot{\phi}_0\operatorname{div}(\dot{\phi}_0\nabla_g) + \operatorname{div}(\ddot{\phi}_0\nabla_g) \end{aligned} \tag{5.1.11}$$

where $\tilde{\Delta}_0^{(1)} = \frac{\partial}{\partial\epsilon}(\Delta_\epsilon)\big|_{\epsilon=0}$ and $\tilde{\Delta}_0^{(2)} = \frac{\partial^2}{\partial\epsilon^2}(\Delta_\epsilon)\big|_{\epsilon=0}$ respectively.

Note though that

$$\operatorname{Tr}(\tilde{\Delta}_\epsilon) = \operatorname{Tr}(T^{-1}\Delta_\epsilon T) = \operatorname{Tr}((\Delta_\epsilon T)T^{-1}) = \operatorname{Tr}(\Delta_\epsilon TT^{-1}) = \operatorname{Tr}(\Delta_\epsilon). \tag{5.1.12}$$

Thus, it is sufficient to work with (5.1.6) and (5.1.7) in this thesis.

5.2. Change in the volume and volume form

From (5.1.1), it follows that

$$dV_\epsilon = \sqrt{|g_\epsilon|}dx = e^{\frac{n}{2}\phi_\epsilon}\sqrt{|g|}dx = e^{\frac{n}{2}\phi_\epsilon}dV_g; \tag{5.2.1}$$

where $|g|$ is the determinant of the metric g.

Suppose now that the volume of (M, g_ϵ) is fixed to be a constant V for the conformal family $\{g_\epsilon = e^{\phi_\epsilon}g\}$. Then $\int_M dV_\epsilon = V$, we have

$$\int_M dV_\epsilon = \int_M e^{\frac{n}{2}\phi_\epsilon}dV_g = V,$$

so we can observe that

$$\begin{aligned} 0 = \frac{\partial}{\partial\epsilon}\int_M e^{\frac{n}{2}\phi_\epsilon}dV_g &= \frac{n}{2}\int_M \frac{\partial}{\partial\epsilon}(\phi_\epsilon)e^{\frac{n}{2}\phi_\epsilon}dV_g \\ &= \int_M \frac{\partial}{\partial\epsilon}(\phi_\epsilon)dV_\epsilon. \end{aligned}$$

We may note that it is not difficult to construct a family of volume preserving conformal metrics on a compact manifold, M. Assume for ease of notation that (M, g) is a Riemannian manifold with volume $Vol(M, g) = 1$. Let ρ_ϵ be a smooth family of functions on M as before, and $\tilde{g}_\epsilon = e^{\rho_\epsilon}g$. Then the volume $Vol_{g_\epsilon} := Vol(M, g_\epsilon)$ is a smooth positive function of ϵ, so we can define the family $g_\epsilon := Vol_{g_\epsilon}^{-1}\tilde{g}_\epsilon$, which is a smooth conformal family of unit volume metrics on M.

We make the following observation.

Observation 5.2.1.

(1.) Since $\int_M \dot{\phi}_\epsilon dV_\epsilon = 0$, *it follows that* $\int_M \dot{\phi}_0 dV_g = 0$.

(2.) Also note that

$$\begin{aligned} 0 &= \frac{\partial^2}{\partial\epsilon^2}\int_M dV_\epsilon = \int_M \frac{\partial}{\partial\epsilon}\Big[\frac{n}{2}\frac{\partial}{\partial\epsilon}(\phi_\epsilon)e^{\frac{n}{2}\phi_\epsilon}\Big]dV_g = \int_M \Big[\frac{n^2}{4}(\dot{\phi}_\epsilon)^2 e^{\frac{n}{2}\phi_\epsilon} + \frac{n}{2}\ddot{\phi}_\epsilon e^{\frac{n}{2}\phi_\epsilon}\Big]dV_g \\ &\Rightarrow \int_M \ddot{\phi}dV_\epsilon = -\frac{n}{2}\int_M (\dot{\phi}_\epsilon)^2 dV_\epsilon \text{ and } \int_M \ddot{\phi}dV_g = -\frac{n}{2}\int_M (\dot{\phi}_0)^2 dV_g. \end{aligned}$$

We make the following salient lemmas.

Lemma 5.2.2. *The following properties hold*

- *The expectation values of the commutator* $[\Delta_g, \dot{\phi}_0]$ *with respect to eigenfunctions is zero; and*
- $\mathrm{Tr}(\Delta_g \circ \dot{\phi}_0 e^{-t\Delta_g}) = \mathrm{Tr}(\dot{\phi}_0 \Delta_g e^{-t\Delta_g})$

where $A \circ B$ *denotes composition of the two operators* A *and* B.

Proof. Let ψ_k be an orthonormal basis of eigenfunction of Δ_g, then the expectation value of $[\Delta_g, \dot{\phi}_0]$ on ψ_k is

$$\begin{aligned}
\Big\langle [\Delta_g, \dot{\phi}_0] \Big\rangle_{\psi_k} &:= \langle (\Delta_g \dot{\phi}_0 - \dot{\phi}_0 \Delta_g)\psi_k, \psi_k \rangle_g \\
&= \langle \Delta_g \dot{\phi}_0 \psi_k, \psi_k \rangle_g - \langle \dot{\phi}_0 \Delta_g \psi_k, \psi_k \rangle_g \\
&= \langle \dot{\phi}_0 \psi_k, \Delta_g \psi_k \rangle_g - \langle \Delta_g \psi_k, \dot{\phi}_0 \psi_k \rangle_g \\
&= 0,
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Tr}(\Delta_g \circ \dot{\phi}_0 e^{-t\Delta_g}) &= \sum_{k=0}^{\infty} \langle \Delta_g \circ \dot{\phi}_0 e^{-t\Delta_g} \psi_k, \psi_k \rangle_g = \sum_{k=0}^{\infty} \langle \dot{\phi}_0 e^{-t\Delta_g} \psi_k, \Delta_g \psi_k \rangle_g \\
&= \sum_{k=0}^{\infty} \langle \dot{\phi}_0 e^{-t\lambda_k} \psi_k, \lambda_k \psi_k \rangle_g = \sum_{k=0}^{\infty} \langle \dot{\phi}_0 \lambda_k e^{-t\lambda_k} \psi_k, \psi_k \rangle_g \\
&= \sum_{k=0}^{\infty} \langle \dot{\phi}_0 \Delta_g e^{-t\Delta_g} \psi_k, \psi_k \rangle_g = \mathrm{Tr}(\dot{\phi}_0 \Delta_g e^{-t\Delta_g})
\end{aligned}$$

the lemma follows □

Lemma 5.2.3. $\Delta_g e^{-t\Delta_g} = e^{-t\Delta_g} \Delta_g$ *for all smooth functions on* M.

Proof. Let $\omega(x)$ be a smooth function on M. Then, (dropping the subscript g) we have

$$\begin{aligned}
(\Delta e^{-t\Delta} \omega)(x) &= \Delta_x \Big[\int_M K(t,x,y)\omega(y) dV(y) \Big] = \int_M \Delta_x K(t,x,y)\omega(y) dV(y) \\
&= -\int_M \frac{\partial}{\partial t} K(t,x,y)\omega(y) dV(y).
\end{aligned}$$

Also by the symmetry of the kernel $K(t,x,y)$ in x and y, we have

$$\begin{aligned}(e^{-t\Delta}\Delta\omega)(x) &= \int_M K(t,x,y)\Delta_y\omega(y)dV(y) = \int_M \Delta_y K(t,x,y)\big|_{(x,y)=(y,x)}\omega(y)dV(y)\\ &= -\int_M \frac{\partial}{\partial t}K(t,y,x)\omega(y)dV(y) = -\int_M \frac{\partial}{\partial t}K(t,x,y)\omega(y)dV(y).\end{aligned}$$

Hence,

$$\Delta_g e^{-t\Delta_g} = e^{-t\Delta_g}\Delta_g$$

□

Take $\omega = \dot\phi_0$ to also have that

$$\Delta_g e^{-t\Delta_g}\dot\phi_0 = e^{-t\Delta_g}\Delta_g\dot\phi_0 \quad \text{and} \quad \mathrm{Tr}\left(\Delta_g\dot\phi_0 e^{-t\Delta_g}\right) = \mathrm{Tr}\left(\dot\phi_0 e^{-t\Delta_g}\Delta_g\right).$$

5.3. Variation of spectral zeta function and the Casimir energy

The Casimir energy $\zeta_g(-\frac{1}{2})$ for the n-sphere has been computed explicitly via various regularization procedures of the spectral zeta function as shown in the previous chapter. We proceed now to look at the variations of the zeta function and the Casimir energy. In this study, one of the concerns is a study of the effect of conformal perturbation of the round metric of S^n on the Casimir energy $\zeta_g(-\frac{1}{2})$.

We need the following lemma for our proof of the variational formula for the spectral zeta function:

Lemma 5.3.1.

$$\mathrm{Tr}(\dot\phi_0 e^{-t\Delta g}) = \int_M \dot\phi_0(x)K(t,x,x)dV_g(x)$$

and

$$\mathrm{Tr}(\mathrm{div}(\dot\phi_0\nabla_g)e^{-t\Delta g}) = \frac{1}{2}\int_M \mathrm{div}(\dot\phi_0\nabla_g)K(t,x,x)dV_g(x).$$

Proof. of Lemma (5.3.1)

Let ψ_k be orthonormal basis of eigenfunctions of Δ_g, we have

$$\begin{aligned}
\mathrm{Tr}(\dot{\phi}_0 e^{-t\Delta g}) &= \sum_{k=0}^{\infty} \langle \dot{\phi}_0 e^{-t\Delta_g} \psi_k, \psi_k \rangle_{L^2} \\
&= \sum_{k=0}^{\infty} \int_M \dot{\phi}_0(x) e^{-\lambda_k t} \psi_k(x) \cdot \bar{\psi}_k(x) dV_g(x) \\
&= \int_M \dot{\phi}_0(x) \cdot \sum_{k=0}^{\infty} e^{-\lambda_k t} (\psi_k(x))^2 dV_g(x) \\
&= \int_M \dot{\phi}_0(x) K(t,x,x) dV_g(x).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathrm{Tr}(\mathrm{div}(\dot{\phi}_0 \nabla_g) e^{-t\Delta g}) &= \sum_{k=0}^{\infty} \langle \mathrm{div}(\dot{\phi}_0 \nabla_g) \cdot e^{-t\Delta g} \psi_k, \psi_k \rangle_{L^2} \\
&= \sum_{k=0}^{\infty} \langle \sum_{i,j=1}^{n} g^{ij} \partial_j \dot{\phi}_0 \partial_i \cdot e^{-t\Delta g} \psi_k, \psi_k \rangle_{L^2} \\
&= \sum_{i,j=1}^{n} \sum_{k=0}^{\infty} \langle g^{ij} \partial_j \dot{\phi}_0 \partial_i \cdot e^{-t\Delta g} \psi_k, \psi_k \rangle_{L^2} \\
&= \sum_{i,j=1}^{n} \sum_{k=0}^{\infty} \int_M g^{ij} \partial_j \dot{\phi}_0(x) \int_M \partial_i K(t,x,y) \psi_k(y) dV_g(y) \big|_{x=y} \cdot \bar{\psi}_k(x) dV_g(x);
\end{aligned}$$

where $\partial_i = \frac{\partial}{\partial x_i}$ and $\partial_j = \frac{\partial}{\partial x_j}$ act on functions of x and y respectively.

Now, using the symmetry of $K(t,x,y)$ i.e $K(t,x,y) = K(t,y,x)$ which implies that

$$\partial_i K(t,x,x) = \left[\partial_i K(t,x,y) + \partial_i K(t,y,x)\right]\big|_{x=y} = 2\partial_i K(t,x,y)\big|_{x=y}$$

we have

$$\begin{aligned}
& \sum_{k=0}^{\infty} \langle \mathrm{div}(\dot{\phi}_0 \nabla_g) \cdot e^{-t\Delta g} \psi_k, \psi_k \rangle_{L^2} \\
= \quad & \sum_{i,j=1}^{n} \sum_{k=0}^{\infty} \int_M g^{ij} \partial_j \dot{\phi}_0(x) \int_M \partial_i K(t,x,y)\big|_{x=y} \psi_k(y) dV_g(y) \cdot \bar{\psi}_k(x) dV_g(x) \\
\Rightarrow \quad & \mathrm{Tr}(\mathrm{div}(\dot{\phi}_0 \nabla_g) e^{-t\Delta g}) = \frac{1}{2} \sum_{i,j=1}^{n} \sum_{k=0}^{\infty} \int_M g^{ij} \partial_j \dot{\phi}_0(x) \partial_i e^{-\lambda_k t} (\psi_k(x))^2 dV_g(x) \\
= \quad & \frac{1}{2} \int_M \sum_{i,j=1}^{n} g^{ij} \partial_j \dot{\phi}_0(x) \partial_i \cdot \sum_{k=0}^{\infty} e^{-\lambda_k t} (\psi_k(x))^2 dV_g(x) = \frac{1}{2} \int_M \mathrm{div}(\dot{\phi}_0 \nabla_g) K(t,x,x) dV_g(x).
\end{aligned}$$

□

Theorem 5.3.2. . *Let (M, g) be smooth, compact and connected Riemannian manifold and Δ_g the Laplacian (defined earlier) on it with eigenvalues $\{\lambda_k\}$ listed according to their multiplicities. Let*

$$\{g_\epsilon = e^{\phi_\epsilon} g\}$$

be a family of volume-preserving conformal metrics. Then the spectral zeta function of Δ_ϵ, given by

$$\zeta_{g_\epsilon}(s) = \sum_{k=1}^{\infty} \frac{1}{\big(\Lambda_k(\epsilon)\big)^s} \tag{5.3.1}$$

varies as

$$\zeta_g^{(1)}(s) = s\int_M \dot{\phi}_0(x)\zeta_g(s,x,x)dV_g + \frac{1}{2}(\frac{n}{2}-1)s\int_M (\Delta_g\dot{\phi}_0(x))\zeta(s+1,x,x)dV_g \tag{5.3.2}$$

(c.f: [45] and [47]). We denote this variation evaluated at $\epsilon = 0$ by $\zeta_g^{(1)}(s)$.

Proof. Recall

$$\zeta_{g_\epsilon}(s) = \frac{1}{\Gamma(s)}\int_0^\infty (\mathrm{Tr}(e^{-t\Delta_\epsilon}) - 1)t^{s-1}dt$$

so,

$$\zeta_g^{(1)}(s) = \frac{\partial}{\partial\epsilon}\zeta_{g_\epsilon}(s)|_{\epsilon=0} = \frac{\partial}{\partial\epsilon}\Big|_{\epsilon=0}\Big(\frac{1}{\Gamma(s)}\int_0^\infty (\mathrm{Tr}(e^{-t\Delta_\epsilon}) - 1)t^{s-1}dt\Big). \tag{5.3.3}$$

In line with Ray and Singer ([45]), one gets

$$\begin{aligned}
\zeta_g^{(1)}(s) &= -\frac{1}{\Gamma(s)}\int_0^\infty \mathrm{Tr}\big(\Delta_\epsilon^{(1)} e^{-t\Delta_g}\big)t^s dt \\
&= -\frac{1}{\Gamma(s)}\int_0^\infty \mathrm{Tr}\big(\big[-\dot{\phi}_0\Delta_g + (1-\frac{n}{2})\mathrm{div}(\dot{\phi}_0\nabla_g\big](e^{-t\Delta_g}\big)t^s dt
\end{aligned}$$

where we have used the variation of Δ_ϵ in (5.1.6).

So,

$$\begin{aligned}
\zeta_g^{(1)}(s) &= \frac{1}{\Gamma(s)}\int_0^\infty \mathrm{Tr}\big(\dot{\phi}_0\Delta_g e^{-t\Delta_g}\big)t^s dt - (1-\frac{n}{2})\frac{1}{\Gamma(s)}\int_0^\infty \mathrm{Tr}\big(\mathrm{div}(\dot{\phi}_0\nabla_g e^{-t\Delta_g}\big)t^s dt \\
&= -\frac{1}{\Gamma(s)}\int_0^\infty \frac{\partial}{\partial t}\mathrm{Tr}\big(\dot{\phi}_0 e^{-t\Delta_g}\big)t^s dt + (\frac{n}{2}-1)\frac{1}{\Gamma(s)}\int_0^\infty \mathrm{Tr}\big(\mathrm{div}(\dot{\phi}_0\nabla_g e^{-t\Delta_g}\big)t^s dt.
\end{aligned}$$

Integrating by parts in the first term, gives

$$\begin{aligned}
\zeta_g^{(1)}(s) &= \frac{s}{\Gamma(s)}\int_0^\infty \mathrm{Tr}\big(\dot{\phi}_0(e^{-t\Delta_g} - \frac{1}{V})\big)t^{s-1}dt \\
&+ (\frac{n}{2}-1)\frac{1}{\Gamma(s)}\int_0^\infty \mathrm{Tr}\big(\mathrm{div}(\dot{\phi}_0\nabla_g(e^{-t\Delta_g} - \frac{1}{V})\big)t^s dt
\end{aligned}$$

where $\frac{1}{V}$ denotes $f \mapsto \frac{1}{V}\int_M f dV$ and V is the volume of (M, g).

Hence to complete the proof of Theorem (5.3.2), using lemma (5.3.1), we have the variation of the zeta function as

$$\begin{aligned}\zeta_g^{(1)}(s) &= \frac{s}{\Gamma(s)}\int_0^\infty \int_M \dot{\phi}_0(x)(K(t,x,x) - \frac{1}{V}))dV_g t^{s-1}dt \\ &+ \frac{1}{2}(\frac{n}{2}-1)\frac{1}{\Gamma(s)}\int_0^\infty \int_M \operatorname{div}(\dot{\phi}_0\nabla_g)(K(t,x,x) - \frac{1}{V}))dV_g t^s dt.\end{aligned}$$

Since

$$\int_M \left(\dot{\phi}_0(x)K(t,x,x) - \frac{1}{V}\right) dV_g(x) \to 0$$

decays exponentially fast as $t \to \infty$, one can use the Fubini - Tonelli theorem (see chapter 2 section (2.8)). Also, recognizing that $\frac{1}{\Gamma(s)} = \frac{s}{\Gamma(s+1)}$ we have

$$\begin{aligned}\zeta_g^{(1)}(s) &= s\int_M \dot{\phi}_0(x)\left\{\frac{1}{\Gamma(s)}\int_0^\infty (K(t,x,x) - \frac{1}{V}))t^{s-1}dt\right\}dV_g \\ &+ \frac{1}{2}(\frac{n}{2}-1)s\int_M \operatorname{div}(\dot{\phi}_0\nabla_g)\left\{\frac{1}{\Gamma(s+1)}\int_0^\infty (K(t,x,x) - \frac{1}{V})t^s dt\right\}dV_g.\end{aligned}$$

Therefore,

$$\begin{aligned}\zeta_g^{(1)}(s) &= s\int_M \dot{\phi}_0(x)\zeta_g(s,x,x)dV_g \\ &+ \frac{1}{2}(\frac{n}{2}-1)s\int_M \operatorname{div}(\dot{\phi}_0\nabla_g)\zeta(s+1,x,x)dV_g.\end{aligned}$$

By Green's formula, we have

$$\zeta_g^{(1)}(s) = s\int_M \dot{\phi}_0(x)\zeta_g(s,x,x)dV_g + \frac{1}{2}(\frac{n}{2}-1)s\int_M (\Delta_g\dot{\phi}_0(x))\zeta_g(s+1,x,x)dV_g$$

which completes the proof. □

Again, note that the first-order variation $\zeta_g^{(1)}(s)$ given by the formula (5.3.2) is true for large s. The right-hand-side of equation (5.3.2) is meromorphic in s with simple poles given in equation (3.4.1) and residues given by equation (3.4.2).

Observation 5.3.3. *(1.) For 2-dimensional Riemannian manifold, the variation of the spectral zeta function reduces to*

$$\zeta_g^{(1)}(s) = s\int_M \dot{\phi}_0(x)\zeta_g(s,x,x)dV_g$$

which agrees with known results; see for example ([43] and [45]).

(2.) Note that

$$\lim_{s\to 0}\frac{\partial}{\partial s}(\frac{1}{\Gamma(s)}) = \lim_{s\to 0}\frac{\partial}{\partial s}(\frac{s}{\Gamma(s+1)}) = \lim_{s\to 0}\{s\frac{\partial}{\partial s}(\frac{1}{\Gamma(s+1)}) + \frac{1}{\Gamma(s+1)}\} = 1.$$

Therefore by equation (5.3.2), the Casimir energy has the first order variation at $\epsilon = 0$, $\mathrm{FP}[\zeta_g^{(1)}(s)]\big|_{s=-\frac{1}{2}}$, given by

$$\begin{aligned} \mathrm{FP}[\zeta_g^{(1)}(s)]\big|_{s=-\frac{1}{2}} &= -\frac{1}{2}\int_M \dot{\phi}_0(x)\mathrm{FP}[\zeta_g(s,x,x)]\big|_{s=-\frac{1}{2}} dV_g \\ &- \frac{1}{4}(\frac{n}{2}-1)\int_M (\Delta_g\dot{\phi}_0(x))\mathrm{FP}[\zeta_g(s+1,x,x)]\big|_{s=-\frac{1}{2}} dV_g. \end{aligned} \tag{5.3.4}$$

Remark 5.3.4. *Write $\{\tilde{\Lambda}_k(\epsilon)\}$ for the spectrum (including possible multiplicities) of the operator $\Delta_\epsilon + c$ with $c = \frac{n-1}{2}$ on S^n under the family of volume-preserving conformal metrics $\{g_\epsilon = e^{\phi_\epsilon} g\}$ then set*

$$\tilde{\Lambda}_k(\epsilon) = \Lambda_k(\epsilon) + c.$$

We have its associated zeta function given by

$$Z_\epsilon(s) = \sum_{k=1}^{\infty}\frac{1}{(\tilde{\Lambda}_k(\epsilon))^s}$$

with $\Re(s) \gg 0$.

Define the function of positive time t by

$$\begin{aligned} f(t) &= \sum_{k=1}^{\infty} e^{-(\Lambda_k(\epsilon)+c)t} \quad \text{as} \quad t\searrow 0 \\ &= e^{-ct}\sum_{k=1}^{\infty} e^{-\Lambda_k(\epsilon)t} \quad \text{as} \quad t\searrow 0 \end{aligned}$$

so that the Mellin transform of $\zeta_{g_\epsilon}(s)$ equals that of $f(t)$ up to a multiple of an infinitesimal positive constant, namely $0 < e^{-ct} \le 1$ as $t \searrow 0$ and $c \to \infty$.

Then, we have that the first, second and higher order variations of $\zeta_{g_\epsilon}(s)$ is equivalent to that of $Z_\epsilon(s)$.

Definition 5.3.5. *The metric g is called a critical point of the Casimir energy $\zeta_g(-\frac{1}{2})$ with respect to all variations $\{g_\epsilon = e^{\phi_\epsilon} g\}$, if the variation $\zeta_g^{(1)}(-\frac{1}{2})$ vanishes for all g_ϵ.*

Another result of this work is the following:

Theorem 5.3.6. *Let Δ_ϵ be the Laplacian on (M, g_ϵ) with zeta kernel $\zeta_g(s,x,y)$. Then, g is a critical point of the Casimir energy $\zeta_g(-\frac{1}{2})$ for all constant-volume conformal variations of the metric if* $\mathrm{FP}[\zeta_g^{(1)}(-\frac{1}{2},x,x)]$ *is constant in x.*

Proof. By the definition of critical point above and the variation of the Casimir energy (5.3.4), consider the function

$$F_{\dot{\phi}_0}(x) := (-\frac{1}{2}\dot{\phi}_0(x)\mathrm{FP}[\zeta_g(s,x,x)] - \frac{1}{4}(\frac{n}{2}-1)(\Delta_g\dot{\phi}_0)\mathrm{FP}[\zeta_g(s+1,x,x)])\big|_{s=-\frac{1}{2}} \quad (5.3.5)$$

where of course,

$$\zeta_g(s,x,x)\big|_{s=-\frac{1}{2}} = \frac{1}{\Gamma(s)}\int_0^\infty \big[K(t,x,x) - \frac{1}{V}\big]t^{s-1}dt\big|_{s=-\frac{1}{2}}.$$

We have a critical point if

$$\begin{aligned}\int_M F_{\dot{\phi}_0}(x)\mathrm{d}V_x &= 0 \\ &\forall \quad \dot{\phi}_0 \in C^\infty(M) \ \text{ such that } \ \int_M \dot{\phi}_0(x)\mathrm{d}V_x = 0. \qquad (5.3.6)\end{aligned}$$

Now, suppose $\mathrm{FP}[\zeta_g(-\frac{1}{2},x,x)]$ is constant. Then one gets

$$\begin{aligned}\int_M F_{\dot{\phi}_0}\mathrm{d}V_x &= -\frac{1}{2}\mathrm{FP}[\zeta_g(-\frac{1}{2},x,x)]\int_M \dot{\phi}_0(x)\mathrm{d}V_x \\ &- \frac{1}{4}(\frac{n}{2}-1)\int_M(\Delta_g\dot{\phi}_0(x))\mathrm{FP}[\zeta_g(\frac{1}{2},x,x)]\mathrm{d}V_x \\ &= -\frac{1}{4}(\frac{n}{2}-1)\int_M(\Delta_g\dot{\phi}_0(x))\mathrm{FP}[\zeta_g(\frac{1}{2},x,x)]\mathrm{d}V_x\end{aligned}$$

since $\int_M \dot{\phi}_0(x)\mathrm{d}V_x = 0$. Now by the self-adjointness of Δ_g, we have

$$\int_M F_{\dot{\phi}_0}(x)\mathrm{d}V_x = -\frac{1}{4}(\frac{n}{2}-1)\int_M \dot{\phi}_0(x)\Delta_g\mathrm{FP}[\zeta_g(\frac{1}{2},x,x)]\mathrm{d}V_x = 0$$

since the Laplacian of a constant function is zero. □

We get a corollary by considering homogeneous manifolds.

Definition 5.3.7. *([8]) A Riemannian manifold (M, g) is called homogeneous if for any two points $x, y \in M$, there exists an isometry $I : M \to M$ with $I(x) = y$. That is to say, I acts transitively on M. More generally, a smooth Riemannian manifold (M, g) endowed with transitive smooth action of a Lie group G is called a G-homogeneous manifold.*

Note: The fact that the action of the group is transitive means the manifold "looks the same" everywhere.

A nice class of homogeneous manifolds comes from quotients of Lie groups with left-invariant metrics. For example, the natural action of $SO(n+1)$ on the n-sphere S^n is transitive, hence $S^n \approx SO(n+1)/SO(n)$ is a homogeneous manifold.

Corollary 5.3.8. *The metrics on homogeneous smooth Riemannian manifolds are critical points of the variation of the Casimir energy $\zeta_g(-\frac{1}{2})$ under fix-volume conformal variation of the metric.*

Proof. Since $\zeta_g(s, x, x)$ is an invariant under isometries on homogeneous manifolds, one can map any point to another point via isometry. Hence, $\zeta_g(s, x, x)$ is a constant $\square$

Remark 5.3.9. *The round metric g on S^n is a critical point for the Casimir energy $\zeta_g(-\frac{1}{2})$ over the constant-volume conformal class $\{g_\epsilon = e^{\phi_\epsilon} g\}$ because S^n is a homogeneous manifold.*

We say that the metric g on M is **critical** for the heat kernel $K_g(t, x, x)$ at the time t, if for any volume-preserving deformation $\{g_\epsilon = e^{\phi_\epsilon} g\}$,

$$\frac{d}{d\epsilon} K_\epsilon(t, x, x)|_{\epsilon=0} = 0. \tag{5.3.7}$$

The spectral zeta function of the Laplacian Δ_ϵ for $\Re(s) > \frac{n}{2}$ is

$$\zeta_{g_\epsilon}(s) = \sum_{k=1}^{\infty} \frac{1}{\Lambda_k^s(\epsilon)} = \frac{1}{\Gamma(s)} \int_0^\infty (K_\epsilon(t, x, x) - \frac{1}{V}) t^{s-1} dt. \tag{5.3.8}$$

For sufficiently large $\Re(s)$, one has

$$\frac{d}{d\epsilon} \zeta_{g_\epsilon}(s)|_{\epsilon=0} = \frac{1}{\Gamma(s)} \int_0^\infty \frac{d}{d\epsilon} \big(K_\epsilon(t, x, x)\big|_{\epsilon=0}\big) t^{s-1} dt. \tag{5.3.9}$$

If g is critical for the heat kernel at any time $t > 0$, then its derivative in ϵ vanishes at $\epsilon = 0$ for all s, so also

$$\lim_{s \to -1/2} \left[\frac{d}{d\epsilon} \zeta_{g_\epsilon}(s)|_{\epsilon=0}\right] = 0. \tag{5.3.10}$$

Hence, g is a critical point for the variation of the Casimir energy $\zeta_g(-\frac{1}{2})$ of the Laplacian on S^n. Consequently, we write this result as the lemma below:

Lemma 5.3.10. *If g is a critical metric for the heat kernel at any time $t > 0$, then it is also a critical metric for the Casimir energy $\zeta_g(-\frac{1}{2})$ under all volume-preserving conformal perturbations $\{g_\epsilon\}$.*

Proposition 5.3.11. *The following conditions hold on all closed homogeneous Riemannian manifolds (M, g):*

(1.) The metric g is critical for the heat kernel at any time $t > 0$ under all volume-preserving conformal deformations.

(2.) The metric g is critical for the Casimir energy $\zeta_g(-\frac{1}{2})$ under all volume-preserving conformal deformations.

(3.) For all $t > 0$, $K_\epsilon(t, x, x)$ is constant on M.

Proof. The proposition follows from the lemma (5.3.10) above. □

CHAPTER 6

Hessians of $\zeta_g(s)$ on Homogeneous manifolds

Now, we proceed to decide the extremal nature of the canonical metric g on M for the volume-preserving conformal deformation $\{g_\epsilon\}$. To do that, we compute the second order variation of the spectral zeta function.

6.1. The case of Δ_g on M.

We have the following result for the second variation of the spectral zeta function of Δ_g on a closed homogeneous Riemannian manifold M.

Theorem 6.1.1. *Let M be a closed homogeneous manifold with the canonical metric g scaled to volume V. Let $\{g_\epsilon = e^{\phi_\epsilon} g\}$ be a family of volume-preserving conformal metrics on M where*

$$\int_M \dot{\phi}_0(x) dV_g(x) = 0$$

and

$$\int_M (\dot{\phi}_0(x))^2 dV_g(x) > 0.$$

Then the second order variation, $\zeta_g^{(2)}(s)$, of the spectral zeta function $\zeta_{g_\epsilon}(s)$ on M at $\epsilon = 0$ is *given by*

$$
\begin{aligned}
\zeta_g^{(2)}(s) &= s\int_M\int_M \dot{\phi}_0(x)\Psi_{s-1}(x,y)\dot{\phi}_0(y)dV_g(x)dV_g(y)\\
&- (1-\frac{n}{2})s\int_M\int_M \dot{\phi}_0(x)\Psi_s(x,y)(\Delta_g\dot{\phi}_0(y))dV_g(x)dV_g(y)\\
&+ \frac{(n-2)^2}{16}s\int_M\int_M(\Delta_g\dot{\phi}_0(x))\Psi_{s+1}(x,y)(\Delta_g\dot{\phi}_0(y))dV_g(x)dV_g(y)\\
&- \frac{1}{8}(n+2)^2 s\frac{1}{V}\int_M\int_M \dot{\phi}_0(x)\dot{\phi}_0(y)\zeta(s,x,y)dV_g(x)dV_g(y)\\
&- \frac{1}{8}(n-2)^2 s\zeta_n(s+1)\frac{1}{V}\int_M \dot{\phi}_0(x)(\Delta_g\dot{\phi}_0(x))dV_g(x)\\
&+ (1-\frac{n}{2})s\zeta_n(s)\frac{1}{V}\int_M(\dot{\phi}_0(x))^2dV_g(x), \qquad (6.1.1)
\end{aligned}
$$

where for $\Re(s)$ sufficiently large, we define

$$\Psi_s(x,y) = \frac{1}{\Gamma(s)}\int_0^\infty\int_0^\infty(K(u,x,y)-\frac{1}{V})(K(v,x,y)-\frac{1}{V})(u+v)^{s-1}\mathrm{d}u\mathrm{d}v. \qquad (6.1.2)$$

Proof. Recall from equation (3.1.12) of Chapter 3 that $\zeta_{g_\epsilon}(s) = \mathrm{Tr}(\Delta_\epsilon^{-s})$. Let P_ϵ be the projection with respect to the metric g_ϵ onto the kernel of Δ_ϵ. Then, since the kernel for all values of ϵ is the constant functions, we get

$$
\left.
\begin{aligned}
P_\epsilon^2 &= P_\epsilon\\
\dot{\left(P_\epsilon^2\right)} &= P_\epsilon\dot{P}_\epsilon = P_\epsilon\dot{P}_\epsilon + \dot{P}_\epsilon P_\epsilon = \dot{P}_\epsilon\\
\Delta_\epsilon P_\epsilon &= 0.
\end{aligned}
\right\} \qquad (6.1.3)
$$

Differentiating, we get

$$
\left.
\begin{aligned}
P_\epsilon\dot{\Delta}_\epsilon P_\epsilon &= -P_\epsilon\Delta_\epsilon\dot{P}_\epsilon = P_\epsilon\Delta_\epsilon P_\epsilon\dot{P}_\epsilon = 0\\
\text{i.e, }\ \dot{\Delta}_\epsilon P_\epsilon &= -\Delta_\epsilon\dot{P}_\epsilon = 0.
\end{aligned}
\right\} \qquad (6.1.4)
$$

Taking the trace in the second equation above,

$$-\,\mathrm{Tr}(P_\epsilon\dot{P}_\epsilon) = \mathrm{Tr}(\dot{P}_\epsilon) = 0. \qquad (6.1.5)$$

Now for $\Re(s) < -1$,

$$\mathrm{Tr}\Big(\dot{P}_\epsilon\Delta_\epsilon^{-s-1}\Big) = -\mathrm{Tr}\Big(P_\epsilon\dot{P}_\epsilon\Delta_\epsilon^{-s-1}\Big) = 0, \qquad (6.1.6)$$

and so by analytic continuation, this is true for all s. From Lemma (3.3.1) and Equation 6.1.5,

$$\zeta_{g_\epsilon}(s) = \mathrm{Tr}\Big(\Delta_\epsilon^{-s} - P_\epsilon\Big) = \frac{1}{\Gamma(s)}\int_0^\infty\mathrm{Tr}\Big(\mathrm{e}^{-t\Delta_\epsilon} - P_\epsilon\Big)t^{s-1}\mathrm{d}t.$$

Now the decay in the integrand allows us to bring the derivative inside, and we obtain:

$$\begin{aligned}\frac{\partial}{\partial\epsilon}\big(\zeta_{g_\epsilon}(s)\big) &= -\frac{s}{\Gamma(s+1)}\int_0^\infty \mathrm{Tr}\Big(\dot\Delta_\epsilon\big(\mathrm{e}^{-t\Delta_\epsilon}-P_\epsilon\big)\Big)t^s\mathrm{d}t - \frac{1}{\Gamma(s)}\int_0^\infty \mathrm{Tr}(\dot P)t^s\,dt \\ &= -\frac{1}{\Gamma(s)}\int_0^\infty \mathrm{Tr}\Big(\dot\Delta_\epsilon\big(\mathrm{e}^{-t\Delta_\epsilon}-P_\epsilon\big)\Big)t^s\mathrm{d}t \qquad (6.1.7)\end{aligned}$$

since by (6.1.6) the last term vanishes.

Hence at $\epsilon = 0$, we get

$$\zeta_g^{(1)}(s) = -\frac{1}{\Gamma(s)}\int_0^\infty \mathrm{Tr}\Big(\Delta_0^{(1)}\big(\mathrm{e}^{-t\Delta_0}-P\big)\Big)t^s\mathrm{d}t$$

($P \equiv P_0$) which we can check agrees with the first-order variation of the spectral zeta function (5.3.3) of Chapter 5.

Differentiating (6.1.7) a second time, we get

$$\begin{aligned}\frac{\partial^2}{\partial\epsilon^2}\big(\zeta_{g_\epsilon}(s)\big) &= -s\mathrm{Tr}\Big(\ddot\Delta_\epsilon\Delta_\epsilon^{-s-1}\Big) - s\mathrm{Tr}\Big(\dot\Delta_\epsilon\big(\frac{\partial}{\partial\epsilon}(\Delta_\epsilon^{-s-1}\big)\Big) \\ &= -\frac{s}{\Gamma(s+1)}\int_0^\infty \mathrm{Tr}\Big(\ddot\Delta_\epsilon \mathrm{e}^{-t\Delta_\epsilon}\Big)t^s\mathrm{d}t \\ &- \frac{s}{\Gamma(s+1)}\int_0^\infty \mathrm{Tr}\Big(\dot\Delta_\epsilon\big(\frac{\partial}{\partial\epsilon}(\mathrm{e}^{-t\Delta_\epsilon}) - P_\epsilon)\big)\Big)t^s\mathrm{d}t. \qquad (6.1.8)\end{aligned}$$

By Duhamel's formula (2.7.1),

$$\frac{\partial}{\partial\epsilon}\mathrm{e}^{-t\Delta_\epsilon} = -\int_0^t \mathrm{e}^{-u\Delta_\epsilon}\dot\Delta_\epsilon \mathrm{e}^{-(t-u)\Delta_\epsilon}\mathrm{d}u$$

for times $0 < u < t < T$.

Therefore, again using (6.1.4), the right side of equation (6.1.8) becomes

$$-\frac{1}{\Gamma(s)}\int_0^\infty \mathrm{Tr}\Big(\ddot\Delta_\epsilon\big(\mathrm{e}^{-t\Delta_\epsilon}-P_\epsilon\big)\Big)t^s\mathrm{d}t + \frac{1}{\Gamma(s)}\int_0^\infty\int_0^t \mathrm{Tr}\Big(\dot\Delta_\epsilon \mathrm{e}^{-u\Delta_\epsilon}\dot\Delta_\epsilon \mathrm{e}^{-(t-u)\Delta_\epsilon}\Big)t^s\mathrm{d}u\mathrm{d}t.$$

At $\epsilon = 0$, we have

$$\begin{aligned}\zeta_g^{(2)}(s) &= \frac{1}{\Gamma(s)}\int_0^\infty\int_0^t \mathrm{Tr}\Big(\Delta_0^{(1)}\mathrm{e}^{-u\Delta_g}\Delta_0^{(1)}\mathrm{e}^{-(t-u)\Delta_g}\Big)t^s\mathrm{d}u\mathrm{d}t \\ &- \frac{1}{\Gamma(s)}\int_0^\infty \mathrm{Tr}\Big(\Delta_0^{(2)}\big(\mathrm{e}^{-t\Delta_g}-P_0\big)\Big)t^s\mathrm{d}t.\end{aligned}$$

We write this for simplicity as

$$\begin{aligned}
\zeta_g^{(2)}(s) &= \mathrm{var}_1(s) + \mathrm{var}_1(s) \;\; \text{where} \\
\mathrm{var}_1(s) &:= \frac{1}{\Gamma(s)} \int_0^\infty \int_0^t \mathrm{Tr}\Big(\Delta_0^{(1)} \mathrm{e}^{-u\Delta_g} \Delta_0^{(1)} \mathrm{e}^{-(t-u)\Delta_g}\Big) t^s \mathrm{d}u \mathrm{d}t \;\; \text{and} \\
\mathrm{var}_2(s) &:= -\frac{1}{\Gamma(s)} \int_0^\infty \mathrm{Tr}\Big(\Delta_0^{(2)} \mathrm{e}^{-t\Delta_g} - P_0\Big) t^s \mathrm{d}t.
\end{aligned}$$

For easier handling of the computation of the terms of this second-order variation, we rewrite the operator Δ_ϵ defined by (5.1.5) in terms of the Laplacian as follows. Define the operator

$$\mathcal{G}_{\dot\phi_0} = \langle \nabla \dot\phi_0, \nabla \cdot \rangle_g \tag{6.1.9}$$

and immediately observe from relation (ii) of Proposition (2.4.6) that for any $f \in C^\infty(M)$

$$\mathcal{G}_{\dot\phi_0} f = \frac{1}{2}(\Delta \dot\phi_0) f + \frac{1}{2}\dot\phi_0 \Delta f - \frac{1}{2}(\Delta \circ \dot\phi_0) f. \tag{6.1.10}$$

Thus, for any $\psi \in C^\infty(M)$, $\Delta_\epsilon \psi$ can be written as

$$\begin{aligned}
\Delta_\epsilon \psi &= e^{-\phi_\epsilon} \Delta_g \psi + \frac{1}{2}(1 - \frac{n}{2}) e^{-\phi_\epsilon} (\Delta_g \phi_\epsilon) \psi \\
&+ \frac{1}{2}(1 - \frac{n}{2}) e^{-\phi_\epsilon} \phi_\epsilon (\Delta_g \psi) - \frac{1}{2}(1 - \frac{n}{2}) e^{-\phi_\epsilon} (\Delta_g \circ \phi_\epsilon) \psi.
\end{aligned} \tag{6.1.11}$$

Note again that $\mathcal{G}_1 \circ \mathcal{G}_2$ denotes composition of operators; e.g

$$(e^{-\phi_\epsilon} (\Delta_g \circ \phi_\epsilon) \psi)(x) = e^{-\phi_\epsilon} \Delta_g (\phi_\epsilon(x) \psi(x)).$$

Similarly, one can re-write (5.1.6) as

$$\Delta_0^{(1)} = -\frac{1}{2}(\frac{n}{2} + 1)\dot\phi_0 \Delta_g - \frac{1}{2}(\frac{n}{2} - 1)(\Delta_g \dot\phi_0) + \frac{1}{2}(\frac{n}{2} - 1)\Delta_g \circ \dot\phi_0. \tag{6.1.12}$$

6.1.1. Computation of $\mathrm{var}_1(s)$

We now compute the term

$$\mathrm{var}_1(s) := \frac{1}{\Gamma(s)} \int_0^\infty \int_0^t \mathrm{Tr}\Big(\Delta_0^{(1)} \mathrm{e}^{-u\Delta_g} \Delta_0^{(1)} \mathrm{e}^{-(t-u)\Delta_g}\Big) t^s \mathrm{d}u \mathrm{d}t. \tag{6.1.13}$$

First we make a few notes. In what follows, we will often drop the subscript g of Δ_g and write $K(u,x,y)$ and $K(t-u,y,x)$ for the kernels of the operators $\mathrm{e}^{-u\Delta_g}$ and $\mathrm{e}^{-(t-u)\Delta_g}$ respectively.

If A and B are differential operators, then their compositions with smoothing operators are bounded, so by the vanishing of trace on commutators of bounded operators, and by change of variables,

$$\begin{aligned}\int_0^t \mathrm{Tr}\Big(Ae^{-u\Delta}Be^{-(t-u)\Delta}\Big)du &= \int_0^t \mathrm{Tr}\Big(Be^{-(t-u)\Delta}Ae^{-u\Delta}\Big)du \\ = \int_0^t \mathrm{Tr}\Big(Ae^{-(t-u)\Delta}Be^{-u\Delta}\Big)du &= \int_0^t \mathrm{Tr}\Big(Be^{-u\Delta}Ae^{-(t-u)\Delta}\Big)du. \end{aligned} \tag{6.1.14}$$

Observe that for any $f \in C^\infty(M)$,

$$\int_M \mathrm{div}(\dot{\phi}_0\nabla_g)fdV_g(x) = \int_M \langle\nabla_g\dot{\phi}_0, \nabla_g f\rangle_g dV_g(x) = \int_M (\Delta_g\dot{\phi}_0)fdV_g(x). \tag{6.1.15}$$

We denote $K(u,x,y) - \frac{1}{V}$ by $\tilde{K}_u$ and $K(v,x,y) - \frac{1}{V}$ by $\tilde{K}_v$. For $f_1, f_2 \in C^\infty(M)$ we also define the notation

$$\mathrm{Tr}[f_1\tilde{K}_u f_2\tilde{K}_v] := \int_M\int_M f_1(x)\tilde{K}_u f_2(y)\tilde{K}_v dV_g(x)dV_g(y). \tag{6.1.16}$$

Now we can begin our calculation of $\mathrm{var}_1(s)$. Using (6.1.12), we get the terms inside the trace in the formula for $\mathrm{var}_1(s)$ by expanding the function

$$\begin{aligned}\mathrm{Tr}\bigg(\Big(&-\frac{1}{2}(\frac{n}{2}+1)\dot{\phi}_0\Delta_g - \frac{1}{2}(\frac{n}{2}-1)(\Delta_g\dot{\phi}_0) + \frac{1}{2}(\frac{n}{2}-1)\Delta_g\circ\dot{\phi}_0\Big)\tilde{K}_u \\ &\cdot\Big(-\frac{1}{2}(\frac{n}{2}+1)\dot{\phi}_0\Delta_g - \frac{1}{2}(\frac{n}{2}-1)(\Delta_g\dot{\phi}_0) + \frac{1}{2}(\frac{n}{2}-1)\Delta_g\circ\dot{\phi}_0\Big)\tilde{K}_v\bigg).\end{aligned}$$

The resulting terms of the expansion are simplified via the following lemmata.

Lemmata 6.1.2.

$$\mathrm{Tr}[\dot{\phi}_0(\Delta\tilde{K}_u)(\Delta\dot{\phi}_0)\tilde{K}_v] = -\frac{\partial}{\partial u}\mathrm{Tr}[\dot{\phi}_0\tilde{K}_u(\Delta\dot{\phi}_0)\tilde{K}_v] \tag{$\blacktriangleleft$}$$

$$\mathrm{Tr}[\dot{\phi}_0(\Delta\tilde{K}_u)\dot{\phi}_0(\Delta\tilde{K}_v)] = \frac{\partial^2}{\partial u\partial v}\mathrm{Tr}[\dot{\phi}_0\tilde{K}_u\dot{\phi}_0\tilde{K}_v] \tag{$\clubsuit$}$$

$$\mathrm{Tr}[\dot{\phi}_0(\Delta\tilde{K}_u)\Delta\circ\dot{\phi}_0\tilde{K}_v] = \frac{\partial^2}{\partial u^2}\mathrm{Tr}[\dot{\phi}_0\tilde{K}_u\dot{\phi}_0\tilde{K}_v] \tag{$\bigstar$}$$

$$\mathrm{Tr}[\Delta\circ\dot{\phi}_0\tilde{K}_u\Delta\circ\dot{\phi}_0\tilde{K}_v] = \frac{\partial^2}{\partial u\partial v}\mathrm{Tr}[\dot{\phi}_0\tilde{K}_u\dot{\phi}_0\tilde{K}_v] \tag{$\blacklozenge$}$$

Proof. To see (◄), observe that

$$\begin{aligned}\mathrm{Tr}[\dot\phi_0(\Delta\tilde K_u)(\Delta\dot\phi_0)\tilde K_v] &= \int_M\int_M \dot\phi_0(x)(\Delta_x\tilde K_u)(\Delta_y\dot\phi_0(y))\tilde K_v dV_g(x)dV_g(y)\\ &= -\frac{\partial}{\partial u}\int_M\int_M \dot\phi_0(x)\tilde K_u(\Delta_y\dot\phi_0(y))\tilde K_v dV_g(x)dV_g(y).\end{aligned}$$

If Δ_y is also on $\tilde K_v$ in the proceeding expression, we have (♣). Similarly for (★),

$$\begin{aligned}\mathrm{Tr}[\dot\phi_0(\Delta\tilde K_u)\Delta\circ\dot\phi_0\tilde K_v] &= \int_M\int_M \dot\phi_0(x)(\Delta_x\tilde K_u)\Delta_y(\dot\phi_0(y)\tilde K_v)dV_g(x)dV_g(y)\\ &= -\frac{\partial}{\partial u}\int_M \dot\phi_0(x)\{\int_M \tilde K_u\Delta_y(\dot\phi_0(y)\tilde K_v)dV_g(y)\}dV_g(x)\\ &= -\frac{\partial}{\partial u}\int_M \dot\phi_0(x)\{\int_M (\Delta_y\tilde K_u)\dot\phi_0(y)\tilde K_v dV_g(y)\}dV_g(x)\end{aligned}$$

by self-adjointness of Δ_y. Similarly (★) gives

$$\mathrm{Tr}[\dot\phi_0(\Delta\tilde K_u)\Delta\circ\dot\phi_0\tilde K_v] = \frac{\partial^2}{\partial u^2}\int_M\int_M \dot\phi_0(x)\tilde K_u\dot\phi_0(y)\tilde K_v dV_g(x)dV_g(y).$$

Finally for (♦), expand $\mathrm{Tr}[\Delta_x\circ\dot\phi_0\tilde K_u\Delta_y\circ\dot\phi_0)\tilde K_v]$ as follows:

$$\begin{aligned}&\mathrm{Tr}[\Delta_x\circ\dot\phi_0\tilde K_u\Delta_y\circ(\dot\phi_0)\tilde K_v] = \int_M\int_M \Delta_x\big\{\dot\phi_0(x)\tilde K_u\big\}\cdot\Delta_y\big\{(\dot\phi_0(y))\tilde K_v\big\}dV_g(x)dV_g(y)\\ &= \int_M\int_M[(\Delta_x\dot\phi_0(x))\tilde K_u + \dot\phi_0(x)(\Delta_x\tilde K_u) - 2(\nabla_x\dot\phi_0(x))(\nabla_x\tilde K_u)]\cdot\Delta_y\big\{(\dot\phi_0(y))\tilde K_v\big\}dV_g(x)dV_g(y)\\ &= \int_M\int_M[(\Delta_x\dot\phi_0(x))\tilde K_u + \dot\phi_0(x)(\Delta_x\tilde K_u) - (\Delta_x\dot\phi_0(x))\tilde K_u - \dot\phi_0(x)(\Delta_x\tilde K_u) + \Delta_x\big\{\dot\phi_0(x)\tilde K_u\big\}]\\ &\cdot\Delta_y\big\{(\dot\phi_0(y))\tilde K_v\big\}dV_g(x)dV_g(y)\\ \Rightarrow\quad &\mathrm{Tr}[\Delta_x\circ\dot\phi_0\tilde K_u\Delta_y\circ(\dot\phi_0)\tilde K_v] = \int_M\int_M[(\Delta_x\dot\phi_0(x))\Delta_y\tilde K_u + \dot\phi_0(x)(\Delta_x\Delta_y\tilde K_u)\\ -\quad &(\Delta_x\dot\phi_0(x))\Delta_y\tilde K_u - \dot\phi_0(x)(\Delta_y\Delta_x\tilde K_u) + \Delta_x\big\{\dot\phi_0(x)\Delta_y\tilde K_u\big\}]\cdot(\dot\phi_0(y))\tilde K_v dV_g(x)dV_g(y).\end{aligned}$$

$$\text{Thus,}\quad \mathrm{Tr}[\Delta_x\circ\dot\phi_0\tilde K_u\Delta_y\circ(\dot\phi_0)\tilde K_v] = \int_M\int_M \dot\phi_0(x)(\Delta_y\tilde K_u)\cdot\dot\phi_0(y)(\Delta_x\tilde K_v)dV_g(x)dV_g(y)$$

□

By a change of coordinates (u,t) to $(u,v) := (u, t-u)$, the double integral in $\mathrm{var}_1(s)$ becomes a double integral over the first quadrant in u and v. Further, because the derivatives we obtain through applying Lemmata (6.1.2) are applied to functions that are constant in the other variable, these derivatives carry through under the coordinate change to derivatives with respect to u and v.

Now collecting like terms gives

$$\begin{aligned}
\mathrm{var}_1(s) &= \frac{1}{8}(n^2+4)\frac{1}{\Gamma(s)}\int_0^\infty\int_0^\infty \frac{\partial^2}{\partial u\partial v}\mathrm{Tr}[\dot{\phi}_0\tilde{K}_u\dot{\phi}_0\tilde{K}_v](u+v)^s\mathrm{d}u\mathrm{d}v \\
&- \frac{1}{8}(n^2-4)\frac{1}{\Gamma(s)}\int_0^\infty\int_0^\infty \frac{\partial^2}{\partial u^2}\mathrm{Tr}[\dot{\phi}_0\tilde{K}_u\dot{\phi}_0\tilde{K}_v](u+v)^s\mathrm{d}u\mathrm{d}v \\
&+ (1-\frac{1}{2})\frac{1}{\Gamma(s)}\int_0^\infty\int_0^\infty \frac{\partial}{\partial u}\mathrm{Tr}[\dot{\phi}_0\tilde{K}_u(\Delta\dot{\phi}_0)\tilde{K}_v](u+v)^s\mathrm{d}u\mathrm{d}v \\
&+ \frac{1}{16}(n-2)^2\frac{1}{\Gamma(s)}\int_0^\infty\int_0^\infty \mathrm{Tr}[(\Delta\dot{\phi}_0)\tilde{K}_u(\Delta\dot{\phi}_0)\tilde{K}_v](u+v)^s\mathrm{d}u\mathrm{d}v \\
&:= T_1+T_2+T_3+T_4
\end{aligned} \tag{6.1.17}$$

where we have used (6.1.14) to treat terms involving $\frac{\partial^2}{\partial u^2}$ and $\frac{\partial^2}{\partial v^2}$ as like terms and those involving $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial v}$ similarly.

To simplify the T_1 term, we proceed as follows.

$$T_1 := \frac{1}{8}(n^2+4)\frac{1}{\Gamma(s)}\int_0^\infty\int_0^\infty \frac{\partial^2}{\partial u\partial v}\mathrm{Tr}[\dot{\phi}_0\tilde{K}_u\dot{\phi}_0\tilde{K}_v](u+v)^s\mathrm{d}u\mathrm{d}v$$

integrating by parts in u and using the fact that

$$\mathrm{Tr}\big(K(t,x,y)-\frac{1}{V}\big)\to 0 \tag{6.1.18}$$

exponentially fast as time $t\to\infty$, we obtain

$$\begin{aligned}
T_1 &= -\frac{1}{8}(n^2+4)\frac{1}{\Gamma(s)}\int_0^\infty \frac{\partial}{\partial v}\mathrm{Tr}[\dot{\phi}_0(\delta(x,y)-\frac{1}{V})\dot{\phi}_0\tilde{K}_v]v^s\mathrm{d}v \\
&- \frac{1}{8}(n^2+4)\frac{s}{\Gamma(s)}\int_0^\infty\int_0^\infty \frac{\partial}{\partial v}\mathrm{Tr}[\dot{\phi}_0\tilde{K}_u\dot{\phi}_0\tilde{K}_v](u+v)^{s-1}\mathrm{d}u\mathrm{d}v.
\end{aligned}$$

Since K_0 is the Dirac δ distribution and $\tilde{K} = K - \frac{1}{V}$, this is

$$\begin{aligned}
T_1 &= -\frac{1}{8}(n^2+4)\frac{1}{\Gamma(s)}\int_0^\infty \frac{\partial}{\partial v}\mathrm{Tr}[(\dot{\phi}_0(x))^2 K_v]v^s \mathrm{d}v \\
&+ \frac{1}{8}(n^2+4)\frac{1}{\Gamma(s)}\frac{1}{V}\int_0^\infty \frac{\partial}{\partial v}\mathrm{Tr}[\dot{\phi}_0(x)\dot{\phi}_0(y)\tilde{K}_v]v^s \mathrm{d}v \\
&- \frac{1}{8}(n^2+4)\frac{s}{\Gamma(s)}\int_0^\infty\int_0^\infty \frac{\partial}{\partial v}\mathrm{Tr}[\dot{\phi}_0\tilde{K}_u\dot{\phi}_0\tilde{K}_v](u+v)^{s-1}\mathrm{d}u\mathrm{d}v.
\end{aligned}$$

Similarly, integrating by parts in v and using $\tilde{K} = K - \frac{1}{V}$, we obtain:

$$\begin{aligned}
T_1 &:= \frac{1}{8}(n^2+4)\frac{s}{\Gamma(s)}\int_0^\infty \mathrm{Tr}[(\dot{\phi}_0(x))^2 K_v]v^{s-1}\mathrm{d}v \\
&- \frac{1}{8}(n^2+4)\frac{s}{\Gamma(s)}\frac{1}{V}\int_0^\infty \mathrm{Tr}[\dot{\phi}_0(x)\dot{\phi}_0(y)\tilde{K}_v]v^{s-1}\mathrm{d}v \\
&+ \frac{1}{8}(n^2+4)\frac{s}{\Gamma(s)}\int_0^\infty \mathrm{Tr}[(\dot{\phi}_0)^2 K_u]u^{s-1}\mathrm{d}u \\
&- \frac{1}{8}(n^2+4)\frac{s}{\Gamma(s)}\frac{1}{V}\int_0^\infty \mathrm{Tr}[\dot{\phi}_0(x)\tilde{K}_u\dot{\phi}_0(y)]u^{s-1}\mathrm{d}u \\
&+ \frac{1}{8}(n^2+4)\frac{s}{\Gamma(s-1)}\int_0^\infty\int_0^\infty \mathrm{Tr}[\dot{\phi}_0\tilde{K}_u\dot{\phi}_0\tilde{K}_v](u+v)^{s-2}\mathrm{d}v\mathrm{d}v
\end{aligned}$$

Now expanding the trace and using (3.3.2) relating the zeta and heat kernels, and combining terms, we get finally

$$\begin{aligned}
T_1 &= \frac{1}{8}(n^2+4)s\int_M\int_M \dot{\phi}_0(x)\Psi_{s-1}(x,y)\dot{\phi}_0(y)dV_g(x)dV_g(y) \\
&- \frac{1}{4}(n^2+4)s\frac{1}{V}\int_M\int_M \dot{\phi}_0(x)\dot{\phi}_0(y)\zeta(s,x,y)dV_g(x)dV_g(y) \\
&+ \frac{1}{4}(n^2+4)s\zeta_n(s)\frac{1}{V}\int_M (\dot{\phi}_0(x))^2 dV_g(x)
\end{aligned}$$

where $\Psi_s(x,y)$ is defined in (6.1.2).

The calculation for the term T_2 is the same as the one for T_1, except the following lemmas used to reduce terms involving $\Delta_x\zeta(s+1,x,y)$ to those containing $\zeta(s,x,y)$ only and to deal with the second derivative in u.

Lemma 6.1.3.

$$\int_M \int_M \dot\phi_0(x)(\Delta_g \dot\phi_0(y))\zeta(s+1,x,y)dV_g(x)dV_g(y)$$
$$= \int_M \int_M \dot\phi_0(x)\dot\phi_0(y)\zeta(s,x,y)dV_g(x)dV_g(y).$$

Proof.

$$\begin{aligned}
&\int_M \int_M \dot\phi_0(x)(\Delta_g \dot\phi_0(y))\zeta(s+1,x,y)dV_g(x)dV_g(y)\\
&= \int_M \dot\phi_0(x)\Big(\int_M \Delta_g\dot\phi_0(y)\zeta(s+1,x,y)dV_g(y)\Big)dV_g(x)\\
&= \int_M \dot\phi_0(x)\Big(\int_M \dot\phi_0(y)\Delta_g\zeta(s+1,x,y)dV_g(y)\Big)dV_g(x)\\
&= \int_M \int_M \dot\phi_0(x)\dot\phi_0(y)\zeta(s,x,y)dV_g(x)dV_g(y)
\end{aligned}$$

since

$$\begin{aligned}
\Delta_x\zeta(s+1,x,y) &= \frac{1}{\Gamma(s+1)}\int_0^\infty \Delta_x((K(t,x,y)-\frac{1}{V}))t^s \mathrm{d}t\\
&= -\frac{1}{\Gamma(s+1)}\int_0^\infty \frac{\partial}{\partial t}((K(t,x,y)-\frac{1}{V}))t^s \mathrm{d}t\\
&= \frac{s}{\Gamma(s+1)}\int_0^\infty ((K(t,x,y)-\frac{1}{V}))t^{s-1}\mathrm{d}t = \zeta(s,x,y)
\end{aligned}$$

□

The lemma to deal with the second derivatives in u is the following.

Lemma 6.1.4.

$$\begin{aligned}
&\frac{s}{\Gamma(s)}\int_0^\infty \int_0^\infty \frac{\partial}{\partial u}\mathrm{Tr}[\dot\phi_0(x)\tilde K_u \dot\phi_0(y)\tilde K_v](u+v)^{s-1}\mathrm{d}u\mathrm{d}v\\
= \;& -s\int_M\int_M \dot\phi_0(x)\Psi_{s-1}(x\cdot y)\dot\phi_0(y)dV_g(x)dV_g(y)\\
+ \;& s\frac{1}{V}\int_M\int_M \dot\phi_0(x)\dot\phi_0(y)\zeta(s,x,y)dV_g(x)dV_g(y)\\
- \;& s\int_M \dot\phi_0(x)\dot\phi_0(x)\zeta(s,x,x)dV_g(x).
\end{aligned}\tag{6.1.19}$$

Proof. Integrating by parts in u and using the fact that (6.1.18) decays exponentially fast as time $t \to \infty$ we have

$$\begin{aligned}
& \frac{s}{\Gamma(s)} \int_0^\infty \int_0^\infty \frac{\partial}{\partial u} \mathrm{Tr}[\dot{\phi}_0(x)\tilde{K}_u \dot{\phi}_0(y)\tilde{K}_v](u+v)^{s-1}\mathrm{d}u\mathrm{d}v \\
= & -\frac{s}{\Gamma(s)} \int_0^\infty \mathrm{Tr}[\dot{\phi}_0(x)(\delta(x,y) - \frac{1}{V}))\dot{\phi}_0(y)\tilde{K}_v]v^{s-1}\mathrm{d}v \\
- & \frac{s(s-1)}{\Gamma(s)} \int_0^\infty \int_0^\infty \mathrm{Tr}[\dot{\phi}_0(x)\tilde{K}_u \dot{\phi}_0(y)\tilde{K}_v](u+v)^{s-2}\mathrm{d}u\mathrm{d}v.
\end{aligned}$$

Using the Fubini-Tonelli theorem gives

$$\begin{aligned}
& \frac{s}{\Gamma(s)} \int_0^\infty \int_0^\infty \frac{\partial}{\partial u} \mathrm{Tr}[\dot{\phi}_0(x)\tilde{K}_u \dot{\phi}_0(y)\tilde{K}_v](u+v)^{s-1}\mathrm{d}u\mathrm{d}v \\
= & -s \int_M (\dot{\phi}_0(x))^2 \Big(\frac{1}{\Gamma(s)} \int_0^\infty (K(v,x,x) - \frac{1}{V})v^{s-1}\mathrm{d}v \Big) dV_g(x) \\
+ & s\frac{1}{V} \int_M \int_M \dot{\phi}_0(x)\dot{\phi}_0(y) \Big(\frac{1}{\Gamma(s)} \int_0^\infty (\tilde{K}_v)v^{s-1}\mathrm{d}v \Big) dV_g(x)dV_g(y) \\
- & s \int_M \int_M \dot{\phi}_0(x)\dot{\phi}_0(y) \Big(\frac{1}{\Gamma(s-1)} \int_0^\infty \int_0^\infty \tilde{K}_u\tilde{K}_v(u+v)^{s-2}\mathrm{d}v\mathrm{d}v \Big) dV_g(x)dV_g(y)
\end{aligned}$$

which yields (6.1.19) □

We also remark that on differentiating by parts in u in T_2, the first time, we used (6.1.10) to write $\frac{1}{\Gamma(s)}\int_0^\infty \mathrm{Tr}[\dot{\phi}_0(\Delta_x(\delta(x,y) - \frac{1}{V}))\dot{\phi}_0\tilde{K}_v]v^s\mathrm{d}v$ as

$$\begin{aligned}
& \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{Tr}[\dot{\phi}_0(\Delta_x(\delta(x,y) - \frac{1}{V}))\dot{\phi}_0\tilde{K}_v]v^s\mathrm{d}v \\
= & -\frac{1}{\Gamma(s)} \int_0^\infty \mathrm{Tr}[\delta(x,y)(\Delta_x\dot{\phi}_0(x))\dot{\phi}_0(y)\tilde{K}_v]v^s\mathrm{d}v \\
- & \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{Tr}[\delta(x,y)\dot{\phi}_0(x)\dot{\phi}_0(y)(\Delta_x\tilde{K}_v)]v^s\mathrm{d}v \\
+ & \frac{1}{2}\frac{1}{\Gamma(s)} \int_0^\infty \mathrm{Tr}[\delta(x,y)\dot{\phi}_0(y)(\nabla_x\dot{\phi}_0(x))(\nabla_x\tilde{K}_v)]v^s\mathrm{d}v.
\end{aligned}$$

Integrating by parts in u the second time and simplifying terms with the aids of (6.1.4) gives

$$\begin{aligned}
T_2 = & -\frac{1}{8}(n^2-4)s \int_M \int_M \dot{\phi}_0(x)\Psi_{s-1}(x,y)\dot{\phi}_0(y)dV_g(x)dV_g(y) \\
+ & \frac{1}{8}(n^2-4)s\frac{1}{V} \int_M \int_M \dot{\phi}_0(x)\dot{\phi}_0(y)\zeta(s,x,y)dV_g(x)dV_g(y) \\
- & \frac{1}{8}(n^2-4)s\zeta_n(s+1)\frac{1}{V} \int_M \dot{\phi}_0(x)(\Delta_g\dot{\phi}_0(x))dV_g(x) \\
- & \frac{1}{4}(n^2-4)s\zeta_n(s)\frac{1}{V} \int_M (\dot{\phi}_0(x))^2dV_g(x).
\end{aligned}$$

For the term T_3, integrating in u using the same procedure as for the case of T_1 and using lemma (6.1.3) yields

$$\begin{aligned}
T_3 &= -(1-\frac{n}{2})s\int_M\int_M \dot{\phi}_0(x)\Psi_s(x\cdot y)(\Delta_g\dot{\phi}_0(y))dV_g(x)dV_g(y) \\
&+ (1-\frac{n}{2})s\frac{1}{V}\int_M\int_M \dot{\phi}_0(x)(\Delta_g\dot{\phi}_0(y))\zeta(s,x,y)dV_g(x)dV_g(y) \\
&- (1-\frac{n}{2})s\zeta_n(s+1)\frac{1}{V}\int_M \dot{\phi}_0(x)(\Delta_g\dot{\phi}_0(x))dV_g(x).
\end{aligned}$$

Finally, the fourth term

$$\begin{aligned}
T_4 &= \frac{1}{16}(n-2)^2\frac{1}{\Gamma(s)}\int_0^\infty\int_0^\infty \mathrm{Tr}[(\Delta\dot{\phi}_0)\tilde{K}_u(\Delta\dot{\phi}_0)\tilde{K}_v](u+v)^s\mathrm{d}u\mathrm{d}v \\
&= \frac{1}{16}(n-2)^2 s\int_M\int_M (\Delta_x\dot{\phi}_0(x))\Psi_{s+1}(x\cdot y)(\Delta_y\dot{\phi}_0(y))dV_g(x)dV_g(y).
\end{aligned}$$

Putting Terms T_1, T_2, T_3 and T_4 together gives

$$\begin{aligned}
\mathrm{var}_1(s) &= s\int_M\int_M \dot{\phi}_0(x)\Psi_{s-1}(x,y)\dot{\phi}_0(y)dV_g(x)dV_g(y) \\
&- (1-\frac{n}{2})s\int_M\int_M \dot{\phi}_0(x)\Psi_s(x,y)(\Delta_g\dot{\phi}_0(y))dV_g(x)dV_g(y) \\
&+ \frac{1}{16}(n-2)^2 s\int_M\int_M (\Delta_g\dot{\phi}_0(x))\Psi_{s+1}(x,y)(\Delta_g\dot{\phi}_0(y))dV_g(x)dV_g(y) \\
&- \frac{1}{8}(n+2)^2 s\frac{1}{V}\int_M\int_M \dot{\phi}_0(x)\dot{\phi}_0(y)\zeta(s,x,y)dV_g(x)dV_g(y) \\
&- \frac{1}{8}(n-2)^2 s\zeta_n(s+1)\frac{1}{V}\int_M \dot{\phi}_0(x)(\Delta_g\dot{\phi}_0(x))dV_g(x) \\
&+ 2s\zeta_n(s)\frac{1}{V}\int_M (\dot{\phi}_0(x))^2 dV_g(x). \qquad (6.1.20)
\end{aligned}$$

6.1.2. Computation of $\mathrm{var}_2(s)$.

Next, we compute the term

$$\mathrm{var}_2(s) = -\frac{1}{\Gamma(s)} \int_0^\infty \mathrm{Tr}\Big(\Delta_\epsilon^{(2)}(e^{-t\Delta_g} - P)\Big) t^s dt. \tag{6.1.21}$$

Substituting the expression for $\Delta_0^{(2)}$ given in Equation (5.1.7) into (6.1.21) gives

$$\begin{aligned}
\mathrm{var}_2(s) &= \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{Tr}\Big(\ddot{\phi}_0 \Delta_g (e^{-t\Delta_g} - P)\Big) t^s dt \\
&- \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{Tr}\Big((\dot{\phi}_0)^2 \Delta_g (e^{-t\Delta_g} - P)\Big) t^s dt \\
&- (n-2)\frac{1}{\Gamma(s)} \int_0^\infty \mathrm{Tr}\Big(\dot{\phi}_0 \langle \nabla_g \dot{\phi}_0, \nabla_g \cdot \rangle_g (e^{-t\Delta_g} - P)\Big) t^s dt \\
&- (1-\frac{n}{2})\frac{1}{\Gamma(s)} \int_0^\infty \mathrm{Tr}\Big(\langle \nabla_g \ddot{\phi}_0, \nabla_g \cdot \rangle_g (e^{-t\Delta_g} - P)\Big) t^s dt.
\end{aligned}$$

Using the same argument as in the calculation of $\mathrm{var}_1(s)$ to replace Δ_g by $-\frac{\partial}{\partial t}$ in the first two terms, we get:

$$\begin{aligned}
\mathrm{var}_2(s) &= -\frac{1}{\Gamma(s)} \int_0^\infty \frac{\partial}{\partial t}\mathrm{Tr}\Big(\ddot{\phi}_0 (e^{-t\Delta_g} - P)\Big) t^s dt \\
&+ \frac{1}{\Gamma(s)} \int_0^\infty \frac{\partial}{\partial t}\mathrm{Tr}\Big((\dot{\phi}_0)^2 (e^{-t\Delta_g} - P)\Big) t^s dt \\
&- (n-2)\frac{1}{\Gamma(s)} \int_0^\infty \mathrm{Tr}\Big(\dot{\phi}_0 \langle \nabla_g \dot{\phi}_0, \nabla_g \cdot \rangle_g (e^{-t\Delta_g} - P)\Big) t^s dt \\
&- (1-\frac{n}{2})\frac{1}{\Gamma(s)} \int_0^\infty \mathrm{Tr}\Big(\langle \nabla_g \ddot{\phi}_0, \nabla_g \cdot \rangle_g (e^{-t\Delta_g} - P)\Big) t^s dt.
\end{aligned}$$

Integrating by parts in t in the first two terms gives

$$\begin{aligned}
\mathrm{var}_2(s) &= \frac{s}{\Gamma(s)} \int_0^\infty \mathrm{Tr}\Big(\ddot{\phi}_0 (e^{-t\Delta_g} - P)\Big) t^{s-1} dt \\
&- \frac{s}{\Gamma(s)} \int_0^\infty \mathrm{Tr}\Big((\dot{\phi}_0)^2 (e^{-t\Delta_g} - P)\Big) t^{s-1} dt \\
&- (n-2)\frac{1}{\Gamma(s)} \int_0^\infty \mathrm{Tr}\Big(\dot{\phi}_0 \langle \nabla_g \dot{\phi}_0, \nabla_g \cdot \rangle_g (e^{-t\Delta_g} - P)\Big) t^s dt \\
&- (1-\frac{n}{2})\frac{1}{\Gamma(s)} \int_0^\infty \mathrm{Tr}\Big(\langle \nabla_g \ddot{\phi}_0, \nabla_g \cdot \rangle_g (e^{-t\Delta_g} - P)\Big) t^s dt.
\end{aligned}$$

Rewriting the trace as an integral, as in the computation of the first order variation $\zeta_\epsilon^{(1)}(s)$, we have

$$\begin{aligned}
\mathrm{var}_2(s) &= s\int_M \ddot{\phi}_0(x)\Big(\frac{1}{\Gamma(s)}\int_0^\infty (K(t,x,x)-\frac{1}{V})t^{s-1}dt\Big)dV_g(x)\\
&- s\int_M (\dot{\phi}_0(x))^2\Big(\frac{1}{\Gamma(s)}\int_0^\infty (K(t,x,x)-\frac{1}{V})t^{s-1}dt\Big)dV_g(x)\\
&- (n-2)s\int_M \dot{\phi}_0(x)\langle\nabla_g\dot{\phi}_0(x),\nabla_g\cdot\rangle_g\Big(\frac{1}{\Gamma(s+1)}\int_0^\infty (K(t,x,x)-\frac{1}{V})t^{s}dt\Big)dV_g(x)\\
&- (1-\frac{n}{2})s\int_M \langle\nabla_g\ddot{\phi}_0,\nabla_g\cdot\rangle_g\Big(\frac{1}{\Gamma(s+1)}\int_0^\infty (K(t,x,x)-\frac{1}{V})t^{s}dt\Big)dV_g(x).
\end{aligned}$$

Hence,

$$\begin{aligned}
\mathrm{var}_2(s) &= s\int_M \ddot{\phi}_0(x)\zeta(s,x,x)dV_g(x)\\
&- s\int_M (\dot{\phi}_0(x))^2\zeta(s,x,x)dV_g(x)\\
&- (n-2)s\int_M \dot{\phi}_0(x)\langle\nabla_g\dot{\phi}_0(x),\nabla_g\zeta(s+1,x,x)\rangle_g dV_g(x)\\
&- (1-\frac{n}{2})s\int_M \langle\nabla_g\ddot{\phi}_0,\nabla_g\zeta(s+1,x,x)\rangle_g dV_g(x).
\end{aligned}$$

The homogeneity of M implies that $\zeta(s+1,x,x)$ is constant in x, so the third and fourth terms here vanish. Further, using the identity $\int_M \ddot{\phi}_0(x)dV_g = -\frac{n}{2}\int_M(\dot{\phi}_0(x))^2dV_g$ (by observation (5.2.1)), we get that $\mathrm{var}_2(s)$ simplifies to

$$\mathrm{var}_2(s) = -\frac{(n+2)s}{2V}\zeta_n(s)\int_M(\dot{\phi}_0(x))^2dV_g(x). \tag{6.1.22}$$

Combining (6.1.20) and (6.1.22), we get that the second order variation $\zeta_g^{(2)}(s)$ of the spectral zeta function $\zeta_{g_\epsilon}(s)$ on M is given by (6.1.1) for $\Re(s)$ large enough. Thus, the proof of the Theorem (6.1.1) is complete □

6.2. Analysis of the distribution Ψ_s on S^n

In order to be able to evaluate the second variation of the spectral zeta function (6.1.1) at any given point $s = s_0$, we need to analytically continue $\zeta_0^{(2)}(s)$ to the whole of the complex s-plane. Define Ψ_s as a distribution in $\mathcal{D}'(M \times M)$ by

$$\Psi_s(f_1 \otimes f_2) \quad := \quad \int_M \int_M \Psi_s(x,y) f_1(x) f_2(y) \mathrm{d}V_g(x) \mathrm{d}V_g(y) \tag{6.2.1}$$

if f_1, f_2 are in $(\ker \Delta_g)^{\perp}$ and $\Psi_s(f_1 \otimes f_2) = 0$ if either f_1 or f_2 is a constant. We then construct the meromorphic continuation of the distribution Ψ_s which appeared in the second variation of $\zeta(s)$; where $\Psi_s(x,y)$ is defined in equation (6.1.2).

To distinguish Ψ_s when used as a function from when used as a distribution, we write $\Psi_s(f_1 \otimes f_2)$ to mean the pairing

$$\langle \Psi_s, f_1 \otimes f_2 \rangle$$

where $f_1 \otimes f_2$ is understood as a function in $C^\infty(M \times M)$ by

$$(f_1 \otimes f_2)(x,y) = f_1(x) \cdot f_2(y).$$

By the density of $C^\infty(M) \times C^\infty(M)$ in $C^\infty(M \times M)$, this indeed defines a distribution in $\mathcal{D}'(M \times M)$.

Following our notation (6.1.16), in the case $f_1, f_2 \in (\ker \Delta_g)^{\perp}$, we can as well write

$$\Psi_s(f_1 \otimes f_2) \quad = \quad \frac{1}{\Gamma(s)} \int_0^\infty \int_0^\infty (u+v)^{s-1} \mathrm{Tr}[f_1 \tilde{\mathcal{K}}_u f_2 \tilde{\mathcal{K}}_v] \mathrm{d}u \mathrm{d}v \tag{6.2.2}$$

where $\mathcal{K}_u := \mathrm{e}^{-u\Delta_g}$; and $\tilde{\mathcal{K}}_u := \mathrm{e}^{-u\Delta_g} - P$.

We remark the following property of (6.2.2).

Theorem 6.2.1. *The function $\Psi_s(f_1 \otimes f_2)$ is symmetric in f_1 and f_2.*

Proof. Assume $f_1, f_2 \in (\ker \Delta_g)^{\perp}$. Using the cyclic permutation of $\tilde{\mathcal{K}}_u, \tilde{\mathcal{K}}_v, f_1, f_2$ under $\mathrm{Tr}[\cdot]$

and change of variables in u and v, (see (6.1.14)), we have that

$$\begin{aligned}
\Psi_s(f_1 \otimes f_2) &= \frac{1}{\Gamma(s)} \int_0^\infty \int_0^\infty (u+v)^{s-1} \mathrm{Tr}[f_1 \tilde{\mathcal{K}}_u f_2 \tilde{\mathcal{K}}_v] \mathrm{d}u \mathrm{d}v \\
&= \frac{1}{\Gamma(s)} \int_0^\infty \int_0^\infty (u+v)^{s-1} \mathrm{Tr}[f_2 \tilde{\mathcal{K}}_v f_1 \tilde{\mathcal{K}}_u] \mathrm{d}u \mathrm{d}v \\
&= \frac{1}{\Gamma(s)} \int_0^\infty \int_0^\infty (u+v)^{s-1} \mathrm{Tr}[f_2 \tilde{\mathcal{K}}_u f_1 \tilde{\mathcal{K}}_v] \mathrm{d}u \mathrm{d}v.
\end{aligned}$$

Thus, $\Psi_s(f_1 \otimes f_2) = \Psi_s(f_2 \otimes f_1)$. □

From now on, we assume $f_1, f_2 \in (\ker \Delta_g)^\perp$.

We now break the integral $\Gamma(s)\Psi_s(f_1 \otimes f_2)$ into four parts, namely

$$\begin{aligned}
\Gamma(s)\Psi_s(f_1 \otimes f_2) &= \int_0^1 \int_0^1 (u+v)^{s-1} \mathrm{Tr}[f_1 \tilde{\mathcal{K}}_u f_2 \tilde{\mathcal{K}}_v] \mathrm{d}u \mathrm{d}v \\
&+ \int_1^\infty \int_0^1 (u+v)^{s-1} \mathrm{Tr}[f_1 \tilde{\mathcal{K}}_u f_2 \tilde{\mathcal{K}}_v] \mathrm{d}u \mathrm{d}v \\
&+ \int_0^1 \int_1^\infty (u+v)^{s-1} \mathrm{Tr}[f_1 \tilde{\mathcal{K}}_u f_2 \tilde{\mathcal{K}}_v] \mathrm{d}u \mathrm{d}v \\
&+ \int_1^\infty \int_1^\infty (u+v)^{s-1} \mathrm{Tr}[f_1 \tilde{\mathcal{K}}_u f_2 \tilde{\mathcal{K}}_v] \mathrm{d}u \mathrm{d}v.
\end{aligned}$$

Due to the exponential decay properties of $\tilde{\mathcal{K}}_u$ and $\tilde{\mathcal{K}}_v$, in u and v as $u, v \to \infty$, we see that the last three integrals are holomorphic in s. So we are left with only the first integral to consider. We denote it by

$$\Gamma(s)\tilde{\Psi}_s(f_1 \otimes f_2) = \int_0^1 \int_0^1 (u+v)^{s-1} \mathrm{Tr}[f_1 \tilde{\mathcal{K}}_u f_2 \tilde{\mathcal{K}}_v] \mathrm{d}u \mathrm{d}v.$$

Observe that

$$\begin{aligned}
\Gamma(s)\tilde{\Psi}_s(f_1 \otimes f_2) &= \int_0^1 \int_0^1 (u+v)^{s-1} \mathrm{Tr}[f_1 (\mathcal{K}_u - P) f_2 (\mathcal{K}_v - P)] \mathrm{d}u \mathrm{d}v \\
&= \int_0^1 \int_0^1 (u+v)^{s-1} \mathrm{Tr}[f_1 \mathcal{K}_u f_2 \mathcal{K}_v] \mathrm{d}u \mathrm{d}v \\
&- \int_0^1 \int_0^1 (u+v)^{s-1} \mathrm{Tr}[P f_1 \tilde{\mathcal{K}}_v f_2] \mathrm{d}u \mathrm{d}v \\
&- \int_0^1 \int_0^1 (u+v)^{s-1} \mathrm{Tr}[f_1 P f_2 \tilde{\mathcal{K}}_v] \mathrm{d}u \mathrm{d}v \\
&+ \int_0^1 \int_0^1 (u+v)^{s-1} \mathrm{Tr}[f_1 P f_2 P] \mathrm{d}u \mathrm{d}v \\
&:= \Gamma(s)\tilde{\Psi}_s^1(f_1 \otimes f_2) + \Gamma(s)\tilde{\Psi}_s^{2A}(f_1 \otimes f_2) \\
&+ \Gamma(s)\tilde{\Psi}_s^{2B}(f_1 \otimes f_2) + \Gamma(s)\tilde{\Psi}_s^3(f_1 \otimes f_2).
\end{aligned}$$

The last integral simplifies as follows:

$$\begin{aligned}\tilde{\Psi}_s^3(f_1\otimes f_2) &= \int_0^1\int_0^1 (u+v)^{s-1}\mathrm{Tr}[f_1Pf_2P]\mathrm{d}u\mathrm{d}v \\ &= \int_0^1\int_0^1 (u+v)^{s-1}\frac{1}{V^2}\big(\int_M\int_M f_1(x)f_2(y)\mathrm{d}V_g(x)\mathrm{d}V_g(y)\big)\mathrm{d}u\mathrm{d}v \\ &= \int_0^1\int_0^1 (u+v)^{s-1}\frac{1}{V^2}\big(\int_M f_2(y)\big[\int_M f_1(x)\mathrm{d}V_g(x)\big]\mathrm{d}V_g(y)\big)\mathrm{d}u\mathrm{d}v = 0\end{aligned}$$

since $f_1, f_2 \in (\ker\Delta_g)^\perp$.

For the second integral $\tilde{\Psi}_s^{2A}(f_1\otimes f_2)$, we have

$$\Gamma(s)\tilde{\Psi}_s^{2A}(f_1\otimes f_2) = \int_M\int_M \zeta(s+1,x,y)f_1(x)f_2(y)\mathrm{d}V_g(x)\mathrm{d}V_g(y) + G_s; \tag{6.2.3}$$

where G_s is an entire function.

To see this, observe that

$$\begin{aligned}\Gamma(s)\tilde{\Psi}_s^{2A}(f_1\otimes f_2) &= -\int_0^1\int_0^1 (u+v)^{s-1}\mathrm{Tr}[f_1Pf_2\tilde{\mathcal{K}}_v]\mathrm{d}u\mathrm{d}v \\ &= -\frac{1}{s}\int_0^1 (1+v)^s\mathrm{Tr}[f_1Pf_2\tilde{\mathcal{K}}_v]\mathrm{d}v \\ &+ \frac{1}{s}\int_0^1 v^s\mathrm{Tr}[f_1Pf_2\tilde{\mathcal{K}}_v]\mathrm{d}v;\end{aligned}$$

where

$$-\frac{1}{s}\int_0^1 (1+v)^s\mathrm{Tr}[f_1Pf_2\tilde{\mathcal{K}}_v]\mathrm{d}v = -\frac{1}{s}\int_0^1 (1+v)^s\cdot f_v\mathrm{d}v$$

with f_v being a smooth function of v on $[0,1]$. Moreover, observe that $\Gamma(s)\tilde{\Psi}_s^{2A}(f_1\otimes f_2)$ at $s=0$ vanishes. This is because

$$\begin{aligned}\tilde{\Psi}_s^{2A}(f_1\otimes f_2)\Big|_{s=0} &= \frac{1}{\Gamma(s+1)}\int_0^1\big\{-(1+v)^s+v^s\big\}\mathrm{Tr}[f_1Pf_2\tilde{\mathcal{K}}_v]\mathrm{d}v\Big|_{s=0} \\ &= \frac{1}{\Gamma(s+1)}\int_0^1\big\{-\exp(s\cdot\log(1+v))+\exp(s\cdot\log(v))\big\}\cdot f_v\mathrm{d}v\Big|_{s=0} = 0.\end{aligned}$$

So,

$$\Gamma(s)\tilde{\Psi}_s^{2A}(f_1\otimes f_2) = \frac{1}{s}\int_0^1 v^s\mathrm{Tr}[f_1Pf_2\tilde{\mathcal{K}}_v]\mathrm{d}v + G_s.$$

Thus, (by the exponential decay of $\tilde{K}_v$)

$$\begin{aligned}\tilde{\Psi}_s^{2A}(f_1 \otimes f_2) &= \frac{1}{\Gamma(s+1)} \int_0^\infty \mathrm{Tr}[f_1 P f_2 \tilde{\mathcal{K}}_v] v^s \mathrm{d}v \\ &= \mathrm{Tr}[f_1 P f_2 \Delta_g^{s+1}] \\ &= \int_M \int_M \zeta(s+1, x, y) f_1(x) f_2(y) \mathrm{d}V_g(x) \mathrm{d}V_g(y).\end{aligned}$$

Therefore, (6.2.3) follows.

Since $\tilde{\Psi}_s^{2A}(f_1 \otimes f_2) = \tilde{\Psi}_s^{2B}(f_1 \otimes f_2)$ by theorem (6.2.1), it implies $\tilde{\Psi}_s^{2B}(f_1 \otimes f_2)$ has the same result as $\tilde{\Psi}_s^{2A}(f_1 \otimes f_2)$ and hence we have $\hat{\Psi}_s(f_1 \otimes f_2) := \tilde{\Psi}_s^{2A}(f_1 \otimes f_2) + \tilde{\Psi}_s^{2B}(f_1 \otimes f_2)$ given by

$$\hat{\Psi}_s(f_1 \otimes f_2) = 2 \int_M \int_M \zeta(s+1, x, y) f_1(x) f_2(y) \mathrm{d}V_g(x) \mathrm{d}V_g(y). \tag{6.2.4}$$

We remark that the zeta kernel $\zeta(s, x, y)$ as a bi-distribution is an entire function. This follows from the fact that

$$\int_M \int_M \zeta_g(s, x, y) f_1(x) f_2(y) \mathrm{d}V_g(x) \mathrm{d}V_g(y) = \sum_{k=1}^{\infty} \lambda_k^{-s} \langle f_1, \psi_k \rangle_{L^2(M)} \cdot \langle f_2, \psi_k \rangle_{L^2(M)} \tag{6.2.5}$$

which converges absolutely for all s since,

$$|\langle f_1, \psi_k \rangle_{L^2(M)}| \leq c_N (1+k)^{-N}$$

and

$$|\langle f_2, \psi_k \rangle_{L^2(M)}| \leq c_N (1+k)^{-N}.$$

Proposition 6.2.2. *Values of*

$$\int_M \int_M \zeta_g(s, x, y) f_1(x) f_2(y) \mathrm{d}V_g(x) \mathrm{d}V_g(y)$$

at all points can be computed explicitly.

Proof. The proposition follows directly from the definition (6.2.5); e.g:

$$\int_M \int_M \zeta_g(s,x,y) f_1(x) f_2(y) \mathrm{d}V_g(x) \mathrm{d}V_g(y) = \begin{cases} \int_M f_1(x) f_2(x) \mathrm{d}V_g(x) & \text{at} \quad s = 0 \\ \int_M f_1(x) \Delta_g f_2(x) \mathrm{d}V_g(x) & \text{at} \quad s = -1 \\ \int_M f_1(x) \Delta_g^2 f_2(x) \mathrm{d}V_g(x) & \text{at} \quad s = -2 \\ \int_M f_1(x) \Delta_g^3 f_2(x) \mathrm{d}V_g(x) & \text{at} \quad s = -3 \\ \vdots & \end{cases} \tag{6.2.6}$$

□

Consequently, up to an entire function, the only integral left to consider is

$$\Gamma(s)\tilde{\Psi}_s^1(f_1 \otimes f_2) = \int_0^1 \int_0^1 (u+v)^{s-1} \mathrm{Tr}[f_1 K_u f_2 K_v] \mathrm{d}u \mathrm{d}v.$$

To do this, we need knowledge of the explicit formula of the heat kernel on a choice of manifold. We treat this for the case of the n-spheres. Let $\mathrm{d}(x,y)$ be the distance between $x, y \in S^n$ and denote by c_n the unique constant such that

$$c_n \int_0^\pi (\sin\theta)^{n-1} \mathrm{d}\theta = \mathrm{Vol}(S^n). \tag{6.2.7}$$

Then for any function $f \in C_0^\infty(0,\infty)$ we have

$$\int_{S^n} f(\mathrm{d}(x,0)) \mathrm{dVol}(S^n)_x = c_n \int_0^\pi f(\theta)(\sin\theta)^{n-1} \mathrm{d}\theta. \tag{6.2.8}$$

For integral kernels and distributional kernels that depend only on the distance between x and y, and time; this means

$$\int_{S^n \times S^n} F(\mathrm{d}(x,y)) f_1(x) f_2(y) \mathrm{dVol}(S^n)_x \mathrm{dVol}(S^n)_y = c_n \int_0^\pi F(\theta) h(\theta) (\sin\theta)^{n-1} \mathrm{d}\theta \tag{6.2.9}$$

where

$$h(\theta) = \frac{1}{V(\theta)} \int_{S^n \times S^n, \mathrm{d}(x,y)=\theta} f_1(x) f_2(y) \mathrm{dVol}(S^n)_x \mathrm{dVol}(S^n)_y \tag{6.2.10}$$

with

$$V(\theta) = \int_{S^n \times S^n, \mathrm{d}(x,y)=\theta} 1 \mathrm{dVol}(S^n)_x \mathrm{dVol}(S^n)_y \tag{6.2.11}$$

is the average of $f_1(x)f_2(y)$ over all pairs x, y such that $\mathrm{d}(x,y) = \theta$. Note that $h \in C_0^\infty(0,\pi)$. We call such kernels point-wise invariants.

The heat kernel $K(t,x,y)$ on S^n (4.1.25) has the asymptotic expansion

$$K(t,x,y) = \big(4\pi t\big)^{-n/2} \mathrm{e}^{-\mathrm{d}^2(x,y)/4t} \sum_{j=0}^{N} t^j a_j(t,x,y) + O\Big(t^{N+1-n/2}\Big); \quad t \to 0 \tag{6.2.12}$$

uniformly for all $x, y \in S^n$ where time $t \in [0,T]$; the heat coefficients $a_j \in C^\infty(S^n \times S^n \times [0,T])$ and $\mathrm{d}(x,y)$ is the geodesic distance between $x, y \in S^n$, see e.g Chavel [16].

Due to the homogeneity of S^n, the heat kernel depends only on time t and on the distance between any x, y on S^n denoted by $\theta = \mathrm{d}(x,y)$. Consequently, it is valid to denote the kernel by $K(t,\theta)$. Recall that explicitly, $\theta = \mathrm{d}(x,y) = \arccos\langle x,y\rangle_g \in [0,\pi]$. For example, if $S^n \ni x_1 = \cos\vartheta$, then the distance between $(x_1, x_2, \cdots, x_{n+1})$ and $(1,0,0,\cdots,0)$ is ϑ.

There are more explicit forms of (6.2.12) e.g:

$$K_{n=2m+1}(t,\theta) = \frac{\mathrm{e}^{m^2 t}}{(2\pi)^m (4\pi t)^{1/2}} \Big(-\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\Big)^m \exp(-\frac{\theta^2}{4t}) \tag{6.2.13}$$

and

$$K_{n=2m+2}(t,\theta) = \frac{\mathrm{e}^{(2m+1)^2 t/4}}{(2\pi)^{m+3/2}\ t^{3/2}} \int_\theta^\pi \sqrt{\cos(\theta) - \cos(\varrho)} \Big(\frac{\partial}{\partial s}\frac{-1}{\sin s}\Big)^{m+1} \varrho \mathrm{e}^{-\varrho^2/4t} \mathrm{d}\varrho \tag{6.2.14}$$

for $t > 0$ and where θ is the geodesic distance between $x, y \in S^n$; see e.g Camporesi [10], Wogu [64] and Nagase [40].

Thus, from (6.2.13), we have e.g for S^1, S^3, S^5 :

$$K_1(t,\theta) \sim \frac{1}{2\sqrt{\pi}}\ t^{-1/2}\ \mathrm{e}^{-\frac{\theta^2}{4t}}, \qquad t \to 0^+; \tag{6.2.15}$$

$$K_3(t,\theta) \sim \frac{1}{8\pi^{3/2}} t^{-3/2}\ \mathrm{e}^t\ \frac{\theta}{\sin\theta}\ \mathrm{e}^{-\frac{\theta^2}{4t}}, \qquad t \to 0^+; \tag{6.2.16}$$

$$K_5(t,\theta) \sim \frac{1}{16\pi^{5/2}} t^{-5/2}\ \mathrm{e}^{4t}\ \frac{\theta^2}{\sin^2\theta}\ \mathrm{e}^{-\frac{\theta^2}{4t}} - \frac{1}{8\pi^{5/2}}\ t^{-3/2}\ \mathrm{e}^{4t}\ \frac{1}{\sin^2\theta}\ \mathrm{e}^{-\frac{\theta^2}{4t}}, \qquad t \to 0^+. \tag{6.2.17}$$

These can be used to construct the meromorphic continuation. We will illustrate the method on S^3 to avoid lengthy computations, and yet remain explicit.

The case of 3-sphere

Now, we will discuss the case of S^3. The idea can be extended to other odd-dimensional unit spheres in a similar way. This means for this case that

$$K_3(t,\theta) - \frac{1}{8\pi^{3/2}} t^{-3/2} \mathrm{e}^{t} \mathrm{e}^{-\frac{\theta^2}{4t}} \frac{\theta}{\sin\theta} = O(t^{\infty})$$

as $t \to 0^+$.

In the following, we fix f_1 and f_2 and again set

$$h(\theta) = \frac{1}{V(\theta)} \int_{S^n \times S^n, \mathrm{d}(x,y)=\theta} f_1(x) f_2(y) \mathrm{dVol}(S^n)_x \mathrm{dVol}(S^n)_y$$

where $V(\theta)$ is as defined in (6.2.11). We need to consider, up to a holomorphic term

$$\Gamma(s)\tilde{\Psi}^1_s(h) = c \int_0^1 \int_0^1 \int_0^{\pi} (u+v)^{s-1} (uv)^{-3/2} \mathrm{e}^{u+v} \mathrm{e}^{-\frac{\theta^2}{4}\left(\frac{u+v}{uv}\right)} \theta^2 h(\theta) \mathrm{d}\theta \mathrm{d}u \mathrm{d}v. \tag{6.2.18}$$

Note $\frac{\theta^2}{\sin^2\theta} \times \sin^2\theta = \theta^2$ and that $c = \left(\frac{1}{8\pi^{3/2}}\right)^2$.

The error term (which constitutes the holomorphic term) to (6.2.18) is

$$\begin{aligned}
\mathcal{R} &= c \int_0^1 \int_0^1 \int_0^{\pi} (u+v)^{s-1} K_3(u,\theta) R(v,\theta) \theta^2 h(\theta) \mathrm{d}\theta \mathrm{d}u \mathrm{d}v \\
&+ c \int_0^1 \int_0^1 \int_0^{\pi} (u+v)^{s-1} K_3(v,\theta) R(u,\theta) \theta^2 h(\theta) \mathrm{d}\theta \mathrm{d}u \mathrm{d}v \\
&+ c \int_0^1 \int_0^1 \int_0^{\pi} (u+v)^{s-1} R(u,\theta) R(v,\theta) \theta^2 h(\theta) \mathrm{d}\theta \mathrm{d}u \mathrm{d}v
\end{aligned}$$

where

$$R(u,\theta) := K_3(u,\theta) - \frac{1}{8\pi^{3/2}} t^{-3/2} \mathrm{e}^{u} \mathrm{e}^{-\frac{\theta^2}{4u}} \frac{\theta}{\sin\theta}$$

and $R(v,\theta)$ defined similarly. Note, $\mathcal{R}$ is holomorphic due to fast decay of $R(u,\theta)$ and $R(v,\theta)$ as $(u,v) \to (0,0)$ to arbitrarily high order; i.e we have used that

$$\int_0^{\pi} \mathrm{e}^{-\frac{\theta^2}{4t}} \theta^2 F(\theta) \mathrm{d}\theta$$

is $O(t^{\infty})$ as $t \to 0^+$ for any smooth function F.

Let $h(\theta)$ be a suitable test function. Define the following cut-off function:

$$\chi(x) := \begin{cases} 1 & \text{when } 0 \leq x < \frac{1}{4} \\ 0 & \text{when } x > \frac{1}{2}; \end{cases}$$

then,

$$h(\theta) = \chi(\theta)(h(\theta)) + (1 - \chi(\theta))h(\theta).$$

This means in particular that $(1 - \chi(\theta))h(\theta)$ vanishes in a neighbourhood of the origin.

So we have

$$\begin{aligned} \tilde{\Psi}_s^1(h) &= \frac{\tilde{c}}{\Gamma(s)} \sum_{k=0}^{\infty} \frac{1}{k!} \int_0^1 \int_0^1 \int_{\mathbb{R}} (u+v)^{s+k-1} (uv)^{-3/2} \mathrm{e}^{-\frac{\theta^2}{4}\left(\frac{u+v}{uv}\right)} \theta^2 \chi(\theta) h(\theta) \mathrm{d}\theta \mathrm{d}u \mathrm{d}v \\ &+ \frac{\tilde{c}}{\Gamma(s)} \sum_{k=0}^{\infty} \frac{1}{k!} \int_0^1 \int_0^1 \int_{\frac{1}{4}}^{\pi} (u+v)^{s+k-1} (uv)^{-3/2} \mathrm{e}^{-\frac{\theta^2}{4}\left(\frac{u+v}{uv}\right)} \theta^2 (1 - \chi(\theta)) h(\theta) \mathrm{d}\theta \mathrm{d}u \mathrm{d}v \end{aligned} \tag{6.2.19}$$

with $\tilde{c} = c_3 \times c = \frac{1}{16\pi^2}$; where $c_n|_{n=3} = c_3 = 4\pi$. The second term of (6.2.19) is holomorphic in s due to the fast decay rate of $\mathrm{e}^{-\frac{\theta^2}{4}\left(\frac{u+v}{uv}\right)}$ as $(u+v) \to 0$ since θ is away from 0.

Now we define the operator

$$\mathcal{P} = -2 \frac{uv}{u+v} \frac{\mathrm{d}}{\mathrm{d}\theta} \frac{1}{\theta}$$

and use the fact that

$$(\mathcal{P}^*)^m \mathrm{e}^{-\frac{\theta^2}{4}\left(\frac{u+v}{uv}\right)} = \mathrm{e}^{-\frac{\theta^2}{4}\left(\frac{u+v}{uv}\right)}; \quad m \in \mathbb{Z}^+$$

to simplify the computations. Note that with the operator $\mathcal{P}$,

$$\tilde{c} \sum_{k=0}^{\infty} \frac{1}{k!} \int_0^1 \int_0^1 \int_{\mathbb{R}} (u+v)^{s+k-1} (uv)^{-3/2} (\mathcal{P}^*)^m \left(\mathrm{e}^{-\frac{\theta^2}{4}\left(\frac{u+v}{uv}\right)} \right) \theta^2 \chi(\theta) h(\theta) \mathrm{d}\theta \mathrm{d}u \mathrm{d}v.$$

We follow the regularization procedure of Gel'fand and Shilov [25] i.e subtracting enough terms of the l^{th} Taylor's expansion of the test function (and compensating) until we are left with a convergent remainder. That is,

$$\begin{aligned}
\Gamma(s)\tilde{\Psi}^1_s(h) &= \tilde{c}\sum_{k=0}^{\infty}\frac{1}{k!}\int_0^1\int_0^1\int_{\mathbb{R}}(u+v)^{s+k-1}(uv)^{-3/2}(\mathcal{P}^*)^m\Big(\mathrm{e}^{-\frac{\theta^2}{4}(\frac{u+v}{uv})}\Big)\theta^2\chi(\theta)\big[h(0)+\theta h'(0)\\
&+ \frac{\theta^2}{2!}h''(0)+\frac{\theta^3}{3!}h'''(0)+\frac{\theta^4}{4!}h^{(iv)}(0)+\cdots+\frac{\theta^{l-1}}{(l-1)!}h^{(l-1)}(0)\big]\mathrm{d}\theta\mathrm{d}u\mathrm{d}v\\
&+ \tilde{c}\sum_{k=0}^{\infty}\frac{1}{k!}\int_0^1\int_0^1\int_{\mathbb{R}}(u+v)^{s+k-1}(uv)^{-3/2}(\mathcal{P}^*)^m\Big(\mathrm{e}^{-\frac{\theta^2}{4}(\frac{u+v}{uv})}\Big)\theta^2\chi(\theta)\big[\chi(\theta)h(\theta)\\
&- \big\{h(0)+\theta h'(0)+\frac{\theta^2}{2!}h''(0)+\frac{\theta^3}{3!}h'''(0)+\frac{\theta^4}{4!}h^{(iv)}(0)+\cdots+\frac{\theta^{l-1}}{(l-1)!}h^{(l-1)}(0)\big\}\big]\mathrm{d}\theta\mathrm{d}u\mathrm{d}v\\
&= \tilde{c}\sum_{k=0}^{\infty}\frac{1}{k!}\int_0^1\int_0^1\int_{\mathbb{R}}(u+v)^{s+k-1}(uv)^{-3/2}\mathrm{e}^{-\frac{\theta^2}{4}(\frac{u+v}{uv})}\theta^2\chi(\theta)\big[h(0)+\theta h'(0)\\
&+ \frac{\theta^2}{2!}h''(0)+\frac{\theta^3}{3!}h'''(0)+\frac{\theta^4}{4!}h^{(iv)}(0)+\cdots+\frac{\theta^{l-1}}{(l-1)!}h^{(l-1)}(0)\big]\mathrm{d}\theta\mathrm{d}u\mathrm{d}v\\
&+ \tilde{c}\sum_{k=0}^{\infty}\frac{1}{k!}\int_0^1\int_0^1\int_{\mathbb{R}}(u+v)^{s+k-1}(uv)^{-3/2}\mathrm{e}^{-\frac{\theta^2}{4}(\frac{u+v}{uv})}(\mathcal{P})^m\big(\theta^2\big[\chi(\theta)h(\theta)\\
&- \big\{h(0)+\theta h'(0)+\frac{\theta^2}{2!}h''(0)+\frac{\theta^3}{3!}h'''(0)+\frac{\theta^4}{4!}h^{(iv)}(0)+\cdots+\frac{\theta^{l-1}}{(l-1)!}h^{(l-1)}(0)\big\}\big]\big)\mathrm{d}\theta\mathrm{d}u\mathrm{d}v.
\end{aligned}$$

So,

$$\begin{aligned}
\Gamma(s)\tilde{\Psi}^1_s(h) &= \tilde{c}\sum_{k=0}^{\infty}\frac{1}{k!}\int_0^1\int_0^1\int_{\mathbb{R}}(u+v)^{s+k-1}(uv)^{-3/2}\mathrm{e}^{-\frac{\theta^2}{4}(\frac{u+v}{uv})}\theta^2\big[h(0)+\theta h'(0)\\
&+ \frac{\theta^2}{2!}h''(0)+\frac{\theta^3}{3!}h'''(0)+\frac{\theta^4}{4!}h^{(iv)}(0)+\cdots+\frac{\theta^{l-1}}{(l-1)!}h^{(l-1)}(0)\big]\mathrm{d}\theta\mathrm{d}u\mathrm{d}v\\
&+ \tilde{c}\sum_{k=0}^{\infty}\frac{(-2)^m}{k!}\int_0^1\int_0^1\int_{\mathbb{R}}(u+v)^{s+k-m-1}(uv)^{-3/2+m}\mathrm{e}^{-\frac{\theta^2}{4}(\frac{u+v}{uv})}\ (\frac{\mathrm{d}}{\mathrm{d}\theta}\frac{1}{\theta})^m\big(\theta^2\big[h(\theta)\\
&- \big\{h(0)+\theta h'(0)+\frac{\theta^2}{2!}h''(0)+\frac{\theta^3}{3!}h'''(0)+\frac{\theta^4}{4!}h^{(iv)}(0)+\cdots+\frac{\theta^{l-1}}{(l-1)!}h^{(l-1)}(0)\big\}\big]\big)\mathrm{d}\theta\mathrm{d}u\mathrm{d}v
\end{aligned} \tag{6.2.20}$$

for any $m, l \in \mathbb{N}_0$.

Again, the second integral of (6.2.20) is holomorphic in s for l large enough. To see this, first take $m = 3$ and observe that

$$\begin{aligned}
&(\frac{\mathrm{d}}{\mathrm{d}\theta}\frac{1}{\theta})^2\frac{\mathrm{d}}{\mathrm{d}\theta}\Big\{\theta\big[h(0)+\theta h'(0)+\frac{\theta^2}{2!}h''(0)+\frac{\theta^3}{3!}h'''(0)\\
&+\frac{\theta^4}{4!}h^{(iv)}(0)+\cdots+\frac{\theta^{l-1}}{(l-1)!}h^{(l-1)}(0)\Big)\Big\}\\
=\ &(\frac{\mathrm{d}}{\mathrm{d}\theta}\frac{1}{\theta})\frac{\mathrm{d}}{\mathrm{d}\theta}\Big\{\frac{1}{\theta}\big[h(0)+2\theta h'(0)+\frac{3\theta^2}{2!}h''(0)+\frac{4\theta^3}{3!}h'''(0)\\
&+\frac{5\theta^4}{4!}h^{(iv)}(0)+\cdots+\frac{l\theta^{l-1}}{(l-1)!}h^{(l-1)}(0)\Big)\Big\}
\end{aligned}$$

will only make sense if $h(0) = 0$ and continuing this way we arrive at a smooth function only if up to $h''(0) = 0$. So,

$$
\begin{aligned}
& \frac{\mathrm{d}}{\mathrm{d}\theta}\Big[\frac{8}{3!}h'''(0) + \frac{15\theta}{4!}h^{(iv)}(0) + \cdots + \frac{l(l-2)(l-4)\theta^{l-4}}{(l-1)!}h^{(l-1)}(0)\Big) \\
= \; & \frac{15}{4!}h^{(iv)}(0) + \cdots + \frac{l(l-2)(l-4)\theta^{l-5}}{(l-1)!}h^{(l-1)}(0) \\
= \; & \sum_{l\geq 5}\frac{l(l-2)(l-4)\theta^{l-5}}{(l-1)!}h^{(l-1)}(0).
\end{aligned}
$$

In general, if h vanishes of order $2m$ in the sense that $h^{(k)}(\theta) = O\big(\theta^{2m-k}\big)$ then $\mathcal{P}^m h$ is well-defined and smooth.

Now for our case $m = 3$, (6.2.20) reduces to

$$
\begin{aligned}
\Gamma(s)\tilde{\Psi}^1_s(h) \;=\; & \tilde{c}\sum_{k=0}^{\infty}\frac{1}{k!}\int_0^1\int_0^1\int_{\mathbb{R}}(u+v)^{s+k-1}(uv)^{-3/2}\mathrm{e}^{-\frac{\theta^2}{4}(\frac{u+v}{uv})}\theta^2\big[h(0) + \theta h'(0) \\
& + \frac{\theta^2}{2!}h''(0) + \frac{\theta^3}{3!}h'''(0) + \frac{\theta^4}{4!}h^{(iv)}(0)\big]\mathrm{d}\theta\mathrm{d}u\mathrm{d}v + \tilde{G}_s
\end{aligned}
$$

where $\tilde{G}_s$ is holomorphic in the strip $\Re(s) > -\frac{5}{2} - k$.

Integrating first in θ, then in u and v we obtain

$$
\begin{aligned}
\tilde{\Psi}^1_s(h) \;=\; & \frac{1}{2\pi^{3/2}\Gamma(s)}\sum_{k=0}^{\infty}\frac{1}{k!}\Big\{\frac{2^{s+k+1/2}-4}{(2s+2k-3)(2s+2k-1)}\cdot h(0) \\
& + \frac{3\times 2^{s+k+1/2}(2s+2k-7)+48}{(2s+2k-5)(2s+2k-3)}\cdot h''(0) \\
& + \frac{30\big(2^{s+k+1/2}[4k^2+8k(s-4)+4s(s-8)+71]\big)}{(2s+2k-7)(2s+2k-5)(8s+8k+12)}\cdot h^{(\mathrm{iv})}(0)\Big\}.
\end{aligned}
\tag{6.2.21}
$$

$\tilde{\Psi}^1_s(h)$ has simple poles exactly at

$$
s = \frac{1}{2} - k, \quad k = 0, 1, 2, \cdots . \tag{6.2.22}
$$

with residues at each pole $s = s_0$ with $s_0 = \frac{1}{2} - k$ given by

$$
\mathrm{Res}_{s=s_0}\tilde{\Psi}^1_s(h) = \lim_{s\to s_0}\frac{1}{\Gamma(s_0)}\big[(s-s_0)f(s,k)\big] \tag{6.2.23}
$$

where

$$\begin{aligned} f(s,k) &= \frac{1}{2\pi^{3/2}}\frac{1}{k!}\Big\{\frac{2^{s+k+1/2}-4}{(2s+2k-3)(2s+2k-1)}\cdot h(0) \\ &+ \frac{3\times 2^{s+k+1/2}(2s+2k-7)+48}{(2s+2k-5)(2s+2k-3)}\cdot h''(0) \\ &+ \frac{30\big(2^{s+k+1/2}[4k^2+8k(s-4)+4s(s-8)+71]\big)}{(2s+2k-7)(2s+2k-5)(8s+8k+12)}\cdot h^{(\mathrm{iv})}(0)\Big\}. \end{aligned} \tag{6.2.24}$$

For example, $s=-\frac{1}{2}$ is a simple pole at exactly $k=1$ and $k=0$ with residue $\frac{1}{16\pi^2}h(0)$ and $\frac{1}{16\pi^2}h''(0)$ respectively. Thus, $FP[\tilde{\Psi}^1(-\frac{1}{2})]=0.00688067$ (to the nearest 8 decimal places).

What we have just proved in this section of the thesis, on combining (6.2.4) and (6.2.21), is the following theorem:

Theorem 6.2.3. *As a distribution,* $\Psi_s(f_1\otimes f_2) := \hat{\Psi}_s(f_1\otimes f_2)+\tilde{\Psi}^1_s$ *has a meromorphic continuation to the whole complex s-plane given by*

$$\begin{aligned} \langle \Psi_s(f_1\otimes f_2),h\rangle &= 2\int_M\int_M \zeta_g(s+1,x,y)f_1(x)f_2(y)\mathrm{d}V_g(x)\mathrm{d}V_g(y) \\ &+ \frac{1}{2\pi^{3/2}\Gamma(s)}\sum_{k=0}^{\infty}\frac{1}{k!}\Big\{\frac{2^{s+k+1/2}-4}{(2s+2k-3)(2s+2k-1)}\cdot h(0) \\ &+ \frac{3\times 2^{s+k+1/2}(2s+2k-7)+48}{(2s+2k-5)(2s+2k-3)}\cdot h''(0) \\ &+ \frac{30\big(2^{s+k+1/2}[4k^2+8k(s-4)+4s(s-8)+71]\big)}{(2s+2k-7)(2s+2k-5)(8s+8k+12)}\cdot h^{(\mathrm{iv})}(0)\Big\} \end{aligned} \tag{6.2.25}$$

where h is a test function. It has simple poles at (6.2.22) with residues (6.2.23). The values of the distribution $\Psi_s(f_1\otimes f_2)$ *at* $s=0,-1,-2,\cdots,-k$*; with* $k\in\mathbb{Z}^+$ *are given by*

$$\Psi_s(f_1\otimes f_2)=\begin{cases} 0 & \text{at} \quad s=0 \\ \int_M f_1(x)f_2(x)\mathrm{d}V_g(x) & \text{at} \quad s=-1 \\ \int_M f_1(x)\Delta_g f_2(x)\mathrm{d}V_g(x) & \text{at} \quad s=-2 \\ \int_M f_1(x)\Delta_g^2 f_2(x)\mathrm{d}V_g(x) & \text{at} \quad s=-3 \\ \int_M f_1(x)\Delta_g^3 f_2(x)\mathrm{d}V_g(x) & \text{at} \quad s=-4 \\ \vdots & \qquad\quad . \end{cases} \tag{6.2.26}$$

6.3. Conclusion

We make a number of concluding remarks. Firstly, we observed that our second variation formula (6.1.1) reduces to a well-known result for the variation of the determinant of the Laplacian on S^3. That is,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\epsilon}\big(\zeta_g^{(2)}(s)\big)\big|_{s=0} &= \frac{1}{16}\mathrm{Tr}\big[\big(\Delta_g\dot{\phi}_0\Delta_g^{-1}\big)^2\big] - \frac{\zeta_{S^3}(1)}{8V}\langle\Delta_g\dot{\phi}_0,\dot{\phi}_0\rangle_{L^3(S^3)}\\ &- \frac{5}{8V}\langle\dot{\phi}_0,\dot{\phi}_0\rangle_{L^3(S^3)}. \qquad (6.3.1)\end{aligned}$$

This is exactly the result of Richardson [47].

Since the Casimir energy is of special significance in Physics, we draw attention to the fact that our formula (6.1.1) can be used to compute its hessian on the n-sphere. From (6.1.1), the second order variation, $\zeta_g^{(2)}(-\frac{1}{2})$, of the Casimir energy, $\zeta_g(-\frac{1}{2})$, of Δ_g on S^n, with respect to the family of volume-preserving conformal metrics $\{g_\epsilon = e^{\phi_\epsilon}g\}$, where g is the round metric is given by

$$\begin{aligned}FP[\zeta_g^{(2)}(-\frac{1}{2})] &= FP[\Psi_{-\frac{3}{2}}\dot{\phi}_0(x)\otimes\dot{\phi}_0(y)] - (1-\frac{n}{2})FP[\Psi_{-\frac{1}{2}}\dot{\phi}_0(x)\otimes\Delta_g(\dot{\phi}_0(y))]\\ &+ \frac{1}{16}(n-2)^2FP[\Psi_{\frac{1}{2}}\Delta_g(\dot{\phi}_0(x))\otimes\Delta_g(\dot{\phi}_0(y))]\\ &+ \frac{1}{16}(n+2)^2\frac{1}{V}\int_M\int_M\dot{\phi}_0(x)\dot{\phi}_0(y)FP[\zeta_{S^n}(-\frac{1}{2},x,y)]dV_g(x)dV_g(y)\\ &+ \frac{1}{16}(n-2)^2FP[\zeta_{S^n}(\frac{1}{2})]\frac{1}{V}\int_M\dot{\phi}_0(x)(\Delta_g\dot{\phi}_0(x))dV_g(x)\\ &- \frac{1}{2}(1-\frac{n}{2})FP[\zeta_{S^n}(-\frac{1}{2})]\frac{1}{V}\int_M(\dot{\phi}_0(x))^2dV_g(x). \qquad (6.3.2)\end{aligned}$$

The finite part scheme (FP) is defined in (4.3.1). One can use that to evaluate $FP[\Psi_s(f_1\otimes f_2)]$, where $\Psi_s(f_1\otimes f_2)$ is given by (6.2.25). Tables (A.1) and (A.2) of the Appendix show values of $\zeta_{S^n}(s)$ and $Z_{S^n}(s)$ respectively for different values of s.

Other special values of s can be computed using the formula. For example, with the aid of Mathematica, we computed

$$\zeta_g^{(2)}(\frac{10}{3}) = 0.0797 \text{ where we choose } \dot{\phi}_0(\theta) = \frac{2}{3}\cos(3\theta) \text{ on } S^3.$$

A numerical check by Strohmaier [55] using 500 eigenvalues confirmed this number.

We hope that the result of this thesis can be used for further numerical and analytical studies of the spectral zeta function and the Casimir energy.

APPENDIX A

Casimir energy of Δ_g and $\Delta_g + \frac{n-1}{2}$ on n-spheres

A.1. Casimir energy of Δ_g and other values of $\mathrm{FP}[\zeta_{S^n}(s)]$

Table A.1: Casimir energy $\mathrm{FP}[\zeta_{S^n}(-\frac{1}{2})]$ and other values.

n	$\mathrm{FP}[\zeta_{S^n}(-\frac{1}{2})]$	$\mathrm{FP}[\zeta_{S^n}(0)]$	$\mathrm{FP}[\zeta_{S^n}(\frac{3}{2})]$	$\mathrm{FP}[\zeta_{S^n}(\frac{1}{2})]$	$\mathrm{FP}[\zeta_{S^n}(1)]$
1	- 0.166667	-1.00000	2.404110	1.154430	3.289870
2	- 0.265096	0	2.070540	-1.754300	0.154430
3	-0.411503	-1.000000	-0.025424	-1.204490	-0.750000
4	-0.150622	-0.595136	0.022700	-1.090330	-0.036680

A.2. Casimir energy of $\Delta_g + \frac{n-1}{2}$ and other values of $\mathrm{FP}[Z_{S^n}(s)]$

Table A.2: Casimir energy $\mathrm{FP}[Z_{S^n}(-\frac{1}{2})]$ and other values.

n	Poles	Residues	$\mathrm{FP}[Z_{S^n}(-\frac{1}{2})]$	$\mathrm{FP}[Z_{S^n}(0)]$	$\mathrm{FP}[Z_{S^n}(\frac{3}{2})]$	$\mathrm{FP}[Z_{S^n}(\frac{1}{2})]$	$\mathrm{FP}[Z_{S^n}(1)]$
1	$\frac{1}{2}$	1	- 0.166667	-1.00000	2.404110	1.154430	3.289870
2	1	1	-0.500000	-0.916667	1.869600	-2.000000	-0.072980
3	$\frac{3}{2}$	$\frac{1}{2}$	-1.000000	-1.083330	0.644934	-1.500000	- 0.422784
4	1, 2	$-\frac{1}{24}, \frac{1}{6}$	-1.500000	-1.005900	-0.707530	-0.666667	- 0.594181

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